

AFFINE IMAGES OF RIEMANNIAN MANIFOLDS

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ABSTRACT. We describe all affine maps from a Riemannian manifold to a metric space and all possible image spaces.

1. INTRODUCTION

A map $f : X \rightarrow Y$ between metric spaces is called **affine** if it preserves the class of linearly parametrized minimizing geodesics. If X is a geodesic metric space then a continuous map f is affine if and only if it sends midpoints to midpoints. A map is called **locally affine** (another word is **totally geodesic**) if the restriction of the map to some neighborhood of any point is affine. Basic examples of (locally) affine maps are (locally) isometric embeddings, rescalings and projections to a factor in a direct product decomposition.

Affine maps arise naturally in questions related to super-rigidity (see the explanations and the literature list in [Oht03]), in the study of Berwald spaces in Finsler geometry ([Sza81]), in the study of product decompositions ([FL08]) and in the study of isometric actions on non-positively curved spaces ([AB98]). It seems to me to be of independent interest, to understand to what extent the geodesics determine the metric of a given space.

We give a complete description of all affine maps $f : M \rightarrow Y$ where M is a smooth Riemannian manifold and Y an arbitrary metric space. First, an important special case of the main result:

Theorem 1.1. *Let M be a connected complete smooth Riemannian manifold. There is a locally affine map $f : M \rightarrow Y$ to some metric space that is not a local homothety if and only if the universal covering of M is a product or a symmetric space of higher rank.*

Here, we say that a map $f : X \rightarrow Y$ is a **local homothety** if, for any point $x \in X$, there is some neighborhood U of x and a non-negative

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number a , such that for all $x_1, x_2 \in U$ the equality $d(f(x_1), f(x_2)) = a \cdot d(x_1, x_2)$ holds true. If X is connected then the number a does not depend on the point x . If $a = 0$ then the map is a constant map. In the non-degenerate case $a > 0$, after a rescaling of the space, such map is locally an isometric embedding.

For the formulation of the main theorem we will need two definitions.

Definition 1.2. *A Riemannian submersion $f : M \rightarrow M_1$ between smooth Riemannian manifolds will be called **flat** if the fibers are totally geodesic and the horizontal distribution is integrable.*

A Riemannian submersion $f : M \rightarrow M_1$ between complete Riemannian manifolds is flat if and only if the lift to the universal coverings $\tilde{f} : \tilde{M} \rightarrow \tilde{M}_1$ is the projection map of a direct product decomposition $\tilde{M} = \tilde{M}_1 \times M_2 \rightarrow \tilde{M}_1$. Another useful formulation is that a Riemannian submersion $f : M \rightarrow M_1$ is flat if and only if the tangent distribution of the fibers is a parallel distribution ([Vil70] or [Sak96]).

Definition 1.3. *Let (M, g) be a Riemannian manifold. We will call a function $|\cdot| : TM \rightarrow \mathbb{R}$ a holonomy invariant Finsler structure if the restriction of $|\cdot|$ to each tangent space $T_p M$ is a (possibly not smooth or not strictly convex) norm and if $|v_1| = |v_2|$ for any two vectors related by the holonomy along some piece-wise smooth curve. For a holonomy invariant Finsler structure $|\cdot|$ we will call the identity map $Id : (M, g) \rightarrow (M, |\cdot|)$ an admissible change of metric.*

Note that an admissible change is a bi-Lipschitz map. Clearly, holonomy invariant Finsler structures are in one-to-one correspondence with norms $|\cdot|$ on a fixed tangent space $T_p M$ that are invariant under the action of the holonomy group Hol_p . We refer to Subsection 4.3 for more information about such norms.

Now we can state the main result:

Theorem 1.4. *Let M be a complete Riemannian manifold. Then any locally affine map $f : M \rightarrow Y$ to a metric space Y is a composition $f = f_i \circ f_a \circ f_p$ of the following factors:*

- (1) *The map $f_p : M \rightarrow M_1$ is flat Riemannian submersion onto a smooth complete Riemannian manifold M_1 ;*
- (2) *The map $f_a : (M_1, g) \rightarrow (M_1, |\cdot|)$ is an admissible change of metric;*
- (3) *The map f_i is a locally isometric embedding.*

Moreover, any map f of this kind is locally affine.

