RIGIDITY OF SPHERICAL BUILDINGS AND JOINS

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Abstract. We prove a rigidity and a characterization result for buildings and spherical joins using sets of antipodal points.

1. Introduction

1.1. **Results and Motivation.** The goal of this paper is the following

Main Theorem. *Let X be a finite dimensional geodesically complete $CAT(1)$ space. If X has a proper closed subset A containing with each $x \in A$ all antipodes of x, i.e. all points $z \in X$ with $d(z,x) \geq \pi$, then X is a spherical join or a building.*

This theorem implies the following result announced in the title:

Corollary 1.1. *Let X be a non-discrete spherical building or a spherical join. If $f : X \to Y$ is a surjective 1-Lipschitz map onto a finite dimensional geodesically complete $CAT(1)$ space, then Y is a spherical building or a spherical join too.*

The above theorem was mainly motivated by an attempt to better understand the following deep rigidity result of Leeb ([Lee97]):

Theorem. *Let H be a geodesically complete locally compact Hadamard space. If the ideal boundary X of H equipped with the Tits metric is a non-discrete irreducible spherical building, then H is an affine building or a symmetric space.*

The connection between our result and the theorem of Leeb is provided by the observation, that for each point x in a Hadamard space H as above, there is a natural (logarithmic) surjective 1-Lipschitz map $f : X \to S_xH$ from the ideal boundary X onto the link at x. Our Corollary 1.1 implies at once that under the conditions of the theorem of Leeb each link of H is itself a spherical join or a building. We hope that our methods will provide a generalization and another proof of the theorem of Leeb.

Another more direct motivation comes from the following theorem of Eberlein ([Ebe96],p.340), that can be used to simplify the proof of

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the higher rank rigidity established in [Bal85] and [BS87] (see [Ebe96] for a detailed exposition):

Theorem. *Let H be a Hadamard manifold, X its ideal boundary. If X contains a subset A, closed in the cone topology and involutive, then the holonomy group of H is not transitive.*

Involutive in the above theorem has essentially the same meaning as the condition in the Main Theorem. The result of Eberlein and the theorem of Berger and Simon imply that H is a product or a symmetric space and therefore X is a building or a spherical join with respect to the Tits metric ([Ebe96]).

One can get more precise statements about surjective Lipschitz maps as far as buildings or spherical joins turn up. The spherical joins are much easier to understand, namely

Corollary 1.2. *Let X be a finite dimensional geodesically complete $CAT(1)$ space. Then X has a unique decomposition $X = S^l * G_1 * ... * G_k * X_1 * .. * X_m$ where G_j are thick irreducible buildings and X_l are irreducible (i.e. indecomposable as a spherical join) but not buildings.*

If $f : X_1 * X_2 \to Y$ is a surjective 1-Lipschitz map, where X_1, X_2 and Y are $CAT(1)$ spaces and Y is geodesically complete, then Y splits as $Y = f(X_1) * f(X_2)$. Moreover f is uniquely determined by the restrictions of f to X_1 and to X_2.

On the side of buildings we first have a universal construction:

Corollary 1.3. *For each finite dimensional geodesically complete $CAT(1)$ space X there is a building $\hat{X}$ with $\dim(\hat{X}) \leq \dim X$ and a bijective 1-Lipschitz map $i_X : \hat{X} \to X$ universal in the following sense: For each surjective 1-Lipschitz map $f : X \to Y$ to another finite dimensional geodesically complete $CAT(1)$ space Y the map $\hat{f} : \hat{X} \to \hat{Y}$ given by $\hat{f} = (i_Y)^{-1} \circ f \circ i_X$ is still surjective and 1-Lipschitz.*

The building $\hat{X}$ arising from X in a functorial way is in fact not really exotic. It is uniquely determined by the following properties: $i_X : \hat{X} \to X$ is an isometry iff X is a building. $\hat{X}$ is discrete iff X is neither a spherical join nor a building of dimension at least 1. Finally the functor $X \to \hat{X}$ respects spherical joins.

To complete the picture we have to describe how complicated surjective 1-Lipschitz maps between buildings can be. There are three basic types of such maps, corresponding to (very coarse) different types of Hadamard spaces.

(1) If G is discrete, then each surjective map $f : G \to X$ is 1-Lipschitz. This describes the fact, that arbitrary $CAT(1)$ spaces can occur as links in negatively curved Hadamard spaces .

(2) There are lots of foldings, i.e. surjective 1-Lipschitz maps $f : G \to G_1$ between buildings of the same type. These essentially combinatorial objects arise for example as natural (logarithmic) maps of the boundary G of an affine building A onto the link of a point $x \in A$.

(3) Finally the most interesting maps are given as follows. Let H be an irreducible symmetric space of higher rank, denote by G its Tits building at infinity. Then the natural projection onto the link (unit sphere in the tangent space) at a point $x \in H$ gives us a bijective (!) 1-Lipschitz map of G onto a Euclidean sphere of a dimension much bigger than that of G. This example can be slightly generalized using isoparametric foliations.

We will prove in [Lyt] that each surjective 1-Lipschitz map $f : G \to X$ of a building G onto a finite dimensional geodesically complete $CAT(1)$ space X is essentially built up of the three types listed above. Here we only prove the much easier compact version, which has an interesting application to the local structure of $CAT(\kappa)$ spaces.

Corollary 1.4. *Let G be a compact spherical building, $f : G \to Y$ a surjective 1-Lipschitz map onto a geodesically complete $CAT(1)$ space Y. Then Y is a building of the same dimension.*

The application we mentioned above is the following:

Corollary 1.5. *Let X be a locally compact geodesically complete $CAT(\kappa)$ space. If the link S_xX is a building, then for each sequence $x_i \in X$ converging to x, the sequence of links S_{x_i} subconverges in the Gromov-Hausdorff topology to a building of the same dimension.*

As Example 2.7 in [Nag00] shows, it is not possible to extend this Gromov-Hausdorff stability to a topological stability, as it was done in [Nag02] in the case of spheres.

1.2. **The Idea and the Plan.** The proof of the Main Theorem starts with the observation that minimal symmetric subsets (i.e. subsets containing all antipodes of each of its points) constitute an equidistant decomposition of X. So we recover a well defined metric quotient (submetry). Moreover a surjective 1-Lipschitz map between geodesically complete $CAT(1)$ spaces defines a submetry between the corresponding quotients. So we get a functor between the category of surjective 1-Lipschitz maps and the category of submetries. The main idea is now that the category of submetries is very small and rigid, i.e. there are very few submetries. Since on the other hand each $CAT(1)$ space has a lot of surjective 1-Lipschitz maps, one gets the desired rigidity.

The paper is organized as follows. After preliminaries given in Sections 2 and 3, we describe in Section 4 a characterization of buildings and spherical joins, getting a direction in which one should work and providing a proof of Corollary 1.4. In Section 5 we define the basic functor, whose properties are studied in the subsequent sections. In Section 10 we give the main argument of the paper, providing the inductive proof of the main theorem under some technical assumptions (always satisfied in the compact case). In Section 11 we use ultrafilter techniques to complete the proof of the Main Theorem. Finally in Section 12 we prove the remaining results mentioned in the introduction up to Corollary 1.5 to be discussed in Section 13.

