LORENTZ MEETS LIPSCHITZ

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ABSTRACT. We show that maximal causal curves for a Lipschitz continuous Lorentzian metric admit a $\mathcal{C}^{1,1}$-parametrization and that they solve the geodesic equation in the sense of Filippov in this parametrization. Our proof shows that maximal causal curves are either everywhere lightlike or everywhere timelike. Furthermore, the proof demonstrates that maximal causal curves for an α-Hölder continuous Lorentzian metric admit a $\mathcal{C}^{1,\frac{\alpha}{4}}$-parametrization.

1. INTRODUCTION

Causality theory is a very active part of mathematical general relativity and Lorentzian geometry, see the current comprehensive review [Min19b]. Recently, in several works [FS12], [CG12], [KSSV14], [Min15], [Säm16], [BS18], [Min19a], [GKSS20] the theory was extended to the setting of non-smooth Lorentzian and Lorentz-Finsler manifolds and towards a synthetic theory [KS18, AGKS19, AHCPS20, CM20], where there might not be a differentiable or manifold structure, analogous to length or Alexandrov spaces in metric geometry. Moreover, issues of low regularity came into focus lately, as they arise in the study of the cosmic censorship conjecture, in particular the (in-)extendibility of spacetimes [Sbi18, GL17, GL18, GLS18, GKS19, LO19a, LO19b, MS19, Sbi20].

The regularity class of $\mathcal{C}^{1,1}$ for the Lorentzian metric turned out to mark a transition between the classical theory and low regularity phenomena. In this regularity, classical results concerning geodesics, causality theory and singularity theorems, remain valid, see [KSS14, Min15, KSSV14, KSSV15, KSV15, GGKS18, SS18], whereas for lower

regularity differences occur [CG12, GKSS20] and concepts diverge. In particular, the notions of maximal curves and geodesics (solutions to the geodesic equations) no longer coincide.

If the metric is $\mathcal{C}^1$ then solutions to the geodesic equation still exist but might not be unique. Nevertheless, in the recent work [Gra20] Graf establishes the Hawking and Penrose singularity theorems in this regularity class and along the way shows that in a (globally hyperbolic) $\mathcal{C}^1$-spacetime any two causally related points are connected by a maximal $\mathcal{C}^2$-geodesic. Building on these results, Schinnerl [Sch20] proved that in a $\mathcal{C}^1$-spacetime any maximal curve is of $\mathcal{C}^2$-regularity and possesses a parametrization solving the geodesic equation. The latter result of Graf and the one of Schinnerl also follow from our work.

Within the low-regularity regime Lipschitz continuous Lorentzian metrics play an especially important role as they cover physically most relevant cases: e.g. spacetimes where there is a loss of regularity on a hypersurface such as shell-crossing singularities, thin shells of matter, and surface layers, cf. [BH03, SSV17, Lak17], or impulsive gravitational waves in the so-called Rosen form, see [GP09, Ch. 20], and especially [PSSŠ15, PSSŠ16], where the Filippov solution concept was employed to establish $\mathcal{C}^1$-regularity of geodesics.

Below this regularity many fundamental statements in causality theory finally cease to hold, e.g. the future of a point need not be open [GKSS20], the push-up principle fails and, most surprisingly, there are Hölder spacetimes where the boundary of a lightcone has positive measure (so-called causal bubbles) [CG12, Ex. 1.11]. On the other hand, several fundamental properties of the causality theory have been verified in continuous spacetimes, [CG12, Säm16, Min19a]. Various other questions concerning causality theory and properties of maximal curves in Lipschitz continuous Lorentzian manifolds have been formulated in [SS18]. The aim of this article is the resolution of most of these questions.

1.1. **Statement of results.** All results discussed in the paper are of local nature, thus we will formulate them only for the case of globally hyperbolic Lorentzian manifolds (cf. [Säm16]).

Given a globally hyperbolic Lorentzian manifold (M, g) with a Lipschitz continuous metric, a causal (Lipschitz continuous) curve $\gamma : [a, b] \to M$ is called maximal if it maximizes the Lorentzian length among all causal curves connecting its endpoints. Our main result for such curves reads as follows.

Theorem 1.1. *Let γ be a maximal curve in a Lipschitz continuous Lorentzian metric. Then γ admits a parametrization making it a $\mathcal{C}^{1,1}$-curve. Moreover, in this parametrization, γ is a solution of the geodesic equation in the sense of Filippov.*

If the Lorentzian metric is assumed to be $\mathcal{C}^1$ then solutions of the geodesic equation in the sense of Filippov are exactly solutions of the geodesic equation in the classical sense and are automatically $\mathcal{C}^2$. Thus, for $\mathcal{C}^1$-regular Lorentzian metrics our results recover [Gra20, Prop. 2.13] and [Sch20].

The proof of Theorem 1.1 shows that for timelike maximal curves, their parametrization with constant Lorentzian velocity satisfies the conclusion of Theorem 1.1, thus provides a solution of the geodesic equation in the sense of Filippov.

Due to a recent result by Graf and Ling ([GL18, Thm. 1.1]) for a Lipschitz continuous Lorentzian metric any maximal curve is either lightlike or timelike almost everywhere. We strengthen this result, see Proposition 9.1, and provide an independent proof of it:

Proposition 1.2. *Let $\gamma : I \to M$ be a maximal curve for a Lipschitz continuous Lorentzian metric. Then γ can be parametrized in a $\mathcal{C}^{1,1}$ way, and either $\gamma'(t)$ is lightlike for all t, or $\gamma'(t)$ is timelike for all t.*

We mention that the proof does not rely on the push-up principle, established for Lipschitz continuous Lorentzian metrics in [CG12]; in fact, the push-up principle can be easily deduced from Proposition 1.2.

1.2. **Additional comments.** We would like to mention a few related statements. First, a direct adaptation of the proof of Proposition 7.2 reveals the following (probably non-optimal) statement:

Proposition 1.3. *Maximal curves for an α-Hölder continuous Lorentzian metric admit a $\mathcal{C}^{1,\frac{\alpha}{4}}$-parametrization.*

The second comment is concerned with the optimality of the $\mathcal{C}^{1,1}$-regularity in Theorem 1.1, whose unclear status was pointed out in [SS18]. In a fixed chart, in which the metric is Lipschitz continuous, the maximal curves do not have to be of class $\mathcal{C}^2$. Indeed, we can start with the Minkowski metric g_0 on $\mathbb{R}^2$ and pull it back by a $\mathcal{C}^{1,1}$-diffeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that the preimage $\gamma^* := \phi^{-1}(\gamma)$ of the y-axes γ is not $\mathcal{C}^2$. Then, γ^* is a maximal curve in the metric $\phi^*(g_0)$, and γ^* is not $\mathcal{C}^2$.

The following question also surely has a negative answer, but a proof would require some considerable amount of new analytic ideas, similar to [DK81], [Sv76] in low regularity: Given a $\mathcal{C}^{0,1}$-Lorentzian metric g

on $U \subset \mathbb{R}^n$, does there exist some coordinates around any point, in which the metric is still $\mathcal{C}^{0,1}$ and all maximal curves are of class $\mathcal{C}^2$?

The third remark concerns the regularity of solutions of the geodesic equation in the sense of Filippov, see [Fil88], [SS18] and Section 3 for a discussion of the corresponding concepts. The following general statement shows that the second part of Theorem 1.1 implies the first and thereby also answers a question posed in [SS18].

