GEODESICALLY COMPLETE SPACES WITH AN UPPER CURVATURE BOUND

ALEXANDER LYTCHAK AND KOICHI NAGANO

Abstract. We study geometric and topological properties of locally compact, geodesically complete spaces with an upper curvature bound. We control the size of the singular subsets, discuss homotopical and measure-theoretic stratifications and the regularity of the metric structure on a large part.

1. Introduction

1.1. **Object of investigations.** Metric spaces with one-sided curvature bounds were introduced A.D. Alexandrov in [Ale57]. After the revival of metric geometry in the eighties, properties and applications of such spaces have been investigated from various points of view, we refer to [Bal95], [BH99], [BBI01], [BS07], [AKP16] and the bibliography therein. Starting with [BGP92] the structure theory of locally compact spaces with a lower curvature bound and finite dimension, the so-called Alexandrov spaces, was developed, see [AKP16] for the huge bibliography.

While many basic geometric and topological features are simpler in the case of upper curvature bounds, due to the local uniqueness of geodesics, finer aspects can be much more complicated. Even a compact tree, hence a topologically 1-dimensional space of non-positive curvature, can have infinite Hausdorff dimension and may not contain any kind of "manifold charts". Also the global topological structure of spaces with upper curvature bounds can be much more complicated than in the case of lower curvature bounds: for instance, any finite-dimensional simplicial complex carries a metric with an upper curvature bound [Ber83].

Without additional assumptions it seems impossible to detect some general regular structures beyond a theorem of B. Kleiner, [Kle99], claiming that the topological dimension coincides with the maximal dimension of a Euclidean ball topologically embedded into the space. What one needs is some assumption, in addition to the local compactness, which would provide a close relation of the local geometry to the infinitesimal geometry of the tangent cone. Such a natural assumption is *(local) geodesic completeness*, also known as *geodesic extension property*. This is the requirement that any compact geodesic can be extended as a local geodesic beyond its endpoints. This condition is stable under natural metric operations and can often be a consequence of purely topological assumptions. For example, it is implied by the non-contractibility of all small punctured neighborhoods of all points. Finally,

2010 *Mathematics Subject Classification.* 53C20, 53C21, 53C23.

Key words and phrases. Curvature bounds, rectifiability, metric measure spaces, stratification, DC-functions.

The first author was partially supported by the DFG grants SFB TRR 191 and SPP 2026. The second author was partially supported by JSPS KAKENHI Grant Numbers 26610012, 21740036, 18740023, and by the 2004 JSPS Postdoctoral Fellowships for Research Abroad.

geodesic completeness plays an important role in geometric group theory, see, for instance, [CM09], [GS17].

The present paper is devoted to the description of basic measure-theoretic, homotopic and analytic properties of such spaces, recovering analogues of most results of [BGP92], [Per94] and [OS94]. Applications to topological questions, geometric group theory and sphere theorems will be discussed in forthcoming papers. Results and ideas of preliminary versions of this work which we have circulated in the last 10 years have already been used, for instance in [Kap07], [Kra11], [BK12], [KK17].

There has been one systematic investigation of the theory of geodesically complete spaces with upper curvature bounds by Otsu and Tanoue, [OT99], announced in [Ots97]. Since [OT99] has never been published and is rather difficult to read, we do not rely on it. In fact, we reprove (simplifying and generalizing) all central results from [OT99].

The special case of two-dimensional topological surfaces has been intensively studied, cf. [Res93]. Some results from [Res93], definitely out of reach in the general case, have been generalized to two-dimensional polyhedra in [BB98].

1.2. **Main results.** From now on, we say that a metric space X is *GCBA*, if X is a locally compact, separable, locally geodesically complete space with curvature bound above.

GCBA spaces have indeed many structural similarities with Alexandrov spaces, see Section 5. Any GCBA space X is locally doubling. For any x in X, the tangent space T_xX and the space of directions Σ_xX are again GCBA. Any compact part of any GCBA admits a biLipschitz embedding into a Euclidean space.

The following theorem already appears in [OT99].

Theorem 1.1. *Let X be GCBA. The topological dimension* $\dim(X)$ *of X coincides with the Hausdorff dimension. It equals the maximal dimension of an open subset of X homeomorphic to a Euclidean ball.*

The local dimension might be non-constant on X, as one observes by looking at simplicial complexes. But the local dimension can be understood by looking on the tangent spaces. For $k = 0, 1, 2, ...$, we call *the k-dimensional part of X*, denoted by X^k, the set of all points $x \in X$ with $\dim(T_xX) = k$. In general, X^k is neither open nor closed in X. However, X^k contains large "regular subsets" open in X, as shown in the next result.

Theorem 1.2. *Let X be GCBA. A point x is contained in the k-dimensional part X^k if and only if all sufficiently small balls around x have dimension k. The Hausdorff measure $\mathcal{H}^k$ is locally finite and locally positive on X^k. There is a subset M^k of X^k, which is open in X, dense in X^k and locally biLipschitz equivalent to $\mathbb{R}^k$. Moreover, the complement $\bar{X}^k \setminus M^k$ of M^k in the closure $\bar{X}^k$ of X^k has Hausdorff dimension at most $k-1$.*

We refer to Section 11 for a stronger statement. The open manifold M^k should be thought as the regular k-dimensional part of X. Its finer geometry is described by the following theorem. We refer to [Per94], [KMS01], [AB15] and Section 14 below for a discussion of the notions of DC-functions, DC-manifolds and functions of bounded variations used in the next theorem.

Theorem 1.3. *For $k = 0, 1, ...$, the manifold $M^k \subset X^k$ in Theorem 1.2 can be chosen to satisfy the following property. The manifold M^k has a unique* DC-*atlas* *such that all convex functions on M^k are* DC*-functions with respect to this atlas. The distance in M^k can locally be obtained from a Riemannian metric tensor g, well-defined and continuous on a complement of a subset $S \subset M^k$ of Hausdorff dimension at most $k-2$. The tensor g locally is of bounded variation on M^k.*

The k-dimensional Hausdorff measure is the natural measure on the k-dimensional part X^k of X. We put these measures together and define the *canonical measure* μ_X of the space X to be the sum of the restrictions of $\mathcal{H}^k$ to X^k, thus

$$\mu_X := \sum_{k=0}^{\infty} \mathcal{H}^k \llcorner X^k \,.$$

By Theorem 1.2 and Theorem 1.3, the restriction of μ_X to M^k is (the Riemannian measure) $\mathcal{H}^k$, and μ_X vanishes on the complement of the open submanifold $\bigcup_{k\geq 0} M^k$. The "canonicity" of μ_X is confirmed by the following two theorems.

Theorem 1.4. *Let X be GCBA. The canonical measure is positive and finite on any open relatively compact subset of X.*

The second theorem tells us that the canonical measure is continuous with respect to the Gromov-Hausdorff topology. We formulate it here for compact spaces and refer to Section 12 for the general local statement.

Theorem 1.5. *Let X_l be a sequence of compact GCBA spaces of dimension, curvature and diameter bounded from above and injectivity radius bounded from below by some constants. The total measures $\mu_{X_l}(X_l)$ are bounded from above by a constant if and only if, upon choosing a subsequence, X_l converge to a compact GCBA space X in the Gromov-Hausdorff topology. In this case, the canonical measures of X_l converge to the canonical measure of X.*

Remark 1.1. In the maximal-dimensional case $X = X^k$, parts of Theorems 1.2, 1.3, 1.5 appear in [OT99] and [Nag02].

Having described the regular parts of X we turn to a stratification of the singular parts $X^k \setminus M^k$ neglected by the canonical measure. The following stratification of X, a weak surrogate of the topological stratification, is motivated by the example of skeletons of a simplicial complex.

For a natural number k, we say that a point $x \in X$ is $(k,0)$*-strained* if its tangent space T_xX admits the Euclidean space $\mathbb{R}^k$ as a direct factor.

Theorem 1.6. *Let X be GCBA and $k \in \mathbb{N}$. Then the set of all points in X which are not $(k,0)$-strained is a countable union of subsets, which are biLipschitz equivalent to some compact subsets of $\mathbb{R}^{k-1}$.*

In particular, the set of not $(k,0)$-strained points is *countably $(k-1)$-rectifiable.* For similar rectifiable stratifications on different classes of metric spaces we refer, for instance, to [MN17] and the literature therein.

1.3. **Main tool and further results.** We are going to introduce the main tool of the paper and a more informal description of further central results. The set of $(k,0)$-strained points is usually not open. As in the theory of Alexandrov spaces developed in [BGP92], there is a natural way to open up the condition of being $(k,0)$-strained.

For any $\delta > 0$, we define an open subset $X_{k,\delta}$ of a GCBA space X which consists of (k,δ)-*strained* points. While the definition of being (k,δ)-strained is slightly technical, see Sections 6 and 7, the meaning is very simple.

A point $x \in X$ is (k,δ)-strained, for a small δ, if its tangent space T_xX is sufficiently close to a space which splits off a direct $\mathbb{R}^k$-factor, see Lemma 6.3. In other words, a point $x \in X$ is (k,δ)-strained if and only if there exist k points $p_1, ..., p_k \in X \setminus \{x\}$, close to x, such that the following holds true. The geodesics p_ix meet in x pairwise at an angle close to $\pi/2$ and the possible branching angles of the geodesics p_ix at x are small ("small" and "close" is expressed in terms of δ).

The subsets $X_{k,\delta}$ are open in X and decrease for fixed k and decreasing δ. The set $X_{k,0}$ of $(k,0)$-strained points is the countable intersection $X_{k,0} = \bigcap_{j=1}^{\infty} X_{k,\frac{1}{j}}$.

Each point $x \in X_{k,\delta}$ comes along with natural maps, the so-called (k,δ)-*strainer maps* $F : V \to \mathbb{R}^k$, defined on a neighborhood V of x. This strainer map is the analog of the orthogonal projection onto a face, defined in a neighborhood of that face in a simplicial complex. The coordinates of F are distance functions to points p_i in $X \setminus \{x\}$, for a k-tuple (p_i) as in the above definition of (k,δ)-strained points. In other words, a point $x \in X$ is (k,δ)-strained if and only if there exists a (k,δ)-strainer map F on a neighborhood V of x.

The basic example of a strainer map, responsible for their abundance, is given by the following observation. For any point p in a GCBA space X, and any $\delta > 0$, the distance function to p is a $(1,\delta)$-strainer map on a small punctured neighborhood V of p, Proposition 7.1.

If δ is small enough, any (k,δ)-strainer map F is similar to a Riemannian submersion: the images of small balls are very close to round balls of the same radius, Section 8. Moreover, the fibers of any strainer map are locally contractible, Theorem 1.8. The following technical result is the base for all further investigations on singular sets:

Theorem 1.7. *Let $F : V \to \mathbb{R}^k$ be a (k,δ)-strainer map on a sufficiently small open subset V of a GCBA space X. Then the set $V \setminus X_{k+1,12\cdot\delta}$ is a union of a countable family of compact subsets K_i such that $F : K_i \to F(K_i)$ is biLipschitz.*

The biLipschitz constant of the restrictions $F_i : K_i \to F(K_i)$ and the total measure $\mathcal{H}^k(V \setminus X_{k+1,12\cdot\delta})$ in Theorem 1.7 are bounded in terms of δ and U, see Theorem 10.3 below. The theorem allows, by a reverse induction on k, a good control of the measures of singular sets. We refer to Section 10 for quantitative versions of the volume estimates, leading to proofs (and more precise versions) of Theorems 1.6, 1.2, 1.4.

The strainer map construction is stable under Gromov-Hausdorff limits, Section 7. This provides us the basic tool for the proof of Theorems 1.4 and 1.5.

The relation to Theorem 1.2 is achieved by defining M^k to be the intersection of the k-dimensional part X^k of X with $X_{k,\delta}$ for sufficiently small δ. The DC-atlas on M^k in Theorem 1.3 is provided by the (k,δ)-strainer maps.

Remark 1.2. If $k = \dim(X)$ then $X_{k,\delta}$ is closely related to sets of *not δ'-branch points* used in [OT99] to analyze the regular part of a GCBA space.

From the homotopy point of view, strainer maps are very close to fibrations, as expressed in the following technical result:

Theorem 1.8. *Let $F : V \to \mathbb{R}^k$ be a (k, δ)-strainer map with $\delta \leq \frac{1}{20 \cdot k}$. Then, for any compact subset V' of V, there is some $\epsilon > 0$ with the following property. For any $x \in V'$, any $0 < r < \epsilon$ and any, possibly degenerated, rectangular box $Q \subset \mathbb{R}^k$ containing $F(x)$, the open ball of radius r around x in $F^{-1}(Q)$ is contractible.*

Here a *rectangular box Q* denotes a direct product of k intervals $\Pi_{i=1}^k [a_i^-, a_i^+]$ for some pairs of real numbers $a_i^- \leq a_i^+$. In the most degenerate case, $a_i^- = a_i^+$ for each i, the preimage $F^{-1}(Q)$ is just the fiber of F through the point x.

Using Theorem 1.8 in the degenerate case, we can apply general results from [Pet90] and obtain homotopical stability of fibers. We refer to Section 13 for exact results and state here the following illuminating special case, originating from the convergence of the rescaling of the given space to the tangent cone at a point.

Theorem 1.9. *Let X be GCBA. For each point $x \in X$ there is some $r_x > 0$ such that for all $r < r_x$ the metric sphere $\partial B_r(x)$ of radius r around x is homotopy equivalent to the space of directions $\Sigma_x X$.*

If the metric sphere in Theorem 1.9 is replaced by a punctured ball, the result is simpler and the extendibility of geodesics does not need to be assumed. This has been observed by Kleiner (unpublished) and appeared in [Kra11]. The term homotopy equivalent in Theorem 1.9 cannot be replaced by "homeomorphic" (Example 15.2 below), as it were the case for Alexandrov spaces, [Per91], [Kap07]. This example shows that there is no hope of obtaining a local conicality theorem or topological stability as in [Per91].

1.4. **Outlook.** In the continuation [LN18] of this paper, we prove, starting with Theorem 1.7, that the local conicality theorem holds for GCBA spaces which are homology manifolds. As a consequence, GCBA spaces which arise as limits of Riemannian manifolds can be very well understood, similarly to [Kap02].

Many natural questions about finer structure of GCBA spaces are barely touched in this paper and will be hopefully addressed elsewhere, using tools developed here. It seems interesting and possible to obtain a good description of singular strata of codimension 1, the analogs of the boundary in Alexandrov spaces. A related much more difficult problem is the existence of a Liouville measure on GCBA spaces and a geodesic flow preserving it, as in [BB95]. The stability of length of DC-curves, proved in Proposition 14.6, can be transferred to Alexandrov geometry and used to simplify [Li15] and prove its analogs for GCBA spaces.

1.5. **Acknowledgments.** Most results presented in the paper have been obtained and presented in talks more than 10 years ago. The authors would like to express their gratitude to people, who have showed interest in our results and whose interest was finally responsible for the finalization of the paper. In particular, we would like to thank Jérôme Bertrand, Pierre-Emmanuel Caprace, Ralf Gramlich, Karsten Grove, Vitali Kapovitch, Urs Lang, Anton Petrunin, Takao Yamaguchi.

2. Preliminaries

2.1. **Spaces and maps.** [BH99], [BBI01] and [Bal04] are general references for this section. By d we denote distances in metric spaces. For a subset A of a metric space X and $r > 0$, we denote by $B_r(A)$ the *open tubular neighborhood of radius r around A*, hence the set of all points with distance less than r from A. By $r \cdot X$

we denote the set X with the metric rescaled by r. A space is *proper* if its closed bounded subsets are compact.

A subset of a metric space is called *r-separated* if its elements have pairwise distances at least r. A metric space X is *doubling* (more precisely, L-doubling) if no ball of radius r in X has an $(r/2)$-separated subset with more than L elements. Equivalently, any r-ball is covered by a uniform number of balls of radius $r/2$.

The *length of a curve* γ in a metric space is denoted by $\ell(\gamma)$. A *geodesic* is an isometric embedding of an interval. A *triangle* is a union of three geodesics connecting three points. A *local geodesic* is a curve $\gamma : I \to X$ in a metric space X defined on an interval I, such that the restriction of γ to a small neighborhood of any $t \in I$ is a geodesic. X is *a geodesic metric space* if any pair of points of X is connected by a geodesic.

A map $F : X \to Y$ between metric spaces is called *L-Lipschitz* if $d(F(x), F(\bar{x})) \leq L \cdot d(x, \bar{x})$, for all $x, \bar{x} \in X$. A map $F : X \to Y$ is called an *L-biLipschitz embedding* if for all $x, \bar{x} \in X$ one has $\frac{1}{L} \cdot d(x, \bar{x}) \leq d(F(x), F(\bar{x})) \leq L \cdot d(x, \bar{x})$.

Let Z be a metric space and $C > 0$. A continuous map $F : Z \to Y$ is called *C-open* if the following condition holds. For any $z \in Z$ and any $r > 0$ such that the closed ball $\bar{B}_{Cr}(z)$ is complete, we have the inclusion $B_r(F(z)) \subset F(B_{Cr}(z))$.

A function $f : X \to \mathbb{R}$ on a metric space X is *convex* if its restriction $f \circ \gamma$ to any geodesic $\gamma : I \to X$ is a convex function on the interval I.

2.2. **Convergence.** On the set of isometry classes of compact metric spaces we will use the Gromov-Hausdorff distance. By an abuse of definition we will identify spaces and their isometry classes. Whenever spaces X, Y at Hausdorff distance smaller δ appear, we will implicitly assume that isometric embeddings of $f : X \to Z$ and $g : Y \to Z$ into some metric space Z are fixed such that the Hausdorff distance between $f(X)$ and $g(Y)$ is smaller than 2δ.

On the set of isometry classes of pointed proper metric spaces we will consider the pointed Gromov-Hausdorff topology (abbreviated as GH-topology), and denote by $(X_l, x_l) \to (X, x)$ a convergent sequence. Each sequence of doubling spaces with a uniform doubling constant has a subsequence converging in the GH-topology. The limit space is doubling with the same doubling constant.

It is often simpler to work with *ultralimits* instead of GH-limits. There are several advantages: the ultralimits are always defined, the limit object is a space and not just an "isometry class" and there is no need to consider subsequences. We refer for details to [AKP16]. We fix an ultrafilter ω and denote by $\lim_\omega (X_i, x_i)$ the ω-ultralimit of pointed metric spaces (X_i, x_i). For C-Lipschitz maps $f_l : (X_l, x_l) \to (Y_l, y_l)$ we will denote by $\lim_\omega f_l$ the ultralimit map $f : (X, x) \to (Y, y)$.

Whenever proper spaces (X_l, x_l) converge in the GH-topology to (X, x), the ultralimit $\lim_\omega (X_l, x_l)$ is (in the isometry class of) (X, x). For the needs of the present paper it is sufficient to work with GH-limits, thus readers not familiar with ultralimits may always choose an appropriate subsequence and consider the corresponding GH-limit.

Let $f_l : (X_l, x_l) \to (Y_l, y_l)$ be C-Lipschitz and C-open maps and assume that the spaces X_l are complete. Then the ultralimit $f = \lim_\omega f_l : (X, x) \to (Y, y)$ is C-Lipschitz and C-open. Moreover, for $\Pi_l := f_l^{-1}(y_l)$ the ultralimit $\lim_\omega \Pi_l \subset (X, x)$ coincides with the fiber $\Pi := f^{-1}(y)$.