In the appendix we review several results of [Lyt01] about submetries of Euclidean spheres, that are used in the proof.

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2. Preliminaries

2.1. **Generalities.** By S^n we shall denote the n-dimensional Euclidean sphere with the usual round metric. We shall denote by d distances in metric spaces. The π-truncated metric d^π on a metric space X is defined by $d^\pi(x,z) = \min\{d(x,z),\pi\}$. A geodesic between x and z is a curve γ of length $d(x,z)$ connecting x and z and parametrized by the arclength. If this geodesic is unique, it will be denoted by xz. A triangle in a metric space is a union of three geodesics closing up. A space is called π-geodesic if each two points with distance less than π are connected by a geodesic. A space X is geodesically complete if it has more than one point and each geodesic of positive length in X can be prolonged to a locally isometric embedding of the whole real line $\mathbb{R}$.

2.2. **Spherical Joins.** By $X_1 * X_2$ we denote the spherical join of metric spaces X_1 and X_2 ([BH99],p.63). It consists of triples (x_1, x_2, t) with $x_i \in X_i, t \in [0, \frac{\pi}{2}]$, where distances are measured as in $S^3 = S^1 * S^1$.

The spaces X_1, X_2 will be called factors of $X_1 * X_2$. If a space can be decomposed as a spherical join, it will be called a reducible space. If the metric on X_i is replaced by the π-truncated metric, the metric on the spherical join remains unchanged. On the other hand (X_i, d^π) are isometrically embedded into $X_1 * X_2$ in a natural way.

Maps $f_i : X_i \to Y_i$ induce a spherical join map $f = f_1 * f_2 : X_1 * X_2 \to Y_1 * Y_2$ extending f_1 and f_2, that sends each geodesic between X_1 and X_2 isometrically onto its image.

2.3. **Submetries.** Two subsets X_1, X_2 of a metric space X are called equidistant, if for each point $x_i \in X_i$ there is a point $x_{i+1} \in X_{i+1}$ with $d(x_1, x_2) = d(X_1, X_2)$. Two subsets X_1 and X_2 are called weakly equidistant if for each $x_i \in X_i$ holds $d(x_i, X_{i+1}) = d(X_i, X_{i+1})$ but the distance is not necessarily realized.

For a decomposition of a space X in closed pairwise (weakly) equidistant subsets $\{X_i | i \in Y\}$ one can define a natural metric on the space Y by $d(i,j) = d(X_i, X_j)$. Then the natural projection $\delta : X \to Y$ becomes a (weak) submetry, a notion invented in [Ber87]:

Definition 2.1. A (weak) submetry $\delta : X \to Y$ between metric spaces is a map that sends for each $x \in X$ and $r \in R^+$ the closed (open) r-ball around x onto the closed (open) r-ball around $\delta(x)$.

On the other hand the fibers of each (weak) submetry $\delta : X \to Y$ constitute a (weakly) equidistant decomposition of X. In a compact space the distance between closed subsets is always realized, therefore each weak submetry $\delta : X \to Y$ of a compact space X is a submetry.

A weak submetry $\delta : X \to Y$ is a 1-Lipschitz map. The composition of two (weak) submetries is a (weak) submetry. If on the other hand the (weak) equidistant decomposition of X defined by the fibers of a (weak) submetry $f : X \to Y$ is finer than the decomposition defined by another (weak) submetry $h : X \to Z$ then the natural map $g : Y \to Z$ with $g \circ f = h$ is a (weak) submetry.

Two points $x, z \in X$ are called δ-*near* for a weak submetry $\delta : X \to Y$ if $d(x,z) = d(\delta(x), \delta(z))$ holds. In this case each geodesic between x and z is mapped isometrically onto a geodesic between $\delta(x)$ and $\delta(z)$. Such geodesics in X are called *horizontal* or more precisely δ-*horizontal*. A weak submetry is a submetry if and only if for each $x \in X$ and each y in Y there is a point z in the fiber $\delta^{-1}(y)$ that is δ-*near* to x.

The reader will find more about submetries in [Lyt01]. Several results about submetries of S^n used in the proof are collected in the appendix.

2.4. $CAT(1)$ **spaces.** We assume the reader to be acquainted with the theory of spaces with upper curvature bound. We refer to [BH99].

Definition 2.2. A metric space X is called $CAT(1)$ if it is complete, π-geodesic and each triangle with perimeter less than 2π is not thicker than the corresponding triangle in the round sphere S^2.

If X is a $CAT(1)$ space then X with the π-truncated metric is $CAT(1)$ too. Since we do not want to worry about points with distances bigger than π, we may always truncate the metric and assume that diameters of all $CAT(1)$ spaces are at most π. (Not necessary in the inner metric!). If you are not happy about this truncation, see Corollary 3.4.

2.5. **Convex subsets.** A subset C in a $CAT(1)$ space X will be called convex if it is π-convex, i.e. if for arbitrary points x_1, x_2 in C with $d(x_1, x_2) < \pi$ the geodesic x_1x_2 is contained in C. If C is closed it is itself a $CAT(1)$ space with respect to the induced metric. Let C be a closed convex subset in a $CAT(1)$ space X, $x \in X$ with $d(x, C) < \frac{\pi}{2}$. Then there is a unique point $p \in C$ with $d(x, p) = d(x, C)$. For each $z \in X$ with $d(z, p) = d(z, x) + d(x, p) \leq \frac{\pi}{2}$ holds $d(z, p) = d(z, C)$ ([BH99],p.178).

A (possibly degenerate) triangle in X is called spherical if it can be isometrically embedded into S^2. The π-convex hull of such a triangle is a convex subset of X isometric to the π-convex hull of a triangle in S^2. The following (lune) lemma is a fundamental tool in finding spherical parts of $CAT(1)$ spaces ([BB99],p.240).

Lemma 2.1. *Let γ_1 and γ_2 be geodesics of length π between x and z in a $CAT(1)$ space X. Then γ_1 and γ_2 span a spherical lune, i.e. the union of γ_1 and γ_2 can be isometrically embedded in S^2.*

2.6. **Links.** For each point x in a $CAT(1)$ space X the link $S_x = S_xX$ at x is the completion of the space of geodesic germs furnished with the angle metric ([BH99]). The link S_x is itself a $CAT(1)$ space. For each $x \in X$ there is a natural (logarithmic) 1-Lipschitz map $p_x : X \to S_x * S^0$, that sends x to south pole of S^0, all points with distance at least π to x are sent to the north pole of S^0 and each other point $z \in X$ is sent to (v, t) where v is the starting direction of the geodesic xz and $t = d(x, z)$.

Definition 2.3. We will call a direction $v \in S_x$ *genuine*, if it is the starting direction of some geodesic.