This global theorem is a consequence of the following local version of this result. To state it we recall the following notation from [Oht03].

Let $f : M \rightarrow Y$ be a locally affine map defined on a Riemannian manifold M . For a tangent vector v to M we denote by γ_v the geodesic in the direction of v and by $|v|^f$ the velocity of the image geodesic $f \circ \gamma_v$. Now the local result reads as follows:

Theorem 1.5. *Let M be a smooth Riemannian manifold and let $f : M \rightarrow Y$ be a locally affine map. Then $|\cdot|^f$ is a continuous family of semi-norms on the tangent bundle of M that is invariant under the parallel translation.*

Remark 1.1. It is important to stress, that we consider all geodesics to be parametrized affinely. A related, but much harder problem, is the study of **projective maps**. These are homeomorphisms $f : X \rightarrow Y$ between metric spaces that preserve geodesics as subsets, disregarding their parametrization. The class of such maps is often considerably larger and, usually, much more difficult to determine. Even if X and Y are Riemannian manifolds, the classification is not completely known. We refer the reader to the following two important contributions to this subject of projective maps: [Mat07] and [Sza86].

The results of this paper generalize previous works [Vil70], [Sza81], [Oht03] and [HL07]. Vilms classifies in [Vil70] all affine maps between smooth Riemannian manifolds. Szabo extended in [Sza81] this result to bijective maps between smooth Riemannian and smooth Finsler manifolds providing a characterization of all Berwald spaces. On the other hand, Ohta ([Oht03]) studied maps from a Riemannian manifold to (locally uniquely geodesic) metric spaces and proved that such maps are compositions of affine maps to continuous Finsler manifolds and isometric immersions. However, one cannot combine results of [Oht03] and [Sza81], since the Finsler metric obtained is only continuous (even point-wise) and the Finsler geometric method used in [Sza81] cannot work.

Our proof is a self-contained combination and simplification of ideas used in [Oht03] and [HL07]. We hope, that our proof may lead to an understanding of affine maps on singular non-positively curved spaces.

The paper is structured as follows. In Section 2 we recall the proof of the semi-continuity of the the semi-Finsler structure $|\cdot|^f$. In Section 3, the core of the paper, we prove Theorem 1.5. In Section 4 we recall basics about invariant norms and Finsler structures. Finally, in Section 5 we prove Theorem 1.1 and Theorem 1.4.

2. BASICS

2.1. Setting. Let M be a Riemannian manifold and let $f : M \rightarrow Y$ be a locally affine map to a metric space Y . We will denote by d the distance in M and by \bar{d} the distance in Y . In this and in the next section we are going to deal with local questions only. Thus we may replace M by a small strictly convex open neighborhood U of a given point x . If U is sufficiently small then $f : U \rightarrow Y$ is affine. By the definition of $|\cdot|^f$, for any geodesic $\gamma_v : [a, b] \rightarrow U$ and all $s, t \in [a, b]$ we have $\bar{d}(f(\gamma_v(s)), f(\gamma_v(t))) = |s - t| \cdot |v|^f$. From this we deduce $|\lambda v|^f = |\lambda| \cdot |v|^f$ for all $\lambda \in \mathbb{R}$.

In this section we are going to prove that f is locally Lipschitz and that $|\cdot|^f$ is a continuous family of semi-norms. With a minor additional assumption it has been proved in [Oht03]. For the convenience of the reader, we recall and slightly simplify Ohta's proof.

2.2. Lipschitz continuity. We start with

Lemma 2.1. *The map f is continuous.*

Proof. We fix $p \in U$ and are going to prove the continuity at the point p . In order to do so, it is enough to show that the homogeneous function $|\cdot|^f$ is bounded on the unit sphere in $T_p^1 M \subset T_p M$.

Choose $v_i \in T_p^1 M$ to be the vertices of a regular simplex Δ . Set $x_i = \exp_p(\epsilon v_i)$, for sufficiently small $\epsilon > 0$. Let $A_0(\epsilon)$ be the union of all x_i . Let $A_k(\epsilon)$ be the set of all points that lie on a shortest geodesic between some pair of points in $A_{k-1}(\epsilon)$. The sets $K_\epsilon := \frac{1}{\epsilon} \exp_p^{-1}(A_{n-1}(\epsilon))$ converge in the Hausdorff topology to the boundary $\partial\Delta$ of Δ . Thus K_ϵ does not contain the origin for small ϵ . On the other hand, K_ϵ contains a small continuous perturbation of $\partial\Delta$. Thus K_ϵ carries a non-trivial homology class of $H_{n-1}(T_p M \setminus \{0\})$. Therefore, K_ϵ intersects all rays starting from the origin. Thus, for sufficiently small ϵ , the compact set $A_{n-1} := A_{n-1}(\epsilon)$ does not contain p , but it intersects any geodesic starting from p .