3. Spaces with an upper curvature bound

3.1. **Definitions and notations.** For $\kappa \in \mathbb{R}$, let R_κ be the diameter of the complete simply connected surface M^2_κ of constant curvature κ. A complete metric space is called CAT(κ) if any pair of its points with distance $< R_\kappa$ is connected by a geodesic and if all triangles with perimeter $< 2R_\kappa$ are not thicker than the comparison triangle in M^2_κ. A complete metric space is called a *space with an upper curvature bound* κ if any point has a CAT(κ) neighborhood. We refer to [BH99], [BBI01], [Bal04] for the basic facts about such spaces.

Any CAT(κ) space is CAT(κ') for $\kappa' \geq \kappa$. By rescaling we may always assume that the curvature bound κ equals 1. Then $R_\kappa = \pi$.

For any CAT(κ) space X, the angle between each pair of geodesics starting at the same point is well defined. The *space of directions* $\Sigma_x = \Sigma_x X$ at each point x, which is the completion of the set of geodesic directions equipped with the angle metric, is a CAT(1) space. The Euclidean cone over Σ_x is a CAT(0) space. It is denoted by $T_x = T_x X$ and called the *tangent space* at x of X. The element w in T_x will be written as $w = tv = (t, v) \in T_x = [0,\infty) \times \Sigma_x / \{0\} \times \Sigma_x$, and its *norm* is defined as $|w| = |tv| := t$.

Let x, y, z be three points at pairwise distance $< R_\kappa$ in a $CAT(\kappa)$ space X. Whenever $x \neq y$, the geodesic between x and y is unique and will be denoted by xy. Its starting direction in Σ_x will be denoted by $(xy)'$ if no confusion is possible. If $y, z \neq x$ the angle at x between xy and xz, hence the distance in Σ_x between $(xy)'$ and $(xz)'$ will be denoted by $\angle yxz$.

For $r < R < R_\kappa/2$ we consider the *contraction map* $c_{R,r} \colon B_R(x) \to B_r(x)$ centered at x, that sends the point y to the point $\gamma(\frac{r}{R} \cdot d(x,y))$, where γ is the unique geodesic from x to y. Due to the CAT(κ) property, the map $c_{R,r}$ is $(2 \cdot \frac{r}{R})$-Lipschitz.

We define the *logarithmic map* $\log_x \colon B_{\frac{1}{2}R_\kappa}(x) \to T_x$ by $\log_x(x) = 0$ and by sending any $y \neq x$ to $tv \in T_x$, with $t = d(x,y)$ and $v = (xy)'$. The CAT(κ) property implies that $\log_x$ is 2-Lipschitz.

Remark 3.1. Indeed, if $\kappa, \epsilon > 0$ are fixed, then for all sufficiently small $r < R$ any contraction map $C_{R,r}$ is $(1+\epsilon) \cdot \frac{r}{R}$-Lipschitz and the logarithmic map log_x : $B_r(x) \to T_x$ is $(1+\epsilon)$-Lipschitz.

3.2. **Basic topological properties.** On spaces with an upper curvature bound, there is a notion of *geometric dimension* invented by Kleiner [Kle99]. It assigns to a discrete set the number 0 and is defined inductively by $\dim X = 1 + \sup_{x\in X} \dim \Sigma_x$. The geometric dimension $\dim X$ is equal to the topological dimension if X is separable [Kle99].

Convexity of all small balls in spaces with upper curvature bounds imply that any space X with an upper curvature bound is an absolute neighborhood retract, see [Ont05], [Kra11]. In particular, each open subset of X is homotopy equivalent to a simplicial complex.

For any CAT(κ) space X the map $\log_x \colon (B_{\frac{1}{2}R_\kappa}(x) \backslash \{x\}) \to T_x \backslash \{0\}$ is a homotopy equivalence, [Kra11]. Note that the embedding $\Sigma_x \to T_x \setminus \{0\}$ is a homotopy equivalence as well.

3.3. **Convergence and semi-continuity.** Let (X_i, x_i) be a sequence of pointed CAT(κ_i) spaces with $\lim_{i\to\infty} \kappa_i = \kappa$. Then $(X,x) = \lim_\omega (X_i, x_i)$ is CAT(κ),

[BH99]. Moreover, $\lim_\omega \dim(X_i, x_i) \geq \dim(X, x)$, [Lyt05b, Lemma 11.1], thus, the geometric dimension does not increase under convergence.

Let $y_i, z_i \in X_i$ be points such that $d(x_i, y_i), d(x_i, z_i)$ are uniformly bounded from above by a constant smaller than R_κ and from below by a positive constant. Then the points $y = \lim_\omega(y_i)$ and $z = \lim_\omega(z_i)$ are well-defined in X and the angles $\angle y_i x_i z_i$ and $\angle yxz$ are well-defined. We have, cf. [BH99], [Lyt05b, p.748]:

$$\limsup \angle y_i x_i z_i \leq \angle yxz \ .$$

4. Geodesic extension property

4.1. **Definition.** Let X be a space curvature $\leq \kappa$. We call X *locally geodesically complete* if any local geodesic $\gamma : [a,b] \to X$, for any $a < b$, extends as a local geodesic to a larger interval $[a-\epsilon, b+\epsilon]$. If any local geodesic in X can be extended to a local geodesic defined on the real line then X is called *geodesically complete.*

In [BH99] local geodesic completeness is called the *geodesic extension property.*

For any local geodesic $\gamma : [a,b] \to X$ in a space X with curvature $\leq \kappa$, we can use Zorn's lemma to find an extension of γ to a local geodesic $\gamma : I \to X$ defined on a maximal interval $I \subset \mathbb{R}$. If X is locally geodesically complete, then such a maximal interval I is open in $\mathbb{R}$. Assume that $t = \sup(I)$ is finite. For any $t_i \in I$ converging to t, the sequence $\gamma(t_i)$ is a Cauchy sequence in X. If $\gamma(t_i)$ converge to a point x in X then the final part $\gamma : [t-\epsilon, t) \to X$ is contained in a $\mathrm{CAT}(\kappa)$ neighborhood U of x. Since local geodesics of length $\leq R_\kappa$ in U are geodesics, the unique extension of γ by $\gamma(t) = x$ is a geodesic on $\gamma : [t-\epsilon, t] \to X$. But then, contrary to our assumption, I is not a maximal interval of definition of γ. Thus we have shown that $\gamma(t_i)$ cannot converge in X. From this we conclude:

Lemma 4.1. *Let X be a locally geodesically complete space with an upper curvature bound. Let the closed ball $\bar{B}_r(x)$ be complete. Then any local geodesic γ in X with $\gamma(0) = x$ can be extended to a local geodesic $\gamma : (-t^-, t^+) \to X$ with $t^\pm > r$.*

Proof. Extend γ to a maximal interval of definition $I = (-t^-, t^+)$. If $t^+ \leq r$ then $\gamma([0, t^+)) \subset \bar{B}_r(x)$. Thus $\lim_{t_i \to t^+} \gamma(t_i)$ exists in $\bar{B}_r(x)$ in contradiction to the observation preceding the lemma. Thus, $t^+ > r$. Similarly, $t^- > r$. $\square$

In particular, a complete, locally geodesically complete space is geodesically complete.

Let the space X with curvature at most κ be locally geodesically complete. Let $x \in X$ be arbitrary. Then some closed ball $K = \bar{B}_{2r}(x)$ with $4r < R_\kappa$ is $\mathrm{CAT}(\kappa)$. Note that any geodesic in K is uniquely determined by its endpoints and any local geodesic in K is a geodesic. Due to Lemma 4.1, for any $y \in B_r(x)$ any geodesic γ starting in y can be extended inside K to a geodesic $\gamma : [-r, r] \to X$.

4.2. **Examples.** The following example, shows that (local) geodesic completeness without further compactness assumption is not of much use.

Example 4.1. Starting with any $\mathrm{CAT}(\kappa)$ space X we glue to all points $x \in X$ a line $\mathbb{R} = \mathbb{R}_x$. The arising "hairy" space $\hat{X}$ is still $\mathrm{CAT}(\kappa)$, geodesically complete and contains X as a convex subset.

Let X be a Euclidean simplical complex with a finite number of isometry classes of simplices and curvature at most 0. Then X is locally geodesically complete if and only if any face of any maximal simplex is a face of at least one other simplex.

For any CAT(1) space Σ, the Euclidean cone $C\Sigma$ over Σ is geodesically complete if and only if Σ is geodesically complete and not a singleton. The direct product of two CAT(0) spaces and the spherical join of two CAT(1) spaces is geodesically complete if and only if it is true for both factors and, in the second case, none of the spherical factors is a point.

There is a simple topological condition implying local geodesic completeness, cf. [LS07, Theorem 1.5]. Namely, if X is a space with an upper curvature bound and if at all points $x \in X$ the local homology $H_*(X, X \setminus \{x\})$ does not vanish then X is locally geodesically complete. In particular, any space with an upper curvature bound which is a (homology) manifold is locally geodesically complete.

Geodesic completeness is preserved under gluings: Let X_1, X_2 be two spaces of curvature $\leq \kappa$ and let $A_i \subset X_i$ be locally convex and are isometric to each other. The space X which arises from gluing of X_1 and X_2 along A_i has curvature $\leq \kappa$, by a theorem of Reshetnyak. It is a direct consequence of the structure of geodesics in X, that if X_1 and X_2 are (locally) geodesically complete then so is X.

Finally, geodesic completeness is preserved under ultralimits:

Example 4.2. Let (X_i, x_i) be locally geodesically complete spaces with curvature $\leq \kappa_i$. Assume that the balls $\bar{B}_{r_i}(x_i) \subset X_i$ are CAT(κ_i), with $2r_i \leq R_{\kappa_i}$. Assume, finally, that $\lim_{i\to\infty}(\kappa_i) = \kappa$ and $\lim_{i\to\infty} r_i > r > 0$. Consider the ultralimit $(X, x) = \lim_\omega(X_i, x_i)$. Then the closed ball $\bar{B}_r(x) \subset X$ is CAT(κ) as an ultralimit of CAT(κ_i) spaces. We claim, that the open ball $B_r(x)$ is locally geodesically complete. Indeed, any geodesic γ in $B_r(x)$ is an ultralimit of the corresponding geodesics in $B_r(x_i)$. Since the latter admit extensions of a uniform size to longer geodesics we obtain an extension of γ as the corresponding ultralimit.

5. Locally compact spaces with upper curvature bounds and extendable geodesics

5.1. **GCBA spaces and their tiny balls.** Now we turn to the main subject of this paper, the structure of locally compact, locally geodesically complete, separable spaces with upper curvature bounds. As in the introduction, we will denote such spaces as GCBA. Then any open subset of a GCBA space is GCBA as well.

Let X be GCBA of curvature $\leq \kappa$. We say that an open ball $U = B_{r_0}(x_0)$ in X is a *tiny ball* if the following holds true. The radius r_0 of U is at most $\min\{1, \frac{1}{100} \cdot R_\kappa\}$ and the closed ball $\tilde{U} = \bar{B}_{10\cdot r_0}(x_0)$ with the same center and radius $10 \cdot r_0$ is compact.

As seen at the end of Subsection 4.1, any geodesic γ with $\gamma(0) \in U$ can be extended to a geodesic $\gamma : [-9 \cdot r_0, 9 \cdot r_0] \to \tilde{U} \subset X$. For any ball $B_r(x)$ contained in $\tilde{U}$ and any $r' < r$ the contraction map $c_{r,r'} : B_r(x) \to B_{r'}(x)$ is surjective.

Any point in X is contained in a tiny ball. Since X is separable, we can write it as a countable union of tiny balls. Any relatively compact subset of X is covered by finitely many tiny balls. All theorems from the introduction will follow once we prove them for all tiny balls in X.

5.2. **Doubling property.** Tiny balls turn out to be doubling.

Proposition 5.1. *Let $U = B_{r_0}(x_0)$ be a tiny ball of radius r_0 in a GCBA space. Let N denote the maximal number of r_0-separated points in the compact ball $\tilde{U} = \bar{B}_{10\cdot r_0}(x_0)$. Then $\bar{B}_{5\cdot r_0}(x_0)$ is N-doubling.*

Proof. It suffices to prove that for any $t > 0$ and any $y \in \bar{B}_{5\cdot r_0}(x_0)$, any $\frac{t}{2}$-separated subset S of $\bar{B}_t(y)$ has at most N elements.

The statement is clear for $t \geq 2r_0$, by the definition of N.

For $t < 2r_0$, consider the $\frac{t}{2\cdot r_0}$-Lipschitz map $c_{4\cdot r_0, t} \colon B_{4\cdot r_0}(y) \to B_t(y)$. The map is surjective, since $B_{4\cdot r_0}(y)$ is contained in $\tilde{U}$. Hence, taking arbitrary preimages of points in S under this contraction map, we obtain an r_0-separated subset of $B_{4\cdot r_0}(y)$ with as many elements as in S. Since $B_{8\cdot r_0}(y) \subset \tilde{U}$, we deduce that S has at most N elements. $\square$

Definition 5.1. For a tiny ball $U = B_{r_0}(x_0)$ of a GCBA space, we say that U has *size bounded by* N if $\bar{B}_{5\cdot r_0}(x_0)$ is N-doubling.

Let X be GCBA. Let $U \subset X$ be a tiny ball of radius r_0 and size bounded by N. Then any open ball contained in U is a tiny ball in X of size bounded by N. Moreover, for any point $x \in U$, the ball $B_s(x)$ is a tiny ball of size bounded by N, for any $s \leq \frac{r_0}{2}$. Finally, for every $s \leq \frac{1}{r_0}$ the rescaled space $s \cdot U$ is a tiny ball in the GCBA space $s \cdot X$ with size bounded by the same N.

5.3. **Distance maps and a biLipschitz embedding.** Let $U \subset \tilde{U}$ be a tiny ball of radius r_0 and size bounded by N as above.

For $p \in \tilde{U}$ we denote by $d_p : \tilde{U} \to \mathbb{R}$ the *distance function* $d_p(x) = d(p, x)$. The function d_p is 1-Lipschitz and convex on $\tilde{U}$. For any m-tuple of points $(p_1, ..., p_m)$ in $\tilde{U}$ the *distance map* defined by the m-tuple is the map $F : \tilde{U} \to \mathbb{R}^m$ with coordinates $f_i(x) = d_{p_i}(x)$. Since any distance function d_{p_i} is 1-Lipschitz, any distance map $F : \tilde{U} \to \mathbb{R}^m$ is $\sqrt{m}$-Lipschitz. Moreover, if we equip $\mathbb{R}^m$ with the *sup-norm*, then F becomes a 1-Lipschitz map $F : \tilde{U} \to \mathbb{R}^m_\infty$.

Let $\gamma : [a, b] \to \tilde{U}$ be a geodesic starting at $x = \gamma(a)$. Let $p \neq x$ and $f = d_p$. The derivative of $f \circ \gamma$ at the a is computed by the *first formula of variation* $(f \circ \gamma)'(a) = -\cos(\alpha)$, where $\alpha \in [0, \pi]$ denotes the angle between γ and the geodesic xp. In particular, $|(f \circ \gamma)'(a)| < \delta$ if $|\alpha - \frac{\pi}{2}| < \delta$. Moreover, $(f \circ \gamma)'(a) > 1 - \delta$ if $\alpha > \pi - \delta$ and $(f \circ \gamma)'(a) < -1 + \delta$ if $\alpha < \delta$.

Denote by $\mathcal{A} \subset \tilde{U}$ the compact subset of all points $p \in \tilde{U}$ with $r_0 = d(p, U)$, thus a distance sphere with radius $2r_0$ around the center of U. Due to the assumptions on r_0 and the curvature bound, for all $\delta > 0$ the following holds true. For every pair of points $p, q \in \mathcal{A}$ with $d(p, q) \leq \delta \cdot r_0$ and any $x \in U$ we have $\angle pxq < \delta$.

For all $\delta > 0$ we choose a maximal $\delta \cdot r_0$-separated subset $\mathcal{A}_\delta$ in $\mathcal{A}$. Due to the doubling property, the number of elements in $\mathcal{A}_\delta$ is bounded by some $m = m(N, \delta)$. Now we obtain:

Proposition 5.2. *For every $\delta > 0$ there exists some natural $m = m(N, \delta)$ and m points $p_1, ..., p_m \in \tilde{U}$ such that the corresponding distance map $F : \bar{U} \to \mathbb{R}^m_\infty$ is a $(1 + \delta)$-biLipschitz embedding. Here $\mathbb{R}^m_\infty$ denotes $\mathbb{R}^m$ with the sup-norm.*

Proof. Consider as above the maximal $\epsilon \cdot r_0$-separated subset $\mathcal{A}_\delta = \{p_1, ..., p_m\}$ in the distance sphere $\mathcal{A}$ and note, that m is bounded in terms of N and δ.

Consider the corresponding distance map $F : \bar{U} \to \mathbb{R}^m_\infty$. As all distance maps, $F : \bar{U} \to \mathbb{R}^m_\infty$ is 1-Lipschitz.

Given arbitrary $x, y \in \bar{U}$, we extend xy beyond y to a point $q \in \mathcal{A}$. We find some $p_j \in \mathcal{A}_\delta$ such that $d(p_j, q) \leq \delta \cdot r_0$ hence $\angle p_j y q < \delta$. Then $\angle p_j y x > \pi - \delta$.

From the first formula of variation the derivative of the distance function d_{p_j} on the geodesic yx at y is at least $(1-\delta)$.

Then $d(p_j,x) - d(p_j,y) \geq (1+\delta)\cdot d(x,y)$, due to the convexity of d_{p_j}. Hence, $|F(x)-F(y)|_\infty \geq (1+\delta)\cdot d(x,y)$. This finishes the proof. □

We let $\delta = 1$ in Lemma 5.2 and obtain a refinement of Proposition 5.1:

Corollary 5.3. *For some $n_0 = n_0(N)$, there exists a biLipschitz embedding $F : U \to \mathbb{R}^{n_0}$. The Hausdorff and the topological dimensions of U are at most n_0.*

5.4. **Almost Euclidean triangles.** The diameter of $\tilde U$ is smaller than $\frac{1}{4}R_\kappa$. Hence $\angle xyz + \angle yxz \leq \pi$ for any triple of pairwise distinct point $x,y,z \in \tilde U$. If, moreover, $d(x,z) = d(y,z)$ then $\angle xyz < \frac{\pi}{2}$.

The following lemma shows that triangles in U with one side fixed and the other side sufficiently small have almost Euclidean angles.

Lemma 5.4. *Let $x \in U$ and $p \in \tilde U$ be arbitrary. For any $\epsilon > 0$ there is some $\delta > 0$ such that for any $y \in B_\delta(x)$ we have $\angle pxy + \angle pyx > \pi - \epsilon$.*

Proof. Assume the contrary and take a sequence y_i converging to x with $\angle pxy_i + \angle py_i x \leq \pi - \epsilon$. Extend the geodesic xy_i beyond y_i up to a point z_i with $d(x,z_i) = r_0$. Choosing a subsequence we may assume that z_i converges to a point z. The semicontinuity of angles gives us

$$\lim \angle pxz_i = \angle pxz \geq \limsup \angle py_i z_i \,.$$

This contradicts $\angle py_i x \geq \pi - \angle py_i z_i$ and finishes the proof. □

5.5. **Tangent spaces and spaces of directions.** Let us fix an arbitrary point $x \in U$ and an arbitrary $r < 5r_0$. Then the logarithmic map $\log_x : B_r(x) \to T_x$ has the ball $B_r(0) \subset T_x$ as its image.