Proposition 1.4. *For any Lorentzian metric g of class $\mathcal{C}^{0,1}$ on an open domain $U \subset \mathbb{R}^n$, any solution $\gamma : [0,1] \to U$ of the geodesic equation in the sense of Filippov is of class $\mathcal{C}^{1,1}$.*

Moreover, the $\mathcal{C}^{1,1}$-norm of γ is bounded in terms of the $\mathcal{C}^{0,1}$-norm of g and the Euclidean length of γ. The proof of this proposition is a direct consequence of a classical result by Filippov [Fil88, Thm. 7.8] and holds true for any pseudo-Riemannian metric g (see also [Ste14]).

The first half of Theorem 1.1 is the analog of the corresponding regularity statement for shortest curves in Riemannian metrics, [LY06], and its proof follows the ideas in [LY06]. The proof that maximal curves in $\mathcal{C}^{0,1}$-Lorentzian metrics are Filippov-geodesics is an extension of the classical "first variation" argument and transfers literally to the Riemannian case (even with some technical issues disappearing). It shows that any shortest curve in a Riemannian $\mathcal{C}^{0,1}$-metric solves the geodesic equation in the sense of Filippov.

Note, however, that solutions of the geodesic equation in the sense of Filippov are neither uniquely defined nor have any extremality properties, even in the Riemannian case, unless the regularity of the metric g is assumed to be $\mathcal{C}^{1,1}$, see [Har50, HW51]. In particular, solutions to the geodesic equations need not be maximal (on any subinterval) nor unique and maximal curves need not be unique, neither.

It seems to be a challenging question to understand the geodesic flow in Lorentzian (or Riemannian) metrics of low regularity, cf. [Amb08], [KLP17]. Finally, it appears to be possible to use our Theorem 1.1 to extend the singularity theorem of Hawking to the setting of $\mathcal{C}^{0,1}$-Lorentzian metrics (this will be explained in a forthcoming article).

1.3. **Structure of the text.** After introducing notations and conventions in Section 2, we discuss properties of solutions of geodesic equations in the sense of Filippov, prove Proposition 1.4 and show that the class of Filippov geodesics is stable under pointwise convergence in Section 3. In Section 4 we show that a maximal curve solves the geodesic equation, if it is already known to be timelike, $\mathcal{C}^{1,1}$ and parametrized with constant Lorentzian velocity.

In Section 5 we state a version of a result of [CH70], [LY06] showing that the property of (not) being a $\mathcal{C}^{1,\alpha}$-curve in a Euclidean space can be characterized in terms of the deviations of the curve from the chords connecting points on the curve.

In Section 6 we provide a quantitative version of the triangle inequality in Minkowski space, which to the best of our knowledge seems be a completely new result.

In the next sections we fix a chart and show that all maximal curves in this chart are $\mathcal{C}^{1,1}$. The basic idea, going back to the Riemannian situation analyzed in [CH70] and [LY06], is that if the maximal curve were too far from a chord, then the chord would have longer Lorentzian length than the maximal curve. One of the arising difficulties is that, a priori, the Euclidean chord does not need to be causal.

In Section 7 we prove Proposition 1.3, relying on the quantitative triangle inequality. Here we compare the Lorentzian length of the curve and the chord with respect to the Minkowski metric of a fixed point on the chord, and then estimate the error terms arising from the non-constancy of the Lorentzian metric. Also in the Riemannian case, this estimate is too coarse to obtain the optimal $\mathcal{C}^{1,1}$-regularity, for Lipschitz continuous metrics. However, for the rest of the argument we actually only need that maximal curves are $\mathcal{C}^1$.

In Section 8 we consider maximal curves whose tangent vectors are timelike everywhere. We compare the lengths of the curve and its chords by estimating the differences of Lorentzian velocities with respect to varying Minkowski metrics. Also this part follows the Riemannian case in [LY06], and supplies us with the $\mathcal{C}^{1,1}$-regularity for uniformly timelike maximal curves.

In Section 9 we prove Proposition 1.2 and finish the proof of the main theorem for timelike maximal curves.

In the final section 10 we show that any lightlike maximal curve can be obtained as a limit of maximal curves which solve the geodesic equation. By stability, it shows that any maximal curve solves the geodesic equation in the sense of Filippov and an application of Proposition 1.4 then finishes the proof of the main theorem.

2. NOTATION AND CONVENTIONS

We use the convention that a Lorentzian metric g has signature $+ - - - \ldots$ and a vector $v \in T_p M$ is *causal* if $g_p(v, v) \geq 0$ ($v \neq 0$), *timelike* if $g_p(v, v) > 0$, *null* or *lightlike* if $g_p(v, v) = 0$ ($v \neq 0$) and *spacelike* if $g_p(v, v) < 0$ or $v = 0$. Corresponding Lorentzian norms are denoted as $|v|_p := \sqrt{|g_p(v, v)|}$. A locally Lipschitz continuous curve

$\gamma\colon I \to M$ is *timelike/causal/lightlike* if $\dot\gamma$ is timelike/causal/lightlike almost everywhere.

On the Euclidean space $\mathbb{R}^n$ we denote by $\|\cdot\|$ the Euclidean norm. We denote the standard Minkowski product on $\mathbb{R}^n$ as $\langle\cdot,\cdot\rangle$,

$$\langle u, v\rangle := u_t v_t - u_x^T v_x \ ,$$

where we denote for a vector $w \in \mathbb{R}^n$ by $w_t \in \mathbb{R}$ its first and by $w_x \in \mathbb{R}^{n-1}$ its last $(n-1)$ coordinates. The Euclidean scalar product will never appear in this text. Otherwise we follow the conventions used in [CG12, GKSS20].

A Lorentzian $\mathcal{C}^{0,1}$-manifold denotes a smooth manifold M with a fixed smooth atlas and a Lorentzian metric g whose coordinates in any chart are Lipschitz continuous.

Most of the time we work locally on an open set $U \subset \mathbb{R}^n$. In such a chart we denote the Lipschitz constant of g as L:

$$|g_x(v, w) - g_y(v, w)| \le L \cdot \|x - y\| \ ,$$

for all unit vectors $v, w \in \mathbb{R}^n$ and all $x, y \in U$.

By making the chart smaller and the Lipschitz constant larger, if needed, we can always assume that for any point $x \in U$ the chart U can be changed, so that g_x coincides with the standard Minkowski product $\langle\cdot,\cdot\rangle$, (while keeping the Lipschitz constant L for g).

Making the chart U smaller we can assume that the first coordinate vector T is future directed timelike. More precisely, we may assume that the first coordinate $t(u) := u_t$ as function on $\mathbb{R}^n$ grows with velocity at least $\frac{1}{2}$ on any future directed causal curve $\gamma : I \to U$ parametrized by Euclidean arclength, i.e.

$$(t \circ \gamma)' \ge \frac{1}{2}$$

almost everywhere. In particular, any causal curve γ in U parametrized by arclength is a bilipschitz curve with bilipschitz constant 2. Our Lorentzian manifold M and therefore also all charts will always be assumed globally hyperbolic (cf. the proof of [SS18, Thm. 2.2]).

By $\mathcal{L}(\gamma) = \mathcal{L}_g(\gamma)$ we denote the Lorentzian length of a causal curve $\gamma : I \to M$:

$$\mathcal{L}(\gamma) = \int_I \sqrt{g(\gamma'(t), \gamma'(t))} \, dt \ .$$

A causal curve $\gamma\colon I \to M$ is called maximal if its Lorentzian length is maximal among all causal curves with the same endpoints. Any subcurve of any maximal curve is maximal.

