AFFINE FUNCTIONS ON $CAT(\kappa)$-SPACES

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Abstract. We describe affine functions on spaces with an upper curvature bound.

1. introduction

A map $f : X \to Y$ between geodesic metric spaces is called affine if it sends geodesics (in this paper always parameterized proportionally to the arclength) to geodesics. Restricting to the case $Y = \mathbb{R}$ we obtain the following definition.

Definition 1.1. *A function $f : X \to \mathbb{R}$ on a geodesic metric space X is called affine, if the restriction $f \circ \gamma$ to each geodesic $\gamma : [a, b] \to X$ is affine, i.e. satisfies $(f \circ \gamma)'' = 0$.*

The easiest example of an affine function is the projection $f = p_{\mathbb{R}} : X \times \mathbb{R} \to \mathbb{R}$ of a direct product onto its factor. In [AB] (see also [In],[Ma1],[Ma2] for earlier results) it is shown, that under some assumptions the existence of a non-constant affine function f on a space X with a one-sided curvature bound forces the space to split as a direct product $X = Y \times \mathbb{R}$. The crucial assumption in [AB] is that the space X does not have boundary in the case of a lower curvature bound or is geodesically complete in the case of an upper curvature bound. Without this assumption one cannot expect that such a splitting exists, as the example of an Euclidean ball shows. A slightly more sophisticated example is a convex subset (for instance a metric ball) in a product of a tree and a real line. The best one can hope for, is the existence of an isometric embedding of X into a product of some space with a real line. Our main result says that this is indeed the case.

We choose a slightly more general formulation that simultaneously takes into account all affine functions on X. First, we restrict ourselves to Lipschitz continuous affine functions, see, however, Theorem 1.6 and Theorem 1.7 below. In the sequel we denote by p_Y and p_H the natural projections from the product $Y \times H$ onto the factors Y and H.

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Theorem 1.2. *Let X be a $CAT(\kappa)$ space. Then there exists a geodesic metric space Y, a Hilbert space H and an isometric embedding $i : X \to Y \times H$ with the following properties:*

1. *Each Lipschitz continuous affine function on Y is constant;*
2. *Each Lipschitz continuous affine function $f : X \to \mathbb{R}$ factors as $f = \hat{f} \circ p_H \circ i$ where $\hat{f} : H \to \mathbb{R}$ is affine and continuous;*
3. *The projection $P_Y = p_Y \circ i : X \to Y$, is surjective;*
4. *Each isometry of X extends to a unique isometry of $Y \times H$.*

The Hilbert space H can be finite or infinite dimensional. The fibers of the projection $P_H = p_H \circ i : X \to H$ consist of points that cannot be separated by Lipschitz continuous affine functions. Each fiber $P_H^{-1}(h)$ is a convex subset of X and projects isometrically into Y. Thus the whole space Y appears as the union of such fibers glued together in some way. This picture suggests that the space Y is $CAT(\kappa)$ as well. Without additional assumptions we can prove this only for $\kappa = 0$.

Proposition 1.3. *Let X be a $CAT(0)$ space and let Y be the space constructed in Theorem 1.2. Then the completion $\bar{Y}$ of Y is $CAT(0)$.*

For a more detailed study one needs the additional concept of interior points. We think that this concept may also be useful in other situations. The motivation comes from the following simple example:

Example 1.1. Let X be a closed convex subset of $\mathbb{R}^n$ with non-empty interior and let O be the set of inner points of X. Then a point x is contained in O if and only if there exists some $\epsilon = \epsilon_x > 0$ such that each geodesic ending in x can be prolongated by the amount ϵ. One can also characterize the set O topologically: $x \in O$ if and only if $X \setminus \{x\}$ is not contractible. Note that O is dense in X and that each dense convex subset of X contains O.

We generalize the two characterizations of inner points to general $CAT(\kappa)$ spaces.

Definition 1.4. *Let X be a $CAT(\kappa)$ space.*

1. *A point $x \in X$ is called a* geometrically inner point *of X if there is some $\epsilon > 0$ with the following property. For each $y \in X$, with $d(x, y) \leq \epsilon$, there is some $z \in X$, with $d(x, z) = \epsilon$, such that x lies on the geodesic connecting y and z.*
2. *A point $x \in X$ is called a* topologically inner point *of X if for all small $\epsilon > 0$ the punctured ball $B_\epsilon(x) \setminus \{x\}$ is not contractible.*

One cannot expect that inner points in a general $CAT(\kappa)$ space have the same properties as in the example above. In an infinite dimensional

Hilbert space H each point is a geometrically inner point, but no topologically inner points exists. Moreover, the intersection of all convex dense subsets of H is empty. In the Hilbert cube no (geometrically or topologically) inner points exist. It is possible to construct a compact tree, in which the set of (geometrically and topologically) inner points is not open. However, under the assumption of finiteness of geometric dimension introduced in [K], inner points exist and there is a relation to convex dense subsets:

Theorem 1.5. *Let X be a $CAT(\kappa)$ space. Then*

(1) *Every topologically inner point is also a geometrically inner point.*
(2) *If X has locally finite geometric dimension, then the set of topologically inner points is dense in X.*
(3) *Every dense convex subset $C \subset X$ contains all topologically inner points of X.*

The existence of topologically inner points has strong consequences for affine functions:

Theorem 1.6. *Let X be a $CAT(\kappa)$ space with a topologically inner point. Then each affine function on X is Lipschitz continuous.*

Under the weaker assumption of the existence of geometrically inner points non-continuous affine functions may exist as the example of non-continuous linear functions on Hilbert spaces shows. However, we obtain the analog of the usual characterization of the continuity of linear functions on Hilbert spaces:

Theorem 1.7. *Let X be a $CAT(\kappa)$ space with at least one geometrically inner point. Let f be an affine function on X. Then the following are equivalent:*

(1) *The function f is Lipschitz continuous;*
(2) *The function f is continuous;*
(3) *All fibers of f are closed;*
(4) *There is a fiber of f that is not dense in X.*

We apply our results in two situations. First we study affine function under the presence of group actions. We show that if X admits a minimal action by a group of isometries, then the existence of an affine function forces the space to split off a line.