By the definition of the link, the set $\tilde{S}_x$ of *genuine* directions is dense in S_x. In a geodesically complete space $CAT(1)$ space X we will denote by $\exp(rv)$ for $v \in \tilde{S}_x, r \in [0, \pi]$ the point $\gamma(r)$ for some geodesic γ starting at x in the direction of v. By definition we have $p_x(\exp(rv)) = (v, r) \in S_x * S^0$.

If X is compact and geodesically complete, then so is each link S_x. Moreover each direction of S_x is *genuine* in this case. We will use

Definition 2.4. A geodesic $\gamma : [a, b] \to X$ branches at $\gamma(b)$ if geodesics $\gamma_1, \gamma_2 : [a, b + \epsilon] \to X$ exist, with $\gamma_1 \cap \gamma_2 = \gamma$. The angle between γ_1 and γ_2 in $S_{\gamma(b)}$ will be called the branching angle (it may be 0).

2.7. **Differentials.** Let $f : X \to Y$ be a 1-Lipschitz map between metric spaces. If for some $x \in X$ and all $z \in X$ one has $d(f(x), f(z)) = d(x, z)$, then each geodesic starting at x is mapped isometrically onto its image and we get a naturally defined 1-Lipschitz differential $D_x f : S_x X \to S_{f(x)} Y$ between the spaces of geodesic germs, if these spaces are defined as in the previous subsection. Moreover if Y is a geodesic space and each geodesic γ in Y starting at y is the unique geodesic between y and $\gamma(\epsilon)$ for a small ϵ, then the surjectivity of f implies the surjectivity of $D_x f$.

Example 2.1. Let $\delta : X \to \Delta$ be a submetry, $x \in X$. Denote by N_x the set of all points δ-*near* to x. Then the restriction $\delta : N_x \to \Delta$ has the property above.

Example 2.2. The canonical projection $p_x : X \to S_x * S^0$ of a $CAT(1)$ space X has the property above. Keep in mind that X always has a π-truncated metric (see Subsection 2.4).

2.8. **Geometric dimension.** The geometric dimension invented in [Kle99] is the smallest function dim assigning to each $CAT(1)$ space a number in $[0, \infty]$ such that $\dim(X) = 0$ iff X is discrete and such that for each $x \in X$ one has $\dim(X) \geq \dim(S_x) + 1$. By [Kle99] $\dim(X)$ is equal to the supremum of topological dimensions of compact subsets $K \subset X$. In particular $\dim(Y) \leq \dim(X)$ for a convex subset Y of X. For $CAT(1)$ spaces X and Y we have $\dim(X * Y) = \dim(X) + \dim(Y) + 1$. If X has finite dimension $n + 1$ then for some $x \in X$ the link S_x contains a sphere S^n ([Kle99]). Since small balls in $CAT(1)$ spaces are $CAT(1)$, this implies that for each $x \in X$ there is a sequence x_i converging to x such that S_{x_i} contains a sphere S^m with $m = \dim(S_{x_j}) \leq n$.

If X is compact and geodesically complete, then X has finite geometric dimension. It coincides with the topological and the Hausdorff dimension ([Ots94]).

2.9. **Buildings.** We refer to [KL97],pp.134-146 for an excellent account on buildings. The results of this paper can be considered as a continuation of the investigations of [CL01]. In order to recall them we need:

Definition 2.5. A $CAT(1)$ space X is called conical if for each $x \in X$ there is an $r > 0$, such that the r-ball around x is isometric to the r-ball around $\bar{x}$ in $\bar{X} = \{\bar{x}\} * S_x X$.

Since a spherical join $X*Y$ of $CAT(1)$ spaces is geodesically complete iff both factors X and Y are geodesically complete, the links of a conical geodesically complete space are geodesically complete.

The next theorem proved in [CL01] will be used in this paper.

Theorem 2.1. *Let X be a connected conical $CAT(1)$ space. If X is 1-dimensional assume further, that the diameter of X in the inner metric is π. If each link of X is a building then X is a building.*

2.10. **Surjective maps.** A map $f : X \to Y$ between $CAT(1)$ spaces will be called an *s.L.m.* if it is 1-Lipschitz and surjective. The world of $CAT(1)$ spaces is full of surjective 1-Lipschitz maps. For example if X is a geodesically complete $CAT(1)$ space, $x \in X$, then the natural map $p_x : X \to S_x * S^0$ is 1-Lipschitz and has a dense image $\tilde{S}_x * S^0$ (compare Definition 2.3). If in addition X is compact, then the map $p_x : X \to S_x * S^0$ is an *s.L.m.*

For a compact geodesically complete $CAT(1)$ space X we get by induction on the dimension, that X admits an *s.L.m.* $f : X \to S^m$ for some $m \leq \dim(X)$. Moreover taking the composition $(f * id) \circ p_x : X \to S_x * S^0 \to S^m * S^0 = S^{m+1}$ for an arbitrary *s.L.m.* $f : S_x \to S^m$ and keeping in mind that our metrics are π-truncated we obtain:

Lemma 2.2. *For each point x in a compact geodesically complete $CAT(1)$ space X there is a number $m = m(x)$ and an* s.L.m. $f^x = f : X \to S^m$ *with $d(f(x), f(z)) = d(x, z)$ for each $z \in X$.*

Remark 2.3. We will see in Section 13, that *s.L.m.* can be used to replace the continuity of tangent spaces in locally compact geodesically complete $CAT(\kappa)$ spaces.

3. Antipodes and symmetric subsets

3.1. **Definitions and examples.** Let X be a $CAT(1)$ space. Two points x, z in X with $d(x, z) \geq \pi$ are called antipodes. The set of all antipodes of x is denoted by $\mathrm{Ant}(x)$. For a subset $A \subset X$ we put $\mathrm{Ant}^1(A) = \mathrm{Ant}(A) = \cup_{x \in A} \mathrm{Ant}(x)$. Inductively we define $\mathrm{Ant}^j(A) = \mathrm{Ant}(\mathrm{Ant}^{j-1}(A))$.

Remark that for $A, B \subset X$ we have $\mathrm{Ant}^j(A \cup B) = \mathrm{Ant}^j(A) \cup \mathrm{Ant}^j(B)$ and $A \subset B$ implies $\mathrm{Ant}^j(A) \subset \mathrm{Ant}^j(B)$. Moreover for $x, z \in X$ the inclusions $x \in \mathrm{Ant}^j(z)$ and $z \in \mathrm{Ant}^j(x)$ are equivalent.

Definition 3.1. For $i = 1, 2$ we call a subset A of a CAT(1) space X i-symmetric if $\mathrm{Ant}^i(A) \subset A$ holds.

Example 3.1. In S^n each subset is 2-symmetric. A subset $A \subset S^n$ is 1-symmetric iff it is invariant under the multiplication with -1, for that reason the name.

Each 1-symmetric subset is also 2-symmetric. If A is 2-symmetric, then $\mathrm{Ant}(A)$ is 2-symmetric and $A \cup \mathrm{Ant}(A)$ is 1-symmetric. If each point $x \in X$ has at least one antipode, then $A \subset \mathrm{Ant}^2(A)$ for each subset A of X. In this case for an i-symmetric subset A holds $A = \mathrm{Ant}^i(A)$. Moreover a 2-symmetric subset is 1-symmetric iff it contains at least one antipode of each of its points.