Indeed, we have $d(0, \log_x(y)) = |\log_x(y)| = d(x,y)$, thus one inclusion is clear. On the other hand, consider any $v \in \Sigma_x$ and write it as a limit of starting direction $(xy_i)'$ of geodesics. We extend xy_i to geodesics xz_i of length r and find a subsequence converging to a geodesic xz. The image of the geodesic xz under $\log_x$ is exactly the set of all tv, with $0 \leq t \leq r$. Thus, $B_r(0) \subset \log_x(B_r(x))$.

The restriction of $\log_x$ to small balls is an almost isometry:

Lemma 5.5. *For any $\epsilon > 0$ there is some $\delta > 0$ (depending on the point x), such that for all $r < \delta$ and all $y_1, y_2 \in B_r(x)$ we have*

$$|d(y_1,y_2) - d(\log_x(y_1), \log_x(y_2))| \leq \epsilon \cdot r \,. \tag{5.1}$$

Proof. We find some finite $\epsilon \cdot r_0$-dense subset $\{p_1, ..., p_m\}$ in $B_{r_0}(x)$. Then the union of geodesics xp_i is $2\epsilon \cdot r$ dense in $B_r(x)$.

By the definition of angles, we find a sufficiently small $\delta > 0$ such that (5.1) holds true for all y_1, y_2 which lie on the union of the finitely many geodesics xp_i. Since the logarithmic map is 2-Lipschitz we conclude (5.1) with ϵ replaced by 9ϵ, for arbitrary $y_1, y_2 \in B_r(x)$. □

Thus, the logarithmic map provides an almost isometry between rescaled small balls in X and corresponding balls in the tangent space. From the definition of GH-convergence this implies:

Corollary 5.6. *For any sequence $t_i \to 0$ the rescaled spaces $(\frac{1}{t_i}\bar U, x)$ converge in the pointed GH-topology to the tangent space $(T_x, 0)$.*

From the stability of the geodesic extension property discussed in Subsection 4.2 and the doubling property of U we see:

Corollary 5.7. *For any $x \in U$ the tangent space T_x is an N-doubling, geodesically complete* CAT(0) *space.*

We derive:

Corollary 5.8. *For any $x \in U$ the space of directions Σ_x is a compact, geodesically complete* CAT(1) *space. Σ_x is N_1-doubling with N_1 depending on N. If U is not a singleton then Σ_x has diameter π.*

Proof. If U is a singleton then Σ_x is empty. Otherwise, there exists at least one geodesic passing through x, hence Σ_x is not empty and has diameter at least π. By the definition of the angle metric, the diameter of Σ_x cannot be larger than π. The doubling property follows from Corollary 5.7, since T_x is the Euclidean cone over Σ_x and the embedding of Σ_x into T_x is 2-biLipschitz. □

5.6. **Precompactness and setting for convergence.** A bound on the numbers and sizes of small balls in a covering is equivalent to precompactness in the GH-topology, once the bounds on the curvature and injectivity radius are fixed:

Proposition 5.9. *Let $\kappa, t > 0$ be fixed. Let X_l be GCBA and assume that in any X_l any ball of radius t is* CAT(κ)*. Let $K_l \subset X_l$ be compact and connected. Then the following are equivalent:*

(1) *There is $r > 0$ such that closed tubular neighborhoods $\bar{B}_r(K_l)$ are uniformly compact, i.e., each one is compact and they constitute a precompact set in the Gromov-Hausdorff topology.*
(2) *There are $r, N > 0$, such that the closed tubular neighborhoods $\bar{B}_r(K_l)$ are compact, have diameter $\leq N$ and are N-doubling.*
(3) *There are some $r, N > 0$ and a covering of $\bar{B}_r(K_l)$ by at most N tiny balls of size bounded by N.*

Proof. The implication (2) to (1) is clear.

Under the assumptions of (3), $\bar{B}_r(K_l)$ is N^3-doubling by the definition of size. Moreover, the diameter of $\bar{B}_r(K)$ can be at most $2 \cdot N$, since the diameter of any tiny ball is at most 2 and $\bar{B}_r(K)$ is connected, at least for all $r \leq t$. Thus (3) implies (2).

Assume (1). We find some $s < \frac{r}{40}$ such that for any $x \in K_l$ the open ball $B_{2s}(x)$ is tiny in X_l. By the assumption of uniform compactness, there is some $N > 0$ such that the maximal s-separated subset in $\bar{B}_r(K_l)$ has at most N elements. Hence, we can cover $\bar{B}_r(K_l)$ by at most N open balls of radius $2s$ and each of these tiny balls has size at most N, due to Proposition 5.1. This implies (3). □

As a consequence of Example 4.2, we see:

Corollary 5.10. *Under the equivalent conditions of Proposition 5.9, the compact subsets $K_l \subset X_l$ converge, upon choosing a subsequence, in the GH-topology to a compact subset K of a GCBA space X. There is some $s > 0$ such that the compact neighborhoods $\bar{B}_{10 \cdot s}(K_l) \subset X_l$ converge in the GH-topology to the compact neighborhood $\bar{B}_{10 \cdot s}(K) \subset X$.*

We can choose s in Corollary 5.10 to be much smaller than 1 and than the injectivity radius t. Then all balls with radius s centered in K_l or in K are tiny balls in X_l and X respectively. Therefore, in all local questions concerning convergence, we can restrict ourselves to a convergence of tiny balls in some GCBA spaces to a tiny ball in some other GCBA space, as described in the following.

Definition 5.2. As the *standard setting for convergence* we will denote the following situation. The sequence $U_l \subset \tilde{U}_l$ of tiny balls in GCBA spaces X_l have the same radius r_0 and the same bound on the size N. The sequence $\tilde{U}_l$ converges in the GH-topology to a compact ball $\tilde{U}$ of radius $10 \cdot r_0$ in a GCBA space X. The closures $\bar{U}_l$ converge to the closure $\bar{U}$ of a tiny ball $U \subset \tilde{U}$ of radius r_0 in X.

5.7. **Semicontinuity of tangent spaces.** For GCBA spaces, semicontinuity of angles discussed in Subsection 3.3 has the following nice formulation.

Lemma 5.11. *Under the standard setting of the convergence as in Definition 5.2, let $x_l \in U_l \subset \tilde{U}_l$ converge to $x \in U \subset \tilde{U}$. Then the sequence of the spaces of directions $\Sigma_{x_l} U_l$ is precompact in the GH-topology. For every limit space Σ' of this sequence there exists a surjective 1-Lipschitz map $P : \Sigma_x U \to \Sigma'$.*

Proof. Corollary 5.8 implies that the sequence $\Sigma_{x_l} U_l$ is uniformly doubling, hence precompact.

In order to prove the second statement, we may replace our sequence U_l by a subsequence and assume that Σ_{x_l} converge to Σ'.

For any direction $v \in \Sigma_x U$ we take a point $y \in \tilde{U}$ with $(xy)' = v$ and $d(x, y) = r_0$. Consider a sequence $y_l \in \tilde{U}_l$ converging to y and put $v_l := (x_l y_l)' \in \Sigma_{x_l} U_l$. Then we choose the limit point $w = \lim_\omega (v_l) \in \Sigma'$ of the sequence (v_l) and set $P(v) := w$.

The semi-continuity of angles discussed in Subsection 3.3 is exactly the statement that the map P is 1-Lipschitz. The surjectivity of P follows from the construction and the fact that any direction $w \in \Sigma'$ is a limit direction of some directions $v_l \in \Sigma_{x_l} U_l$, which are starting directions of geodesics of length r_0 in U_l. $\square$

6. Almost suspensions

6.1. **Spherical and almost spherical points.** In this section let Σ be a compact, geodesically complete CAT(1) space with diameter π. Note, that any space of directions Σ_x of any GCBA space X satisfies this assumption by Corollary 5.8.

Definition 6.1. Let Σ be a compact CAT(1) space which is GCBA and has diameter π. For $v \in \Sigma$ an *antipode* of v is a point $\bar{v}$ with $d(v, \bar{v}) = \pi$. A point $v \in \Sigma$ is called *spherical* if it has only one antipode.

Consider the subset Σ^0 of all spherical points $v \in \Sigma$. Then Σ^0 is a convex subset isometric to some unit sphere $\mathbb{S}^k$ and Σ is a spherical join $\Sigma = \Sigma^0 * \Sigma'$, see, for instance, [Lyt05b, Corollary 4.4]. The Euclidean cone $C\Sigma$ has an $\mathbb{R}^k$-factor if and only if Σ is decomposable as a spherical join of $\mathbb{S}^{k-1}$ and another space. Moreover, the maximal Euclidean factor is $C\Sigma^0 \subset C\Sigma$.

Definition 6.2. Let Σ be as above and let $\delta > 0$ be arbitrary. We call a point $v \in \Sigma$ a *δ-spherical point*, if there exists some $\bar{v} \in \Sigma$ such that for any $w \in \Sigma$

$$d(v, w) + d(w, \bar{v}) < \pi + \delta \,. \tag{6.1}$$

Moreover, we say that v and $\bar{v}$ are *opposite* δ-spherical points.

The triangle inequality and extendability of geodesics to length π directly imply:

Lemma 6.1. *Let Σ be as above. The points $v, \bar{v} \in \Sigma$ are opposite δ-spherical points if and only if $d(\bar{v}, w) < \delta$ for any antipode w of v. In particular, in this case $d(v, \bar{v}) > \pi - \delta$ and the set of all antipodes of v has diameter less than 2δ. Finally, for every antipode v' of v, the pair (v, v') are opposite 2δ-spherical points.*

6.2. **Tuples of δ-spherical points.** We define special positions of pairs of almost spherical points:

Definition 6.3. Let Σ be as above. Let $(v_1, ..., v_k)$ be a k-tuple of points in Σ. We say that (v_i) is a *δ-spherical k-tuple* if there exists another k-tuple $(\bar{v}_i)$ in Σ with the following two properties. For any $i = 1, ..., k$, v_i and $\bar{v}_i$ are opposite δ-spherical points. For any $i \neq j$, we have

$$d(v_i, \bar{v}_j) < \frac{\pi}{2} + \delta \; ; \; d(v_i, v_j) < \frac{\pi}{2} + \delta \; ; \; d(\bar{v}_i, \bar{v}_j) < \frac{\pi}{2} + \delta \, .$$

Moreover, $(\bar{v}_i)$ and (v_i) are called *opposite δ-spherical k-tuples*.

From Lemma 6.1 and the triangle inequality we deduce:

Corollary 6.2. *Let Σ be as above. Let $v_1, ..., v_k \in \Sigma$ be δ-spherical points. If $(v_1, ..., v_k)$ is a δ-spherical k-tuple then, for all $i \neq j$,*

$$\frac{\pi}{2} - 2\delta < d(v_i, v_j) < \frac{\pi}{2} + \delta \, .$$

Assume, on the other hand, that for all $i \neq j$

$$\frac{\pi}{2} - \delta < d(v_i, v_j) < \frac{\pi}{2} + \delta \, .$$

Then, for arbitrary antipodes $\bar{v}_i$ of v_i, the tuples (v_i) and $(\bar{v}_i)$ are opposite 2δ-spherical k-tuples.

It is important to notice that all definitions above only use upper bounds on distances. Thus, due to the semicontinuity of angles, they are suitable to provide open conditions on spaces of directions.

6.3. **Connection with GH-topology.** The existence of almost spherical k-tuples is equivalent to a small distance from a k-fold suspension:

Proposition 6.3. *Let $\mathcal{C}$ be a compact set in the GH-topology of (isometry classes of) compact, geodesically complete* CAT(1) *spaces with diameter π. Let k be a natural number. The following are equivalent for any sequence Σ_i in $\mathcal{C}$.*

1. *Any accumulation point $\Sigma \in \mathcal{C}$ of the sequence Σ_i is isometric to a k-fold suspension $\mathbb{S}^{k-1} * \Sigma'$, with possibly empty Σ'.*
2. *For any $\delta > 0$ and all sufficiently large i, the space Σ_i admits a δ-spherical k-tuple.*

Proof. Choosing a subsequence we may restrict ourselves to the case that Σ_i converges to a space Σ.

A sequence of δ_i-spherical k-tuples in Σ_i with $\delta_i \to 0$ converges to a k-tuple of spherical points in Σ with pairwise distance $\frac{\pi}{2}$. This spherical k-tuple determines a splitting $\Sigma = \mathbb{S}^{k-1} * \Sigma'$, hence (2) implies (1).

On the other hand, if $\Sigma = \mathbb{S}^{k-1} * \Sigma'$, we choose the standard coordinate directions $e_1, ..., e_k \in \mathbb{S}^{k-1} \subset \Sigma$ and consider in Σ_i tuples of points converging to the k-tuple (e_i). These k-tuples satisfy the condition of (2), finishing the proof. $\square$

7. Strainers

7.1. **Strained points.** The following definition, translated from [BGP92] to our setting, is central for all subsequent considerations.

Definition 7.1. Let X be GCBA, k an integer and $\delta > 0$. A point $x \in X$ is (k, δ)-*strained* if the space of directions Σ_x contains some δ-spherical k-tuple.

As in the introduction, we denote by $X_{k,\delta}$ the set of (k, δ)-strained points in X. We have $X_{k,\delta} \subset X_{k-1,\delta} \ldots \subset X_{1,\delta} \subset X_{0,\delta} = X$. Due to Proposition 6.3, $X_{k,0} := \bigcap_{\delta>0} X_{k,\delta}$ is exactly the set of all points $x \in X$, for which the tangent space T_xX splits off the Euclidean space $\mathbb{R}^k$ as a direct factor.

7.2. **Strainers.** As in Section 5 we fix a tiny ball $U \subset \tilde{U} \subset X$.

Definition 7.2. Let $x \in U$ be a point and let $\delta > 0$ be arbitrary. A k-tuple of points $p_i \in \tilde{U} \setminus \{x\}$ is a (k, δ)-*strainer at* x if the k-tuple of the starting directions $((xp_i)')$ is δ-spherical in Σ_x.

Two (k, δ)-strainers (p_i) and (q_i) at x are *opposite* if the δ-spherical k-tuples $((xp_i)')$ and $((xq_i)')$ are opposite in Σ_x.

For a set $V \subset U$, a k-tuple (p_i) of points in $\tilde{U}$ is a (k, δ)-*strainer in* V if (p_i) is a (k, δ)-strainer at all $x \in V$. If (k, δ)-strainers (p_i) and (q_i) are opposite at all points $x \in V$, we say that (p_i) and (q_i) are *opposite* (k, δ)-*strainers* in V.

A point p is a $(1, \delta)$-strainer at x if and only if there is some $v \in \Sigma_x$ such that any continuation of px beyond x as a geodesic encloses an angle smaller than δ with v. The following observation is the most fundamental source of strainers.

Proposition 7.1. *For any $\delta > 0$ and $p \in U$, there is a neighborhood O of p such that the point p is a $(1, \delta)$-strainer in $O \setminus \{p\}$.*

Proof. Otherwise we find points $x_i \neq p$ arbitrary close to p such that $(x_ip)'$ is not δ-spherical. Set $s_i = d(x_i, p)$ and extend px_i by different geodesics to points y_i, z_i with $d(y_i, x_i) = d(z_i, x_i) = s_i$ and $\angle y_ix_iz_i \geq \delta$.

By construction, $d(y_i, p) = d(z_i, p) = 2 \cdot s_i$ and $\log_p(y_i) = \log_p(z_i)$. On the other hand $d(y_i, z_i) \geq \rho \cdot s_i$, where $\rho > 0$ depends only on δ and the curvature bound κ. For $s_i \to 0$, this contradicts Lemma 5.5. $\square$

For any $\delta > \pi$ and any $x \in U$, any k-tuple (p_i) of points in $\tilde{U} \setminus \{x\}$ is a (k, δ)-strainer at x. On the other hand, we have:

Lemma 7.2. *There exists a number $k_0(N)$ with the following property. For any tiny ball U of size bounded by N and any $1 \geq \delta > 0$, there do not exist (k, δ)-strained points in U with $k > k_0$.*

Proof. Let x be a (k, δ)-strained point in a tiny ball U of size bounded by N. By definition, we find in Σ_x a $(\frac{\pi}{2} - \delta)$-separated subset with k points. From the bound on the doubling constant (Corollary 5.8) and the assumption $\frac{\pi}{2} - \delta > \frac{1}{2} > 0$, we deduce that k is bounded from above in terms of N. $\square$

7.3. **Almost Euclidean triangles.** The existence of strainers imply the existence of many almost Euclidean triangles. We will only use the following special case:

Lemma 7.3. *Let $p, q \in \tilde{U}$ be opposite $(1, \delta)$-strainers at points $x \neq y$ in a tiny ball U. Then the following hold true.*

(1) $\pi - 2 \cdot \delta < \angle pxy + \angle pyx < \pi$.
(2) *If* $d(p,x) = d(p,y)$ *then* $\frac{\pi}{2} - 2 \cdot \delta < \angle pxy < \frac{\pi}{2}$.
(3) *If* $\frac{\pi}{2} - 2 \cdot \delta < \angle pxy < \frac{\pi}{2}$ *then* $\frac{\pi}{2} - 2 \cdot \delta < \angle pyx < \frac{\pi}{2} + 2 \cdot \delta$.

Proof. From the assumption on the upper curvature bound and diameter of $\tilde{U}$ we deduce the right hand side inequalities in (1) and in (2). By the same reason

$$\angle qxy + \angle qyx < \pi. \tag{7.1}$$

On the other hand, by the definition of opposite strainers we have

$$\angle pxy + \angle qxy > \pi - \delta \text{ and } \angle pyx + \angle qyx > \pi - \delta\,.$$

Hence the sum of these four angles is at least $2\pi - 2\delta$. Combining with (7.1) we deduce the left hand side of (1).

The remaining statements are direct consequences of (1). □

7.4. **Stability of strainers.** If (p_i) is a (k,δ)-strainer at the point x and if $\hat{p}_i \in \tilde{U} \setminus \{x\}$ is any point on the geodesic xp_i or on an extension of xp_i beyond p_i then $(\hat{p}_i)$ is still a (k,δ)-strainer at x.

From Corollary 6.2 we obtain:

Lemma 7.4. *Let $p_1,...,p_k \in \tilde{U}$ be $(1,\delta)$-strainers at $x \in U$ and let $q_i \in \tilde{U}$ be arbitrary points lying on an extension of the geodesic $p_i x$ beyond x. If $|\angle p_i x p_j - \frac{\pi}{2}| < \delta$, for all $i \neq j$, then (p_i) and (q_i) are opposite $(k, 2\delta)$-strainers at x.*

The definition of strainers is designed to satisfy the following openness condition:

Lemma 7.5. *Let $U_l \subset \tilde{U}_l \subset X_l$ converge to $U \subset \tilde{U} \subset X$ as in our standard setting for convergence in Definition 5.2. Let (p_i) and (q_i) be opposite (k,δ)-strainers at $x \in U$. Let, for $i = 1,..,k$, the sequences $p_i^l, q_i^l, x^l \in \tilde{U}_l$ converge to p_i, q_i and x, respectively.*

Then the k-tuples (p_i^l) and (q_i^l) in $\tilde{U}_l$ are opposite (k,δ)-strainers at the point x^l, for all l large enough.