Corollary 1.8. *Let X be a $CAT(0)$ space with at least one geometrically inner point. Assume that a group Γ acts on X by isometries, such that in X there are no non-trivial, closed, convex, Γ-invariant subsets.*

Then X has the form $X = Y \times H$, where H is a Hilbert space and Y is $CAT(0)$ space, on which no continuous affine functions exist.

Finally we study the situation in the $CAT(-1)$ case and obtain:

Corollary 1.9. *Let X be a $CAT(-1)$ space with a geometrically inner point. If on X a non-constant continuous affine function exists then X is isometric to a subset of $\mathbb{R}$.*

Remark 1.2. Our investigations were mainly motivated by Theorem 1.6 in [AdB], where affine Busemann functions were analyzed and a result similar to our Theorem 1.2 was obtained.

Remark 1.3. We do not know if our results generalize to the case of K-affine functions studied in [AB]

Without any curvature assumption, it is not clear to us what implications the existence of an affine function must have. The best one can hope for is that (in some sense) X looks like a Banach space in some direction:

Example 1.4. On strictly convex Banach spaces each continuous linear function is affine. But Banach spaces usually do not admit non-trivial isometric embeddings into a space with a Euclidean factor. Let Z be any geodesic metric space, Y a strictly convex Banach space, and $||\cdot||$ a strongly convex norm on a two dimensional vector space. Let $X = Y \times_{||\cdot||} Z$ be the non-standard metric product in the sense of [BFS]. Then the projection $p_Y : X \to Y$ is an affine map and composing this map with affine functions on Y we obtain many affine functions on X. To obtain further examples one can take convex subsets of such spaces or glue such spaces together in the right way.

The proofs of splitting results in [In],[Ma1],[Ma2] and [AB] have in common that the non-Euclidean factor can be recognized as a convex subset of X. Our proof is quite different, and we hope that it may find other applications. We construct the factor Y in a very abstract way, that we are going to sketch now.

Assume for a moment that Theorem 1.2 is true and consider the projection $F = p_H \circ i : X \to H$. The metric on Y must satisfy $d(P_Y(y), P_Y(z)) = \sqrt{d(y,z)^2 - ||F(y) - F(z)||^2}$ for all y, z in X. In particular, the right hand side must be a pseudo metric on X and Y must be the metric space corresponding to this pseudo metric one. To prove Theorem 1.2 we go the same way backwards. First we define some Hilbert space H with a natural map $F : X \to H$ that would coincide with $p_H \circ i$, if the theorem was true (Section 3 and Section 4). Then it only remains to prove that the term $\sqrt{d(y,z)^2 - ||F(y) - F(z)||^2}$ is a pseudo metric on X. We prove that this is true infinitesimally (i.e. in the tangent cones) at many regular points, and then use first variation formula to deduce that $\tilde{d}$ is indeed a pseudo metric. The proof also shows that the spaces H and Y and the isometric embedding i in Theorem 1.2 are unique up to isometry.

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2. Preliminaries

2.1. **Spaces.** By d we will denote the distance in metric spaces without an extra reference to the space. By $B_r(x)$ we denote the open metric ball of radius r around a point x. A *pseudo metric* is a metric for which the distance between different points may be zero. Identifying points with pseudo distance 0 one obtains a metric space from it.

A *geodesic* in a metric space is a length minimizing curve parameterized proportionally to arclength. A metric space is *geodesic* if each pair of its points is connected by a geodesic. A subspace of a geodesic space is *convex* if it is geodesic with respect to the induced metric. A $CAT(\kappa)$ space is a complete, geodesic metric space in which triangles are not thicker than in the space of constant curvature κ. We refer to [BH] for a detailed discussion of such spaces. We will need the following estimates that are direct consequences of the $CAT(\kappa)$ property and spherical trigonometry. There are numbers $A = A(\kappa), r = r(\kappa)$ such that for each triple x_1, x_2, x_3 in a $CAT(\kappa)$ space X with $d(x_i, x_j) \leq r$ the following holds. Let m_i be the midpoint between x_j and x_k ($j \neq i \neq k$).

(1) If $d(x_1,x_2), d(x_1,x_3) \leq 2t$ then $d(m_2,m_3) \leq \frac{1}{2}d(x_2,x_3)(1+At^2)$;
(2) If $d(x_1,x_2) = d(x_1,x_3) = \epsilon$ then $d(x_1,m_1) \leq \epsilon - Ad(x_2,x_3)^2$.

2.2. **Functions.** A map $f : X \to Y$ is called *L-Lipschitz* if it satisfies $d(f(x), f(z)) \leq Ld(x,z)$. The smallest L as above is called the optimal Lipschitz constant of f. For a function $f : X \to \mathbb{R}$, we denote by $|\nabla_x f|$ the *absolute gradient* at x which is given by $\max\{0, \limsup_{z\to x} \frac{f(z)-f(x)}{d(x,z)}\}$. If f is L-Lipschitz then all absolute gradients are not larger than L. On the other hand, if the space X is geodesic then the optimal Lipschitz constant is the supremum of all absolute gradients.

2.3. **Gradients of affine functions.** Let X be a geodesic metric space and $f : X \to \mathbb{R}$ an affine function. For a point $x \in X$, the absolute gradient $|\nabla_x f|$ is maximum of 0 and the supremum over $(f \circ \gamma)'$

where γ runs over all unit speed geodesics starting at x. Moreover, we have $|\nabla_x f| = \max\{0, \sup_{z\neq x} \frac{f(z)-f(x)}{d(x,z)}\}$.