Unions and intersections of i-symmetric subsets are i-symmetric. Each subset A of X is contained in a unique smallest i-symmetric subset, given by $A \cup \mathrm{Ant}^i(A) \cup \mathrm{Ant}^{2i}(A)$....

Definition 3.2. For $i = 1, 2$ we will denote for a point $x \in X$ by A^i_x the minimal i-symmetric that contains x.

Since $z \in A^i_x$ and $x \in A^i_z$ are equivalent, the spaces A^i_x define a decomposition of X into minimal i-symmetric subsets. A subset A is i-symmetric iff it is a union of some A^i_x. For each $x \in X$ the set A^2_x is either 1-symmetric or A^1_x has the disjoint decomposition $A^1_x = A^2_x \cup A^2_z$.

The following example shows that in non-geodesically complete spaces lots of symmetric subsets may exist.

Example 3.2. If X has diameter less than π then each subset is 1-symmetric, i.e. $A^1_x = \{x\}$ for each $x \in X$. If X is an interval of length less than 2π then the midpoint is a 1-symmetric subset.

However too big spaces do not have symmetric subsets:

Example 3.3. Let X be a $CAT(1)$ space. Assume that X has two points x_1, x_2, such that for each $z \in X$ and some $j = 1, 2$ holds $d(x_j, z) \geq \pi$. Then $A^1_{x_1} = \mathrm{Ant}(x_1) \cup \mathrm{Ant}(x_2) = X$. From this we see $A^1_z = X$ for each $z \in X$, i.e. X has no proper 1-symmetric subsets. Points x_1, x_2 as above always exist if X is not connected or if X has diameter $\geq 2\pi$ in the inner metric.

Now we describe symmetric subsets in buildings and spherical joins.

Example 3.4. Let $X = G$ be a thick irreducible building ([KL97],p.134), $\Delta = \Delta(G)$ the Coxeter simplex of G, W the corresponding Weyl group, $\theta : G \to \Delta$ the canonical projection that maps each chamber of G isometrically onto Δ. In this case the sets A^i_x were considered in [KL97],p.140. It was proved, that for each $x \in G$ the subset A^2_x intersects each chamber of G (it follows directly from the wellknown fact, that for each pair of chambers there is a chamber opposite to both of them). On the other hand one easily shows by induction on j, that $A^2_x = \cup \mathrm{Ant}^{2j}(x)$ is mapped by θ on a unique point in Δ.

Example 3.5. Let $X = Y * Z$ be a reducible space, $x_0 = (y_0, z_0, t_0)$ and $x_1 = (y_1, z_1, t_1)$ points in X. Then $x_1 \in \mathrm{Ant}(x_0)$ iff $y_1 \in \mathrm{Ant}(y_0), z_1 \in \mathrm{Ant}(z_0)$ and $t_1 = t_0$. If each point $y \in Y$ resp. $z \in Z$ has at least one antipode, then an inductive argument shows that $x_1 \in A^2_{x_0}$ holds if and only if $y_1 \in A^2_{y_0}$, $z_1 \in A^2_{z_0}$ and $t_1 = t_0$.

Before going on to geodesically complete spaces we state the trivial

Lemma 3.1. *Let $f : X \to Y$ be a surjective 1-Lipschitz map between $CAT(1)$ spaces. Then arbitrary preimages of antipodes are antipodes. The images of i-symmetric subsets are i-symmetric.*

3.2. **First rigidity.** Let now X be a geodesically complete $CAT(1)$ space. For x, z in such a space X with $d(x, z) \leq \pi$ we can find an antipode $\bar{x}$ of x with $\pi = d(x, \bar{x}) = d(x, z) + d(\bar{x}, z)$. The whole rigidity is based on the following easy remark:

Lemma 3.2. *Let A be a subset in a geodesically complete $CAT(1)$ space X, $x \in X$, $z \in \mathrm{Ant}(x)$. Then $d(z, \mathrm{Ant}(A)) \leq d(x, A)$. In the case of equality one gets $d(x, z) = \pi$ in the inner metric of X and for each point $p \in A$ with $d(x, A) = d(x, p)$ the equality $d(x, p) + d(p, z) = \pi$ holds, i.e. xpz is a geodesic.*

Proof. For each $p \in A$ we get $d(z, p) \geq d(x, z) - d(x, p) \geq \pi - d(x, p)$. So we find an antipode $\bar{p} \in \mathrm{Ant}(p) \subset \mathrm{Ant}(A)$ with $d(z, \bar{p}) \leq d(x, p)$. The equality statements follow from the proof. □

By induction this lemma implies $d(z, \mathrm{Ant}^i(A)) \leq d(x, A)$ for each $A \subset X$ and $z \in \mathrm{Ant}^i(x)$. We obtain

Lemma 3.3. *Let X be a geodesically complete $CAT(1)$ space, $A \subset X$ an i-symmetric subset, $x \in X$, $z \in A^i_x$. Then $d(z, A) = d(x, A)$ holds. If one distance is realized, then so is the other. In particular the closure of an i-symmetric subset is i-symmetric. Moreover if $d(x, A) = d(x, p)$ for some $p \in A$, then for each antipode $\bar{x}$ of x holds $d(x, p) + d(p, \bar{x}) = \pi$.*

An easy consequence is

Corollary 3.4. *Let $X \neq S^0$ be geodesically complete. If X has a proper 2-symmetric subset then X has diameter π in the inner metric.*

Proof. If such a space $X \neq S^0$ is non-connected, then for each $x \in X$ holds $\mathrm{Ant}^2(x) = X$. Thus we may assume that X is connected. Let x, z in X satisfy $d(x, z) = \pi + \epsilon > \pi$ with respect to the inner metric. Then A^2_x contains the ϵ-ball around x. If $A^2_x \neq X$, there must be a point $\bar{z} \in X \setminus A^2_x$ and some $\bar{x} \in A^2_x$ with $d(\bar{z}, A^2_x) \leq d(\bar{z}, \bar{x}) < \epsilon$. Thus we would get $d(\bar{x}, A^2_{\bar{z}}) < d(x, A^2_{\bar{z}})$ in contradiction to Lemma 3.3. □

If the spaces X and Y are geodesically complete, symmetric subsets are much more rigid under surjective 1-Lipschitz maps.

Lemma 3.5. *Let $f : X \to Y$ be an* s.L.m. *between geodesically complete spaces, $x \in X$ an arbitrary point, A an i-symmetric subset of X.*

Then $d(x, A) = d(f(x), f(A))$ holds. If the distance between $f(x)$ and $f(A)$ is realized then the distance between x and A is realized too. In particular $f^{-1}(f(A)) = A$ holds and the set A is closed if and only if the set $f(A)$ is closed.