Set $l = \max\{\bar{d}(f(p), f(x_i))\}$. By induction on k and the triangle inequality, $\bar{d}(f(p), f(x)) \leq 2^k l$ for all $x \in A_k$. Since A_{n-1} is compact, the number $\rho = d(p, A_{n-1})$ is positive. Thus for any $x \in A_{n-1}$ we get $\bar{d}(f(p), f(x)) \leq C d(p, x)$ with $C = 2^k \cdot l/\rho$. Since A_{n-1} intersects any geodesic starting in p , we deduce that $|\cdot|^f$ is bounded by C on $T_p^1 M$. \square

Lemma 2.2. *The map $|\cdot|^f$ is continuous and f is locally Lipschitz continuous.*

Proof. Choose vectors $v_n \in T_{p_n}U$ converging to $v \in T_pU$. Set $x_n = \exp(v_n)$; $x = \exp(v)$. Then x_n converges to x and by continuity of f , the images $f(p_n)$ converge to $f(p)$ and the images $f(x_n)$ converge to $f(x)$. Thus $|v_n|^f = \bar{d}(f(p_n), f(x_n)) \rightarrow \bar{d}(f(p), f(x)) = |v|^f$.

The Lipschitz constant of f at p is bounded by the supremum of $|\cdot|^f$ on the unit sphere in T_pU . Thus, the continuity of $|\cdot|^f$ implies the Lipschitz continuity of f . \square

2.3. Finsler structure. Now we claim:

Lemma 2.3. *The function $|\cdot|^f$ restricted to any tangent space T_pU is a semi-norm.*

Proof. One can follow the proof in ([Oht03]). Another short proof goes as follows. The function $|\cdot|^f$ coincides with the metric differential defined by Kirchheim ([Kir94]) for any locally Lipschitz continuous map from a manifold to a metric space. Kirchheim proved that this metric differential is a semi-norm at almost all points. By continuity of $|\cdot|^f$ on TU , in our case this is a semi-norm at all points. \square

3. MAIN ARGUMENT

We are going to use the notations and assumptions from the last section. In this section we are going to prove that the semi-norms $|\cdot|^f$ are invariant under parallel translations along arbitrary piecewise \mathcal{C}^1 curves in U . Approximating such a curve by a geodesic polygon in the \mathcal{C}^1 -topology and using the fact that parallel transport behaves continuously with respect to such approximations, we deduce that it is enough to prove that the semi-norms are parallel along any geodesic polygon. Therefore, it is enough to prove that $|\cdot|^f$ is parallel along any geodesic γ in U .

Thus let us fix a geodesic $\gamma : [-a, a] \rightarrow U$ parametrized by the arclength. Set $p = \gamma(0)$ and denote by $P_t : T_pM \rightarrow T_{\gamma(t)}M$ the parallel transport along γ .

We call a vector $h \in T_xU$ **regular** if for all $v \in T_xU$

$$\lim_{t \rightarrow 0} \frac{|h + tv|^f + |h - tv|^f - 2|h|^f}{t} = 0$$

A vector $h \in T_xU$ is regular if and only if the semi-norm $|\cdot|^f : T_xU \rightarrow \mathbb{R}$ is differentiable at h . By the Theorem of Rademacher almost all vectors in T_xM are regular. Note that h is regular if and only if rh is regular, for all real non-zero r .

We call a vector $h \in T_pU$ **good** if, for almost all $t \in [-a, a]$, the vector $h_t = P_t(h)$ is a regular vector (in $T_{\gamma(t)}U$). Applying Fubini's

theorem we deduce that almost all $h \in T_p U$ are good. By continuity, it suffices to show that for any good vector h , the function $l(t) = |h_t|^f$ is constant. Since P_t and $|\cdot|^f$ are positively homogeneous, we may assume after rescaling that $\exp(h_t) \in U$ exists for all $t \in [-a, a]$.