This implies that for each convergent sequence $x_i \to x$ in X such that $f(x_i)$ converges to $f(x)$, we have $\liminf |\nabla_{x_i} f| \geq |\nabla_x f|$ (compare [P]). If f is continuous then $|\nabla_x f|$ is semi-continuous in x.

We are going to show that in the $CAT(\kappa)$ case the gradient is constant along geodesics.

Lemma 2.1. *Let X be a $CAT(\kappa)$ space and $f : X \to \mathbb{R}$ be an affine function. Let $\gamma : [a,b) \to X$ be a local geodesic. Then $p(t) = |\nabla_{\gamma(t)} f|$ is constant p_0 on (a,b) and we have $p(a) \leq p_0$.*

Proof. The restriction of f to γ is continuous. Thus $\liminf_{t\to a} p(t) \geq p(a)$. It remains to prove that p is constant on (a,b). The statement is local, therefore we may assume that γ is parameterized by the arclength and has length smaller than $\frac{r}{2}$. Here and below r and A are chosen as in Subsection 2.1.

We claim that for each $s \in (a,b)$ and each t with $|t| \leq \{s-a, b-s\}$ one has $p(s+t) \leq p(s)(1+At^2)$. Observe that the claim implies that p is either constant ∞ on (a,b), or it is finite everywhere on (a,b) and locally Lipschitz. Moreover, in the last case the differential of p vanishes at each point, hence p is constant as well.

In order to prove the claim, choose an arbitrary point z close to $\gamma(s+t)$ such that $f(z) \geq f(\gamma(s+t))$. Consider the midpoint m between z and $\gamma(s-t)$. Since f is affine, $f(m) - f(\gamma(s)) = \frac{1}{2}(f(z) - f(\gamma(s+t)))$. On the other hand, we have $d(m, \gamma(s)) \leq \frac{1}{2} d(z, \gamma(s+t))(1+At^2)$. Hence

$$\frac{f(m) - f(\gamma(s))}{d(m,\gamma(s))} \geq \frac{f(z) - f(\gamma(s+t))}{d(z,\gamma(s+t))(1+At^2)}.$$

We deduce $p(s+t) \leq p(s)(1+At^2)$. $\square$

Let X be a $CAT(\kappa)$ space and $f : X \to \mathbb{R}$ be an affine function. Then for each $t \geq 0$ the set of all points $x \in X$ with $|\nabla_x f| > t$ is convex in X. Moreover, it is dense in X if it is non-empty. By semi-continuity this subset is open, if f is continuous.

Let $x \in X$ be a geometrically inner point. Then no local geodesic ends in x. From the last lemma we deduce $|\nabla_x f| \geq |\nabla_y f|$ for any point $y \in X$. Thus the function f is L-Lipschitz if and only if $|\nabla_x f| \leq L$.

3. The space of affine functions

Let X be an arbitrary geodesic metric space. The set of all Lipschitz continuous affine functions on X is a vector space and will be denoted by $\tilde{\mathcal{A}}(X)$. It always contains the one-dimensional subspace $Const(X)$

of constant functions. By $\mathcal{A}(X)$ or simply $\mathcal{A}$ we will denote the quotient vector space $\tilde{\mathcal{A}}(X)/Const(X)$. For $f \in \tilde{\mathcal{A}}$ we denote by $[f] \in \mathcal{A}$ the corresponding element of $\mathcal{A}$. The optimal Lipschitz constant defines a norm on the space $\mathcal{A}$. Equipped with this norm $\mathcal{A}$ is a normed vector space. It is complete (even if X is not complete), hence it is a Banach space.

Consider the evaluation map $E : X \times X \to \mathcal{A}^*$ from the product $X \times X$ to the (Banach) dual space of $\mathcal{A}$ given by $E(x,y)([f]) = f(y) - f(x)$. By definition $||E(x,y)|| \leq d(x,y)$. Moreover, the map E is strongly affine in the sense that it maps geodesics to linear intervals of the Banach space $\mathcal{A}^*$. Observe that $E(x,y) = 0$ if and only if the points x and y cannot be separated by a Lipschitz continuous affine function on X.

By $E_x : X \to \mathcal{A}^*$ we denote the restriction $E_x(y)([f]) = f(y) - f(x)$. We have $E_y = E_x - E(x,y)$.

Lemma 3.1. *The evaluation map $E_x : X \to \mathcal{A}^*$ is 1-Lipschitz. For each affine function $f \in \tilde{\mathcal{A}}$ we have $[E_x(.)([f])] = [f]$ in $\mathcal{A}$.*

Proof. The second claim is a consequence of the definition. The first one follows from $||E_x(y) - E_x(z)|| = ||E(z,y)|| \leq d(y,z)$. □

4. Normalization

4.1. **Basic splitting results.** The following splitting results will be basic tools in detecting infinitesimal splittings.