Proof. The claim is a consequence of the semicontinuity of angles under convergence, Subsection 3.3 (see also Lemma 5.11), and the definition of δ-spherical k-tuples, which only involves non-strict upper bounds on distances. □

Restricting to the case $U_l = U$ for all l, we see from Lemma 7.5

Corollary 7.6. *For k-tuples (p_i) and (q_i) in $\tilde{U}$, the set of points $x \in U$ at which (p_i) and (q_i) are opposite (k,δ)-strainers is an open set.*

The set of points $x \in U$ at which (p_i) is a (k,δ)-strainer is open.

7.5. **Straining radius.** We will need some uniformity in the choice of opposite strainers and the diameters of strained neighborhoods. As before, we denote by r_0 the radius of the tiny ball U.

Lemma 7.7. *Let (p_i) be a (k,δ)-strainer at $x \in U$. Then there exists some number $0 < \epsilon_x < \frac{1}{2} \cdot d(x, \partial U)$ with the following properties. Let $y \in B_{\epsilon_x}(x)$ be arbitrary and let $q_i \in \tilde{U}$ lie on an arbitrary continuation of $p_i y$ beyond y such that $d(q_i, y) = r_0$. Then the k-tuples (q_i) and (p_i) are opposite $(k, 2\delta)$-strainers in the ball $B_{\epsilon_x}(y)$.*

Proof. In order to prove the statement, we assume the contrary and find contradicting sequences $y_l, z_l \to x$ and k-tuples (q_i^l). Thus, $d(y_l, q_i^l) = r_0$, the point y_l is on the geodesic $p_i q_i^l$, and (p_i^l) and (q_i^l) is not an opposite $(k, 2\delta)$-strainer at z_l. Taking limit points we find a k-tuple $(q_i) \in \tilde{U}$ such that x is an inner point of the geodesic $p_i q_i$ for any i.

Due to stability of strainers, Lemma 7.5, (q_i) and (p_i) cannot be opposite $(k, 2\delta)$-strainer at x. But this contradicts Lemma 7.4. □

We will call the maximal number ϵ_x as in Lemma 7.7 above the *straining radius at* x, of the (k,δ)-strainer (p_i). By definition, $\epsilon_y \geq \epsilon_x - d(x,y)$, for all $x, y \in V$. In particular, the map $x \to \epsilon_x$ is continuous.

Note finally, that the proof above literally transfers to the convergence setting from Lemma 7.5. Thus the proof shows:

Lemma 7.8. *Under the assumptions of Lemma 7.5, let ϵ_x and ϵ_{x_l} be the straining radius of (p_i) and (p_i^l) at x and x_l, respectively. Then* $\liminf_{l\to\infty} \epsilon_{x_l} \geq \epsilon_x$.

8. Strainer maps

8.1. **Differentials of distance maps and a criterion for openness.** Let $U \subset \tilde{U} \subset X$ be a tiny ball, as always in our setting. For $p \in \tilde{U}$ we denote by $d_p : \tilde{U} \to \mathbb{R}$ the distance function $d_p(x) = d(p,x)$.

For any point $x \in \tilde{U}$ we collect the directional derivatives of $f = d_p$ to a differential $D_x f : T_x \to \mathbb{R}$. If $x \neq p$, $v \in \Sigma_x$ and $t \geq 0$ then the differential $D_x f(tv)$ is given by the first formula of variation as $D_x f(tv) = -t \cdot \cos(\alpha)$, where α is the distance in Σ_x between v and the starting direction of the geodesic xp.

If $F = (d_{p_1}, ..., d_{p_k}) : \tilde{U} \to \mathbb{R}^k$ is a distance map, we denote as its differential $D_x F : T_x \to \mathbb{R}^k$ the map whose coordinates are the differentials of d_{p_j} at x.

The following criterion is essentially taken from [BGP92, Section 11.5].

Lemma 8.1. *Set $\rho = \frac{1}{4k}$. Let (p_i) be a k-tuple in $\tilde{U}$ and let $f_i = d_{p_i}$ be the corresponding distance functions. Assume that for every x in an open subset V of U and any $1 \leq i \leq k$ there are directions $v_i^{\pm} \in \Sigma_x$ with*

$$\pm D_x f_i(v_i^{\pm}) > 1 - \rho \ \text{ and } \ |D_x f_j(v_i^{\pm})| < 2 \cdot \rho, \text{ for } i \neq j \,. \tag{8.1}$$

Then the distance map $F = (f_1, ..., f_k) : V \to \mathbb{R}_1^k$ is locally 2-open if we equip $\mathbb{R}^k$ with the L^1-norm $|(t_i)|_1 = \sum_{i=1}^k |t_i|$.

Proof. The arguments could be transferred from [BGP92]. We rely on [Lyt05a] instead.

Let $x \in V$ be arbitrary and $r > 0$ such that $\bar{B}_{2r}(x)$ is complete. Let $\mathfrak{t} = (t_1, ..., t_k) \in \mathbb{R}^k$ with $s := |\mathfrak{t} - F(x)|_1 < r$ be fixed. In order to find $y \in B_{2r}(x) \cap F^{-1}(\mathfrak{t})$ we consider the function $h : V \to \mathbb{R}$ given by

$$h(z) := |\mathfrak{t} - F(x)|_1 - |\mathfrak{t} - F(z)|_1 \,.$$

Then $h(x) = 0$ and we are looking for $y \in B_{2r}(x)$ with $h(y) = s$.

For every $z \in V$ with $h(z) < s$ there is some $i = 1, ..., k$ such that $t_i \neq f_i(z)$. On the geodesic γ starting at such z in the direction $v_i^{\pm}$ (depending on the sign of $t_i - f_i(z)$), the value of $|t_i - f_i(z)|$ decreases (infinitesimally) with velocity larger than $1-\rho$ while the values of $|t_j - f_j(z)|$ for $j \neq i$ increase with velocity less than 2ρ. Therefore the *norm of the gradient of* h at any $z \in V \setminus F^{-1}(\mathfrak{t})$ satisfies

$$|\nabla_z h| := \limsup_{y \to z} \frac{h(y) - h(z)}{d(y,z)} \geq (1-\rho) - (k-1)\cdot(2\cdot\rho) > 1 - 2\cdot k \cdot \rho \geq \frac{1}{2} \, .$$

Due to [Lyt05a, Lemma 4.1], for any $s' < s$ we find some $z \in B_{2s}(x)$ with $h(z) = s'$. We let s' go to s and use compactness of $\bar{B}_{2r}(x)$ to find the desired point y.

This shows $B_r(F(x)) \subset F(B_{2r}(x))$ and finishes the proof. □

8.2. **Strainer maps.** Let $V \subset U$ be a subset, let (p_i) be a k-tuple in $\tilde{U}$, and let $F = (d_{p_1}, ..., d_{p_k}) : V \to \mathbb{R}^k$ be the corresponding distance map. We say that F is a (k,δ)-*strainer map* in V if (p_i) is a (k,δ)-strainer in V. In this case, we denote by the *straining radius of* F at $x \in V$ the straining radius of the (k,δ)-strainer (p_i) at x. We say that distance maps F, G are *opposite* (k,δ)-*strainer maps* in V if their defining k-tuples are opposite (k,δ)-strainers in V.

Let $F : V \to \mathbb{R}^k$ be a (k,δ)-strainer map with coordinates $f_i = d_{p_i}$ defined on an open subset V of U. For any $x \in V$, we find some distance map $G = (d_{q_i})$ such that F and G are opposite (k,δ)-strainer maps at x. Choose $v_i^{\pm} \in \Sigma_x$ to be the starting directions of xp_i and xq_i, respectively. By the definition of strainers and the first formula of variation, we see that (8.1) hold true at the point x with ρ replaced by δ. Replacing the L^1-norm by the Euclidean norm we get:

Lemma 8.2. *If $\delta \leq \frac{1}{4\cdot k}$ then any (k,δ)-strainer map $F : V \to \mathbb{R}^k$ on any open non-empty set $V \subset U$ is L-Lipschitz and L-open with $L = 2\sqrt{k}$. In particular, the Hausdorff dimension of V is at least k.*

Proof. The Lipschitz property is true for any distance map. The openness constant follows from Lemma 8.1. The bound on the Hausdorff dimension follows, since the image $F(V)$ is open in $\mathbb{R}^k$. □

Similarly to the corresponding result in [BGP92, Section 11.1], one can derive from Lemma 8.2 that the intrinsic metric on the fibers of F is locally equivalent to the induced metric. Since it is not used in the sequel, we do not provide the proof.

8.3. **Convergence of maps and an improvement of constants.** We are going to prove that for small δ, the constant L in Lemma 8.2 can be chosen arbitrary close to 1. These results will only be used in the sequel on the set of regular points in the proof of Theorem 1.5, where one can rely instead on results from [Nag02]. For this reason, the argument in this subsection will be sketchy. Readers not familiar with ultralimits may restrict to the case of tiny balls with uniformly bounded size and replace ultralimits by GH-limits of a subsequence, using Corollary 5.7.

Let $F : V \to \mathbb{R}^k$ be a (k,δ)-strainer map on an open subset V of some tiny ball U. For $\delta \leq \frac{1}{4\cdot k}$, the map F is L-open and L-Lipschitz on V with $L = 2\sqrt{k}$. Therefore, the differential $D_x F : T_x \to \mathbb{R}^k$ which is a limit of the rescalings of F is L-Lipschitz and L-open.

Let $F_l : V_l \to \mathbb{R}^k$ be a sequence of (k,δ_l)-strainer maps with $\lim_{l\to\infty} \delta_l = 0$. Let $x_l \in V_l$ be arbitrary and consider the sequence of differentials $D_{x_l} F_l : T_{x_l} \to \mathbb{R}^k$.

As in the proof of Proposition 6.3, we see that the ultralimit $T = \lim_\omega T_{x_l}$ is a Euclidean cone which splits as $T = \mathbb{R}^k \times T'$. Moreover, the ultralimit $P = \lim_\omega D_{x_l} F_l : T \to \mathbb{R}^k$ is just the projection of T onto the direct factor $\mathbb{R}^k$.

Since the maps $D_{x_l}F_l$ are all L-open, any fiber $P^{-1}(w)$ with $w \in \mathbb{R}^k$ coincides with the ultrlimit of fibers $\lim_\omega (D_{x_l}F_l)^{-1}(w_l)$ for any sequence w_l converging to w. For any unit vector $w \in \mathbb{S}^{k-1}$ the fiber $P^{-1}(w)$ has distance 1 to the origin of the cone T. Therefore, for any $\epsilon > 0$, the distances of infinitely many of the fibers $(D_{x_l}F_l)^{-1}(w_l)$ to the origin must be between $1-\epsilon$ and $1+\epsilon$.

Arguing by contradiction we conclude:

Lemma 8.3. *For every $k \in \mathbb{N}$ and $L > 1$ there exists some $\delta = \delta(L,k) > 0$ such that the following holds true. For any (k,δ)-strainer map F at a point x in a tiny ball U of a GCBA space X the differential $D_xF : T_x \to \mathbb{R}^k$ satisfies:*

(1) *$|D_xF(v)| < L$, for any $v \in \Sigma_x \subset T_x$.*
(2) *For any $u \in \mathbb{S}^{k-1} \subset \mathbb{R}^k$, there exists $v \in T_x$ with $D_xF(v) = u$ and $|v| < L$.*

The infinitesimal characterization of L-Lipschitz and L-open maps, [Lyt05a, Theorem 1.2], now directly implies:

Corollary 8.4. *For any $L > 1$ there is some $\delta = \delta(L,k) > 0$ such that the following holds true. Any (k,δ)-strainer map $F : V \to \mathbb{R}^k$ is L-open and L-Lipschitz, whenever V is an open convex subset of a tiny ball of size bounded by N.*

9. Fibers of strainer maps

9.1. **Local contractibility.** In the topological sense a strainer map is close to being a fibration:

Theorem 9.1. *Let $F : \tilde{U} \to \mathbb{R}^k$ be a distance map and $0 < \delta \leq \frac{1}{20k}$. Assume that F is a (k,δ)-strainer map with straining radius ϵ_x at some $x \in U$. Let $Q = \Pi_{i=1}^k [a_i^-, a_i^+]$ be a rectangular box in $\mathbb{R}^k$ which contains $F(x)$. Then, for any $\epsilon_x > r > 0$, the ball $B_r(x) \cap F^{-1}(Q)$ is contractible.*

Proof. Let $f_i = d_{p_i}$ be the coordinates of F. By definition of ϵ_x we find $q_i \in \bar{U}$ for $i = 1, ..., k$, such that x lies on the geodesic p_iq_i and such that $(p_i), (q_i)$ are opposite $(k, 2\delta)$-strainers in $W := B_{\epsilon_x}(x)$.

We set $\hat{\Pi} := F^{-1}(Q)$. We are going to construct a homotopy retraction $\Psi^\infty : W \to W \cap \hat{\Pi}$ such that $\Psi^\infty(B_r(x)) = B_r(x) \cap \hat{\Pi}$, for all $r < \epsilon_x$. Since $B_r(x)$ is contractible, this will prove the theorem.

For $i = 1, ..., k$, let Π_i be the set of all points z with $a_i^- \leq f_i(z) \leq a_i^+$. First, we define flows $\phi_i : W \times [0,1] \to U$. For a point $y \in W$ with $f_i(y) \geq f_i(x)$, the flow ϕ_i moves y with velocity 1 along the geodesic yp_i until it reaches Π_i and then the flow stops for all times. For a point y with $f_i(y) \leq f_i(x)$, the flow moves y along the geodesic yq_i until it reaches Π_i and stops there.

Since x is on the geodesic p_iq_i, the CAT(κ) condition directly implies that the flow ϕ_i does not increase the distance to the point x, in particular it leaves the ball $B_r(x)$ invariant.

By the first formula of variation, the value of f_i changes along the flow lines of ϕ_i with velocity at least $1 - 2\delta$ until the point reaches the set Π_i. Moreover, for $j \neq i$, the value of f_j changes along the flow lines of ϕ_i with velocity at most 4δ.

Consider the function $M_i, M : U \to \mathbb{R}$ defined by

$$M_i(y) := \max\{0, f_i(y) - a_i^+, a_i^- - f_i(y)\} \text{ and } M(y) := \max_{1\leq i\leq k} M_i(y)\,.$$

Note that $M(y) = 0$ if and only if $y \in \hat{\Pi}$.

The above observation shows that the flow line $\phi_i(y,t)$ reaches Π_i at latest at $t=(1-2\delta)^{-1}\cdot M_i(y)$. Due to the first formula of variation and $\delta \leq \frac{1}{20}$, we have for all $j\neq i$ and all $1\geq t\geq 0$:

$$M_j(\phi_i(y,t)) \leq M_j(y) + \frac{4\delta}{1-2\delta}\cdot M_i(y) \leq M_j(y) + 5\delta \cdot M(y)\,. \tag{9.1}$$

Consider the concatenation Ψ of the flows $\phi_1,...,\phi_k$. Thus $\Psi : W\times[0,k]\to W$ is a homotopy which moves on the time interval $[j-1,j]$ the point $\Psi(y,j-1)$ along the flow lines of ϕ_j to Π_j. Applying (9.1) k-times we see

$$M(\Psi(y,t)) \leq (1+5\delta)^k\cdot M(y)\,,$$

for all $(y,t)\in W\times[0,k]$. By construction $M_j(\Psi(y,j))=0$. Applying (9.1) again, for all j, we improve the last inequality to

$$M(\Psi(y,k)) \leq k\cdot 5\cdot\delta\cdot(1+5\cdot\delta)^k\cdot M(y) \leq \frac{1}{4}\cdot(1+\frac{1}{4k})^k\cdot M(y)\,.$$

Since $(1+\frac{1}{x})^x$ is increasing and converges to the Euler number e, we see

$$M(\Psi(y,k)) \leq \frac{1}{4}\cdot e^{\frac{1}{4}}\cdot M(y) < \frac{1}{4}\cdot 2\cdot M(y) = \frac{1}{2}\cdot M(y)\,.$$

Moreover, the flow line of the homotopy Ψ of a point y has length at most

$$k\cdot\frac{1}{1-2\delta}\cdot(1+5\cdot\delta)^k\cdot M(y)\leq 4\cdot k\cdot M(y)\,.$$

Putting the last two observation together, we see (inductively) that the m-fold concatenation $\Psi_m : W\times[0,k\cdot m]\to W$ of the homotopy Ψ satisfies the inequality $M(\Psi_m(y,k\cdot m))\leq 2^{-m}\cdot M(y)$, for any $y\in W$. Moreover, the Ψ_m-flow line of y has length at most $8k\cdot M(y)$. Therefore, reparametrizing Ψ_m we obtain a limit homotopy $\Psi^\infty\colon W\times[0,1]\to W$ that leaves all balls around x invariant and retracts W onto $W\cap\hat{\Pi}$. □

In the special case that Q just consists of one point $F(x)$ we deduce that all small balls in all fibers of F are contractible.

Proof of Theorem 1.8. Under the assumptions of Theorem 9.1, let $V'\subset V$ be compact. Since the straining radius depends continuously on the point, we find some $\epsilon>0$ smaller than the straining radius at any $x\in V'$ and smaller than $d(V',\partial V)$. By Theorem 9.1, for any $x\in V'$, any $r<\epsilon$ and any rectangular box Q containing $F(x)$, the ball $B_r(x)\cap F^{-1}(Q)$ is contractible. □

9.2. **Dichotomy.** The openness of strainer maps and local connectedness of their fibers implies a dichotomy in the behavior of strainer maps. First a local result:

Lemma 9.2. *Let F be a (k,δ)-strainer map at $x\in U$ with $\delta\leq\frac{1}{20\cdot k}$. Let $3r$ be not larger than the the straining radius of F at x. Then either*

- *$F : B_r(x)\to\mathbb{R}^k$ is injective, or*
- *For all $y\in B_r(x)$ the fiber $\Pi := F^{-1}(F(y))\cap B_r(y)$ is a connected set of diameter at least r.*

Proof. Fix $y\in B_r(x)$ and the fiber $\Pi := F^{-1}(F(y))\cap B_r(y)$. Due to Theorem 9.1, Π is connected. Assume that Π is not a singleton.

If the diameter of Π is smaller than r we find a point $z \in \Pi$ which has in Π maximal distance $s < r$ from y. Consider a point z', such that z is on the geodesic $\eta = yz'$ with sufficiently small $l := d(z, z')$.

Let $(p_1, ..., p_k)$ be the k-tuple of points defining F. For $i = 1, ..., k$, we get from the assumption on r and Lemma 7.3

$$\frac{\pi}{2} - 4\delta < \angle p_i y z < \frac{\pi}{2}\,.$$

Another application of Lemma 7.3 shows that for any $y' \neq y$ on η

$$\frac{\pi}{2} - 4\delta < \angle p_i y' y < \frac{\pi}{2} + 4\delta\,.$$

By the first formula of variation we see

$$|F(z') - F(z)| \leq 4\delta \cdot \sqrt{k} \cdot l\,.$$

Since the map F is $2 \cdot \sqrt{k}$-open, we find a point z_0 with

$$F(z_0) = F(z) \;\text{ and }\; d(z_0, z) \leq 2 \cdot \sqrt{k} \cdot 4 \cdot \delta \cdot \sqrt{k} \cdot l < l\,.$$

Therefore, $d(y, z_0) > d(y, z) = s$. If l has been small enough, then z_0 is contained in $B_r(y)$ in contradiction to the choice of z.