3. Filippov geodesics

3.1. Filippov geodesics, their regularity and stability. Let M be a Lorentzian manifold with Lorentzian metric g of class $C^{0,1}$. Then there exists a unique Levi-Civita connection ∇^g on M. This connection assigns to any pair of locally Lipschitz vector fields X, Y a locally bounded vector field $\nabla^g_X Y$ so that the usual rules (torsion freeness and metric property) are satisfied, see, for instance, [Hon14].

In any chart $U \subset \mathbb{R}^n$ the Levi–Civita connection differs from the directional derivative $DY(X)$ by a tensor Γ, called the Christoffel symbol, which is a bounded symmetric $(1,2)$-tensor on U.

The usual Koszul formula provides a formula for Γ via the partial derivatives of g, which exist almost everywhere on U. More precisely, the value of Γ is defined in all points in which g is differentiable.

Let U be a chart as in Section 2. We say that a curve $\gamma : I \to U$ is a (Filippov-) geodesic if γ' exists almost everywhere, is absolutely continuous, and γ is a solution of the differential equation

$$\gamma''(t) + \Gamma_{\gamma(t)}(\gamma'(t), \gamma'(t)) = 0$$

in the sense of differential inclusions of Filippov [Fil88], [SS18]. This is the case if for almost every $t \in I$ the point $-\gamma''(t)$ is contained in the essential convex hull

$$\hat{\Gamma}_{\gamma(t)}(\gamma'(t), \gamma'(t)) := \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \mathcal{K}(\delta, N),$$

where μ is the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}^n$ and $\mathcal{K}(\delta, N)$ is the closed convex hull

$$\mathcal{K}(\delta, N) := \mathrm{co}\{\Gamma_x(w, w) \ : \ (x, w) \in (U \times \mathbb{R}^n) \setminus N \ ;$$
$$\|(x, w) - (\gamma(t), \gamma'(t))\| < \delta\} \, .$$

Due to the continuity of $\Gamma_x(v, v)$ in the second variable v for almost all $x \in U$, it suffices in the definition of the essential convex hull $\hat{\Gamma}_\gamma(\gamma', \gamma')$ to consider subsets N with $\mu(N) = 0$ of the form $N = N_0 \times \mathbb{R}^n$.

The notion is invariant under smooth changes of coordinates and therefore, we can unambiguously talk about Filippov-geodesics in the Lorentzian manifold M, by requiring that the intersection of the curve with any chart is a Filippov-geodesic in this chart.

Remark 3.1. Filippov-geodesics are invariant under $\mathcal{C}^2$-changes of coordinates. However, it is not clear to us if Filippov geodesics are invariant under $\mathcal{C}^{1,1}$-changes of coordinates, the most natural class in the context of $\mathcal{C}^{0,1}$-metrics.

We can now provide:

Proof of Proposition 1.4. The statement is local, thus we may assume that the neighborhood U is small enough as in Section 2, in particular, g is globally Lipschitz continuous on U. Let $\gamma : [0, 1] \to U$ be a solution of the geodesic equation in the sense of Filippov. Then γ' is absolutely continuous, in particular, $\|\gamma'(t)\|$ is bounded by some constant C_1.

For almost every $x \in U$ the Christoffel symbol Γ_x has a norm (as a bilinear map) bounded by a constant C_2 depending on the Lipschitz norm of g. Hence, for almost all $x \in U$ and all $t \in [0, 1]$ we have

$$\|\Gamma_x(\gamma'(t), \gamma'(t))\| \leq C_1^2 \cdot C_2 \ .$$

Thus the same bound is true for any convex hull of vectors of the form $\Gamma_x(\gamma'(t), \gamma'(t))$. The definition of being a solution of the geodesic equation now implies for almost all $t \in [0, 1]$

$$\|\gamma''(t)\| \leq C_2 \cdot C_1^2 \ .$$

Thus $\gamma'(t)$ is Lipschitz continuous and γ is of class $\mathcal{C}^{1,1}$. $\qquad\square$

As the proof shows, the $\mathcal{C}^{1,1}$-norm of γ depends only on the Lipschitz constant of g and an upper bound on the velocity $\|\gamma'\|$ of γ. Thus, due to the next observation, the $\mathcal{C}^{1,1}$-norm is even controlled by the length:

Lemma 3.1. *Let $\gamma : [a, b] \to U$ be a solution of the geodesic equation in the sense of Filippov. Then, for all $t \in [a, b]$, we have*

$$\|\gamma'(t)\| \leq \frac{e^{Cr} - 1}{C} \cdot \frac{1}{b - a} \ ,$$

where C is some constant depending only on U and where r denotes the Euclidean length of γ.

Proof. By the form of the geodesic equation, the $\mathcal{C}^{1,1}$-curve γ satisfies for almost all $t \in [a, b]$ the differential inequality

$$\|\gamma''(t)\| \leq C \cdot \|\gamma'(t)\|^2 \ ,$$

where the constant $C > 0$ depends only on the Lipschitz constant of the Lorentzian metric g.

Thus, the Lipschitz continuous non-negative function $s(t) := \|\gamma'(t)\|$ satisfies almost everywhere

$$|s'(t)| \leq C \cdot s^2(t) \ .$$

If $s(t) \neq 0$, this is equivalent to

$$\left| \left(\frac{1}{s(t)} \right)' \right| \leq C \ .$$

Thus, on the open set of points t at which s is positive, the functions $\frac{1}{s(t)}$ is C-Lipschitz continuous. By continuity, either s is constantly equal to 0 or $s(t) > 0$, for all t.

In the first case, γ is constant and the conclusion clearly holds. In the second case, the function $\frac{1}{s(t)}$ is C-Lipschitz continuous on $[a, b]$. Let, in the second case, s assume its maximum s_0 on $[a, b]$ at the point t_0. Setting $\epsilon := \frac{1}{s_0}$, we get

$$\frac{1}{s(t)} \leq \epsilon + C \cdot |t - t_0| \,,$$

$$s(t) \geq \frac{1}{\epsilon + C \cdot |t - t_0|} \,.$$

Thus, we can estimate the Euclidean length of γ as

$$r = \int_a^b s(t)\, dt \geq \int_a^b \frac{1}{\epsilon + C \cdot |t - t_0|}\, dt \geq \int_a^b \frac{1}{\epsilon + C \cdot (t - a)}\, dt =$$

$$= \frac{1}{C} \cdot (\log(C(b - a) + \epsilon) - \log(\epsilon)) = \frac{1}{C} \cdot \log(1 + \frac{C \cdot (b - a)}{\epsilon}) \,.$$

Therefore,

$$e^{Cr} - 1 \geq \frac{C \cdot (b - a)}{\epsilon}$$

Recalling that $\frac{1}{\epsilon}$ is the maximum of $\|\gamma'\|$ completes the proof. $\qquad\square$

As a consequence we derive:

Corollary 3.2. *Let $\gamma_i : [a, b] \to U$ be a sequence of causal curves converging pointwise to $\gamma : [a, b] \to U$. If all curves γ_i are Filippov-geodesics then so is γ.*

Proof. By our assumption on U, all causal curves in U have Euclidean length bounded by a uniform constant r_0. By Lemma 3.1, this implies that all γ_i have uniformly bounded first derivatives. As has been shown in Proposition 1.4, this implies that the curves γ_i are uniformly $\mathcal{C}^{1,1}$. Then, the pointwise convergence implies that γ_i converges uniformly to γ and γ_i' converges uniformly to γ'. Therefore we obtain the conclusion by applying stability theorems for solutions of differential inclusions [Fil88, Cor. 1, Ch. 2, §7, p. 77]. $\qquad\square$

4. FIRST VARIATIONAL FORMULA

In the remainder of this section we are going to prove the following statement. The proof is the standard variational argument, known as "first variation formula of length" or the du Bois-Reymond trick. The only non-classical point in the proof is to carry out all arguments not for

a single variation but for a "full-dimensional" family of variations. This allows us to use the properties of the Levi-Civita connection, which in our setting are valid only almost everywhere.