Lemma 4.1. *Let X be a $CAT(0)$ space and $f : X \to \mathbb{R}$ be an affine function. If for some line γ in X we have $\infty > (f \circ \gamma)' = ||f|| > 0$ then X splits as $X = Z \times \mathbb{R}$ and f is given by $f(z,t) = ||f||t$.*

Proof. By rescaling and adding a constant we may assume that $||f|| = 1$ and $f(\gamma(t)) = t$. Let $x \in X$ be arbitrary. For the rays γ_x^+ and γ_x^- starting at x and asymptotic to γ^+ and γ^-, respectively, we get $(f \circ \gamma_x^+)' = 1$ and $(f \circ \gamma_x^-)' = -1$. Therefore $|f(\gamma_x^+(1)) - f(\gamma_x^-(1))| = 2$. Since f is 1-Lipschitz we deduce $d(\gamma_x^+(1), \gamma_x^-(1)) = 2$. Hence the concatenation of γ_x^+ and γ_x^- is a line γ_x which is parallel to γ. Therefore, through each point $x \in X$, there exists a line parallel to γ and the well known splitting theorem ([BH]) says that γ defines a line factor of X. Now the last statement is clear, too. □

Lemma 4.2. *Let X be a $CAT(0)$ case and $F : X \to \mathbb{R}^n$ be a 1-Lipschitz affine map with coordinates F_i. Assume that there is a point $x \in X$ and lines $\gamma_1, ..., \gamma_n$ through x such that $(F_i \circ \gamma_i)' = 1$. Then X splits as $X = Z \times \mathbb{R}^n$ such that F is the projection onto the $\mathbb{R}^n$ factor.*

Proof. We may assume $F(x) = 0$. The case $n = 1$ is done by the last lemma. Proceeding by induction on n, we assume that X splits as $\tilde{Z} \times \mathbb{R}^{n-1}$ such that $(F_1, ..., F_{n-1})$ is the projection onto $\mathbb{R}^{n-1}$. Since F is 1-Lipschitz, we must have $(F_i \circ \gamma_n)' = 0$ for all $i \leq n-1$. Therefore, γ_n is contained in $\tilde{Z}$. Applying Lemma 4.1 to the function $F_n : \tilde{Z} \to \mathbb{R}$ we see that γ_n splits of in $\tilde{Z}$. We obtain the desired splitting $X = \tilde{Z} \times \mathbb{R}^{n-1} = Z \times \mathbb{R}^n$. □

4.2. **Differentials.** Let X be a $CAT(\kappa)$ and $f : X \to \mathbb{R}$ an affine Lipschitz function with optimal Lipschitz constant $||f||$. Due to Subsection 2.3, the set X_ϵ of all points $x \in X$ with $|\nabla_x f| > ||f|| - \epsilon$ is open, dense and convex in X, for each $\epsilon > 0$. From the theorem of Baire we deduce:

Corollary 4.3. *Let f_j be a sequence of affine functions. Then the set X_0 of points x such that $|\nabla_x(-f_j)| = |\nabla_x f_j| = ||f_j||$, for all j, is convex and dense in X.*

For each point $x \in X$ there is a tangent cone C_x at x (the Euclidean cone over the space of directions) that is a $CAT(0)$ space. The Lipschitz continuous affine function $f : X \to \mathbb{R}$ has a well defined directional derivative $D_x f : C_x \to \mathbb{R}$, that is itself a Lipschitz continuous affine function. If f is L-Lipschitz then $D_x f$ is L-Lipschitz too, i.e. $||D_x f|| \leq ||f||$. If $|\nabla_x f| > 0$ then there is a unique unit vector $v \in C_x$ with $D_x f(v) = |\nabla_x f|$ (see, for instance [K] or [P]).

Let now $x \in X$ be a point with $||f|| = |\nabla_x f| = |\nabla_x(-f)|$. Consider the unit vectors $v^\pm$ in C_x with $D_x f(v^\pm) = \pm||f||$. Since $D_x f$ is $||f||$-Lipschitz, we must have $d(v^+, v^-) = 2$, i.e. the concatenation of the homogeneous rays $\gamma^\pm(t) = tv^\pm$ is a line γ in C_x. Moreover, by construction, $(D_x f \circ \gamma)' = ||f|| = ||D_x f||$.

Lemma 4.4. *Let X be a $CAT(\kappa)$ space. Then the Banach space $\mathcal{A}(X)$ of Lipschitz continuous affine functions on X is a Hilbert space.*

Proof. Let f, g be two Lipschitz continuous affine function on X. We have to prove the parallelogram equality $||f+g||^2 + ||f-g||^2 = 2(||f||^2 + ||g||^2)$. Due to Corollary 4.3, there exists a point $x \in X$ such that $|\nabla_x h| = ||h||$ for the eight functions $h = \pm f, \pm g, \pm(f+g), \pm(f-g)$. We have seen above that this implies $||h|| = ||D_x h||$ and that for each h there is a line γ_h in C_x through 0, with $(h \circ \gamma_h)' = ||h||$.

Since $||h|| = ||D_x h||$, we may replace X by C_x and h by $D_x h$. Thus we may assume that X is a $CAT(0)$ space and that there are lines γ_f and γ_g in X through a point x, with $(h \circ \gamma_h)' = ||h||$ for $h = f, g$. Moreover, $f(x) = g(x) = 0$.

Due to Lemma 4.1, the lines γ_h (for $h = f, g$) split of as line factors of the space $X = Z^h \times \mathbb{R}$ such that h is given by $h(z, t) = ||h||t$ with respect to this decomposition. The line factors γ_f and γ_g give rise to a splitting $X = Z \times E$ where E is a one- or two-dimensional Euclidean space, such that $h = \hat{h} \circ p_E$ (for $h = f, g$), where p_E is the projection of X onto E and where $\hat{h}$ is the restriction of h to E.

Replacing h by $\hat{h}$ we reduce the situation to the case, where X is the Euclidean space E. In this case the statement is clear, since the dual of a Hilbert space is a Hilbert space. $\square$

4.3. **Normalized maps and their regular points.** We are going to discuss an important property of the evaluation map now.

Definition 4.5. *Let X be a $CAT(\kappa)$ space, H be a Hilbert space and $F : X \to H$ an affine map. We call F normalized, if F is 1-Lipschitz and for each unit vector $h \in H$ the affine function $F^h : X \to \mathbb{R}$ given by $F^h(x) = \langle F(x), h\rangle$ satisfies $||F^h|| = 1$.*

For example an affine function $f : X \to \mathbb{R}$ is normalized if and only if it has norm 1.