Hence, for any $y \in B_r(x)$ the fiber $\Pi_y = F^{-1}(F(y)) \cap B_r(y)$ is a connected set that is either a point or has diameter at least r.

Since the map F is open, we deduce that the set of points y at which fiber Π_y is a singleton is an open and closed subset of $B_r(x)$. Therefore, this set is either empty or the whole ball $B_r(x)$. This finishes the proof. □

As a direct consequence of this local statement, the openness of strainer maps and a standard connectedness argument we get the following global statement:

Proposition 9.3. *For any (k, δ)-strainer map $F : V \to \mathbb{R}^k$ with $\delta \leq \frac{1}{20 \cdot k}$ and connected, open V the following dichotomy holds true. Either no fiber of F in V contains an isolated point, or all fibers of F in V are discrete.*

10. Finiteness results

10.1. **Notations.** In this section we continue to use the previous notations for a tiny ball $U \subset \tilde{U} \subset X$. Let again N denote a bound on the size of $U \subset \tilde{U}$ and let $r_0 \leq 1$ be the diameter of U. Let $\delta > 0$ be arbitrary.

As in Subsection 5.3 we denote by $\mathcal{A}$ the distance sphere of radius r_0 around U and by $\mathcal{A}_\delta$ a fixed maximal $\delta \cdot r_0$-separated subset of $\mathcal{A}$. Let $m = m(N, \delta)$ be an upper bound on the number of elements in $\mathcal{A}_\delta$.

Let k be a natural number. Denote by $\mathcal{F}_\delta$ the set of distance maps $F : \tilde{U} \to \mathbb{R}^k$, whose coordinates are distance functions to points $p_j \in \mathcal{A}_\delta$. The number of elements in $\mathcal{F}_\delta$ is bounded from above by the constant m^k depending on N, δ and k.

10.2. **Bounding straining sequences.** It turns out, that in the investigations of the (k, δ)-strained points we may restrict the attention to the finitely many maps from $\mathcal{F}_\delta$. In the notations above we have:

Lemma 10.1. *Let $F : \tilde{U} \to \mathbb{R}^k$ be a distance map which is a (k, δ)-strainer map at $x \in U$. Then there exist maps $F_1, F_2 \in \mathcal{F}_\delta$ such that the pairs (F, F_2) and (F_1, F_2) are opposite $(k, 3 \cdot \delta)$-strainer maps at x.*

Proof. Let F be given by the k-tuple (p_i) and find at x an opposite (k,δ)-strainer map G given by a k-tuple (q_i). By the definition of $\mathcal{A}_\delta$ we find k-tuples (p'_i) and (q'_i) in $\mathcal{A}_\delta$ such that

$$\angle p'_i x p_i < \delta \text{ and } \angle q'_i x q_i < \delta .$$

Due to the triangle inequality and the definition of strainers, the distance maps $F_1, F_2 \in \mathcal{F}_\delta$ given by the k-tuples (p'_i) and (q'_i) have the required properties. □

10.3. **Bounding bad sequences.** The following definition can be considered as a counterpart of straining sequences:

Definition 10.1. A subset T of U is called *δ-bad* if no point $x \in T$ is a $(1,\delta)$-strainer of another point $y \in T$.

We derive the following uniform bound:

Proposition 10.2. *There is a number $C_0 = C_0(N,\delta)$ such that each δ-bad subset of U has at most C_0 elements.*

Proof. Since the claim is scaling invariant, we may rescale U and assume that $r_0 = 1$. Then the curvature bound κ is at most $\frac{1}{10}$. Moreover, we may assume $\delta \leq \pi$.

Using comparison of quadrangles, we find some $r_1 > 0$ depending only on δ such that the following holds true for all $y_1, x_1, x_2, y_2 \in \tilde{U}$. If $d(x_1,y_1) = d(x_2,y_2) = 1$ and the angles satisfy $\angle y_1 x_1 x_2 \geq \delta/2$ and $\angle y_2 x_2 x_1 \geq \pi - \delta/4$ then the distance between y_1 and y_2 is at least r_1.

We fix some number L_0 depending only on δ (and the curvature bound $\frac{1}{10}$), such that for all pairwise distinct points $x,y,z \in U$ the inequality $d(x,z) \geq L_0 \cdot d(x,y)$ implies $\angle xzy \leq \frac{\delta}{4}$.

By the doubling property there is a constant $C = C(N)$, such that any set T in U of diameter $0 < D < \infty$ is covered by at most C balls of diameter less than $\frac{D}{4L_0}$.

Assume now that the Proposition does not hold. Then there are arbitrary large δ-bad subsets, possibly in different tiny balls U (in different GCBA spaces), but of the same bound on the size N.

We claim that there are δ-bad sets $\{x_1,...,x_M\}$ with arbitrary large M, such that $d(x_i,x_{i+1}) \geq L_0 \cdot d(x_i,x_k)$ for all $1 \leq k \leq i < M-1$.

In order to find such a δ-bad subset, we start with an arbitrary δ-bad finite subset $T_1 \subset U$ with at least $M \cdot C^M$ elements, where C is as above. If T_1 has diameter D_1 then, by the choice of C, we find a subset T_2 of T_1 with at least $M \cdot C^{M-1}$ elements and diameter $D_2 < \frac{D_1}{4L_0}$. Proceeding by induction we find for $k = 3,...,M$ a subset T_k of T_{k-1} such that T_k has at least $M \cdot C^{M-k}$ elements and such that the diameter D_k of T_k is smaller than $\frac{D_{k-1}}{4L_0}$.

We start with an arbitrary $x_1 \in T_M$. We choose by induction for $k = 2,...,M$, an arbitrary $x_k \in T_{M-k}$ such that $d(x_{k-1},x_k)$ is not smaller than half of the diameter of T_{M-k}. By construction, the arising sequence has the desired property. This proves the claim.

Let $x_1,...,x_M$ be as above. Denote by $v_{i,j} \in \Sigma_{x_i}$ the starting direction of the geodesic $x_i x_j$. For each $i \geq 2$, we use that x_1 is not a $(1,\delta)$-strainer at x_i to find antipodes $w_i^+, w_i^- \in \Sigma_{x_i}$ of $v_{i,1}$ such that $d(w_i^+, w_i^-) \geq \delta$.

We proceed as follows. For each $i \geq 3$ the distance in Σ_{x_2} between $v_{2,i}$ and either w_2^+ or w_2^- is at least $\delta/2$. Hence we can find a subsequence $x_1, x_2, x_{l_3}, x_{l_4}, \ldots, x_{l_k}$

of the tuple (x_i) with at least $M/2$ elements such that for one of the directions $w_2^{\pm}$, say w_2^+, and for each $i \geq 3$ we have $d(w_2^+, v_{2,l_i}) \geq \delta/2$. Denote this direction w_2^+ by w_2 and replace our original tuple $x_1, ..., x_M$ by this subsequence.

We repeat the procedure at x_3 and continue inductively. In this way we obtain a δ-bad sequence $x_1, \ldots, x_s$ with $s \geq \log_2 M$ and, for each $i \geq 2$, a direction $w_i \in \Sigma_{x_i}$, such that the following two conditions hold:

(1) $d(x_i, x_{i+1}) \geq L_0 \cdot d(x_i, x_k)$, for all $1 \leq k \leq i < s$;
(2) The direction w_i is antipodal to $v_{i,1}$. For all $j > i$, we have $d(v_{i,j}, w_i) \geq \delta/2$.

For $2 \leq i \leq s$ choose a geodesic γ_i in $\tilde{U}$ of length 1 starting at x_i in the direction w_i and set $y_i = \gamma_i(1)$. Thus, $d(y_i, x_i) = 1$.

Let $2 \leq i < j \leq s$ be arbitrary. By construction, $\angle y_i x_i x_j \geq \delta/2$. On the other hand, by the choice of L_0, we have $\angle x_1 x_j x_i \leq \delta/4$ and therefore, $\angle y_j x_j x_i \geq \pi - \delta/4$. Due to the first statement in the proof, we have $d(x_j, x_i) \geq r_1$.

Therefore, the doubling constant of $\tilde{U}$ (and hence the size of U) bounds the number s in our sequence, providing a contradiction. $\square$

10.4. **Extension of strainer maps.** We now prove the following central result:

Theorem 10.3. *There exists $C_1 = C_1(N, \delta) > 0$ with the following properties.*

Let $F \colon V \to \mathbb{R}^k$ be a (k, δ)-strainer map on an open subset V of a tiny ball U of size bounded by N. Let E denote the set of points in V at which F cannot be extended to a $(k+1, 12 \cdot \delta)$-strainer map $\hat{F} = (F, f)$ using some distance function $f = d_{p_{k+1}}$ as last coordinate.

Then E intersects each fiber Π of F in V in at most C_1 points. E is a countable union of compact subsets E_j, such that the restriction $F : E_j \to F(E_j)$ is C_1-biLipschitz. Moreover,

$$\mathcal{H}^k(E) \leq C_1^k \cdot \mathcal{H}^k(F(E)) \leq C_1^{2k} \cdot 10 \cdot r_0^k \,. \tag{10.1}$$

Proof. If $\delta > \frac{\pi}{12}$ then E is empty, and the statement is clear. Thus, we may assume $\delta \leq \frac{\pi}{12}$. Due to Lemma 7.2, there is a number $k_0 = k_0(N)$ such that $k \leq k_0$.

Let F be defined by a k-tuple $(p_1, ..., p_k)$. By Lemma 10.1, there is a finite set $\mathcal{F}_\delta$ of distance maps $G : U \to \mathbb{R}^k$ with at most $C = C(N, \delta)$ elements and the following property. If V_G denotes the set of points in V at which F and G are opposite $(k, 3 \cdot \delta)$-strainer maps, then the open set $\bigcup \{ V_G \mid G \in \mathcal{F}_\delta \}$ covers V. Since $\mathcal{F}_\delta$ has at most C elements, we may replace V by one of the sets V_G and assume that on the whole set V there exists an opposite $(k, 3 \cdot \delta)$-strainer map G to F.

Let Π be a fiber of the map F on V. For any pair of points $x, y \in V \cap \Pi$ we deduce from Lemma 7.3 that $|\angle p_i x y - \frac{\pi}{2}| < 6\delta$. Therefore, if x were a $(1, 6 \cdot \delta)$-strainer at y then the $(k+1)$-tuple $(p_1, \ldots, p_k, x)$ is a $(k+1, 12 \cdot \delta)$-strainer at x, as follows from Corollary 6.2.

By definition, this implies that the subset $E \cap \Pi$ must be 6δ-bad. Due to Proposition 10.2, $E \cap \Pi$ can have at most $C_0(N, 6 \cdot \delta)$ elements, thus proving the first statement of the Theorem.

Assume now that $x_l \in E$ is a sequence converging to some $x \in E$ and violating

$$\liminf_{l \to \infty} \frac{||F(x) - F(x_l)||}{d(x, x_l)} \geq \delta \,. \tag{10.2}$$

Then, replacing x_l by a subsequence and applying the first formula of variation we deduce, for any $i = 1, ..., k$ and all large l, $|\angle p_i x x_l - \frac{\pi}{2}| < 2\delta$. Fix an opposite

(k, δ)-strainer (q_i) to (p_i) at x. Then (q_i) and (p_i) are opposite (k, δ)-strainers at x_l, for all l large enough, Corollary 7.6. Applying Lemma 7.3, we deduce that $|\angle p_i x_l x - \frac{\pi}{2}| < 4\delta$, for all sufficiently large l and all $1 \leq i \leq k$. But, due to Proposition 7.1, the point x is a $(1, 4 \cdot \delta)$-strainer at x_l, for all l large enough. Hence, $(p_1, \ldots, p_k, x)$ is a $(k+1, 8\delta)$-strainer at x_l (Corollary 6.2) in contradiction to the assumption $x_l \in E$. This finishes the proof of (10.2).

The remaining claims are consequences of this infinitesimal property. More precisely, $k \leq k_0 = k_0(N)$, by Lemma 7.2. We set $C_1 := \max\{\frac{4}{\delta}, 2\sqrt{k_0}\}$, The restriction of F to E is $2\sqrt{k}$-Lipschitz, hence also C_1-Lipschitz, as any distance map. The set E is closed in V, hence locally complete. The implication that E is a union of compact subsets E_j to which F restricts as a C_1-biLipschitz map is shown in [Lyt05a, Lemma 3.1], as a consequence of (10.2).

The set E is a union of a countable number of Lipschitz images of compact subsets of $\mathbb{R}^k$, hence E is countably k-rectifiable, [AK00]. An application of the co-area formula, [AK00], together with (10.2) proves the first inequality in (10.1). The second inequality in (10.1) follows from the fact that $F(E)$ is contained in a Euclidean k-dimensional ball of radius $C_1 \cdot r_0$, and the fact that the volumes of Euclidean unit balls in any dimension are smaller than 10.

This finishes the proof. □

10.5. **Conclusions.** Note that Theorem 10.3 is a quantitative version of Theorem 1.7. Thus the proof of Theorem 1.7 is finished as well.

In order to derive Theorem 1.6, we prove the following localized more precise version of it. Let again U be a tiny ball of radius r_0 and size bounded by N as above. As in the introduction, we denote by $U_{k,\delta}$ the set of all points x in U at which there exists some (k, δ)-strainer.

Proposition 10.4. *There exists a function $C_2 = C_2(N, \delta) > 0$ with the following properties. The set $U \setminus U_{k,\delta}$ is a union of countably many images of biLipschitz maps $G_j : A_j \to U$, with A_j compact in $\mathbb{R}^{k-1}$. Moreover, $\mathcal{H}^{k-1}(U \setminus U_{k,\delta}) < C_2 \cdot r_0^{k-1}$.*

Proof. If δ decreases, the sets $U_{k,\delta}$ increase, thus in all subsequent considerations we may assume that δ is sufficiently small.

We proceed by induction on k. The set $U \setminus U_{1,\delta}$ has at most $C_1(\delta, N)$ elements, due to Proposition 10.2. This proves the statement for $k = 1$.

Assuming the result is true for k, we are going to prove it for $k+1$. By the inductive assumption, the set $U \setminus U_{k,\delta/50}$ is a countable union of images of biLipschitz maps defined on compact subsets of $\mathbb{R}^{k-1}$.

Thus it suffices to represent $K := U_{k,\delta/50} \setminus U_{k+1,\delta}$ as a union of biLipschitz images and to estimate its k-dimensional Hausdorff measure.

Any point $x \in K$ admits a $(k, \delta/12)$-strainer map $F \in \mathcal{F}_{\delta/12}$, due to Lemma 10.1. Thus, we have a finite number of $(k, \delta/12)$-strainer maps $F_j : V_j \to \mathbb{R}^k$ defined on open subsets $V_j \subset U$ such that the union of V_j covers K and such that the number of V_j is bounded by some $C_3(N, \delta)$.

Applying now Theorem 10.3 to the maps $F_j : V_j \to \mathbb{R}^k$ and observing that $K_j := K \cap V_j$ is contained in the set E from the formulation of Theorem 10.3 we deduce the following.

Each K_j is a countable union of biLipschitz images of compact subsets of $\mathbb{R}^k$ and $\mathcal{H}^k(K_j)$ is bounded by $C_4 \cdot r_0^k$ for some $C_4 = C_4(N, \delta)$. Summing up, we deduce the required bound on the volume $\mathcal{H}^k(K)$ and the fact that K is the union of countably many images of biLipschitz maps defined on compact subsets of $\mathbb{R}^k$. $\square$

Now we obtain:

Proof of Theorem 1.6. We cover X by a countable number of tiny balls U, using the separability of X. The set $U \setminus X_{k,0}$ of not $(k,0)$-strained points in U is the union of the complements $U \setminus U_{k,\delta}$ where δ runs over all sufficiently small rational numbers. Applying Proposition 10.4, we deduce that $X \setminus X_{k,0}$ is a countable union of compact subsets biLipschitz equivalent to subsets of $\mathbb{R}^{k-1}$. $\square$

11. Dimension

11.1. **Topological and Hausdorff dimension.** We can now prove a quantitative version of the first part of Theorem 1.1.

Proposition 11.1. *There is some $C(N) > 0$ such that the following holds true for any tiny ball U of radius r_0 and size bounded by N.*

If n is the topological dimension of U then $0 < \mathcal{H}^n(U) < C \cdot r_0^n$. In particular, the Hausdorff dimension of U equals n. Moreover, n is the largest number such that some tangent cone T_xU is isometric to $\mathbb{R}^n$. Finally, n is the largest number, such that there are $(n, \frac{1}{4\cdot n})$-strained points in U.

Proof. We already know that the topological dimension n of U is finite. Then $\mathcal{H}^n(U) > 0$ by general results in dimension theory.

By [Kle99], the geometric dimension of U is n as well. Therefore, there are no points in U at which the tangent space T_xU contains an $(n+1)$-dimensional Euclidean space. In particular, all points of U are contained in $Sing_{n+1}(X)$.

Due to Theorem 1.6, U is a countable union of biLipschitz images of subsets of $\mathbb{R}^n$. Therefore, the Hausdorff dimension of U is at most n.

Due to Lemma 8.2, there are no $(n+1, \frac{1}{4\cdot(n+1)})$-strainer maps defined on subsets of U. Thus, there are no $(n+1, \frac{1}{4\cdot(n+1)})$-strained points in U. Due to Proposition 10.4, $\mathcal{H}^n(U) < C \cdot r_0^n$, for some C depending only on N.

Applying Theorem 1.6 again, we find a point $x \in X$ such that the tangent space T_x has $\mathbb{R}^n$ as a direct factor. It T_x is not equal to $\mathbb{R}^n$ then it contains $\mathbb{R}^n \times [0,\infty)$. But this is impossible, since the geometric dimension of X is n. Therefore, $T_x = \mathbb{R}^n$.

This finishes the proof. $\square$

From now on we fix some bound $n_0 = n_0(N)$ on the dimension on U provided by Corollary 5.3 and set $\delta_0 = \delta_0(N) = \frac{1}{50\cdot n_0}$. We can now relate the dichotomy observed in Proposition 9.3 to the dimension.

Corollary 11.2. *Let $F : V \to \mathbb{R}^k$ be a (k,δ)-strainer map on a connected open subset V of a tiny ball U. If $\delta < \delta_0(N)$ then one of the following possibilities occurs:*

1. *No fiber of F in V has isolated points. Then $\dim(W) > k$, for every open subset $W \subset V$.*
2. *V is a k-dimensional topological manifold. Then for every $x \in V$ and every r, such such $3r$ is smaller than the straining radius of F at x, the map $F : B_r(x) \to F(B_r(x))$ is L-biLipschitz, where L goes to 1 as δ goes to 0.*

Proof. By Proposition 9.3 either no fiber of F has isolated points or the map F is locally injective. In the second case, for any $x \in V$ and $r > 0$ are as in the statement above, we deduce from Lemma 9.2 and Lemma 8.2 that $F : B_r(x) \to F(B_r(x))$ is L-biLipschitz with $L = 2\sqrt{k}$. The statement that L can be chosen close to 1 if δ goes to 0 follows from Corollary 8.4. Since $F(V)$ is open in $\mathbb{R}^k$, we see that V is a k-dimensional manifold.