Proposition 4.1. *Let g be a Lipschitz continuous Lorentzian metric on a domain $U \subset \mathbb{R}^n$. Let $\gamma : [0,1] \to U$ be a $\mathcal{C}^{1,1}$-timelike maximal curve with constant Lorentzian speed $|\gamma'(t)|_g$. Then γ solves the geodesic equation in the sense of Filippov.*

Assume that the statement is wrong. We then find a Lebesgue point $t_0 \in I$ of the L^∞-function $\gamma'' : I \to U$, a negligible set $N \subset U$ (containing the set of all points where g is not differentiable) and some $\delta > 0$ such that the following holds true. The closed convex hull K of the set of all vectors of the form $\Gamma_x(v,v)$ with $(x,v) \in B_\delta(\gamma(t_0), \gamma'(t_0)) \setminus (N \times \mathbb{R}^n)$ does not contain $-\gamma''(t_0)$. Thus we find a unit vector h and some $\epsilon > 0$ such that

(4.1) $$g_{x_0}(h, \gamma''(t_0) + k) > \epsilon$$

for all $k \in K$, where $x_0 = \gamma(t_0)$. In fact, (4.1) holds for all h in a small nonempty open set O. Shrinking O we can assume that all $h \in O$ are linearly independent of $\gamma'(t_0)$.

Reparametrizing γ and changing coordinates, we may assume that $t_0 = 0$, $\gamma(0) = 0$ and that γ is parametrized on an interval $[-r, r]$ for some small r. We further assume that g_0 is the standard Minkowski product on $\mathbb{R}^n$. By continuity of γ' we may further assume that the inequality (4.1) holds for all k of the form $\Gamma_x(\gamma'(t), \gamma'(t))$, where $t \in [-r, r]$ and x in $B_\delta(0) \setminus N$ are arbitrary.

In the following we can and will assume that $\sup_{u_1, u_2 \in U} \|u_1 - u_2\| < 1$. Let H be a hyperplane containing h and transversal to $\gamma'(0)$. Since 0 is a Lebesgue point of γ'', we have for all small $r_0 > 0$

(4.2) $$\int_{-r_0}^{r_0} \|\gamma''(t) - \gamma''(0)\| \, dt \le \frac{\epsilon}{8L} \cdot r_0,$$

where L is the Lipschitz constant of g.

We fix such an r_0 and choose a smooth function $f \colon [-r_0, r_0] \to [0, 1]$ vanishing only in $\pm r_0$ with $f(t) \ge 1/2$ on $[-r_0/2, r_0/2]$. For a small neighborhood W of 0 in H, we consider the map $F : W \times [-r_0, r_0] \to U$ given by

$$F(w, t) = -f(t) \cdot w + \gamma(t) \ .$$

For any $w \in W$, the curve $\gamma_w(t) := F(w, t)$ has the same endpoints as γ. By assumption $|\gamma'(t)|_g$ is a positive constant. Hence, if W is small enough, then all curves γ_w are timelike.

The map F is a $\mathcal{C}^{1,1}$-diffeomorphism onto the image outside the boundary points $\gamma(\pm r_0)$. For sufficiently small W the map

$$w \to \mathcal{L}_w := \mathcal{L}(\gamma_w) := \int_{-r_0}^{r_0} \sqrt{g(\gamma_w'(t), \gamma_w'(t))}\, dt$$

is Lipschitz continuous.

We can follow the classical computation for the first derivative of the length, cf. e.g. [O'N83, Prop. 10.2] or [Gro02, Ch. 3]. We see that for almost all small $w \in W$ the derivative $\frac{\partial}{\partial s}|_{s=\rho}\mathcal{L}_{sw}$ exists for almost all small ρ and can be computed by the usual formula

$$\frac{\partial}{\partial s}\bigg|_{s=\rho} \mathcal{L}_{sw} = \int_{-r_0}^{r_0} f(t)g(w, \tilde{v}_{\rho w}'(t) + \Gamma_{\gamma_{\rho w}(t)}(\tilde{v}_{\rho w}(t), \gamma_{\rho w}'(t)))dt$$

where $\tilde{v}_{\rho w}(t) := \frac{\gamma_{\rho w}'(t)}{|\gamma_{\rho w}'(t)|}$.

We claim that for almost all $w \in O$ this expression is positive for almost all small $\rho > 0$ (and we only consider such ρ with $\rho \cdot O \subset W$). Indeed for such w we have

$$\frac{\partial}{\partial s}\bigg|_{s=\rho} \mathcal{L}_{sw} \geq \int_{-r_0}^{r_0} f(t)g(w, \tilde{v}_0'(0) + \Gamma_{v_{\rho w}(t)}(\tilde{v}_{\rho w}(t), \gamma_{\rho w}'(t)))dt$$
$$- L\Big(\int_{-r_0}^{r_0} \big\|\tilde{v}_0'(t) - \tilde{v}_0'(0)\big\|\, dt - \int_{-r_0}^{r_0} \big\|\tilde{v}_{\rho w}'(t) - \tilde{v}_0'(t)\big\|\, dt\Big).$$

By continuity of $\tilde{v}_{\rho w}'(t)$ in ρ (at times of existence) and the constant Lorentzian speed $|v_t|$ of γ, the integrand of the last term can be made smaller than $\epsilon/16|v_t|$ by choosing ρ small enough. The second term can be estimated with (4.2). Hence

$$\frac{\partial}{\partial s}\bigg|_{s=\rho} \mathcal{L}_{sw} \geq \frac{1}{2|v_t|} \int_{-r_0/2}^{r_0/2} g\left(w, \tilde{v}_0'(0) + \Gamma_{v_{\rho w}(t)}(\gamma_{\rho w}'(t), \gamma_{\rho w}'(t))\right) dt - C\frac{r_0\epsilon}{|v_t|}$$

with $C < \frac{1}{2}$. By Lipschitz continuity of g and boundedness of Γ the same estimate holds for g_0 instead of g, if we choose r_0 much smaller than ϵ. By continuity in ρ and our choice of w we obtain

$$\frac{\partial}{\partial s}\bigg|_{s=\rho} \mathcal{L}_{sw} \geq \frac{r_0\epsilon}{2|v_t|} - C\frac{r_0\epsilon}{|v_t|} > 0$$

for almost all sufficiently small $\rho > 0$. Thus, $\mathcal{L}_{\rho w} = \mathcal{L}(\gamma_{\rho w}) > \mathcal{L}(\gamma)$ for almost all sufficiently small ρ, in contradiction to the maximality of γ. This finishes the proof of Proposition 4.1.

5. $\mathcal{C}^{1,\alpha}$ CURVES

The following lemma is a variant of [CH70, Lem. 2.1], modified for our purposes.