Example 4.1. Let $H_0 \subset H$ be a Hilbert subspace. Then the orthogonal projection $p : H \to H_0$ is normalized. If $F : X \to H$ is normalized then so is the composition $p \circ F$.

Using the natural isometry $\mathcal{A}^* = \mathcal{A}$ for the Hilbert space $\mathcal{A}$ we deduce from Lemma 3.1:

Lemma 4.6. *For each point $x \in X$ the evaluation map $E_x : X \to \mathcal{A}^*$ is normalized.*

Given an L-Lipschitz continuous affine map $F : X \to H$ to a finite dimensional Hilbert space H one can define directional differentials $D_x F : C_x \to C_{(f(x))} H = H$ by setting $D_x F(v) = (F \circ \gamma)'$, for a geodesic γ starting at x in the direction v. The differentials are again L-Lipschitz and affine.

Definition 4.7. *Let X be a $CAT(\kappa)$ space, H a finite dimensional Hilbert space and $F : X \to H$ a normalized affine map. We call a point $x \in X$ regular (with respect to F) if C_x has the splitting $C_x = C'_x \times H_x$, with a Hilbert space H_x, such that $D_x F : C_x \to H$ is a composition of the projection of C_x to H_x and an isometry.*

An affine function $f : X \to \mathbb{R}$ is normalized if and only its optimal Lipschitz constant is 1. In this case a point $x \in X$ is regular with respect to f if and only if $|\nabla_x f| = |\nabla_x(-f)| = 1$ as the discussion preceding Lemma 4.4 shows.

Lemma 4.8. *Let X be a $CAT(\kappa)$ space and $F : X \to \mathbb{R}^n$ a normalized affine map. Then the set of regular points (with respect to F) is convex and dense in X.*

Proof. Let F_i, $i = 1, ..., n$, be the coordinates of F. If a point $x \in X$ is regular, we must have $1 = |\nabla_x F_i| = |\nabla_x(-F_i)|$ for all i. On the other hand, such a point x is regular, due to Lemma 4.2 and the observations preceding Lemma 4.4. The result now follows from Corollary 4.3. $\square$

5. Proof of Theorem 1.2

5.1. **The pseudo metric.** In the proof we will need the following first variation formula, compare [L1].

Lemma 5.1. *Let X be a $CAT(\kappa)$ space, y, x two point in X. Let γ be a geodesic starting at x and let η be a geodesic from x to y, both parameterized by the arclength. Let β be the angle between η and γ. Then for the function $b(t) = d(y, \gamma(t))$, we have $b'(0) \leq -\cos(\beta)$, i.e. $b(t) \leq b(0) - t\cos(\beta) + o(t)$.*

The proof is a direct consequence of the definition of the angle and the triangle inequality $d(y, \gamma(t)) \leq d(y, \eta(s)) + d(\eta(s), \gamma(t))$.

The proof of the main theorem will be a direct consequence of the following result:

Theorem 5.2. *Let X be a $CAT(\kappa)$ space, H be a Hilbert space and $F : X \to H$ be a normalized affine map. Then $\tilde{d} : X \times X \to [0, \infty)$ given by $\tilde{d}(y, z) = \sqrt{d(y, z)^2 - ||F(y) - F(z)||^2}$ defines a pseudo metric on X.*

Proof. By definition $\tilde{d}$ is symmetric. Since F is 1-Lipschitz, $\tilde{d}$ is non-negative. Moreover, $\tilde{d}(x, x) = 0$ for all $x \in X$. It remains to prove the triangle inequality.

Since F is 1-Lipschitz, the map $\tilde{d} : X \times X \to \mathbb{R}$ is continuous and it is locally Lipschitz outside the set $\tilde{\Delta}$ of pairs (y, z) with $\tilde{d}(y, z) = 0$.

Assume now that the triangle inequality does not hold for $\tilde{d}$ and take three points $x, y, z \in X$ with $\tilde{d}(y, z) > \tilde{d}(y, x) + \tilde{d}(x, z)$. Denote by H_0 the linear hull of three points $F(x), F(y), F(z)$ in H and consider the composition $F_0 = p \circ F : X \to H_0$, where $p : H \to H_0$ is the orthogonal projection. Since $||F(x) - F(y)|| = ||F_0(x) - F_0(y)||$, $||F(x) - F(z)|| = ||F_0(x) - F_0(z)||$ and $||F(y) - F(z)|| = ||F_0(y) - F_0(z)||$ we see that Theorem 5.2 is wrong for F_0 too. Thus to prove Theorem 5.2 it is enough to consider the case of finite dimensional H. From now on we will assume $\dim(H) < \infty$.

Due to Corollary 4.8 the set of regular points is dense in X. Thus moving our three points x, y, z slightly, we may assume that they are regular. Since $\tilde{d}$ is non-negative we must have $\tilde{d}(y,z) > 0$. Choose a unit speed geodesic $\gamma : [0, d(x,z)] \to X$ from x to z. Due to Corollary 4.8 all points on γ are regular. Set $\bar{A} = (F \circ \gamma)' \in H$ and let $A = \sqrt{1 - ||\bar{A}||^2}$. ($\bar{A}$ is well defined since $F \circ \gamma$ is a geodesic). For all s, t we have $\tilde{d}(\gamma(s), \gamma(t)) = A|t - s|$.

We set $a(t) = \tilde{d}(y, \gamma(t))$. The function $a(t)$ is continuous, non-negative and locally Lipschitz, whenever it is positive. For the function $h(t) = \tilde{d}(y, \gamma(t)) + \tilde{d}(z, \gamma(t)) = a(t) + A(d(y,z) - t)$ we have $h(0) < h(d(y,z))$ by our assumption. Therefore, we can find some t_0 such that $a(t_0) > 0$ and $a'(t_0) > A$. Without loss of generality we may assume $t_0 = 0$ and $F(\gamma(t_0)) = F(x) = 0$.