Proof. Assume the contrary and choose a point $p \in A$ with $r = d(f(p), f(x)) < d(x, A)$. Choose an antipode y of $f(p)$ and let $z \in X$ be a preimage of y. Then $d(y, f(x)) \geq \pi - r$ and so $d(x, z) \geq \pi - r$. Now take an antipode $\bar{z}$ of z with $d(\bar{z}, x) \leq r$. Due to $\bar{z} \in \mathrm{Ant}^2(p) \subset A$ we arrive at a contradiction. The equality statements follow from the proof. $\square$

A direct important corollary is:

Corollary 3.6. *Let $f : X \to Y$ be an* s.L.m. *between geodesically complete spaces and let $A, B \subset X$ be i-symmetric subsets. Then A and B are (weakly) equidistant iff the images $f(A)$ and $f(B)$ are (weakly) equidistant. Moreover $d(A, B) = d(f(A), f(B))$.*

4. Two important special cases

Before we give a first characterization of reducible spaces and buildings, we will prove the following folklore lemma:

Lemma 4.1. *Let C_1, C_2 be closed convex subsets of a $CAT(1)$ space X. Assume that for arbitrary $x_i \in C_i$ holds $d(x_1, x_2) = \frac{\pi}{2}$. Then the natural map $f : C_1 * C_2 \to X$ sending the geodesic between $x_1 \in C_1$ and $x_2 \in C_2$ isometrically to the geodesic between x_1 and x_2 in X is an isometric embedding.*

Proof. Take two points $y = (x_1, x_2, t)$ and $\bar{y} = (z_1, z_2, s)$ in $C_1 * C_2$. If $d(x_1, z_1) \geq \pi$ or $d(x_2, z_2) \geq \pi$ then from the lune lemma we conclude that y and $\bar{y}$ are mapped by f isometrically. So we may assume $d(x_i, z_i) < \pi$. Let γ_i be the geodesic between x_i and z_i in C_i. Restricting f we may assume $C_i = \gamma_i$. In this case $\gamma_1 * \gamma_2$ is a convex subset of S^3 and therefore it has curvature bounded below (!) by 1. From the main result of [LS97] we see that f is 1-Lipschitz.

Since triangles in X with all vertices on γ_1 or γ_2 are spherical by Toponogov, all geodesics in $\gamma_1 * \gamma_2$ that start in γ_1 or in γ_2 are mapped isometrically onto their images. As in Subsection 2.7 we get a differential $D_{x_1} f : S_{x_1}(\gamma_1 * \gamma_2) \to S_{x_1} X$. Moreover $S_{x_1}(\gamma_1 * \gamma_2) = \{v\} * \gamma_2$, where v is the starting direction of γ_1, and for each $w \in \gamma_2 \subset S_{x_1}(\gamma_1 * \gamma_2)$ one has $\frac{\pi}{2} = d(v, w) = d(D_{x_1} f(v), D_{x_1} f(w))$. Since Lemma 4.1 is certainly true in the case $C_1 = \{v\}$ (again by Toponogov), we see that the differential $D_{x_1} f : S_{x_1}(\gamma_1 * \gamma_2) \to S_{x_1} X$ is an isometric embedding.

Now in the triangle $x_1x_2f(\bar{y}) \subset X$ the lengths of the sides are the same as in the spherical triangle $x_1x_2\bar{y}$. Due to the considerations above the angles at x_1 also coincide. Thus the triangle $x_1x_2f(\bar{y})$ is spherical by Toponogov. This implies $d(y,\bar{y}) = d(f(y), f(\bar{y}))$. □

In a spherical join $Z = X * Y$ both factors X and Y are closed convex and 1-symmetric. On the other hand the existence of such subsets characterize reducibility:

Proposition 4.2. *Let C be a non-trivial closed convex 1-symmetric subset in a geodesically complete $CAT(1)$ space X. Then X splits as $X = C * \mathrm{Pol}(C)$, where $\mathrm{Pol}(C)$ is the set of all $x \in X$ with $d(x, C) \geq \frac{\pi}{2}$.*

Proof. For each $x \in X$ and $z \in C$ with $d(x,z) \geq \frac{\pi}{2}$ we can choose an antipode $\bar{z} \in \mathrm{Ant}(z) \subset C$ with $d(\bar{z}, x) \leq \pi - d(x,z)$. This implies that C is $\frac{\pi}{2}$-dense in X and for each $x \in \mathrm{Pol}(C)$ and each $z \in C$ holds $d(x,z) = \frac{\pi}{2}$. Moreover for $x_1, x_2 \in \mathrm{Pol}(C)$ the lune lemma says that the triangle x_1x_2z is spherical for each $z \in C$. From this we conclude that $\mathrm{Pol}(C)$ is convex.

By Lemma 4.1 we get a natural isometric embedding $i : C * \mathrm{Pol}(C) \to X$. However for each $x \in X \setminus (C \cup \mathrm{Pol}(C))$ we can take the unique projection z of x to C and extend the geodesic zx up to a point in $\mathrm{Pol}(C)$ (Subsection 2.5). This shows that the set $\mathrm{Pol}(C)$ is not empty and that the embedding i is surjective. □

We are already in position to discuss rigidity of spherical joins and to prove the second part of Corollary 1.2:

Proposition 4.3. *Let K_1, K_2 be arbitrary $CAT(1)$ spaces and X a geodesically complete one. Then for each* s.L.m. *$f : K = K_1 * K_2 \to X$ the space X splits as $X = f(K_1) * f(K_2)$. Moreover f splits as $f_1 * f_2$ where f_i is the restriction $f : K_i \to f(K_i)$.*

Proof. Let $\tilde{K}_i$ be the set K_i with a discrete metric. The images $f(K_i)$ are 1-symmetric subsets, in particular each K_i has more than one point. Then the space $\tilde{K} = \tilde{K}_1 * \tilde{K}_2$ is geodesically complete. Replacing K_i by $\tilde{K}_i$ we may assume that the space K is geodesically complete.

From Corollary 3.6 we derive that $f(K_i)$ are closed 1-symmetric subsets of X with $d(f(K_1), f(K_2)) = \frac{\pi}{2}$. Since f is 1-Lipschitz, each point from $\mathrm{Pol}(f(K_1))$ must be in $f(K_2)$. We get $\mathrm{Pol}(f(K_1)) = f(K_2)$. As in the proof of Proposition 4.2 we see that $f(K_1)$ and $f(K_2)$ are convex. Now we can apply Proposition 4.2. Finally the splitting of f is trivial by construction. □

Proposition 4.2 has a nice special case. If a point x in a geodesically complete $CAT(1)$ space X has only one antipode $\bar{x}$, then for each $z \in X$

must hold $d(x,z)+d(z,\bar{x})=\pi$. We get $\mathrm{Ant}(\bar{x})=\{x\}$. Therefore $\{x,\bar{x}\}$ is a closed convex and 1-symmetric subset of X. Proposition 4.2 implies $X=\{x,\bar{x}\}*\mathrm{Pol}(\{x,\bar{x}\})$. Form this one directly arrives at

Corollary 4.4. *Let X be a geodesically complete $CAT(1)$ space. The set S of all points in X which have only one antipode is closed,* 1-*symmetric and convex and X splits as $X=S*\mathrm{Pol}(S)$.*

Remark 4.1. It is easy to see that in the factor S each triangle is spherical and so S is isometric to the unit sphere in some Hilbert space.