Lemma 5.1. *Let $\gamma : [a,b] \to \mathbb{R}^n$ be a bilipschitz curve parametrized by arclength. Let $0 < \alpha \leq 1$ be fixed. Then the following are equivalent:*

(1) *The curve γ is $\mathcal{C}^{1,\alpha}$.*
(2) *There exists some $C \geq 1$ such that for all $0 \leq h \leq \frac{1}{C}$ and all intervals $[t, t+h] \subset [a,b]$ the image of the restriction $\gamma([t,t+h])$ is contained in the tube $B_{C \cdot h^{1+\alpha}}([\gamma(t), \gamma(t+h)])$ of radius $C \cdot h^{1+\alpha}$ around the linear segment between $\gamma(t)$ and $\gamma(t+h)$.*

Proof. If γ is $\mathcal{C}^{1,\alpha}$, then there exists some $C > 1$ such that for all small h and all $r \in [t, t+h]$ we have (see [LY06, Lemma 2.1])

$$\| h \cdot \gamma'(r) - (\gamma(t) - \gamma(t-h)) \| \leq C \cdot h^{1+\alpha} \ .$$

Dividing by h and integrating from t to r we deduce

$$\gamma(r) = \gamma(t) + \frac{r-t}{h} \cdot (\gamma(t+h) - \gamma(t)) + q(r) \ ,$$

where $\|q(r)\| \leq C \cdot h^{1+\alpha}$. Since $q(r)$ bounds the distance to the segment between $\gamma(t)$ and $\gamma(t+h)$, this proves that (1) implies (2).

Assume (2) and let L denote the biLipschitz constant of γ, i.e. $|t_2 - t_1| \leq Ld(\gamma(t_1), \gamma(t_2))$. Set $X = \gamma([a,b])$ and $\epsilon := h/L$. Consider arbitrary points $x_1, x_2 \in X$ with $\|x_1 - x_2\| \leq \epsilon$.

Considering the projection of the part X' of γ between x_1 and x_2 onto the segment $[x_1, x_2]$ we find a point p in X' which is projected onto the midpoint m between x_1 and x_2. By assumption $d(p, m) \leq C \cdot d^{1+\alpha}(x_1, x_2)$.

By Pythagoras, we deduce that for $i = 1, 2$

$$d(x_i, p) \leq \frac{1}{2} d(x_1, x_2) \cdot (1 + 4C^2 d^{2\alpha}(x_1, x_2))$$

By [Lyt05, Proposition 1.1] and [Lyt05, Theorem 1.2] there exists some $\bar{\epsilon} > 0$ such that all pairs of points in X at distance less than $\bar{\epsilon}$ are connected in X by a $\mathcal{C}^{1,\alpha}$-curve parametrized by arclength. Since γ is injective, this curve must coincide with γ, thus finishing the proof. $\square$

6. QUANTITATIVE TRIANGLE INEQUALITY

Recall that vectors $u, v \in C_+$ in the positive causal cone

$$C_+ = \{ v = (v_t, v_x) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1} \mid \langle v, v \rangle \geq 0, \ v_t > 0 \}$$

of the standard Minkowski space $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$ with Minkowski product $\langle u, v \rangle = u_t v_t - u_x^T v_x$ and norm $|v| = \sqrt{|\langle v, v \rangle|}$ satisfy the reversed triangle inequality

$$(6.1) \qquad |u + v| \geq |u| + |v|$$

as well as the reversed Cauchy-Schwarz inequality $\langle u, v \rangle \geq |u| \, |v|$, see e.g. [O'N83, Prop. 5.30]. Here we describe a quantitative version of the reversed triangle inequality:

Lemma 6.1. *For $A = \frac{1}{10}$ and for all $u, v \in C_+$, the inequality*

$$(6.2) \qquad |u + v|^2 \geq |u + v| \, (|u| + |v|) + A \cdot D^2$$

holds, where D is the Euclidean distance from u and v to the Euclidean line spanned by $u + v$.

Proof. By (6.1) we can suppose that u and v are linearly independent. By homogeneity and by the $O(n)$-invariance in the spatial part of $\mathbb{R}^{n+1}$ we can assume that $n = 2$ and that

$$u = (t, x, y), \quad v = (1, z, 0)$$

with $0 \leq t \leq 1$, $0 \leq z \leq 1$ and $x^2 + y^2 \leq t^2$. Using the Cauchy-Schwarz and the reversed triangle inequality we compute

$$
\begin{aligned}
|u + v| - |u| - |v| &= \frac{|u + v|^2 - (|u| + |v|)^2}{|u + v| + |u| + |v|} \\
&\geq \frac{\langle u, v \rangle - |u| \, |v|}{|u + v|} \\
&\geq \frac{\langle u, v \rangle^2 - |u|^2 \, |v|^2}{2 \, |u + v| \, \langle u, v \rangle} \\
&= \frac{(t - xz)^2 - (t^2 - x^2 - y^2)(1 - z^2)}{2 \, |u + v| \, \langle u, v \rangle} \\
&= \frac{(tz - x)^2 + y^2(1 - z^2)}{2 \, |u + v| \, \langle u, v \rangle}.
\end{aligned}
$$

We also observe that for such u and v

$$D^2 = \frac{\|u \times v\|^2}{\|u + v\|^2} = \frac{(tz - x)^2 + y^2(1 + z^2)}{\|u + v\|^2} \geq (tz - x)^2 + y^2(1 + z^2),$$

because $\|u + v\| \geq 1$. We deduce that

$$(6.3) \qquad \frac{|u + v|^2 - |u + v|(|u| + |v|)}{D^2} \geq \frac{(tz - x)^2 + y^2(1 - z^2)}{(tz - x)^2 + y^2(1 + z^2)} \cdot \frac{1}{2(t - zx)}$$

For fixed t, z, x, the term on the right is monotone decreasing in y^2. Thus, we may assume that y^2 is as large as possible, $y^2 = t^2 - x^2$. Then

$$(tz - x)^2 + y^2(1 - z^2) = (t - zx)^2$$

and the right hand side of (6.3) is at least

$$\frac{t - zx}{(tz - x)^2 + (t^2 - x^2)(1 + z^2)} \geq \frac{1}{2} \cdot \frac{t - zx}{|tz - x| + (t - x) \cdot 2 \cdot 2} .$$

Since $t - zx \geq |tz - x|$ and $t - zx \geq t - x$ we arrive at

$$\frac{|u + v|^2 - |u + v|(|u| + |v|)}{D^2} \geq \frac{t - zx}{10 \cdot (t - zx)} = \frac{1}{10} .$$

This finishes the proof. $\qquad\square$

We can now easily derive the following conclusion:

Corollary 6.2. *For any compact set $\mathcal{S}$ of Lorentzian bilinear forms $\lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ there exists a constant $A = A(\mathcal{S})$ such that for any $\lambda \in \mathcal{S}$ the following holds true:*

If $\gamma : [a, b] \to \mathbb{R}^n$ is an λ-timelike curve, if w is the vector $\gamma(b) - \gamma(a)$ and if D denotes the maximal Euclidean distance of a point on γ from the Euclidean line through $\gamma(a)$ and $\gamma(b)$, then

$$(6.4) \qquad |w|_\lambda \geq \mathcal{L}_\lambda(\gamma) + \frac{A \cdot D^2}{|w|_\lambda}$$

Proof. Any λ in $\mathcal{S}$ can be brought by a bounded linear transformation to the standard Minkowski product. By compactness, all these transformations change the Euclidean norm (and thus the distance D appearing in the formula) by a uniformly bounded factor. Thus it suffices to prove the statement for the standard Minkowski product $\langle \cdot, \cdot \rangle$.

We first observe that in Minkowski space all chords of a timelike curve are timelike as well. Now, by homogeneity, we can assume $\gamma(a) = 0$. We consider the point u on γ with maximal distance to the line through 0 and $\gamma(b) = w$. Then for $v = w - u$ we can apply Lemma 6.1, to deduce

$$|w| \geq |u| + |v| + \frac{AD^2}{|w|} .$$

Since straight lines in Minkowski space maximize Lorentzian length, we have $\mathcal{L}(\gamma) \leq |u| + |v|$, and this finishes the proof. $\qquad\square$

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7. First step to regularity

We now embark on the proof of Theorem 1.1. The statement is local and we will restrict from now on to a chart U as in Section 2.