Lemma 3.1. *The function $l(t)$ is Lipschitz continuous.*

Proof. For all $t, s \in (-a, a)$, by the triangle inequality $||h_t|^f - |h_s|^f| = |\bar{d}(f(\gamma(t)), f(\exp(h_t))) - \bar{d}(f(\gamma(s)), f(\exp(h_s)))| \leq \bar{d}(f(\gamma(t)), f(\gamma(s))) + \bar{d}(f(\exp(h_t)), f(\exp(h_s)))$.

The parallel transport and the exponential map are smooth and f is Lipschitz continuous, thus we can estimate $||h_t|^f - |h_s|^f|$ from above by $A \cdot |t - s|$, for some $A > 0$. \square

Since $l(t)$ is Lipschitz continuous, it is enough to prove that the derivative of $l(t)$ is 0 almost everywhere. Thus, it suffices to prove that if h_{t_0} is regular and if the derivative $l'(t)$ exists at t_0 , this derivative must be zero. To prove this claim, we may assume without loss of generality (reparametrizing γ) that $t_0 = 0$ and that this derivative is non-negative. Thus, we have reduced our task to proving the following claim:

Proposition 3.2. *Let γ be a geodesic starting in p . Let h be a regular vector in $T_p U$. Let h_t be the parallel translates of h along γ and let the function $l(t) = |h_t|^f$ be differentiable at 0. Then $l'(0) \leq 0$.*

Proof. Again, we may assume that $\exp(h_t)$ exists for all t . Assume $l'(0) = 2\delta > 0$. Then, for all small positive t , we have $|h_t|^f \geq |h|^f + \delta t$.

Denote by η_t the geodesic $\eta_t(r) = \exp(rh_t)$. Set $x_{t,r} := \eta_t(r)$ and $p = x_{0,0} = \gamma(0)$. Finally, set $\mu_{t,r} := \exp_p^{-1}(x_{t,r}) \in T_p U$. The triangle inequality gives us

$$\bar{d}(f(x_{t,r}), f(x_{t,-r})) = 2r|h_t|^f \leq \bar{d}(f(p), f(x_{t,r})) + \bar{d}(f(p), f(x_{t,-r})) = |\mu_{t,r}|^f + |\mu_{t,-r}|^f$$

Therefore

$$2r|h|^f + 2r\delta t \leq |\mu_{t,r}|^f + |\mu_{t,-r}|^f$$

Denote by $v_r \in T_p M$ the derivative $v_r := \frac{d}{dt}|_{t=0}\mu_{t,r}$. The proof of the following differential geometric lemma will be given below.

Lemma 3.3. *In the above notations one has $\lim_{r \rightarrow 0} (||v_r - v_{-r}||/r) = 0$.*

Given this lemma, it is easy to derive a contradiction: We find and fix a small r with $|v_r - v_{-r}|^f \leq \delta r/4$. By definition, $\mu_{t,\pm r} = (\pm r) \cdot h + t \cdot v_{\pm r} + o(t)$. Thus $|\mu_{t,-r}|^f \leq |-rh + t \cdot v_r|^f + \delta r t/2$, for small t . Hence $|\mu_{t,r}|^f + |\mu_{t,-r}|^f \leq |r \cdot h + t \cdot v_r|^f + |rh - t \cdot v_r|^f + \delta r t$.

By assumption, h is regular. Thus $|rh + tv_r|^f + |rh - tv_r|^f = 2|rh|^f + o(t)$. For small t , this contradicts to $|\mu_{t,r}|^f + |\mu_{t,-r}|^f \geq 2r|h|^f + 2r\delta t$. \square

It remains to provide:

Proof of Lemma 3.3. Let $\eta = \eta_0$. Denote by Y the Jacobi field $Y(r) = \frac{d}{dt}\eta_t(r)$ along η . Let V_r be the Jacobi field $V_r(s) := \frac{d}{dt}|_{t=0}\exp_p(\frac{\mu_{t,r}}{r}s)$.