Set $r = d(x,y)$ and choose a geodesic $\eta : [0, r] \to X$ from x to y. Let v and w denote the starting directions of γ and η, respectively. We denote by x_1 the origin 0 of the tangent cone C_x. We set $y_1 = rw$ and consider the ray $\gamma_1(t) = tv$. We denote by d_1 the distance in the tangent cone C_x and by $F_1 : C_x \to H$ the differential $F_1 = D_x F$ of F.

By assumption, the point x is regular with respect to F, hence F_1 is the projection onto a Euclidean factor of C_x. Therefore the function $\tilde{d}_1(p,q) = \sqrt{d_1(p,q)^2 - ||F_1(p) - F_1(q)||^2}$ is a pseudo metric on C_x.

Finally we set $a_1(t) = \tilde{d}_1(y_1, \gamma_1(t))$. The definition of F_1 implies that $F(\gamma(t)) = F_1(\gamma_1(t))$ and $F(y) = F_1(y_1)$. Moreover, we have $d(x_1, y_1) = d(x, y) = r$. The fact that $\tilde{d}_1$ is a pseudo metric together with $\tilde{d}_1(\gamma_1(s), \gamma_1(t)) = A|s - t|$ implies $a_1'(0) \leq A$. On the other hand, for $b(t) = d(y, \gamma(t))$ and $b_1(t) = d_1(y_1, \gamma_1(t))$ we have $b'(0) \leq b_1'(0)$, due to Lemma 5.1. Since $a^2(t) - a_1^2(t) = b^2(t) - b_1^2(t)$, this implies $a'(0) \leq a_1'(0)$ and we get a contradiction. $\square$

5.2. **Main Theorem.** Let $F : X \to H$ be an affine normalized map and let $\tilde{d}$ be the pseudo metric on X defined in Theorem 5.2. Let $Y = X/\tilde{d}$ be the induced metric space. A point in Y is an equivalence class $[x]$ where $x \sim x'$ iff $\tilde{d}(x, x') = 0$. The definition of $\tilde{d}$ implies that the map $X \to Y \times H$, $x \mapsto ([x], F(x))$ is an isometric embedding.

For each unit speed geodesic γ in X we have seen, that γ is a geodesic of velocity $\sqrt{1 - ||(F \circ \gamma)'||^2}$ with respect to $\tilde{d}$. Hence the space Y is geodesic.

Let now $g : X \to X$ be an isometry such that $||F(g(x)) - F(g(y))|| = ||F(x) - F(y)||$ for all $x, y \in X$. Then g induces a map $\tilde{g}$ on Y through $\tilde{g}([x]) = [g(x)]$ and this map is an isometry.

For the proof of Theorem 1.2 we use the affine map $F = E_o : X \to \mathcal{A}^*$, where E_o is the evaluation map for some base point $o \in X$. Due to Lemma 4.6, the map E_o is affine and normalized. Due to Theorem 5.2, the function $\tilde{d}$ on $X \times X$ given by $\tilde{d}(x,y) := \sqrt{d^2(x,y) - ||E_o(x) - E_o(y)||^2}$ defines a pseudo metric on X. Since the value of $||E_o(x) - E_o(y)||$ does not depend on o, the definition of this pseudo metric does not depend on the point o.

The above discussion shows that the metric space Y defined by this pseudo metric is a geodesic space and that X has a natural isometric embedding $i : X \to Y \times \mathcal{A}^*$, $x \mapsto ([x], E_o(x))$. Each isometry g of X sends affine functions to affine functions and preserves the Lipschitz constants. Hence g induces an isometry on $\mathcal{A}^*$. Above we have seen that in such a case g also induces an isometry on Y. By construction g is the restriction of the induced isometry on $Y \times \mathcal{A}^*$.

Finally, let $f \in \tilde{\mathcal{A}}(X)$ be a Lipschitz continuous affine function on X. Define $\hat{f} : \mathcal{A}^* \to \mathbb{R}$ by $\hat{f}(\xi) := \xi([f]) + f(o)$, where $[f]$ is the class of f in $\mathcal{A}(X)$. Then $\hat{f}$ is an affine function on $\mathcal{A}^*$ and

$$\hat{f}(E_o(x)) = E_o(x)([f]) + f(o) = f(x) - f(o) + f(o) = f(x)$$

hence $\hat{f} \circ p_{\mathcal{A}^*} \circ i = f$ as required.

5.3. **Hadamard spaces.** We are going to prove Proposition 1.3.

Proof of Proposition 1.3. We use the notations of the previous subsection. It is enough to prove for all $x, y, z \in X$ and for the midpoint m of the geodesic yz the following inequality (see e.g. [BH] p.163):

$$\tilde{d}^2(x,m) \le \frac{1}{2}\tilde{d}^2(x,y) + \frac{1}{2}\tilde{d}^2(x,z) - \frac{1}{4}\tilde{d}^2(y,z) \tag{5.1}$$

Since X is $CAT(0)$ we have

$$d^2(x,m) \le \frac{1}{2}d^2(x,y) + \frac{1}{2}d^2(x,z) - \frac{1}{4}d^2(y,z)$$

and since F is affine and H a Hilbert space we see

$$||F(x)-F(m)||^2 = \frac{1}{2}||F(x)-F(y)||^2 + \frac{1}{2}||F(x)-F(z)||^2 - \frac{1}{4}||F(y)-F(z)||^2.$$

Subtracting the two formulas we obtain inequality (5.1). □

6. Inner points

Before embarking on the proof of Theorem 1.5 we make some general topological remarks which we will use later. Let X be a $CAT(\kappa)$ space. All balls of radius $\le \frac{\pi}{2\sqrt{\kappa}}$ are totally convex in X. Hence all intersections of such balls are either empty or contractible. Thus X has arbitrary fine coverings such that all intersections of the members of each covering are either empty or contractible. This implies that X is an absolute neighborhood retract (ANR), since the criterion of Theorem 1.1 (b) of [To] is satisfied. Since X is an ANR, each open subset of X is homotopy equivalent to a simplicial complex.