We will need to compare Lorentzian lengths with respect to different Lorentzian metrics and will rely on the following observation, a direct analog of [LY06, Lemma 3.1]. The proof is obtained by integration.

Lemma 7.1. *Let $\varepsilon > 0$ be sufficiently small and let g_1, g_2 be Lorentzian metrics on U such that $|g_1(v, v) - g_2(v, v)| < \varepsilon$ for all points in U and all unit vectors v. Let $\gamma : [0, \delta] \to U$ be a curve parametrized by Euclidean arclength which is causal with respect to g_1 and g_2. Then the Lorentzian lengths of γ with respect to g_1 and g_2 can be compared as $|\mathcal{L}_1(\gamma) - \mathcal{L}_2(\gamma)| \le \delta \cdot \sqrt{\varepsilon}$.*

In this section we are going to prove

Proposition 7.2. *Any maximal curve $\gamma : I \to U$ in U parametrized by Euclidean arclength is $\mathcal{C}^{1, \frac{1}{4}}$.*

Proof. Assume the contrary. By Lemma 5.1 we find for any arbitrarily large $C > 0$, an arbitrarily small $h > 0$, some subinterval $[t, t + h] \subset I$, and some point q on $\gamma([t, t + h])$ such that

$$(7.1) \qquad D \ge Ch^{1 + \frac{1}{4}},$$

where D denotes the Euclidean distance from q to the Euclidean line between $\gamma(t)$ and $\gamma(t + h)$.

The vector

$$u := \frac{\gamma(t + h) - \gamma(t)}{h}$$

satisfies $\|u\| \le 1$ since γ is 1-Lipschitz.

We consider the restriction $\gamma_{t,h} = \gamma_{|[t, t+h]}$ and the curve $\tilde{\gamma} = \tilde{\gamma}_{t,h} : [0, h] \to U$ given by $\tilde{\gamma}(s) := \gamma(t) + s \cdot u$. The curve $\tilde{\gamma}$ has the same endpoints as the maximal curve $\gamma_{t,h}$. A contradiction proving Proposition 7.2 will be provided by showing that for $C \ge C(L)$ large enough the curve $\tilde{\gamma}$ is timelike and has larger Lorentzian length than $\gamma_{t,h}$.

We may assume that $t = 0$, that $\gamma(t) = 0 \in U$ and that g_0 is the standard Minkowski product $g_0 = \langle \cdot, \cdot \rangle$, see Section 2. We set $\gamma_h := \gamma_{t,h}$. We consider the auxiliary Lorentzian products on $\mathbb{R}^n$ given by

$$g^h(v, w) = \langle v, w \rangle + 4 \cdot L \cdot h \cdot v_t \cdot w_t ,$$

where as before v_t denotes the first coordinate of $v \in \mathbb{R}^n$.

Then, for all x in $\mathbb{R}^n$ with $\|x\| \le h$ and for all unit vectors $v \in \mathbb{R}^n$ we have

(1) $|g_x(v, v) - g^h(v, v)| \le 5Lh$, and

(2) if $g_x(v, v) > 0$ then $g^h(v, v) > 0$,

compare [CG12, Thm. 1.15]. In particular, any g-causal curve in the ball $B_h(0) \subset \mathbb{R}^n$ is g^h-causal.

Hence, the curve γ_h is timelike with respect to the Minkowski product g^h. Thus also u and the straight segment $\tilde{\gamma}$ are g^h-timelike.

Note that all our auxiliary Lorentzian products g^h are contained in a compact set of Lorentzian products, fixed independently of h. Thus, we find a constant $A > 0$ such that the conclusion of Corollary 6.2 is valid for all such g^h, and so we deduce:

$$(7.2) \qquad \mathcal{L}_{g^h}(\tilde{\gamma}) = |\gamma(h) - \gamma(0)|_{g^h} = h \cdot |u|_{g^h} \geq \mathcal{L}_{g^h}(\gamma) + \frac{A \cdot D^2}{h \cdot |u|_{g^h}} .$$

Inserting $D \geq C \cdot h^{1+\frac{1}{4}}$, we obtain

$$|u|_{g^h} \geq \frac{\sqrt{A} \cdot D}{h} \geq \sqrt{A} \cdot C \cdot h^{\frac{1}{4}} .$$

Thus, for all $x \in B_h(0)$ we have

$$g_x(u, u) \geq AC^2 \cdot h^{\frac{1}{2}} - 5L \cdot h \geq \frac{1}{2}AC^2 \cdot h^{\frac{1}{2}}$$

for all sufficiently small h. Therefore the curve $\tilde{\gamma}$ is g-timelike. Moreover, we have

$$\mathcal{L}(\tilde{\gamma}) - \mathcal{L}(\gamma_h) = (\mathcal{L}_{g^h}(\tilde{\gamma}) - \mathcal{L}_{g^h}(\gamma_h))$$
$$+ \underbrace{(\mathcal{L}(\tilde{\gamma}) - \mathcal{L}_{g^h}(\tilde{\gamma})) + (\mathcal{L}_{g^h}(\gamma_h) - \mathcal{L}(\gamma_h))}_{=:R} .$$

By Lemma 7.1, we deduce that the absolute value of R is at most $2\sqrt{5L}h^{\frac{3}{2}}$. Since γ is maximal , $\mathcal{L}(\tilde{\gamma}) - \mathcal{L}(\gamma_h) \leq 0$. Applying (7.2), we find a constant $C_1 > 0$ such that

$$\frac{AD^2}{h|u|_{g^h}} \leq C_1 \cdot h^{\frac{3}{2}}$$

$$AD^2 \leq C_1 \cdot h^{\frac{5}{2}} \cdot |u|_{g^h}$$

Since $|u|_{g^h} \leq 2$ we deduce that $D \leq C_2 \cdot h^{1+\frac{1}{4}}$ for some constant C_2 (independent of γ, t and h), and thus a contradiction, if C has been chosen large enough. This finishes the proof. $\square$

Easily adapting the arguments we provide

Proof of Proposition 1.3. The proof follows literally as above by redefining g^h as

$$g^h(v, w) = \langle v, w \rangle + 4 \cdot L \cdot h^\alpha \cdot v_t \cdot w_t ,$$

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and replacing $\frac{1}{4}$ by $\frac{\alpha}{4}$ in (7.1). $\hfill\square$

8. Regularity away from lightlike vectors

In this section we prove, again following [LY06]:

Theorem 8.1. *Let $\gamma : I \to U$ be a maximal causal curve parametrized by Euclidean arclength. If $|\gamma'(t)|_g > 0$ for all $t \in I$, then γ is $\mathcal{C}^{1,1}$.*

Proof. The statement is local and we may assume that I is compact. By continuity of γ', Proposition 7.2, we find some $e > 0$ such that $|\gamma'(t)|_g > 2 \cdot e$, for all $t \in I$. Restricting γ to a sufficiently small subinterval, we see that all linear segments between points on γ are timelike curves.

Assume that γ is not $\mathcal{C}^{1,1}$. Relying on Lemma 5.1 and arguing as in the proof of Proposition 7.2, we arrive at the following situation. For an arbitrarily large $C > 0$ we find an arbitrarily small $h > 0$ with the following property. The restriction of γ to an interval $[a, a + h] \subset I$ contains a point at distance

$$(8.1) \qquad\qquad D \geq C \cdot h^2$$

from the linear segment η connecting $\gamma(a)$ and $\gamma(a + h)$. We may assume without loss of generality that $a = 0$ and $\gamma(0) = 0 \in U$.