In the first case, let $x \in V$ be arbitrary. Since the fiber of F through x is not finite, we apply Theorem 10.3 and find points arbitrary close to x which are $(k+1, 12 \cdot \delta)$-strained. Then, by Proposition 8.2, the dimension of any small ball around x is at least $k+1$. □

Now we can finish

Proof of Theorem 1.1. Given any GCBA space X we cover X by a countable number of tiny balls U and reduce all statements to the case of tiny balls. For any tiny ball U, the topological dimension n equals the Hausdorff dimension by Proposition 11.1. Moreover, by Proposition 11.1 there exists an (n, δ)-strainer map $F : V \to \mathbb{R}^n$ for arbitrary small δ and some $V \subset U$. Applying Corollary 11.2, we see that V is a topological manifold. Hence, n equal the maximal dimension of a Euclidean ball which embeds into U as an open set. □

11.2. **Lower bound on the measure.** The Euclidean spheres are the smallest GCBA spaces with the same dimension and curvature bound, compare [Nag02]:

Proposition 11.3. *Let Σ be a compact GCBA space, which is* CAT(1) *and of dimension n. Then there exists a 1-Lipschitz surjection $P : \Sigma \to \mathbb{S}^n$.*

Proof. By Proposition 11.1 we find a point $x \in \Sigma$ with T_x isometric to $\mathbb{R}^n$. Then one can define a surjective 1-Lipschitz map $P : \Sigma \to \mathbb{S}^0 * \Sigma_x = \mathbb{S}^n$ as the "spherical logarithmic map", i.e. the composition of the logarithmic map in Σ and the exponential map in $\mathbb{S}^n$, see [Lyt05b, Lemma 2.2]. □

11.3. **Dimension and convergence.** We are going to describe possible behaviour of dimension under convergence.

Lemma 11.4. *Let $\tilde{U}_l$ converge to $\tilde{U}$ as in the standard setting for convergence. Let $x \in U$ be a limit point of $x_l \in U_l$. If $\dim(T_x) = n$ then there exists some $\epsilon > 0$ and $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$ the ball $B_\epsilon(x_l)$ has dimension n.*

In particular, $\dim(T_{x_l}) \leq n$, for all l large enough.

Proof. Assume first that there is some $\epsilon > 0$, such that $\dim(B_\epsilon(x_l)) < n$ for infinitely many l. Due to the semicontinuity of the geometric dimension under convergence (cf. [Lyt05b, Lemma 11.1]) we conclude that $\dim(\bar{B}_r(x)) < n$ for any $r < \epsilon$. But then $\dim(T_x) < n$, by the definition of geometric dimension, in contradiction to our assumption.

Assuming that the statement of the lemma is wrong, we can therefore choose a subsequence and assume that $\dim(B_{\frac{1}{l}}(x_l)) = m + 1 > n$, for some fixed m (since the dimensions in question are bounded by Proposition 5.9). Since the dimension equals the geometric dimension, we find some $y_l \in B_{\frac{1}{l}}(x_l)$ with $\dim(\Sigma_{y_l}) = m$.

Due to Proposition 11.3, any Σ_{y_l} and then also any limit space Σ' of this sequence, admits a surjective 1-Lipschitz map onto $\mathbb{S}^m$. Therefore, the Hausdorff dimension of Σ' is at least m. Due to Lemma 5.11, Σ_x admits a surjective 1-Lipschitz map onto Σ', since y_l converge to x. Hence the Hausdorff dimension of Σ_x is at least m as well. But this contradicts $\dim(T_x) = n \leq m$.

This contradiction finishes the proof. □

Let X again be GCBA . As in the introduction we consider the *k-dimensional part* X^k of X as the set of all points $x \in X$ with $\dim(T_x) = k$. Applying Lemma 11.4 to the case of constant sequence $X_l = X$ we directly see:

Corollary 11.5. *A point $x \in X$ is contained in X^k if and only if there is some $\epsilon > 0$, such that for all $r < \epsilon$ we have $\dim(B_r(x)) = k$. The closure of X^k in X does not contain points from X^m with $m < k$.*

In the strained case we get more stability:

Lemma 11.6. *In the notations of Lemma 11.4 above, assume the point $x \in U$ is (k,δ)-strained. Then for all sufficiently large l, we have $\dim(T_{x_l}) \geq k$.*

If $\dim(T_{x_l}) = k$ for all l large enough then $n = k$, hence $\dim(T_{x_l}) = \dim(T_x)$ for all l large enough.

Proof. We find some (k,δ)-strainer map F in a neighborhood of x, defined by a k-tuple (p_i). We approximate this tuple by k-tuples in $\tilde{U}_l$ and obtain distance maps $F_l : \tilde{U}_l \to \mathbb{R}^k$ converging to F. Moreover, for all l large enough, F_l is a (k,δ)-strainer map at x with a uniform lower bound $3r$ on the straining radii of F_l at x_l, Lemma 7.5 and Lemma 7.8. Due to Lemma 8.2, the dimension of any ball around x_l must be at least k, hence $\dim(T_{x_l}) \geq k$.

Assume $\dim(T_{x_l}) = k$, for all l large enough. Due to Corollary 11.2, the restriction of the strainer maps F_l to the ball $B_r(x_l)$ is L-biLipschitz. Therefore, so is the restriction of F to $B_r(x)$. Applying Corollary 11.2 again, we see that $B_r(x)$ is a k-dimensional manifold, hence $n = k$. □

11.4. **Regular parts.** We fix now some $\delta \leq \delta_0$. By the *k-regular part* of U we denote the set of (k,δ)-strained points $x \in U$ with $\dim(T_x) = k$. We now easily see:

Corollary 11.7. *Let U be a tiny ball of radius r_0 and size bounded by N. Then for every k the set $Reg_k(U)$ of k-regular points is open in U. This set is a Lipschitz manifold and dense in the k-dimensional part U^k of U. The topological boundary $\partial Reg_k(U) := U \cap (\bar{U}^k \setminus Reg_k(U))$ of $Reg_k(U)$ in U does not contain (k,δ)-strained points. Moreover,*

$$\mathcal{H}^{k-1}(\bar{U}^k \setminus Reg_k(U)) < C \cdot r_0^{k-1} \text{ and } \mathcal{H}^k(U^k) < C \cdot r_0^k \,,$$

for some constant C depending only N and the choice of δ.

Proof. Any point x in $Reg_k(U)$ admits a (k,δ)-strainer map F. Due to Corollary 11.2, the restriction of F to a small ball around x is biLipschitz onto an open subset of $\mathbb{R}^k$. Hence, this ball is contained in U^k and consists of (k,δ)-strained points. Therefore, $Reg_k(U)$ is open in X and locally biLipschitz to $\mathbb{R}^k$.

Let $x \in U^k$ be arbitrary. Then any sufficiently small ball W around x has dimension k, Corollary 11.5. Hence, W contains (k,δ)-strained points, therefore points from $Reg_k(U)$. Thus, $Reg_k(U)$ is dense in U^k.

Assume that $x \in \bar{U}^k$ is (k,δ)-strained. Writing x as a limit of points $x_l \in Reg_k(U)$ and applying Lemma 11.6, we see $\dim(T_x) = k$. Hence $x \in Reg_k(U)$.

No point in U^k is $(k+1,\delta)$-strained, due to Lemma 8.2. Thus the bounds on measures are contained in Theorem 10.3. □

11.5. **Conclusions.** We finish the proofs of two theorems from the introduction.

Proof of Theorem 1.2. Thus let X be GCBA and k a natural number. As we have seen in Corollary 11.5, a point $x \in X$ is in the k-dimensional part X^k if and only if all sufficiently small balls around x have dimension k.

Cover X by a countable collection of tiny balls. For each of these tiny balls U consider its k-regular part and let $M^k \subset X^k$ denote the union of these k-regular parts. Due to Corollary 11.7, this subset M^k is open in X, dense in X^k and locally biLipschitz to $\mathbb{R}^k$. Moreover, $\bar{X}^k \setminus M^k$ is a countable union of subsets of finite $(k-1)$-dimensional Hausdorff measure.

Every nonempty $V \subset X^k$ which is open in X^k, contains an open non-empty subset of M^k hence $\mathcal{H}^k(V) > 0$. From Corollary 11.7, we deduce that the measure $\mathcal{H}^k(X^k \cap U)$ is finite for every tiny ball U.

This finishes the proof. □

Recall from the introduction that the *canonical measure* μ_X on X is the sum over all $k = 0, 1, ...$ of the restrictions of $\mathcal{H}^k$ to X^k.

Proof of Theorem 1.4. Let X again be GCBA. If $x \in X$ satisfies $\dim(T_x) = k$, thus $x \in X^k$, then the measure $\mathcal{H}^k \llcorner X^k$ is positive on any neighborhood V of x, due to Theorem 1.2. Hence $\mu_X(V) > 0$.

On the other hand, the dimension of any tiny ball U in X is finite, hence only finitely many of the measures $\mathcal{H}^k \llcorner X^k$ can be non-zero on U. Due to Corollary 11.7, the measure $\mathcal{H}^k(X^k \cap U)$ is finite, hence so is $\mu_X(U)$.

Therefore, the measure μ_X is finite on any relatively compact subset of X. □

12. Stability of the canonical measure

12.1. **Setting and Preparations.** We are going to prove here Theorem 1.5 and its local generalization. First we recall the notion of measured Gromov-Hausdorff convergence, sufficient for our purposes.

Let Z_l be a sequence of compact spaces GH-converging to a compact set Z. Let $\mathcal{M}_l$ be a Radon measure on Z_l and let $\mathcal{M}$ be a Radon measure on Z. *The measures $\mathcal{M}_l$ converge to $\mathcal{M}$* if for any compact sets $K_l \subset Z_l$ converging to $K \subset Z$ the following holds true:

$$\lim_{\epsilon \to 0} (\liminf_{l \to \infty} \mathcal{M}_l(B_\epsilon(K_l)) = \lim_{\epsilon \to 0} (\limsup_{l \to \infty} \mathcal{M}_l(B_\epsilon(K_l)) = \mathcal{M}(K) \,. \tag{12.1}$$

By general results, any sequence of Radon measures $\mathcal{M}_l$ on Z_l contains a converging subsequence if the total measures $\mathcal{M}_l(Z_l)$ are uniformly bounded.

We continue working in the standard setting for convergence as in Definition 5.2. We fix some $k = 0, 1....$ and restrict our attention to the k-dimensional part $\mu_U^k = \mathcal{H}^k \llcorner U^k$ of μ.

The aim of this section is the following:

Theorem 12.1. *Under the GH-convergence $\tilde{U}_l \to \tilde{U}$ the k-dimensional parts of the canonical measures $\mathcal{M}_l := \mu_{U_l}^k$ converge to $\mathcal{M} := \mu_U^k$ locally on U. Thus,* (12.1) *holds for all compact $K \subset U$.*

We know that $\mathcal{M}_l(\bar{U}_l)$ is uniformly bounded by a constant C, Corollary 11.7. Thus, by general compactness of measures, we may choose a subsequence and assume that the measures $\mathcal{M}_l$ converge to a finite Radon measure $\mathcal{N}$ on $\bar{U}$. We need to verify that $\mathcal{M} = \mathcal{N}$ on U.

It suffices to prove that $\mathcal{N}$ coincides with $\mathcal{H}^k$ on the regular part $Reg_k(U)$ and that $\mathcal{N}$ vanishes on the complement $U \setminus Reg_k(U)$. We fix δ as in Subsection 11.4.

12.2. **Regular part.** In order to prove that $\mathcal{N}$ and $\mathcal{H}^k$ coincide on the regular part $Reg_k(U)$ we note that $\mathcal{N}$ satisfies $\mathcal{N}(B_r(x)) \leq C \cdot r^k$ whenever $\bar{B}_r(x) \subset U$. Indeed, this inequality is true for all the approximating measures, by Corollary 11.7. Thus $\mathcal{N}$ is absolutely continuous on the Lipschitz manifold $Reg_k(U)$ with respect to $\mathcal{H}^k$. By general measure theory, it suffices to prove that for $\mathcal{H}^k$-almost every point $x \in Reg_k(U)$ the density

$$b(x) := \lim_{r\to 0} \frac{\mathcal{N}(B_r(x))}{\mathcal{H}^k(B_r(x))},$$

exists and is equal to 1.

Due to Theorem 1.6, $\mathcal{H}^k$-almost every point in $Reg_k(U)$ has as tangent space $T_x = \mathbb{R}^k$. Let x be such a point and let x_l be a sequence of points in U_l converging to x. We take points $p_1, ..., p_k \in \tilde{U}$ such that the directions $(xp_i)'$ are pairwise orthogonal in $T_x = \mathbb{R}^k$. Then the distance map $F : U \to \mathbb{R}^k$ defined by the k-tuple (p_i) is a (k, δ)-strainer map at x for any $\delta > 0$. Consider a sequence of distance maps $F_l : U_l \to \mathbb{R}^k$ converging to F.

For any $\delta > 0$ we find some $r > 0$ and some $l_0 > 0$ such that F and F_l, for $l \geq l_0$, are (k, δ)-strainer maps with straining radius at least $3r$ at x and x_l respectively. Then the maps $F : B_r(x) \to \mathbb{R}^k$ and $F_l : B_r(x_l) \to \mathbb{R}^k$ are L-biLipschitz onto their images and L goes to 1 as δ goes to 0, due to Lemma 11.6 and Corollary 11.2. Moreover, by Corollary 8.4, the images contain balls with radius $r/2$ around $F(x)$ and $F_l(x_l)$ respectively.

Thus, for any $s < \frac{r}{10}$ and all sufficiently large l the volumes $\mathcal{H}^k(B_s(x)), \mathcal{H}^k(B_s(x_l))$ are bounded between $L^{-2k} \cdot \omega_k \cdot s^k$ and $L^{2k} \cdot \omega_k \cdot s^k$, where ω_k denotes the volume of the k-dimensional Euclidean unit ball.

Since L goes to 1, as δ goes to 0, we conclude $b(x) = 1$ at every point $x \in U$ with $T_x = \mathbb{R}^k$.

12.3. **Singular part.** The support S of $\mathcal{N}$ in U is contained in the limit set of the supports of $\mu^k_{U_l}$, thus in the set of all points $x \in U$ which are limits of a sequence of k-regular points $x_l \in U_l$. Due to Lemma 11.6, any such point x which is not in $Reg_k(U)$ cannot be (k, δ)-strained.

Therefore, $T := S \setminus Reg_k(U)$ is a closed subset of $U \setminus U_{k,\delta}$ of points which are not (k, δ)-strained. Note that $\mathcal{H}^k(T) = 0$, by Theorem 1.6. It is enough to prove that $\mathcal{N}(K) = 0$ for any compact subset K of T.

Fix a compact subset K in T and a sequence of compact $K_l \subset U_l$ converging to K. Let finally $t > 0$ be arbitrary. It suffices to find some $s = s(t) > 0$ such that for all sufficiently large l

$$\mu^k_{U_l}(B_s(K_l)) = \mathcal{H}^k(B_s(K_l) \cap Reg_k(U_l)) < t\,.$$

As in Subsection 5.3 denote by $\mathcal{A}_\delta \subset \tilde{U}$ some maximal $\delta \cdot r_0$-separated subset in the distance sphere $\mathcal{A}$ of radius r_0 around U. Numerate the elements of $\mathcal{A}_\delta$ as $\mathcal{A}_\delta = \{p_1, ..., p_m\}$ and approximate any p_i by points p^l_i in the distance sphere of radius r_0 around U_l in $\tilde{U}_l$.

For all l large enough, the points $\{p^l_1, ..., p^l_m\}$ are $\delta \cdot r_0$-dense in the distance sphere of radius r_0 around U_l. Denote by $\mathcal{F}_\delta$ the set of distance maps $F : \tilde{U} \to \mathbb{R}^k$

defined by k-tuples in $\mathcal{A}_\delta$. Denote by $\mathcal{F}^l_\delta$ the corresponding lifts to distance maps $F_l : \tilde{U}_l \to \mathbb{R}^k$. We numerate the elements of F_δ and $\mathcal{F}^l_\delta$ as $G_1, ..., G_j, ...$ and $G^l_1, ..., G^l_j, ...$, respectively. These are finite sets (with m^m elements). For any j, the distance maps G^l_j converge to G_j.

The argument in Lemma 10.1 shows, that for all l large enough the following holds true: If a point x_l in U_l is (k,δ)-strained then there exists some G^l_j which is a $(k, 3\cdot\delta)$-strainer map at x_l.

From Theorem 10.3 and the finiteness of the elements in $\mathcal{F}_\delta$, we get a number $C > 0$ such that for any measurable subset $Y \subset Reg_k(U_l)$ we have

$$\mathcal{H}^k(Y) \leq C \cdot \max_j \mathcal{H}^k(G^j_l(Y)) \,.$$

Since all the maps G^l_j are $2\sqrt{k}$-Lipschitz, the image $G^l_j(B_s(K_l))$ is contained in the $2\sqrt{k}\cdot s$-tubular neighborhood around $G^j_l(K_l)$. Thus, for all l large enough, $G^l_j(B_s(K_l))$ is contained in the $3\sqrt{k}\cdot s$-tubular neighborhood around $G^j(K)$. But $\mathcal{H}^k(K) = 0$, hence $\mathcal{H}^k(G^j(K))$ is 0 for all j. Thus, for all sufficiently small s_0, the $3\cdot\sqrt{k}\cdot s_0$-tubular neighborhood around the compact set $G^j(K)$ has $\mathcal{H}^k$-measure less than t.

By the previous considerations, for such s_0 we have $\mu^k(B_{s_0}(K_l)) \leq C\cdot t$. Since t was arbitrary, this proves the claim.

12.4. **Conclusions.** We can now finish the

Proofs of Theorem 12.1 and Theorem 1.5. The proof of Theorem 12.1 follows from the combination of the two Subsections above.

In order to prove Theorem 1.5, assume that X_l are compact GCBA spaces with uniform bounds on dimension, curvature and injectivity radius. If X_l converge in the GH-topology to a space X then they are covered by a uniform number of uniformly bounded tiny balls by Proposition 5.9. Thus the total canonical measures $\mu_{X_l}(X_l)$ are uniformly bounded by Corollary 11.7. Hence, upon choosing a subsequence we may assume that μ_{X_l} converges to a measure $\mathcal{M}$ on the limit space X. Applying the local statement of Theorem 12.1, we see that $\mathcal{M}$ coincides with the canonical measure μ_X.

Therefore, it remains only to show that a uniform upper bound on the total measures $\mu_{X_l}(X_l)$ implies that the sequence is precompact in the GH-topology.

Assume the contrary. Then, applying Proposition 5.9, we find tiny balls U_l in X_l of the same radius r_0 such that $\mu_{U_l}(U_l)$ converges to 0.

From the uniform upper bound on the dimension, the 2-Lipschitz property of the logarithmic maps and Proposition 11.3, we find for any $s > 0$ some $\epsilon > 0$ such that the following holds true for any l and any $x_l \in U_l$. If T_{x_l} is k-dimensional and if the ball $B_s(x_l)$ is contained in U_l then $\mathcal{H}^k(B_s(x_l)) \geq \epsilon$.