Recall that a subset Z of a metric space X is called locally homotopically negligible (also known under the name Z-subset) if for each open subset U of X the inclusion $i : U \setminus Z \to U$ is a weak homotopy equivalence. In [To], Theorem 2.3 it is shown that Z is locally homotopically negligible if each $x \in X$ has arbitrary small neighborhoods V such that $V \setminus Z \to V$ is a weak homotopy equivalence. In [To], Corollary 2.6 it is shown that each subset of a locally homotopically negligible subset is locally homotopically negligible.

Small balls $B_\epsilon(x)$ in a $CAT(\kappa)$ space X are contractible. Since $B_\epsilon(x) \setminus \{x\}$ is homotopy equivalent to a simplicial complex, we deduce that a point $x \in X$ is a topologically inner point of X if and only if the subset $\{x\}$ of X is not locally homotopically negligible.

Remark 6.1. It can be shown that x is a topologically inner point if and only if the space of directions S_x is not contractible (compare [LN]).

Proof of Theorem 1.5. (3) Let $C \subset X$ a dense convex subset. We set $Z = X \backslash C$. For each point $x \in X$ and a small ball $B_\epsilon(x)$ the intersection of $B_\epsilon(x)$ with $C = X \setminus Z$ is totally convex and not empty, hence it is contractible. By the criterion mentioned in the general remarks above, Z is a homotopically negligible subset of X, and each point in $Z = X \backslash C$ is not a topologically inner point of X.

(1) Let X be a $CAT(\kappa)$ space and x a topologically inner point of X. For all small $r > 0$, the punctured ball $B_r(x) \setminus \{x\}$ is not contractible. Since the punctured ball is homotopy equivalent to a simplicial complex, there is some $j \geq 0$ such that the j-th homotopy group $\pi_j(B_r(x) \setminus \{x\}) \neq 1$. Take some j with this property. Then some map $F : \mathbb{S}^j \to B_r(x) \setminus \{x\}$ is not contractible. Since $\mathbb{S}^j$ is compact the image $F(\mathbb{S}^j)$ has distance $\geq \epsilon$ from x for some $\epsilon > 0$.

Assume that for some $y \in B_\epsilon(x)$ there is no point z with $\epsilon = d(x, z) = d(y, z) - d(y, x)$. Consider the homotopy $\tilde{F}$ in $B_r(x)$ from F to the point y along the geodesics starting at y. The assumption on y shows that the homotopy $\tilde{F}$ does not meet the point x, thus $\tilde{F}$ gives a contraction of F to a point inside $B_r(x) \setminus \{x\}$. This is a contradiction, that shows that x must be a geometrically inner point of X.

(2) In [K] it is shown, that in a $CAT(\kappa)$ space of a finite geometric dimension n, there are points $x \in X$ such that the local homology $H_n(X, X \setminus \{x\})$ does not vanish. Such a point x is then a topologically

inner point of X. Since the same argument applies to arbitrary small balls in X, we see that the set of topologically inner point in a locally finite dimensional space X is dense in X. $\square$

7. Inner points and affine functions

7.1. **Continuity and Lipschitz continuity.** Let X be a $CAT(\kappa)$ space and assume that x is a geometrically inner point of X. By Definition 1.4 there exists $\epsilon > 0$ and a (not necessarily continuous) map $I : \bar{B}_\epsilon(x) \to \bar{B}_\epsilon(x)$ that sends a point $y \in \bar{B}_\epsilon(x)$ to a point z with $\epsilon = d(x,z) = d(y,z) - d(y,x)$. We fix ϵ and I for the rest of this section. We are going to prove Theorem 1.7 and Theorem 1.6 now.

Proof of Theorem 1.7. The implications (1) $\to$ (2) $\to$ (3) $\to$ (4) are clear. In order to prove that (4) implies (1) let us assume that some fiber $f^{-1}(t)$ is not dense in X.

Small balls $B_r(y)$ around each point are convex, hence so are the images $f(B_r(y)) \subset \mathbb{R}$. If for some $y \in X$ and for arbitrary small r, we have $f(B_r(y)) = \mathbb{R}$ then for each other point $z \in X$ we have $f(B_r(z)) = \mathbb{R}$ too. To see this, choose $0 < r < \frac{d(y,z)}{2}$ and connect z by geodesics with all points in $B_r(y)$. Consider the points on this geodesics with distance r from z. Since f is affine, it has arbitrary large and arbitrary small values on these points. Hence $f(B_r(z))$ is the whole real line. This contradicts to the assumption that $f^{-1}(t)$ is not dense in X. Hence there is some $r > 0$ such that $f(B_r(x)) \neq \mathbb{R}$. By the convexity of the image this implies that f is bounded on $B_r(x)$ from above or from below.

By making ϵ smaller, if necessary, we may assume $\epsilon < r$. Assume for a moment that $|\nabla_x f|$ is unbounded Then there is a sequence $x_j \to x$ with $\frac{f(x_j)-f(x)}{d(x_j,x)} \to \infty$. Since F is affine this implies $\frac{f(I(x_j))-f(x)}{d(I(x_j),x)} \to -\infty$ and $\frac{f(I(I(x_j)))-f(x)}{d(I(I(x_j)),x)} \to \infty$. However, $d(I(x_j),x) = d(I(I(x_j)),x) = \epsilon$, hence f is not bounded from below nor from above on $B_r(x)$. This contradiction proves that $|\nabla_x f| = L < \infty$. By the last observation in Section 2 the function f is L-Lipschitz.