Proposition 4.5. *Let X be a geodesically complete finite dimensional $CAT(1)$ space. If for each $x\in X$ the set* $\mathrm{Ant}(x)$ *is discrete, then X is a building.*

Proof. If X is not connected, it must be discrete. If X contains two points x,z with $d(x,z)>\pi$ in the inner metric, then x must be an isolated point and X cannot be connected. Thus we may assume that X is connected and has diameter π in the inner metric.

Given $x\in X$ choose an antipode $\bar{x}$ of x. Since x is an isolated point in $\mathrm{Ant}(\bar{x})$, we can find an $r>0$ such that the ball of radius $3r$ around x contains no other points of $\mathrm{Ant}(\bar{x})$. For each z from the r-ball around x the geodesic $\bar{x}z$ prolonged to an antipode of $\bar{x}$ must end in x. For each z_1,z_2 in the r-ball around x the triangle xz_1z_2 is spherical, due to Lemma 2.1. Therefore X is a conical space.

We proceed by induction over the dimension of X. In dimension 1 we are done by Theorem 2.1. If a point $v\in S_x$ has a sequence of antipodes w_j converging to w, then the points $z_j=\exp(rw_j)$ converge to $z=\exp(rw)$ and z_j are all antipodes of $\exp((\pi-r)v)$, providing a contradiction. By induction all links are buildings and we are done by Theorem 2.1. □

Now we are already in position to prove Corollary 1.4:

Proof. Since G is a building the set of antipodes $\mathrm{Ant}(x)$ is discrete for each $x\in G$. But G is compact and so $\mathrm{Ant}(x)$ must be finite. Due to Lemma 3.1 the set $\mathrm{Ant}(y)$ is finite for each y in Y. By Proposition 4.5 the space Y must be a building. Since f is 1-Lipschitz, the Hausdorff dimension of Y is not bigger than $\dim(G)$. On the other hand we will see in Proposition 6.1 that each chamber of G is mapped isometrically by f. Therefore $\dim(G)=\dim(Y)$. □

5. The functor

5.1. **Spaces.** From now on let X be a geodesically complete $CAT(1)$ space. For a point $x \in X$ we denote by B^i_x the closure of A^i_x. By Lemma 3.3 it is an i-symmetric subset. Moreover

Corollary 5.1. *For $x, z \in X$ holds $d(B^i_x, z) = d(B^i_x, B^i_z) = d(x, B^i_z)$. Especially $x \in B^i_z$ and $z \in B^i_x$ are equivalent.*

We see that each set B^i_x is a minimal closed i-symmetric subset of X. Much more important is that the sets B^i_x constitute a weakly equidistant decomposition of X. So we get two corresponding weak submetries $\delta^i_X = \delta^i : X \to \Delta^i_X$. The decomposition in the subsets B^2_x is finer than the decomposition defined by the subsets B^1_x, so the weak submetry δ^1 must factorize as $\delta^{\frac{1}{2}} \circ \delta^2$, where $\delta^{\frac{1}{2}}$ is a submetry from Δ^2_X to Δ^1_X, whose fibers have at most two elements.

As in Example 3.3 we see that for a non-connected geodesically complete space $X \neq S^0$ holds $\Delta^2_X = \{\mathrm{pt}\}$. If X is connected then Δ^2_X is connected too. Since the fibers of $\delta^{\frac{1}{2}} : \Delta^2_X \to \Delta^1_X$ have at most two elements, we get

Lemma 5.2. *For $X \neq S^0$ the following statements are equivalent:*

1. *X has a nontrivial closed 1-symmetric subset;*
2. *X has a nontrivial closed 2-symmetric subset;*
3. *Δ^2_X contains more than one point;*
4. *Δ^1_X contains more than one point.*

We can now reformulate our Main Theorem as follows:

Theorem 5.1. *For an irreducible finite dimensional geodesically complete $CAT(1)$ space X holds $\Delta^2_X = \{\mathrm{pt}\}$ unless X is a building.*

We describe now the shape of some Δ^i_X. For $X = S^n$ we have $\Delta^2_{S^n} = S^n$; $\Delta^1_{S^n}$ is the projective space $\mathbb{R}P^n$.

By Example 3.4 we see that for a thick irreducible building G the canonical projection $\theta : G \to \Delta$ coincides with the map $\delta^2 : G \to \Delta^2_G$. Further one easily sees, that Δ^1_G is equal to Δ if the Weyl group W of G contains the element $-\mathrm{Id}$. If it is not the case, then Δ^1_G is the quotient of Δ under the natural isometric operation of $Z_2 = \{-\mathrm{Id}, \mathrm{Id}\}$.

A very pleasant property of Δ^2 is the compatibility with spherical joins. Namely for a reducible space $X = Y * Z$ one derives from Example 3.5 the equality $\Delta^2_X = \Delta^2_Y * \Delta^2_Z$. Especially for a thick reducible building G its Weyl polyhedron is still canonically isometric to Δ^2_G.

To simplify the notations we will call two points x, z in X *i-near* if they are *δ^i_X-near*, i.e. if $d(x, z) = d(B^i_x, B^i_z)$ holds. A geodesic will be called *i-horizontal*, if it is *δ^i_X-horizontal.*

Example 5.1. In S^n each point is 2-*near* to each other point. In a spherical join $X = Y * Z$ each point $y \in Y$ is 1-*near* to each point $z \in Z$. A point $x_0 = (y_0, z_0, t)$ is 2-*near* to $x_1 = (y_1, z_1, t_1)$ with $0 < t_0, t_1 < \frac{\pi}{2}$ iff y_0 is 2-*near* to y_1 and z_0 is 2-*near* to z_1.

Example 5.2. In a thick spherical building G two points are 2-*near* to each other iff they are contained in the same chamber.

5.2. **Maps.** Let now X and Y be geodesically complete spaces. Consider an *s.L.m.* $f : X \to Y$. By Corollary 3.6 the sets $f(B^i_x)$ constitute a weakly equidistant decomposition of Y in i-symmetric closed subsets and $d(f(B^i_x), f(B^i_z)) = d(B^i_x, B^i_z)$ holds. So we get a weak submetry $\delta^i_f : Y \to \Delta^i_X$. Since the decomposition of Y in subsets $f(B^i_x)$ is coarser than the decomposition of Y in minimal closed i-symmetric subsets, the weak submetry δ^i_f must factorize over δ^i_Y and so it defines another weak submetry $\Delta^i_f : \Delta^i_Y \to \Delta^i_X$. In this way the following commutative diagram arises, in which all maps but f are weak submetries.

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \delta^i_X}\big\downarrow & \overset{\delta^i_f}{\swarrow} & \big\downarrow{\scriptstyle \delta^i_Y} \\ \Delta^i_X & \xleftarrow{\ \Delta^i_f\ } & \Delta^i_Y \end{array}$$

The map δ^i_f sends a point $y \in Y$ to the set B^i_x, where x is an arbitrary preimage of y. If $g : Y \to Z$ is another *s.L.m.* then $\Delta^i_f \circ \Delta^i_g = \Delta^i_{g \circ f}$ holds. So far we have defined a functor from the category of *s.L.m.* to the category of weak submetries.