Denote by u the Euclidean unit vector

$$u := \frac{\gamma(h) - \gamma(0)}{\|\gamma(h) - \gamma(0)\|}$$

and let the hyperplane H denote the g_0-orthogonal complement of u in $\mathbb{R}^n$. Since u is timelike, the hyperplane H consists of g_0-spacelike vectors. Moreover, by compactness, we find some constant $C_1 > 0$, depending only on e, such that

$$-g_0(w, w) \leq C_1 \cdot \|w\|^2$$

for all $w \in H$. By continuity of g, we may assume, adjusting C_1, that also for all $x \in \mathbb{R}^n$ with $\|x\| \leq h$ and all $w \in H$ we have the inequality

$$(8.2) \qquad\qquad - g_x(w, w) \leq C_1 \cdot \|w\|^2$$

Consider the unique parametrization $\eta : [0, h] \to U$ of the linear segment η such that

$$v(t) := \gamma(t) - \eta(t)$$

is g_0-orthogonal to u. In other words, $\eta(t)$ is the g_0-orthogonal projection of $\gamma(t)$ onto the line spanned by u.

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The curves η and v are well-defined and $\mathcal{C}^{1,\frac{1}{4}}$. Moreover, $v(0) = v(h) = 0$. By the uniform continuity of v', for an arbitrarily small $\rho > 0$, the inequality

$$\|v'(t)\| + \|v(t)\| < \rho$$

holds true for all $t \in [0, h]$ once h has been chosen small enough, compare [LY06, Lemma 2.1].

For sufficiently small h (and thus, sufficiently small ρ), this implies

(8.3) $$\frac{1}{2} < \|\eta'(t)\| < 2 \quad \text{and} \quad |\eta'(t)|_{\gamma(t)} > e .$$

Now we can estimate for all h small enough:

$$|\eta'(t)|_{\eta(t)} - |\gamma'(t)|_{\gamma(t)}$$

$$= |\eta'(t)|_{\eta(t)} - \sqrt{|\eta'(t)|^2_{\gamma(t)} + 2g_{\gamma(t)}(\eta'(t), v'(t)) + g_{\gamma(t)}(v'(t), v'(t))}$$

$$\geq |\eta'(t)|_{\eta(t)} - |\eta'(t)|_{\gamma(t)} - \frac{g_{\gamma(t)}(\eta'(t), v'(t))}{|\eta'(t)|_{\gamma(t)}} - \frac{g_{\gamma(t)}(v'(t), v'(t))}{2\,|\eta'(t)|_{\gamma(t)}}$$

$$\geq -2L \cdot \|v(t)\| - \frac{\left|g_{\gamma(t)}(\eta'(t), v'(t))\right|}{e} - \frac{g_{\gamma(t)}(v'(t), v'(t))}{2e} .$$

Here we have used $\sqrt{1 + x} \leq 1 + \frac{x}{2}$ for $x > -1$ in the first inequality and the L-Lipschitz continuity of g in the second inequality.

We estimate the second and third summand separately. In order to deal with the second term, we observe that for a unit vector $w \in H$:

$$\left|g_{\gamma(t)}(u, w)\right| \leq L \cdot \|\gamma(t)\| \leq L \cdot t$$

due to $g_0(u, w) = 0$, the Lipschitz continuity of g and the 1-Lipschitz continuity of γ. Therefore,

$$- \left|g_{\gamma(t)}(\eta'(t), v'(t))\right| \geq -2L \cdot t \cdot \|v'(t)\| .$$

Estimating the third summand by (8.2) we find

$$|\eta'(t)|_{\eta(t)} - |\gamma'(t)|_{\gamma(t)} \geq -2L \|v(t)\| - \frac{2L}{e} \cdot t \|v'(t)\| + \frac{C_1}{2e} \|v'(t)\|^2$$

Now we integrate over $[0, h]$ and use the maximality of $L(\gamma)$ to deduce that the integral of the right hand side form 0 to h is non-positive. Thus, for a constant $C_2 > 0$ (depending only on e and L)

(8.4) $$\int_0^h \|v'(t)\|^2 \, dt \leq C_2 \int_0^h (\|v(t)\| + \|v'(t)\|\, t) \, dt .$$

Since $v(0) = v(h) = 0$ we use integration by parts to deduce

$$\int_0^h \|v(t)\| \, dt = -\int_0^h (\|v(t)\|)' \cdot t \, dt .$$

Thus, using $|(\|v(t)\|)'| \leq \|v'(t)\|$ we obtain

$$\int_0^h \|v'(t)\|^2 \, dt \leq 2C_2 \cdot \int_0^h \|v'(t)\| \cdot t \, dt .$$

We apply Cauchy-Schwarz to deduce

$$\left(\int_0^h \|v'\|^2 \right)^2 \leq 4C_2^2 \left(\int_0^h \|v'\|^2 \right) \cdot \left(\int_0^h t^2 \, dt \right) = \frac{4}{3} C_2^2 \cdot h^3 \cdot \int_0^h \|v'\|^2 .$$

Thus, we obtain

$$\frac{4}{3} C_2^2 h^3 \geq \int_0^h \|v'\|^2 \geq \frac{1}{h} \cdot \left(\int_0^h \|v'\| \right)^2 \geq \frac{1}{h} \cdot v_0^2 ,$$

with $v_0 = \max_{t \in [0,h]} \|v(t)\|$. Thus, $\|v(t)\| \leq 2C_2 \cdot h^2$ for all $t \in [0, h]$.

This provides a contradiction with (8.1) and finishes the proof of Theorem 8.1. $\qquad \square$

9. Timelike maximal curves

The following result proves and strengthens Proposition 1.2.

Proposition 9.1. *Let* $\gamma : [0, r] \to U$ *be a maximal curve parametrized by Euclidean arclength. If* $\mathcal{L}(\gamma) = 0$ *then* $|\gamma'(t)|_g = 0$ *for all* t.
If $\mathcal{L}(\gamma) > 0$ *then* γ *is* $\mathcal{C}^{1,1}$. *Moreover, for all* $t \in [0, r]$,

(9.1) $$|\gamma'(t)|_g \geq K \cdot \frac{\mathcal{L}(\gamma)}{r} > 0 ,$$

where the positive constant K *depends only on* U. *The curve* γ *admits parametrizations with constant Lorentzian velocity and any such parametrization solves the geodesic equation in the sense of Filippov.*

Proof. By Proposition 7.2, γ is $\mathcal{C}^1$. If $\mathcal{L}(\gamma) = 0$ then $|\gamma'(t)|_g = 0$, for almost all $t \in [0, r]$. By continuity of γ', this equality holds for all t.

Assume $\mathcal{L}(\gamma) > 0$. Let $[a_0, b_0] \subset [0, r]$ be such that $|\gamma'(t)|_g > 0$, for all $t \in [a_0, b_0]$. By Theorem 8.1, the restriction $\gamma : [a_0, b_0] \to U$ is $\mathcal{C}^{1,1}$.

For any interval $[a, b] \subset \mathbb{R}$ and a non-decreasing Lipschitz function $f : [a, b] \to [a_0, b_0]$ the reparametrization

$$\tilde{\gamma} = \gamma \circ f : [a, b] \to U$$

has constant Lorentzian velocity $\ell > 0$ if and only if

(9.2) $$f'(t) = \frac{\ell}{|\gamma'(f(t))|_g} ,$$

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for almost all $t \in [a, b]$. This is a differential equation for f which has a unique maximal solution with $f(a) = a_0$. Integrating we see that $f(b_0) = b$ if and only if

$$\ell \cdot (b - a) = \mathcal{L}(\gamma|_{a_0, b_0]}) .$$

The function $f'(t)$ is Lipschitz. Hence f and, therefore, $\tilde{\gamma}$ are $\mathcal{C}^{1,1}$.