By construction, $Y'(0) = 0$, $V_r(0) = 0$ and $V_r(r) = Y(r)$. Finally, $V_r'(0) = \frac{d}{dt}\frac{\mu_{t,r}}{r} = \frac{v_r}{r}$. Thus, we only need to prove that $|V_r'(0) + V_{-r}'(0)|$ converges to 0, as r goes to 0. Since the Jacobi field $(V_r + V_{-r})$ has value 0 at the point 0, it suffices to prove that $|(V_r + V_{-r})(r)| = o(r)$.

In fact we have $\|Y(t) - Y(0)\| \leq Ct^2$ and $\|V_r(t) - tV_r'(0)\| \leq Ct^2$ for all $|t| \leq r$ and some C independent of r . Together this gives us $\|V_{-r}(r) + V_{-r}(-r)\| \leq Cr^2$ and $\|Y(r) - Y(-r)\| \leq Cr^2$. Together this gives us $\|V_{-r}(r) + V_r(r)\| \leq Cr^2$. This finishes the proof of Lemma 3.3, Proposition 3.2 and Theorem 1.5. \square

4. REMARKS ON INVARIANT NORMS

In this section we collect some elementary observation, probably well known to experts, for which we could not find a reference.

4.1. Invariant norms. We recall a few definitions. A **semi-norm** on a finite-dimensional vector space V is a non-negative, homogeneous, convex function $q : V \rightarrow [0, \infty)$. It is called a **norm** if $q(v) > 0$ for all $v \neq 0$. It is called a Minkowski norm, if q is smooth outside the origin and the Hessian D_x^2q is positive definite at all points $x \neq 0$. The second condition is equivalent to the requirement that q can be expressed as $q = q_0 + q_1$, where q_0 is a scalar product and q_1 is a norm.

For two norms q_1, q_2 on V there is some $L \geq 1$ with $q_1 \leq L \cdot q_2$ and $q_2 \leq L \cdot q_1$. The distance $|q_1 - q_2|$ between q_1 and q_2 is defined to be the infimum of $\log(L)$ taken over the set of all such L . The topology on the set of norms defined by this distance function coincides with the Hausdorff topology on the set of convex centrally-symmetric bodies. It is well known that the set of Minkowski norms is dense in the set of all norms.

Lemma 4.1. *Let V be a Euclidean space and let G be a closed subgroup of the orthogonal group. Then the set of G -invariant Minkowski norms is dense in the set of all G -invariant norms.*

Proof. Let q be a G -invariant norm. Choose some small $\epsilon > 0$. We find a Minkowski norm q_1 with $|q_1 - q| \leq \epsilon$. Let q_2 be the norm obtained from q_1 by the averaging procedure, i.e., $q_2(x) = \int_G q_1(gx)d\mu(g)$, where μ is the unit volume Haar measure on G . Then q_2 is a G -invariant Minkowski norm and we still have $|q - q_2| \leq \epsilon$, since q is G -invariant. \square

Lemma 4.2. *Let V, G be as above. If G acts transitively on the unit sphere then the only G -invariant norms are multiples of the given Euclidean norm. If G does not act transitively on the unit sphere then there are non-Euclidean G -invariant Minkowski norms.*

Proof. The first statement is clear. To prove the second statement we proceed as follows. If G acts irreducibly, consider any orbit Gp and let C be its convex hull. If G acts reducibly, we consider orthogonal G -invariant subspaces $V_1, V_2 \subset V$ with $V_1 \oplus V_2 = V$, and let C be the convex hull of the union of the unit spheres $S_1 \subset V_1$ and $S_2 \subset V_2$. In both cases we obtain a G -invariant convex body that is not strictly convex. Its symmetrization around the origin and the norm q defined by the symmetrized body is still G -invariant and not strictly convex. Due to the previous lemma, we find a G -invariant Minkowski norm q_n arbitrary close to q . Since q is non-Euclidean, q_n is non-Euclidean as well, at least for large n . \square

4.2. Holonomy group. For simplicity, we will restrict ourselves to the complete case. Thus let (M, g) be a complete smooth Riemannian manifold. Let $p \in M$ be a point and let H and H^0 denote the holonomy group Hol_p and its identity component respectively. The tangent space $V = T_p M$ splits under the action of H^0 in a uniquely defined way as $V = V_0 \oplus V_1 \oplus \dots \oplus V_i$, where the action of H on V_0 is trivial and the action on $V_i, i \geq 1$ is irreducible and not trivial. Moreover, this decomposition determines the unique direct product decomposition of the universal covering \tilde{M} of M ([Sak96]). By the uniqueness of this decomposition up to permutation, the action of H preserves this decomposition of V , possibly up to a permutation of summands. In particular, if H^0 acts reducibly then the closure \bar{H} of H acts non-transitively on the unit sphere.