Set $s = \frac{r_0}{2n}$, where n is the common upper bound on the dimensions of U_l. With ϵ as above, our assumption implies $\mu_{U_l}(U_l) < \epsilon$, for all l large enough. Then, by the above estimate, we deduce that for any k and any point x in the k-dimensional part of U_l the ball $B_s(x)$ contains points from some k'-dimensional part with $k' > k$, if $B_s(x) \subset U_l$. Starting in the center of U_l we obtain a finite sequence of points such that the distance of any two consecutive points is less than s and the dimensions of the tangent spaces of the elements of the sequence strictly increase. Thus we will find a point in U with tangent space of dimension larger than n, in contradiction to our assumption. □

12.5. **Additional comments on the measure-theoretic structure of GCBA spaces.** Let X be a GCBA space and k a natural number.

For any point $x \in X$ we consider a tiny ball U around x and apply Theorem 12.1 to the convergence of rescaled spaces $(\frac{1}{r}\tilde{U}, x) \to T_x$. We deduce that μ^k has a well-defined *k-dimensional density at the point x*

$$\lim_{r \to 0} \frac{\mu^k_U(B_r(x))}{r^k} = \mu^k_{T_x}(B_1(0))\,.$$

Let now x be a point in X^k and let $\epsilon > 0$ be as in Corollary 11.5. For every $z \in B_\epsilon(x) \cap X^k$ and every $r < \epsilon - d(x,z)$ the measure $\mu^k(B_r(z))$ is bounded from below by $C(k) \cdot r^k$, due to the Lipschitz property of the logarithmic map and Proposition 11.3. Here $C(k)$ is a positive constant depending only on k. Together with Corollary 11.7, we see that the restriction of μ^k to X^k is *locally Ahlfors k-regular*, cf. [Hei01].

Finally, we note that the Hausdorff dimension and the *rough dimension*, of any open relatively compact subset U in X coincide, we refer to [BGP92] and [Ber02] for the definition of the rough dimension and a discussion of this question. Indeed, a slightly more thorough look into the proof of Proposition 10.4 and Theorem 10.3 reveals the following claim, for any tiny ball U: For fixed $\delta > 0$ and for $\epsilon \to 0$, any ϵ-separated subset of $U \setminus U_{k,\delta}$ has at most $O(\epsilon^{1-k})$ elements. This means, by definition, that the *rough dimension* of the set $U \setminus U_{k,\delta}$ is at most $k-1$. For $k = \dim(U) + 1$, and $\delta < \frac{1}{4k}$, we deduce that $\dim(U)$ coincides with the rough dimension of U.

13. Homotopic stability

Let $U_l \subset \tilde{U}_l$ and $U \subset \tilde{U}$ be as in our standard setting for convergence, Definition 5.2. We have the following general stability result:

Theorem 13.1. *Under the standard setting for convergence let the distance maps $F_l : \bar{U}_l \to \mathbb{R}^k$ converge to the distance map $F : \bar{U} \to \mathbb{R}^k$. Assume that the restriction of F to an open set $V \subset U$ is a (k,δ)-strainer map. Let $\mathfrak{t}_l \to \mathfrak{t}$ be a converging sequence in $\mathbb{R}^k$ and assume that the fiber $\Pi := F^{-1}(\mathfrak{t}) \subset V$ is compact. Let, finally, $K_l \subset U_l$ be compact sets converging to Π.*

Then there exists $r > 0$ such that the following holds true, for all l large enough. The restriction of F_l to $V_l = B_r(K_l)$ is a (k,δ)-strainer map, the fibers $\Pi_l := F_l^{-1}(\mathfrak{t}_l) \subset V_l$ are compact and converge to Π.

Finally, Π_l is homotopy equivalent to Π, for all l large enough.

Proof. By the compactness of Π we find some $r > 0$, such that for any $x \in \Pi$ the straining radius of x with respect to F is larger than $2r$. Due to Lemma 7.5, for all l large enough F_l is a (k,δ)-strainer in V_l. Moreover, for all l large enough and any $x_l \in V_l$ the straining radius of F_l at x_l is at least r, Lemma 7.8.

The maps F_l are $2\sqrt{k}$-open on V_l. This implies that Π_l converges to Π.

For all l large enough, all balls in Π_l of all radii $s < r$ are contractible, due to Theorem 9.1. The homotopy equivalence of Π_l and Π is now a direct consequence of the general homotopy stability theorem [Pet90, Theorem A]. □

We discuss two special cases. The first one is an immediate application of the theorem in the case of a constant sequence $X^l = X$.

Corollary 13.2. *Let $F : V \to \mathbb{R}^k$ be a (k,δ)-strainer map defined on an open subset V of the tiny ball U. Assume that a fiber Π of F is compact. Then all fibers of F, sufficiently close to Π, are homotopy equivalent to Π.*

As a second application we obtain

Proof of Theorem 1.9. Indeed, for any point x in a GCBA space X, we find a tiny ball $U \subset \tilde{U}$ containing x. Consider an arbitrary sequence of positive numbers t_l converging 0 and the metric spheres Π_l of radius t_l around x. Consider the convergence of the rescaled space $(\frac{1}{t_l}\tilde{U}, x) \to (T_x, 0)$, provided by Corollary 5.6. In the tangent cone T_x the origin is a $(1,\delta)$-strainer at any point of $T_x \setminus \{0\}$, for any $\delta > 0$. Moreover, the fiber $F^{-1}(1)$ of the distance function F to the vertex of the cone is exactly $\Sigma_x \subset T_x$. The sphere Π_l is exactly the fiber $d_x^{-1}(1)$ of the distance functions d_x on $\frac{1}{t_l} \cdot \tilde{U}$. From Theorem 13.1, we deduce that Π_l and Σ_x are homotopy equivalent for all l large enough. Since the sequence (t_l) was arbitrary, this finishes the proof. $\square$

Using in addition Lemma 5.5, one could observe, that a homotopy equivalence in Theorem 1.9 is provided by the logarithmic map.

14. The differentiable structure

This section is essentially a rewording of [Per94]. The only result not having a direct analogue in [Per94] is Proposition 14.6.

14.1. **Setting.** In this section we are going to prove Theorem 1.3. The statement is local, so we may restrict ourselves to a tiny ball U and assume that U coincides with its set of k-regular points. Hence we may assume that U is a k-dimensional manifold and that every point $x \in U$ is (k,δ)-strained for a sufficiently small $\delta = \delta(k)$.

14.2. **Euclidean points.** For any point $x \in U$ a neighborhood of x in U is biLipschitz to a k-dimensional Euclidean ball. Therefore, $T_x U$ is biLipschitz to $\mathbb{R}^k$. If $T_x U$ is a direct product $T_x U = \mathbb{R}^{k-1} \times Y$ for some space Y then Y must be $\mathbb{R}$. (Indeed, Y must be a 1-dimensional cone over a finite set. By homological considerations we see that it must be the cone over a two-point space). Therefore, a point $x \in U$ whose tangent cone T_x has $\mathbb{R}^{k-1}$ as a direct factor satisfies $T_x = \mathbb{R}^k$.

We call $x \in U$ with $T_x = \mathbb{R}^k$ a *Euclidean point*. Let $\mathcal{R}$ denote the set of all Euclidean points in U. From Theorem 1.6 and the previous conclusion, we deduce that $U \setminus \mathcal{R}$ has Hausdorff dimension at most $k-2$.

14.3. **Charts, differentials, Riemannian metric.** Let now $V \subset U$ be an open, convex subset and assume that F and G are opposite (k,δ)-strainer maps in V. Let F be defined by the k-tuple (p_i). Due to Corollary 11.2, the map $F : V \to \mathbb{R}^k$ is locally L-biLipschitz with $L \leq 2\sqrt{k}$. Moreover, L goes to 1 as δ goes to 0. Thus, for any $x \in V$, the differential $D_x F : T_x \to \mathbb{R}^k$ is an L-biLipschitz map.

If $F(x) = F(y)$ for some $x, y \in V$ then, using Lemma 7.3 and the first formula of variation, we see

$$D_x F(v) \leq \sqrt{k} \cdot \delta \leq \frac{1}{20\sqrt{k}} \,,$$

where $v \in \Sigma_x \subset T_x$ is the starting direction of the geodesic xy. This contradicts the fact that $D_x F$ is L-biLipschitz. Therefore, $F : V \to F(V)$ is injective.

The preimage F^{-1} is (directionally) differentiable at all points of $\hat{V} := F(V)$ with differentials being the inverse maps of the differentials of F. This differentiability just means, that compositions of any differentiable curve with F^{-1} has well-defined directions in all points.

For any Euclidean point $x \in \mathcal{R}$ the differential $D_x F : T_x \to \mathbb{R}^k$ is a linear map, as directly seen from the first formula of variation. We have the following continuity property in $\mathcal{R}$. Let $v_l \in \Sigma_{x_l}$ be the starting direction of a geodesic γ_l. Let $x_l \in \mathcal{R}$ converge to a point $x \in \mathcal{R}$ and the geodesics γ_l converge to a geodesic γ with starting direction $v \in T_x$. Then $D_{x_l} F(v_l)$ converge in $\mathbb{R}^k$ to $D_x F(v)$. Indeed, this is just the reformulation of the statement that angles between $x_l p_i$, for $i = 1, ..., k$, and γ_l converge to the angle between $p_i x$ and γ. The last statement can be easiest seen as a consequence of Lemma 5.11 and the fact that the concerned spaces of directions are all unit spheres.

We denote the image $\hat{V} = F(V)$ together with the map $F^{-1} : \hat{V} \to V \subset U$ as a *metric chart*. On this metric chart, we have the subset $\hat{\mathcal{R}} := F(\mathcal{R})$ whose complement in $\hat{V}$ has Hausdorff dimension at most $k-2$. For any point $y \in \hat{\mathcal{R}}$ we get a scalar product on its tangent space $T_y \mathbb{R}^k$, given by the pullback (via the linear map $D_y F^{-1}$) of the scalar product on $T_{F^{-1}(y)} U$. Due to the previous considerations, this Riemannian metric g_F is continuous on $\hat{\mathcal{R}}$.

Expressing the length of a Lipschitz curve as an integral of its pointwise velocities, we immediately see that for any Lipschitz curve γ in $\mathcal{R}$ the length of γ coincides with the length of $\bar{\gamma} := F \circ \gamma$ with respect to the Riemannian metric g_F, hence

$$\ell(\gamma) = \int |\bar{\gamma}'(t)|_{g_F} \, dt \, .$$

14.4. **DC-maps in Euclidean spaces.** We refer the reader to [Per94] and [AB15] for more details.

A function $f : V \to \mathbb{R}$ on an open subset V of $\mathbb{R}^m$ is called a DC-*function* if in a neighborhood of each point $x \in V$ one can write f as a difference of two convex functions. The set of DC-functions contains all functions of class $C^{1,1}$ and it is closed under addition and multiplication.

A map $F : V \to \mathbb{R}^l$ is called a DC-map if its coordinates are DC. The composition of DC-maps is again a DC-map. In other words, a map $F : V \to \mathbb{R}^l$ is DC if and only if for every DC-function $g : W \subset \mathbb{R}^l \to \mathbb{R}$, the composition $g \circ F$ is a DC-function on $F^{-1}(W)$.

The last statement is due to the following two easy facts which will play a role below, cf. [Per94].

- For intevals I, J, a convex function $f : I \to J$ and a convex non-decreasing function $g : J \to \mathbb{R}$ the composition $g \circ f$ is convex.
- Any convex L-Lipschitz map $F : U \to \mathbb{R}$ on an open convex set in $\mathbb{R}^n$ can be written as $F = F_1 - F_2$, with $F_2(x) = L \cdot (x_1 + x_2 + \cdots + x_n)$. Then F_1 and F_2 are convex $L \cdot (n+1)$-Lipschitz functions on U, which are increasing in all coordinates.

14.5. **DC-maps on metric spaces.** The following definition is meaningful only if the metric spaces in question are (locally) geodesic.

Definition 14.1. Let Y be a metric space. A function $f : Y \to \mathbb{R}$ is called a DC-*function* if it can be locally represented as difference of two Lipschitz continuous convex functions.

Due to the corresponding statements about DC-functions on intervals, the set of DC functions on Y is closed under addition and multiplication.

Remark 14.1. We refer to [Pet07] for the definition and properties of semi-convexity. Assume that in Y each point x admits a Lipschitz 1-convex function in a small neighborhood V of x. Then each semi-convex function on Y is DC. Such strongly convex functions exist on Alexandrov spaces with lower curvature bound, [Pet07]. On any CAT(κ) space X we get such a function as a scalar multiple of d_x^2.

We use compositions to define DC-maps between metric spaces.

Definition 14.2. A locally Lipschitz map $F \colon Z \to Y$ between metric spaces Z and Y is called a DC-*map* if for each DC-function $f \colon U \to \mathbb{R}$ defined on an open subset U of Y the composition $f \circ F$ is DC on $F^{-1}(U)$. If F is a biLipschitz homeomorphism and its inverse is DC, then we call F a DC-*isomorphism.*

We immediately see that a composition of DC-maps is a DC-map. For a map $F : Z \to \mathbb{R}^l$ we recover the old definition: F is DC if and only if the coordinates of F are DC.

14.6. **Crucial observation.** Let now $U \subset \tilde{U} \subset X$ be again a tiny ball consisting of k-regular points as above. Since all distance functions to points in $\tilde{U}$ are convex, each (k,δ)-strainer map is a DC-map by definition. These strainer maps turn out to be DC-isomorphisms, in direct analogy with [Per91], see also [AB15]. The proof of the following observation is taken from [Per91].

Proposition 14.1. *Let F and G be opposite (k,δ)-strainer maps in an open subset V of a tiny ball U. Then $F : V \to F(V) \subset \mathbb{R}^k$ is a* DC*-isomorphism, if $\delta \leq \frac{1}{4\cdot k^2}$.*

Proof. Denote by $f_i = d_{p_i}$ the coordinates of F. We already know that the map F is a locally biLipschitz DC-map. It remains to prove that the inverse map $F^{-1} \colon F(V) \to V$ is DC too. Thus, given an open subset $O \subset V$ and a convex function $g \colon O \to \mathbb{R}$, we have to show that the function $\overline{g} = g \circ F^{-1}$ is DC on $F(O)$.

We introduce the following auxiliary notion. We say that a convex Lipschitz continuous function $g : O \to \mathbb{R}$ on an open subset $O \subset V$ is α-*special* for some $\alpha \geq 0$ if the following holds true. For any $x \in O$ and any unit vector $v \in T_x$ such that $D_x f_i(v) \geq 0$ for all $i = 1, ..., k$ we have $D_x g(v) \leq -\alpha$.

If g is α-special then, for any Lipschitz curve $\eta : [a,b] \to O$ parametrized by arclength and such that all f_i are non-decreasing on η, the composition $g \circ \eta : [a,b] \to \mathbb{R}$ decreases at least with velocity α.

The proof of the Proposition will follow from two auxiliary statements:

Lemma 14.2. *There is a 1-Lipschitz α-special function g on V with $\alpha = \frac{1}{4\cdot k^2}$.*

Lemma 14.3. *If g is a 0-special function in O then the composition $\bar{g} = g \circ F^{-1}$ is a convex function on $F(O)$.*

Indeed, assuming Lemma 14.2 and Lemma 14.3 to be true we derive:

Corollary 14.4. *In the notations above let $h : O \to \mathbb{R}$ be an L_1-Lipschitz convex function. Then h can be represented as $h = h_1 - h_2$ with h_1 and h_2 being 0-special $L_1 \cdot (1 + \frac{1}{\alpha})$-Lipschitz functions.*

Moreover, $\bar{h} := h \circ F^{-1}$ is the difference of two $C \cdot L_0$-Lipschitz convex functions, with some C depending only on k.

Proof. Indeed, choosing g as in Lemma 14.2, we set $h_2(x) := \frac{L_0}{\alpha} \cdot g(x)$. Then h_2 is L_0-special. Since h is convex and L-Lipschitz, we deduce that the function $h_1 = g + h_2$ is convex and 0-special. The statement about the Lipschitz constants of h_1 and h_2 is clear.

Due to Lemma 14.3, the compositions $\bar{h}_i := h_i \circ F^{-1}$ are convex on $F(O)$. The Lipschitz constants of $\bar{h}_i$ are bounded from above by the product of the Lipschitz constants of h_i and F^{-1}. □

Thus assuming Lemma 14.2 and Lemma 14.3 to be true, we finish the proof of the proposition. □

We turn to the auxiliary lemmas used in Proposition 14.1.

Proof of Lemma 14.2. Let f_i be the coordinates of F and let g_i be the coordiates of G. The functions g_i are convex and 1-Lipschitz, for $i = 1, ..., k$, hence so is $g(x) = \frac{1}{k} \sum_{i=1}^{k} g_i(x)$. We claim that g is $\frac{1}{4 \cdot k^2}$-special.

Indeed, let $x \in V$ be arbitrary and let $v \in \Sigma_x$ be such that $D_x f_i(v) \geq 0$, for all $i = 1, ..., k$. Then $D_x g_i(v) < \delta$ for all $i = 1, ..., k$, as follows directly from the first formula of variation and the definition of opposite strainer maps.

The map $D_x F : T_x \to \mathbb{R}^k$ is $2\sqrt{k}$-biLipschitz, thus $D_x F(v)$ has norm at least $\frac{1}{2\sqrt{k}}$. Therefore, for at least one $1 \leq j \leq k$, we must have $D_x f_j(v) \geq \frac{1}{2k}$.

For this j we get, $D_x g_j(v) \leq -\frac{1}{2k} + \delta$. Summing up, we get

$$D_x g(v) \leq \frac{1}{k} \cdot ((-\frac{1}{2k} + \delta) + (k-1) \cdot \delta) \leq \frac{1}{k} \cdot (-\frac{1}{2k} + \frac{1}{4 \cdot k}) \leq \frac{1}{4 \cdot k^2} \, .$$

This finishes the proof. □

Proof of Lemma 14.3. It follows word by word as in [Per94], since the proof in [Per94] only uses convexity and differentiability and does not use lower curvature bounds. □

14.7. **The Riemannian metric revisted.** As in [Per94], we have:

Lemma 14.5. *For any metric chart $F : V \to \mathbb{R}^k$ as above, the Riemannian metric g_F defined and continuous on the subset $\hat{\mathcal{R}} = F(\mathcal{R})$ is locally of bounded variation. Moreover, g_F is differentiable almost everywhere in $F(\mathcal{R})$.*

The proof follows from [Per94] (see also [AB15]) literally without changes. The main idea is to take a sufficiently large generic set of points q_j in $\bar{U}$. The the distance functions h_j to these points have the following property. The compositions $\bar{h}_j := h_j \circ F^{-1}$ are DC-functions by Proposition 14.1. On the other hand, since h_j are distance functions, the gradients of h_j at all points of $\hat{\mathcal{R}}$ have norm 1 with respect to the Riemannian metric g_F. One obtains a linear equation for the coordinates of g_F and shows that they can be expressed through the first derivatives of the DC-functions $\bar{h}_j$.

14.8. DC**-curves in GCBA spaces.** In order to prove that the Riemannian structure on the set $\mathcal{R}$ determines the metric, we will need a stability statement about variations of DC-curves, which might be of independent interest. In the following definition and Proposition 14.6 we work in general GCBA spaces, and not only in their regular parts as in the rest of this section.

Let $U \subset \tilde{U}$ be a tiny ball as above. We say that a curve $\gamma : I \to \bar{U}$ on a compact interval I is a DC-*curve of norm bounded by* A if γ is A-Lipschitz and for any 1-Lipschitz convex function $f : \bar{U} \to \mathbb{R}$ the restriction $f \circ \gamma$ can be (globally) written as a difference of two A-Lipschitz convex functions on I.