Due to Proposition 4.1, $\tilde{\gamma}$ solves the geodesic equation in the sense of Filippov. Applying Lemma 3.1, for all $t \in [a, b]$,

$$f'(t) = \|\tilde{\gamma}'(t)\| \leq \frac{e^{Cr_0} - 1}{C} \cdot \frac{1}{b - a} ,$$

where the constant $C = C(U) > 0$ depends only on the Lipschitz constant of g and where $r_0 = b_0 - a_0$ denotes the Euclidean length of $\gamma|_{[a_0, b_0]}$. Inserting into (9.2), we observe for all $t_0 \in [a_0, b_0]$

$$|\gamma'(t_0)|_g \geq \ell \cdot (b - a) \cdot \frac{C}{e^{Cr_0} - 1} = \frac{Cr_0}{e^{Cr_0} - 1} \cdot \frac{1}{r_0} \cdot \mathcal{L}(\gamma|_{[a_0, b_0]}) .$$

Note that $\frac{Cr_0}{e^{Cr_0} - 1}$ is decreasing in r_0. Thus, for all $t_0 \in [a_0, b_0]$,

$$(9.3) \qquad |\gamma'(t_0)|_g \geq K \cdot \frac{\mathcal{L}(\gamma|_{[a_0, b_0]})}{r_0} ,$$

where the constant $K = K(U)$ denotes $K = \frac{Cr_{max}}{e^{Cr_{max}} - 1}$ and r_{max} is the maximal Euclidean length of causal curves in U (which is bounded by twice the diameter of U).

It remains to prove that the set T of all times $t \in [0, r]$ with $|\gamma'(t)|_g = 0$ is empty. Assuming the contrary and using $\mathcal{L}(\gamma) > 0$, we find a subinterval $[a_1, b_1] \subset [0, r]$ such that $|\gamma'(t)|_g$ is positive for all $t \in (a_1, b_1)$ and vanishes at one of the boundary points a_1 or b_1.

We let $[a_0, b_0] \subset (a_1, b_1)$ converge to $[a_1, b_1]$. Applying (9.3) to the intervals $[a_0, b_0]$, we find a uniform positive lower bound on $|\gamma'(t)|_g$ for all $t \in (a_1, b_1)$. By continuity, this positive bound is valid at the boundary points a_1 and b_1 as well. This contradiction finishes the proof. $\qquad \square$

Remark 9.1. As the proof shows the constant K can be chosen arbitrarily close to 1, if U is chosen sufficiently small.

10. FINAL ARGUMENTS

Now we are in position to finish the proof of Theorem 1.1. The remaining control of lightlike curves is obtained by a limiting argument, independently discovered in [Sch20]:

Proof of Theorem 1.1. All statements of the theorem are local. Thus we may restrict to a sufficiently small chart U satisfying the assumptions of Section 2.

We call a causal curve γ in U tame if γ is maximal and admits a $\mathcal{C}^{1,1}$-parametrization solving the geodesic equation in the sense of Filippov. We need to prove that all maximal curves are tame. Due to Proposition 9.1, all maximal curves of positive Lorentzian length are tame.

We first claim that any pair $x, y \in U$ of causally related points is connected by a tame curve. Without loss of generality let y be in the future of x. We find a sequence y_n converging to y, such that y and y_n are related by a future directed timelike curve. Then x and y_n are related by future directed causal curves of positive Lorentzian length. By global hyperbolicity of U we find a maximal curve γ_n in U connecting x and y_n. The curve γ_n satisfies $\mathcal{L}(\gamma_n) > 0$. By Proposition 9.1, we can parametrize γ_n on the interval $[0,1]$ with constant Lorentzian speed, so that it solves the geodesic equation in the sense of Filippov.

By global hyperbolicity of U and the uniform bound on velocities provided by Lemma 3.1, the curves γ_n converge pointwise, after choosing a subsequence, to a future directed causal curve $\gamma : [0,1] \to U$ connecting x and y. Due to Corollary 3.2, this limiting curve γ is tame.

Let now γ be an arbitrary maximal curve in U. When parametrized by Euclidean arclength, the curve $\gamma : [0, r] \to U$ is $\mathcal{C}^1$ by Proposition 7.2. If $\mathcal{L}(\gamma) > 0$ then γ is tame by Proposition 9.1.

Thus we may and will assume $L(\gamma) = 0$. For any natural $m > 1$ and $0 \le i \le m$, consider points $x_i = \gamma(\frac{i}{m} \cdot r)$ subdividing γ into m pieces of equal length. By the above, we find for any $i = 0, ..., m-1$ a tame curve γ_m^i connecting x_i and x_{i+1}. Consider the concatenation γ_m of the curves γ_m^i for $i = 0, ..., m-1$. The curve γ_m is a causal curve connecting $x = \gamma(0)$ and $y = \gamma(r)$.

Clearly, for any parametrizations of the curves γ_m with uniformly bounded speed, the curves subconverge pointwise to a reparametrization of γ. Hence, due to Lemma 3.1 and Corollary 3.2, in order to prove that γ is tame, we only need to verify that all curves γ_m are tame.

Since γ is maximal and $\mathcal{L}(\gamma) = 0$, any causal curve connecting x and y is maximal. In particular, γ_m is maximal. By Proposition 7.2, γ_m is $\mathcal{C}^1$ when parametrized by Euclidean arclength. By construction, all γ_m^i admit parametrizations solving the geodesic equation in the sense of Filippov.

Any affine reparametrization of a $\mathcal{C}^{1,1}$-curve solving the geodesic equation in the sense of Filippov is again a solution of the geodesic

equation. Due to Lemma 3.1, we can find such parametrization, having any prescribed positive Euclidean velocity at the starting point.

We now start with any parametrization $\gamma_m^0 : [0, t_1] \to U$ solving the geodesic equation and proceed by induction on i, to find a parametrization $\gamma_m^i : [t_i, t_{i+1}] \to U$ solving the geodesic equation and such that $\|(\gamma_m^i)'(t_i)\| = \|(\gamma_m^{i-1})'(t_i)\|$.

The arising concatenation $\tilde{\gamma}_m : [0, t_m] \to U$ is a reparametrization of the $\mathcal{C}^1$-curve γ_m with the following properties. The restriction of $\tilde{\gamma}_m$ to any of the subintervals $[t_i, t_{i+1}]$ is $\mathcal{C}^{1,1}$ and solves on this subinterval the geodesic equation in the sense of Filippov. At the boundary points t_i the incoming and the outgoing directions of $\tilde{\gamma}_m$ have the same norms. Moreover, since γ_m is $\mathcal{C}^1$, these incoming and outgoing directions are parallel vectors, hence they coincide.

Therefore, the derivative $\tilde{\gamma}_m'$ is continuous. Since $\tilde{\gamma}_m'$ is Lipschitz continuous on the subintervals $[t_i, t_{i+1}]$, the derivative $\tilde{\gamma}_m'$ is Lipschitz continuous on all of $[0, t_m]$. Thus, $\tilde{\gamma}_m$ is a $\mathcal{C}^{1,1}$-curve. Clearly, $\tilde{\gamma}_m$ solves the geodesic equation almost everywhere on any interval $[t_i, t_{i+1}]$, hence almost everywhere on the whole interval of definition $[0, t_m]$.

This shows the tameness of γ_m and finishes the proof. $\qquad\square$

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