$\square$

Proof of Theorem 1.6. Let $f : X \to \mathbb{R}$ be an affine function and let x be a topologically inner point of X. Due to Theorem 1.5 (1) we may apply Theorem 1.7. Thus if f is not continuous then each fiber $f^{-1}(t)$ is dense in X. However, each fiber $f^{-1}(t)$ is convex. Due to Theorem 1.5 (3), each fiber must contain the point x. But this is impossible. $\square$

7.2. **Non-injectivity.** We need the following

Lemma 7.1. *Let X be as above and let $f : X \to \mathbb{R}$ be a normalized affine function. Then x is the midpoint of a geodesic γ of length 2ϵ such that $(f \circ \gamma)' = 1$.*

Proof. We have $|\nabla_x f| = 1$. Let x_j be a sequence convergent to x with $\frac{f(x_j)-f(x)}{d(x_j,x)} \to 1$. Then for the points $z_j = I(I(x_j))$, we still have $\frac{f(z_j)-f(x)}{d(z_j,x)} \to 1$, but now $d(z_j, x) = \epsilon$ for all j.

Hence, for $i, j \to \infty$, the midpoint m_{ij} between z_i and z_j satisfies $f(m_{ij}) \to \epsilon$. If the sequence z_i does not converge then one finds a subsequence z_k of this sequence with $d(z_k, z_{k+1}) \geq \rho$ for some $\rho > 0$ and all k. For the midpoints m_k between z_k and z_{k+1} we had $f(m_k) \to \epsilon$ and $d(x, m_k) \leq \epsilon - A\rho^2$, for some $A > 0$, due to Section 2. For sufficiently large k, we obtain a contradiction to the fact that f is 1-Lipschitz. Therefore the sequence z_i converge to some point z in X. Set $y = I(z)$. Then yz is a geodesic of length 2ϵ with midpoint x and $f(z) - f(x) = d(x, z)$. $\square$

Let X be as above, H a Hilbert space and $F : X \to H$ a normalized affine map. Let $i : X \to Y \times H$ be the isometric embedding as in Section 5. For each point $z \in X$ we denote by H_z the Hilbert space $\{P_Y(z)\} \times H \subset Y \times H$. In general, the intersection $H_z \cap X$ may be very thin (for instance, consists of the point z only). We define the thickness of X at z to be the maximal radius of the ball in H_z centered at z that is contained in X and denote it by $q^F(z)$.

The function q^F is non-negative and the completeness of X implies that it is semi-continuous, i.e. for each convergent sequence $x_i \to x$ in X we have $\liminf q^F(x_i) \geq q^F(x)$. If for some $\rho > 0$ we have $q^F(z) \geq \rho$ for all $z \in X$, then $H_z \cap X = H_z$ for all $z \in X$ and the embedding i is surjective, i.e. an isometry.

Observe that $q^F(z)$ is the largest number $r \geq 0$ such that for each unit vector $h \in H$, there is a unit speed geodesic of length $2r$ with midpoint z such that $(F^h \circ \gamma)' = (\langle F \circ \gamma, h\rangle)' = 1$. From the last lemma we deduce:

Corollary 7.2. *Let x be a geometrically inner point of a $CAT(\kappa)$ space X and let $F : X \to H$ be a normalized affine map. Then the thickness $q^F(x)$ is positive.*

7.3. **The case $\kappa \leq 0$.** Let now X be a $CAT(0)$ space, $F : X \to H$ a normalized affine map. Due to Proposition 1.3, the completion of Y is $CAT(0)$. In particular, geodesics in Y and in $Y \times H$ are uniquely determined by their endpoints and X is totally geodesic in $Y \times H$.

Given two points $z, \bar{z} \in X$ we deduce that the convex hull of the subsets $H_z \cap X$ and $H_{\bar{z}} \cap X$ is contained in the convex hull of H_z and $H_{\bar{z}}$ in $Y \times H$. The last convex hull is just the product $P_Y(\gamma) \times H$, where γ is the geodesic between z and $\bar{z}$. In particular, this convex hull is flat. Therefore, for the midpoint m between z and $\bar{z}$ we can deduce $q^F(m) \geq \frac{1}{2}(q^F(z) + q^F(\bar{z}))$. Hence the subset X_ϵ of all points z with $q^F(z) \geq \epsilon$ is a closed, convex subset of X.

Proof of Corollary 1.8. Let $i : X \to Y \times H$ be the embedding constructed in Theorem 1.2 and let $F = P_H : X \to H$ be the normalized affine map. Since the embedding is invariant under each isometry of X, the thickness q^F is Γ-invariant. Therefore, the closed convex subset X_ϵ defined above are Γ-invariant. In Corollary 7.2 we have seen that X_ϵ is not-empty, for some $\epsilon > 0$. By assumption we must have $X_\epsilon = X$. But this implies that $i : X \to Y \times H$ is an isometry. □

Proof of Corollary 1.9. Thus let X be a $CAT(-1)$ space with a geometrically inner point x and $f : X \to \mathbb{R}$ be a continuous non-constant affine function. Due to Theorem 1.7, the function f is Lipschitz continuous. Hence there is a normalized affine function $F : X \to \mathbb{R}$. Take a non-trivial geodesic xy in the subset $\mathbb{R}_x \cap X$ (it exists by Lemma 7.1). Then for each $z \in X$ the triangle xyz is flat, hence it must be degenerate (since X is $CAT(-1)$). But this implies that z is contained in $\mathbb{R}_x$. Thus the whole space X is the subset of the line R_x. □

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