If the action of H^0 on the unit sphere is irreducible, then either it acts transitively on the unit sphere or the space M is a locally symmetric space of rank at least two, with irreducible universal covering, due to the theorem of Berger-Simons ([Sim62]). In this case, each element of H acts as the differential of a local isometry, thus it preserves the type of a vector in its Weyl chamber. Hence, the action of \bar{H} on the unit sphere is non-transitive as well.

We conclude:

Lemma 4.3. *Let M be a complete Riemannian manifold. Let $p \in M$ be a point, $V = T_p M$ and let H be the holonomy group at the point p . Then the following are equivalent:*

- (1) H does not act transitively on the unit sphere V^1 of V ;

- (2) *The closure of H does not act transitively on V^1 ;*
- (3) *There are non-Euclidean Minkowski norms on V invariant under H ;*
- (4) *The universal covering of M is either a direct product or a symmetric space of higher rank.*

4.3. Description of invariant norms. In this subsection we are going to provide a description of (Minkowski) norms invariant under the action of a continuous holonomy group. Since these statements are not used in the rest of the paper we will omit some details.

Thus let M be a Riemannian manifold and assume that the holonomy group $H = Hol_p$ is connected (this happens, for instance, if M is complete and simply connected). Then (due to the theorems of Berger-Simons and de Rham) the action of H on $V = T_pM$ is polar, i.e., there is a linear subspace Σ of V that intersects all orbits of H with all intersection being orthogonal. The stabilizer $N(\Sigma)$ acts on Σ as a finite Coxeter group W . Then each H -invariant norm q on V restricts to a W -invariant norm \bar{q} on Σ . We have:

Proposition 4.4. *The restriction $q \rightarrow \bar{q}$ is a bijection between the set of H -invariant norms on V and W -invariant norms on Σ . Moreover, q is a Minkowski norm if and only if \bar{q} is a Minkowski norm.*

Proof. If q is a (Minkowski) norm then so is its restriction to any subspace, in particular \bar{q} . On the other hand, each W -invariant function \bar{q} extends to a unique H -invariant function q on V . Now, if \bar{q} is a norm then so is q , due to [HT01], p. 107. Moreover, if one can represent \bar{q} as a sum $\bar{q}_0 + \bar{q}_1$ of a scalar product and a norm, then, averaging, we may assume that \bar{q}_0 and \bar{q}_1 are W -invariant as well. Thus they extend to an H -invariant scalar product and norm q_0 and q_1 with $q = q_0 + q_1$. Finally, an H -invariant function is smooth if and only if its restriction to W is smooth ([Dad82]). Smoothing \bar{q} at the origin, we deduce, that \bar{q} is smooth in $\Sigma \setminus \{0\}$ if and only if q is smooth in $V \setminus \{0\}$. Thus q is a Minkowski norm if and only if \bar{q} is. \square

We illustrate this result by two examples:

Example 4.1. Let M is a symmetric space and let F be a maximal flat through a point p . Let W be the group of all isometries of M that fix p and leaves F invariant. Then the set of parallel (smooth) Finsler structures is in one-to-one correspondence with the set of all W -invariant (Minkowski) norms on T_pM .

Example 4.2. Let $M = M_1 \times M_2$ be the product of two irreducible not locally symmetric spaces. For $p \in M$, choose any unit vector e_1 in

$T_p M_1$ and e_2 in $T_p M_2$. Then one can choose Σ to be the plane generated by e_1 and e_2 . The group W has 4 elements and is generated by the reflections $e_i \rightarrow -e_i$.

5. THE CONCLUSION

5.1. The if part. We are going to prove Theorem 1.4. First we would like to see that each map as described in the theorem is locally affine.

A composition of continuous locally affine map is locally affine. A locally isometric embedding is locally affine. A flat submersion is locally given by a projection onto a direct factor, thus it is locally affine as well. Therefore, it suffices to prove that an admissible change of a metric is locally affine. We are going to reduce the statement to the smooth case, where the result is known (cf. [Sza81]).