Assume now that X admits an *s.L.m.* $f : X \to S^n$. For two points $x, z \in X$ the images $f(B^i_x)$ and $f(B^i_z)$ are closed weakly equidistant subsets in S^n. Since S^n is compact, these subsets must be equidistant. By Corollary 3.6 the subsets B^i_x and B^i_z are equidistant too. In this case the weak submetries $\delta^i : X \to \Delta^i_X$ are submetries. Moreover a new application of Corollary 3.6 gives us, that for such an X and an *s.L.m.* $g : X \to Y$ to another geodesically complete space Y the maps $\delta^i_g : Y \to \Delta^i_X$ are submetries.

Moreover we get (see appendix or [Lyt01],p.47) the following result:

Proposition 5.3. *If X admits an* s.L.m. $f : X \to S^n$ *then Δ^i_X is the base of the submetry $\delta : S^n \to \Delta^i_X$. Therefore Δ^i_X is a compact finite dimensional Alexandrov space with curvature bounded below by* 1. *For each point $q \in \Delta^i_X$ there is an $\epsilon > 0$, such that in each direction of $S_q(\Delta^i_X)$ starts a geodesic of length at least ϵ.*

A direct consequence is:

Corollary 5.4. *For each point x in a space X as above there is an $\epsilon = \epsilon(x) > 0$, such that for each $\bar{x} \in B_x^1$ and each point z with $d(\bar{x}, z) < \epsilon$ that is* 1-near *to $\bar{x}$ the following holds: z lies on a* 1-horizontal *geodesic of length ϵ starting at $\bar{x}$. If z is* 2-near *point to $\bar{x}$ then a starting part of the geodesic $\bar{x}z$ is* 1-horizontal.

6. Almost near points

The last statement of Lemma 3.3 motivates the following

Definition 6.1. Let X be a $CAT(1)$ space, $x_0, x_1 \in X$. We will say that x_1 is *almost near* to x_0 if for each antipode x of x_0 holds $d(x_0, x_1) + d(x_1, x) = \pi$.

Example 6.1. From Lemma 3.3 we conclude that *i-near* points in geodesically complete spaces are *almost near*. The notion of *almost near* points can be considered as an unstable approximation of the notion of 2-*near* points. As the following example shows, to be *almost near* is a much weaker condition.

Example 6.2. Let X be a hemispherix ([BB99],p.236), i.e. the Euclidean sphere S^2 with finitely many hemispheres attached along great circles H_i, such that the intersection of all H_i is empty. The main result of this paper says that different points in X are never 2-*near*. In contrast a lot of *almost near* points exist. Namely the circles H_i divide S in convex regions C_j and one easily shows that each two points in the same region C_j are *almost near*.

Proposition 6.1. *Let X be geodesically complete. If a point x_1 is* almost near *to x_0 then for each $z \in X$ the triangle x_0x_1z is spherical. Therefore x_0 is* almost near *to x_1. Moreover for each* s.L.m. *$f : X \to Y$ onto another geodesically complete space Y, the points x_0, x_1 are mapped isometrically and $f(x_0)$ and $f(x_1)$ are* almost near *in Y.*

Proof. If $d(z, x_0) \geq \pi$ then we are done by Definition 6.1. If $d(z, x_0) < \pi$, we can choose an antipode x of x_0 with $d(x_0, z) + d(z, x) = \pi$. Since x_0 and x_1 are *almost near*, x_0x_1x is a geodesic of length π and from Lemma 2.1 we deduce that x_0x_1z is a spherical triangle.

For an antipode y of $f(x_0)$ take a preimage x of y. Then x is an antipode of x_0 and we get $\pi \leq d(f(x_0), y) \leq d(f(x_0), f(x_1)) + d(f(x_1), y) \leq d(x_0, x_1) + d(x_1, x) = \pi$. This implies the rest. □

From Proposition 6.1 we conclude, that if x and $\exp(rv)$ are *almost near* for some $r \in (0, \pi)$ and a *genuine* direction $v \in S_x$, then $\exp(rv)$ is uniquely defined. Especially the geodesic xz between two *almost near* points x and z cannot branch. Moreover each point m on xz is *almost near* to x and to z. On the other hand we have:

Lemma 6.2. *Let xmz be a geodesic. If x and m are* almost near *and if the geodesic does not branch, then x and z are* almost near *too.*

Proof. Let $\bar{x}$ be an antipode of x. Then $xm\bar{x}$ is a geodesic of length π. Since it cannot branch up to the point z, it must contain z. Hence $d(x,z) + d(z,\bar{x}) = \pi$ holds. □

Almost near points give us many long geodesics in the links:

Lemma 6.3. *Let x, z be* almost near *points in X, let $v \in S_x$ be the starting direction of xz. Let w be a* genuine *direction in $\tilde{S}_x$. If $d(v,w) < \pi$ then the geodesic η between v and w can be continued until an antipode $\bar{v}$ of v, such that for all $t \in [0,\pi]$ the direction $\eta(t)$ is* genuine.

Proof. Take $z_0 = \exp(\epsilon w)$ and an antipode $\bar{z}$ of z with $d(z,z_0) + d(z_0,\bar{z}) = \pi$. Since z and x are *almost near*, $zx\bar{z}$ is a geodesic that spans together with $zz_0\bar{z}$ a spherical lune. The link of this spherical lune at x is the geodesic η. □

Corollary 6.4. *Let $x, z \in X, v \in S_x$ be as above. For each $w \in \mathrm{Ant}(v)$ there is a sequence of* genuine *directions $w_i \in \mathrm{Ant}(v)$ converging to w.*

7. Nice spaces

The following definition is motivated by Lemma 2.2.

Definition 7.1. We call a geodesically complete $CAT(1)$ space X nice, if for each $x \in X$ there is some $n = n(x)$ and an *s.L.m.* $f = f^x : X \to S^n$ with $d(f(x), f(z)) = d(x,z)$ for each $z \in X$.

Recall that we stick to the convention $\mathrm{diam}(X) = \pi$. If one does not like this convention, the above definition should be modified to $d(f(x), f(z)) = \min\{\pi, d(x,z)\}$. Using Proposition 4.3 we see

Lemma 7.1. *A reducible space $X = X_1 * X_2$ is nice iff the factors X_1 and X_2 are nice.*

Directly from Definition 7.1 and Proposition 6.1 we get

Lemma 7.2. *Let X be a nice space, $x, x_0, x_1 \in X$. If x_0 and x_1 are almost near, then there is an* s.L.m. *$f : X \to S^k$ that maps the spherical triangle xx_0x_1 isometrically.*

For the induction step in the proof of Theorem 5.1 we will need the following statement, whose assumptions are related to Lemma 6.3:

Proposition 7.3. *Let X be a nice space. Assume that a link S_x has a decomposition $S_x = H_x * V_x$ such that all directions in H_x are* genuine *and such that for each* genuine *direction $v \in V_x$ and each $h \in H_x$ the geodesic between h and v in S_x consists of* genuine *directions. Then one* *can find an* s.L.m. $f : X \to H_x * S^l$ *satisfying* $d(f(x), f(z)) = d(x, z)$ *for each* $z \in X$.