The following statement is closely related to the well-known fact [AR89], that the lengths is continuous under convergence of curves of uniformly bounded turn in the Euclidean space.

Proposition 14.6. *Let $\gamma_l : I \to \bar{U}$ be* DC-*curves with a uniform bound on the norms. If γ_l converges to γ pointwise then $\lim_{l\to\infty} \ell(\gamma_l) = \ell(\gamma)$.*

Proof. Assuming the contrary and choosing a subsequence, we find $\epsilon > 0$ with

$$(1 + 2 \cdot \epsilon)^2 \cdot \ell(\gamma) < \lim_{l\to\infty} \ell(\gamma_l)\,.$$

Due to Proposition 5.2, we find a distance map $F_\epsilon : \bar{U} \to \mathbb{R}^m_\infty$, which is a $(1 + \epsilon)$-biLipschitz embedding, if $\mathbb{R}^m$ is equipped with the sup-norm $|\cdot|_\infty$. Set $\eta_l = F_\epsilon \circ \gamma_l$ and $\eta = F_\epsilon \circ \gamma$. From the biLipschitz property we obtain a contradiction, once we show that the lengths of η_l converge to the length of η in $\mathbb{R}^m_\infty$.

The i-th coordinate of η_l is the composition of γ_l and a convex distance function d_{p_i}. Thus, this i-th coordinate is a difference of two convex A-Lipschitz functions h_l^+ and h_l^- on I. Adding a constant we may assume that h_l^+ equals 0 at some fixed point on I.

Going to subsequences, we may assume that h_l^+ and h_l^- converge to h^+ and h^- such that $h^+ - h^-$ is the corresponding coordinate of η. Due to the standard results about convergence of convex functions, we see that at almost every $t \in I$, the differentials of h_l^+, h_l^- exists at t and converge to the differentials of h^+, h^- at t. Taking again all coordinates together, we see that for almost every $t \in I$, the differentials $\eta_l'(t) \in \mathbb{R}^m$ exist and converge to $\eta'(t)$.

Expressing the length of η and η_l as integrals of $|\cdot|_\infty$-norms of η' and η_l' over I we finish the proof of the convergence. This finishes the proof of the Proposition. ☐

Coming back to the regular part, we can use this result to prove:

Corollary 14.7. *Let $F : V \to \mathbb{R}^k$ be a metric chart as in Subsection 14.3, with convex $V \subset U$. Let S be a subset of V with $\mathcal{H}^{k-1}(S) = 0$. Then every pair of points $x, y \in V \setminus S$ is connected in $V \setminus S$ by curves of lengths arbitrary close to $d(x, y)$.*

Proof. The statement is well-known and easy to prove for open convex subsets $\hat{V}$ in $\mathbb{R}^k$, connecting x and y concatenations of two segments.

Since the claim is true in $F(V)$ and the map F is biLipschitz, it suffices to prove the following claim. Let $\gamma : I \to V$ be a geodesic. Then there exist curves $\gamma_l : I \to V \setminus S$ converging to γ and such that $\ell(\gamma_l)$ converges to $\ell(\gamma)$. (Once such γ_l are constructed we obtain the desired curves by connecting the endpoints of γ_l with x and y within $V \setminus S$).

In order to find such γ_l we consider the curve $\eta := F \circ \gamma$ in $\hat{V}$. Note that the differentials of η at different points have distance at most $k \cdot \delta$ from each other.

Take a small ball B around the origin in the hyperplane of $\mathbb{R}^k$ orthogonal to the starting direction of η. Then we observe that the map $Q : B \times I \to \mathbb{R}^k$ given by $Q(x,t) = x + \eta(t)$ is a biLipschitz embedding.

This implies that for almost every $x_0 \in B$ the curve $t \to \eta(t) + x_0$ does not meet the set $F(S)$ with vanishing $\mathcal{H}^{k-1}$-measure. Letting x_0 going to 0 we find a sequence of translates $\eta_l(t) = \eta(t) + x_l$ converging to η and disjoint from $F(S)$.

We set $\gamma_l = F^{-1} \circ \eta_l$. It suffices to prove that $\ell(\gamma_l)$ converge to $\ell(\gamma)$.

Clearly, the curves γ_l are uniformly Lipschitz. Let f be a convex 1-Lipschitz function on V. We have $f \circ \gamma_l = f \circ F^{-1} \circ \eta_l$.

Due to Corollary 14.4, $f \circ F^{-1}$ is the difference of two convex A-Lipschitz functions h_1 and h_2 on $F(V)$, where A is independent of f. On the other hand, the curve η is a DC-curve of bounded norm, since its coordinates are convex 1-Lipschitz functions. The curves η_l are then also DC-curves with the same bound on the norm. Together, this implies that $f \circ \gamma_l$ can be written as a difference of two convex B-Lipschitz functions with some B independent of l.

Hence γ_l are DC-curves of uniformly bounded norm and the claim follows from Proposition 14.6. □

14.9. **Conclusions.** Now we can summarize the results to the

Proof of Theorem 1.3. Define as above M^k to be the set of all (k,δ)-strained points in the k-dimensional part X^k, with $\delta \leq \frac{1}{4 \cdot k^2}$. We have seen in Theorem 1.2, that M^k is a Lipschitz manifold.

For any open convex set V with opposite (k,δ)-strainer maps $F, G : V \to \mathbb{R}^k$, the map $F : V \to F(V)$ is a biLipschitz map onto an open subset of $\mathbb{R}^k$. Moreover, F is a DC-isomorphism, Proposition 14.1. Thus, the set of all such charts provides M^k with a DC-atlas.

On the set of Euclidean points $\mathcal{R}$ in M^k we get a Riemannian metric g_F in any chart. Moreover, $M^k \setminus \mathcal{R}$ has Hausdorf dimension at most $k-2$ as shown in Subsection 14.2. Due to the intrinsic definition, this metric is globally well defined on $\mathcal{R}$. As shown in Subsection 14.3, the Riemannian tensor is continuous on $\mathcal{R}$ and due to Lemma 14.5, it is locally of bounded variation.

The length of all curves contained in $\mathcal{R}$ is computed via the Riemannian metric. Finally, the length of all curves in $\mathcal{R}$ locally determines the metric in M^k due to Corollary 14.7. □

14.10. **Second order differentiability of DC-functions.** As in the case of Alexandrov spaces described in [Per94], all DC-functions are almost everywhere twice differentiable, as stated in the following Proposition.

Proposition 14.8. *Let U be a a tiny ball which coincides with its k-regular part and let $f : U \to \mathbb{R}$ be a* DC*-function. Then, for $\mathcal{H}^k$-almost all $x \in \mathcal{R} \cap U$, there exists a bilinear form $H_x = H_x(f) : T_x \times T_x \to \mathbb{R}$, called the* Hessian *of f at x, such that the following holds true. The remainder $R_x : U \to \mathbb{R}$ in the* Taylor formula

$$R_x(y) := f(y) - (f(x) + D_x f(v) + H_x f(v,v)), \tag{14.1}$$

where $v := \log_x(y)$, satisfies

$$\lim_{y \to x} \frac{R_x(y)}{d(x,y)^2} = 0. \tag{14.2}$$

We only sketch the proof and refer for details to [Per94] and [AB15, Section 7.2].

Using a coordinate change to "normal coordinates" as in [Per94] and [AB15], Proposition 14.8 follows directly from the corresponding theorem of Alexandrov in $\mathbb{R}^n$, [EG15, Theorem 6.9], once the following lemma is verified. In the formulation of the lemma and later on, we denote by o as usual the *Landau symbol*.

Lemma 14.9. *Let $G : V \to \mathbb{R}^k$ be a DC-isomorphism on an open subset $V \subset U$, given by a composition of a metric chart F and a diffeomorphism of $\mathbb{R}^k$. Let $x \in \mathcal{R}$ be a Euclidean point with $G(x) = 0$. Assume that the metric tensor g of V expressed on $W = G(V)$ via G satisfies, for all $y \in V \cap \mathcal{R}$,*

$$||g(G(y)) - g(G(x))|| = o(d(x, y)) . \tag{14.3}$$

Then, for all $y \in V$ and the corresponding direction $v = \log_x(y) \in T_x$, we have

$$||G(y) - D_x G(v)|| = o(d(x, y)^2) .$$

Proof. We sketch the proof, referring for details to [Per94] and [AB15, Section 7.2].

From (14.3), and the fact that the Riemannian tensor on $\mathcal{R}$ determines the metric in V, Corollary 14.7, we obtain, for all small r, and all $y, z \in \bar{B}_r(x)$, the estimate

$$|(d(y, z) - ||G(y) - G(z)||)| = o(r^2) . \tag{14.4}$$

Hence, it suffices to prove, that for all $y \in \bar{B}_r(x)$ the angle $\beta(y)$ between $G(y)$ and $D_x G(v)$, (with $v = \log_x(y)$ as in the formulation) satisfies the estimate $\beta(y) = o(r)$.

In order to prove this estimate, it is sufficient to show that for the midpoint m of the geodesic xy the angle $\beta_1(y)$ between $G(y)$ and $G(m)$ satisfies $\beta_1(y) = o(r)$. (Relying only on (14.4) one can show $\beta_1(y) = o(\sqrt{r})$ and as a consequence that $\beta(y) = o(\sqrt{r})$, as done in the course of the proof of [AB15, Proposition 7.8 (d)].) In order to prove the required stronger estimate $\beta_1(y) = o(r)$, we will rely on the curvature bound, similarly to [Per94].

We say that the triangle xyz in $\bar{B}_r(x)$ is *sufficiently non-degenerated*, respectively *very non-degenerated*, if all of its comparison angles are at least $\frac{\pi}{100}$, respectively at least $\frac{\pi}{10}$. For any sufficiently non-degenerated triangle xyz in $\bar{B}_r(x)$, we deduce from (14.4), that the comparison angle $\tilde{\angle} yxz$ differs from the angle between $G(y)$ and $G(z)$ in $\mathbb{R}^k$ by at most $o(r)$.

Given a very non-degenerated triangle xyz in $\bar{B}_r(x)$, we find a point $w \in \bar{B}_r(x)$ such that the triangles xyw and xzw are sufficiently non-degenerated and such that

$$\angle yxz + \angle yxw + \angle wxz = 2\pi .$$

Since the corresponding comparison angles are not smaller and since the three angles between pairs of different vectors in $\{G(y), G(z), G(w)\}$ sum up to at most 2π, we arrive at the following conclusion:

For any very non-degenerated triangle xyz in $\bar{B}_r(x)$ the angle $\angle yxz$ differs from the angle in $\mathbb{R}^k$ between $G(y)$ and $G(z)$ by at most $o(r)$.

Let now $y \in \bar{B}_r(x)$ be arbitrary and let m be the midpoint of xy. We find a point z with $d(x, y) = d(x, z)$, such that $G(z)$ lies in the same plane as $G(y)$ and $G(m)$ and such that $G(z)$ is orthogonal to $G(y)$. Then the difference of the angle between $G(z)$ and $G(y)$ and the angle between $G(z)$ and $G(m)$ is exactly the angle between $G(y)$ and $G(m)$. On the other hand, due to the previous considerations, the angle between $G(z)$ and $G(y)$ (respectively, between $G(z)$ and $G(m)$) coincides with $\angle zxy$ (respectively, with $\angle zxm$) up to $o(r)$. But, by construction, $\angle zxy = \angle zxm$.

Therefore, we have verified the estimate $\beta_1(y) = o(r)$, thus finishing the proof of the Lemma and of Proposition 14.8. $\square$

15. Topological counterexamples

Example 15.1. Let X_n denote a unit circle S with two other unit circles $S_n^{\pm}$ attached to S at points $p_n^{\pm}$ at distance $1/n$ from each other. The sequence X_n converges to the wedge of three unit circles X. Thus, X_n is not homeomorphic to X for no n. This shows that there is no topological stability even in dimension 1.

Example 15.2. Kleiner pointed out in [Kle99] a construction of a locally compact geodesically complete 2-dimensional CAT(0) space that does not admit a triangulation. This example, built by gluing four half-planes together with a complicated intersection pattern (discussed in detail in [Nag00, Example 2.7]), shows that two-dimensional GCBA spaces do not need to be locally conical. Moreover, in this example there is a point $x \in X$ and arbitrary small $r_1, r_2 > 0$ such that the distance spheres $\partial B_{r_1}(x)$ and $\partial B_{r_2}(x)$ are not homeomorphic.

References

[AB15] L. Ambrosio and J. Bertrand. DC calculus. *arXiv:1505.04817*, 2015.

[AK00] L. Ambrosio and B. Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Ann.*, 318(3):527–555, 2000.

[AKP16] S. Alexander, V. Kapovitch, and A. Petrunin. Alexandrov geometry. *preprint, https://www.math.psu.edu/petrunin/papers/alexandrov-geometry/*, 2016.

[Ale57] A. D. Alexandrow. Über eine Verallgemeinerung der Riemannschen Geometrie. *Schr. Forschungsinst. Math.*, 1:33–84, 1957.

[AR89] A. D. Alexandrov and Yu. G. Reshetnyak. *General theory of irregular curves*, volume 29 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1989. Translated from the Russian by L. Ya. Yuzina.

[Bal95] W. Ballmann. *Lectures on spaces of nonpositive curvature*, volume 25 of *DMV Seminar*. Birkhäuser Verlag, Basel, 1995. With an appendix by Misha Brin.

[Bal04] W. Ballmann. On the geometry of metric spaces. *Preprint, lecture notes, http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/sin40827.pdf*, 2004.

[BB95] W. Ballmann and M. Brin. Orbihedra of nonpositive curvature. *Inst. Hautes Études Sci. Publ. Math.*, 82:169–209, 1995.

[BB98] Yu. D. Burago and S. V. Buyalo. Metrics with upper-bounded curvature on 2-polyhedra. II. *Algebra i Analiz*, 10(4):62–112, 1998.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[Ber83] V. Berestovskii. Borsuk's problem on metrization of a polyhedron. *Dokl. Akad. Nauk SSSR*, 268(2):273–277, 1983.

[Ber02] V. Berestovskii. Pathologies in Alexandrov spaces with curvature bounded above. *Siberian Adv. Math.*, 12(4):1–18 (2003), 2002.

[BGP92] Yu. Burago, M. Gromov, and G. Perelman. A. D. Aleksandrov spaces with curvatures bounded below. *Uspekhi Mat. Nauk*, 47(2(284)):3–51, 222, 1992.

[BH99] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[BK12] J. Bertrand and B. Kloeckner. A geometric study of Wasserstein spaces: Hadamard spaces. *J. Topol. Anal.*, 4(4):515–542, 2012.

[BS07] S. Buyalo and V. Schroeder. Spaces of curvature bounded above. In *Surveys in differential geometry. Vol. XI*, volume 11 of *Surv. Differ. Geom.*, pages 295–327. Int. Press, Somerville, MA, 2007.

[CM09] P. Caprace and N. Monod. Isometry groups of non-positively curved spaces: discrete subgroups. *J. Topol.*, 2(4):701–746, 2009.

[EG15] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, revised edition, 2015.

[GS17] A. Genevois and A. Stocker. Barely $CAT(-1)$ groups are cylindrically hyperbolic. *arXiv:1712.04736*, 2017.

[Hei01] J. Heinonen. *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001.

[Kap02] V. Kapovitch. Regularity of limits of noncollapsing sequences of manifolds. *Geom. Funct. Anal.*, 12(1):121–137, 2002.

[Kap07] V. Kapovitch. Perelman's stability theorem. In *Surveys in differential geometry. Vol. XI*, volume 11 of *Surv. Differ. Geom.*, pages 103–136. Int. Press, Somerville, MA, 2007.

[KK17] V. Kapovitch and C. Ketterer. CD meets CAT. *arXiv:1712.02839*, 2017.

[Kle99] B. Kleiner. The local structure of spaces with curvature bounded above. *Math. Z.*, 231:409–456, 1999.

[KMS01] K. Kuwae, Y. Machigashira, and T. Shioya. Sobolev spaces, Laplacian and heat kernel on Alexandrov spaces. *Math. Z.*, 238, 2001.

[Kra11] L. Kramer. On the local structure and homology of $CAT(\kappa)$ spaces and buidlings. *Advances in Geometry*, 11:347–369, 2011.

[Li15] Nan Li. Lipschitz-volume rigidity in Alexandrov geometry. *Adv. Math.*, 275:114–146, 2015.

[LN18] A. Lytchak and K. Nagano. Topological regularity of spaces with an upper curvature bound. *In preparation*, 2018.

[LS07] A. Lytchak and V. Schroeder. Affine functions on $CAT(\kappa)$ spaces. *Math. Z.*, 255:231–244, 2007.

[Lyt05a] A. Lytchak. Open map theorem for metric spaces. *Algebra i Analiz*, 17(3):139–159, 2005.

[Lyt05b] A. Lytchak. Rigidity of spherical buildings and joins. *Geom. Funct. Anal.*, 15(3):720–752, 2005.

[MN17] A. Mondino and A. Naber. Structure theory of metric measure spaces with lower Ricci curvature bounds. *arXiv:1405.2222v3, to appear in J. Eur. Math. Soc.*, 2017.

[Nag00] K. Nagano. Asymptotic rigidity of Hadamard 2-spaces. *J. Math. Soc. Japan*, 52(4):699–723, 2000.

[Nag02] K. Nagano. A volume convergence theorem for Alexandrov spaces with curvature bounded above. *Math. Z.*, 241(1):127–163, 2002.

[Ont05] P. Ontaneda. Cocompact CAT(0) spaces are almost geodesically complete. *Topology*, 44(1):47–62, 2005.

[OS94] Y. Otsu and T. Shioya. The Riemannian structure of Alexandrov spaces. *J. Differential Geom.*, 39(3):629–658, 1994.

[OT99] Y. Otsu and H. Tanoue. The Riemannian structure of Alexandrov spaces with curvature bounded above. *preprint*, 199?

[Ots97] Y. Otsu. Differential geometric aspects of Alexandrov spaces. In K. Grove and P. Petersen, editors, *Comparison Geometry*, volume 30 of *M.S.R.I. Publ.*, pages 135–148. Cambridge Univ. Press, 1997.

[Per91] G. Perelman. A. D. Alexandrov spaces with curvature bounded below II. *preprint*, 1991.

[Per94] G. Perelman. DC-structures on Alexandrov spaces. *preprint, preliminary version*, 1994.

[Pet90] P. Petersen. A finiteness theorem for metric spaces. *J. Differential Geom.*, 31:387–395, 1990.

[Pet07] A. Petrunin. Semiconcave functions in Alexandrov's geometry. In *Surveys in differential geometry. Vol. XI*, volume 11 of *Surv. Differ. Geom.*, pages 137–201. Int. Press, Somerville, MA, 2007.

[Res93] Yu. G. Reshetnyak. Two-dimensional manifolds of bounded curvature. In *Geometry, IV*, volume 70 of *Encyclopaedia Math. Sci.*, pages 3–163, 245–250. Springer, Berlin, 1993.

E-mail address, Alexander Lytchak: `alytchak@math.uni-koeln.de`
E-mail address, Koichi Nagano: `nagano@math.tsukuba.ac.jp`

(Alexander Lytchak)
Mathematisches Institut, Universität Köln
Weyertal 86-90, D-50937 Köln, Germany

(Koichi Nagano)

Institute of Mathematics, University of Tsukuba
Tennodai 1-1-1, Tsukuba, Ibaraki, 305-8571, Japan