Thus let (M, g) be a smooth complete Riemannian manifold and let $|\cdot|$ be a holonomy invariant Finsler structure. Fix a point $x \in M$. Approximate the norm $|\cdot|$ on $T_p M$ by Minkowski norms $|\cdot|^n$ norms on $T_p M$ invariant under the holonomy group Hol_p (Lemma 4.1). Extend each norm $|\cdot|^n$ to a smooth holonomy invariant Finsler structure on M . Due to [Sza81], the identity map $Id : (M, g) \rightarrow (M, |\cdot|^n)$ is affine. Since minimizing geodesics in the smooth Finsler manifold $(M, |\cdot|^n)$ are locally unique, all $(M, |\cdot|^n)$ geodesics are images of (M, g) geodesics (i.e., the map $Id : (M, |\cdot|^n) \rightarrow (M, g)$ is affine as well). We find some L such that the identity $Id : (M, g) \rightarrow (M, |\cdot|^n)$ is L -bilipschitz, for all n . We find some strictly convex ball V of radius r around x , such that any geodesic between points of V that is not contained in V has length at least $4Lr$. It follows, that any (M, g) -geodesic contained in V is a minimizing geodesic in $(M, |\cdot|^n)$, for all n . Since minimizing (!) $(M, |\cdot|^n)$ geodesics converge to minimizing $(M, |\cdot|)$ geodesics, we deduce that $Id : (V, g) \rightarrow (V, |\cdot|)$ sends minimizing geodesics to minimizing geodesics. The linear parametrization of the geodesics is clear by approximation (or by the fact that geodesics are auto-parallel).

5.2. Decomposition of f . Let M be a complete Riemannian manifold and let $f : M \rightarrow Y$ be a locally affine map. Due to Theorem 1.5, the function $|\cdot|^f$ defines a parallel family of semi-norms. For each $q \in M$, we set $V_q = \{v \in T_q M \mid |v|^f = 0\}$. Since the semi-norms are parallel this is a parallel distribution.

Assume first that M is simply connected. Then by the theorem of de Rham this distribution $q \rightarrow V_q$ is the vertical distribution of a projection $f_p : M = M_1 \times M_2 \rightarrow M_1$ onto a direct factor. The map f factors through f_p as $f = \hat{f} \circ f_p$. Moreover, \hat{f} coincides with the restriction of f to any horizontal slice $M_1 \times \{x_2\}$. Thus \hat{f} is again

affine, and the semi-norm $|\cdot|^{\hat{f}}$ coincides with the restriction of $|\cdot|^f$. By the definition of V_q , this semi-norm is a norm. Thus on M_1 we have an admissible change of metric $(M_1, g) \rightarrow (M_1, |\cdot|^f)$ and the induced map $f_i : (M_1, |\cdot|^f) \rightarrow Y$ is by definition of $|\cdot|^f$ a local isometry.

Assume now that M is not simply connected. Consider the lift \tilde{f} of f to the universal covering \tilde{M} . Consider a horizontal slice M'_1 in \tilde{M} . The restriction of \tilde{f} to this slice is locally bi-Lipschitz. On the other hand this restriction factors through the canonical projection of the slice M'_1 to M . This implies that the leaves of the integral distribution V_q in M are closed submanifolds that define a Riemannian submersion $f_p : M \rightarrow M_1$ for some manifold M_1 covered by M'_1 . It is clear that the Riemannian submersion f_p is flat. The rest follows from the simply connected case.

5.3. Proof of Theorem 1.1. To prove Theorem 1.1, first let M be locally irreducible and not locally symmetric of higher rank. Let $f : M \rightarrow Y$ be a locally affine map. Due to Theorem 1.5 and Lemma 4.3, at each point $p \in M$, the semi-norm $|\cdot|^f$ on T_pM is either constant 0 or a positive multiple of the metric g , the multiple not depending on the point. Thus the map f either sends the whole manifold M to a point or rescales distances in any small neighborhood of any given point by the constant number a .

If, on the other hand, M is either locally reducible or locally symmetric of higher rank, then there is an admissible change of metric to a non-Euclidean (smooth) Finsler structure. This change is certainly not a local homothety, but it is a locally affine map by Theorem 1.4.

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