Proof. Take an *s.L.m.* $f^x : X \to S^n$ from Definition 7.1. As in Subsection 2.7 we get a natural differential $D_x f : S_x \to S_{f(x)} = S^{n-1}$ which is a 1-Lipschitz map with $D_x f(\tilde{S}_x) = S^{n-1}$. By Lemma 4.3 the map $D_x f$ splits as $D_x f = f^h * f^v$, where f^h resp. f^v is the restriction of $D_x f$ to H_x resp. to V_x. Moreover the image $f^v(V_x)$ is a factor S^l of S^{n-1}. Therefore the map $g = id * f^v : H_x * V_x \to H_x * S^l$ is an *s.L.m.* of S_x onto $H_x * S^l$. Moreover the restriction of g to the set of *genuine* directions is still surjective. The composition $(g * id) \circ p_x : X \to S_x * S^0 \to H_x * S^l * S^0$ is an *s.L.m.* with the desired properties. □

8. Horizontal spaces

We are going to investigate the structure of the submetries δ_f^i in this section. All proofs are provided by pushing the problems forward to S^n, by special *s.L.m.* and then using results about submetries of Euclidean spheres stated in the appendix.

We will fix nice spaces X, Y (see Definition 7.1) and an *s.L.m.* $f : X \to Y$. To restrict the number of indices we will only consider the case $i = 1$ and write δ_f for δ_f^1. For a point $y \in Y$ we will denote by $N_y^f \subset Y$ the set of all points δ_f-*near* to y. A direction $v \in S_y$ will be called δ_f-*horizontal* if a short δ_f-*horizontal* geodesic starts in this direction. The set of all δ_f-*horizontal* directions will be denoted by $H_y^f \subset S_y Y$. In the case $f = id : X \to X$ we recover the notion of 1-*near* points. We will write $N_x = N_x^{id}$ and $H_x = H_x^{id}$.

Remark 8.1. By definition each δ_f-*horizontal* direction is *genuine*.

If $g : Y \to Z$ is another *s.L.m.*, then δ_f-*near* points are sent by g to $\delta_{g \circ f}$-*near* points, i.e. $g(N_y^f) \subset N_{g(y)}^{g \circ f}$. In particular 1-*near* points in X are sent by $f : X \to Y$ to δ_f-*near* points. For each $x \in X$ we get a natural 1-Lipschitz map $D_x f : H_x \to H_{f(x)}^f$, called the *horizontal* differential of f at x. We will prove in this section that H_y^f is a factor of S_y and the 1-Lipschitz map $D_x f : H_x \to H_{f(x)}^f$ is always surjective.

Since δ_f-*near* points in Y are 1-*near*, they are *almost near* by Example 6.1. By Proposition 5.3 for each $v \in H_y^f$ the geodesic of length $\epsilon = \epsilon(y)$ in the direction of v is δ_f-*horizontal*. The set N_y^f is closed. Therefore H_y^f is closed too.

Lemma 8.1. *The set* N_y^f *is convex.*

Proof. For $y_1, y_2 \in N_y^f$ let t be a point on the geodesic y_1y_2. By Lemma 7.2 we can find an *s.L.m.* $g : Y \to S^n$, that maps the triangle yy_1y_2 isometrically. The points $g(y_1)$ and $g(y_2)$ are $\delta_{g\circ f}$ *-near* to $g(y)$. Since $N_{g(y)}^{g\circ f}$ is convex (see Lemma 14.1), the point $g(t)$ is also $\delta_{g\circ f}$*-near* to $g(y)$. From $d(t,y) = d(g(y), g(t))$ we see that y is δ_f*-near* to t. $\square$

This implies the convexity of H_y^f.

Lemma 8.2. *For $x \in X$, $z \in \mathrm{Ant}(x)$, let γ be a geodesic from x to z, with in- resp. outcoming direction $v \in S_x$ resp. $w \in S_z$. If $v \in H_x$, then for each* genuine *direction $\bar{w} \in \mathrm{Ant}(w)$ holds $\bar{w} \in H_z$.*

Proof. The point $x_1 = \exp(\epsilon v)$ is *near* to x and antipodal to $z_1 = \exp(\epsilon\bar{w})$. We see $d(B^1_{z_1}, B^1_z) = d(B^1_{x_1}, B^1_x) = d(x_1, x) = d(z_1, z)$. $\square$

Lemma 8.3. *If H_x is 1-symmetric, $z \in \mathrm{Ant}(x)$, then H_z is 1-symmetric and there is a natural isometry $I_{zx} : H_z \to H_x$.*

Proof. For each $v \in H_z$ there is a unique geodesic γ between z and x, starting in the direction v. Let $w \in S_x$ be the incoming direction of γ. Choose a *genuine* direction $\bar{w} \in \mathrm{Ant}(w)$. By Lemma 8.2 we know $\bar{w} \in H_x$. Since H_x is 1-symmetric, the direction w must be *horizontal* too. Applying Lemma 8.2 again we see that all *genuine* antipodes of v in S_z are *horizontal.* Since H_z is closed we get by Corollary 6.4 that H_z is 1-symmetric.

Sending the *horizontal* direction v to the *horizontal* direction w, gives us by the lune lemma the desired isometry $I_{zx} : H_z \to H_x$. $\square$

By induction on j we get:

Corollary 8.4. *If for $x \in X$ the space H_x is 1-symmetric, then for each $z \in A_x^1 = \cup \mathrm{Ant}^j(x)$ the subset $H_z \subset S_z$ is 1-symmetric. Moreover H_x and H_z are isometric.*

Lemma 8.5. *Let $x \in X$, $y = f(x)$. Then H_y^f is 1-symmetric in S_y iff H_x is 1-symmetric in S_x. In this case the restriction $f : N_x \to N_y^f$ and the* horizontal *differential $D_x f : H_x \to H_y^f$ are surjective.*

Proof. Let H_x be 1-symmetric, $v \in H_y^f$, and let $w \in \mathrm{Ant}(v)$ be a *genuine* direction. Set $y_1 = \exp(\epsilon v)$, $\bar{y}_1 = \exp((\pi - \epsilon)w)$. Choose a preimage $\bar{z}_1 \in X$ of $\bar{y}_1$. We have $d(\delta(x), \delta(\bar{z}_1)) = d(\delta_f(y), \delta_f(\bar{y}_1)) = d(y, y_1)$, since y_1 is δ_f-near to y. Thus $d(x, \bar{z}_1) \leq \pi - d(x, y_1)$. But f is 1-Lipschitz, hence $d(\bar{z}_1, x) = d(\bar{y}_1, y)$ and the geodesic $x\bar{z}_1$ is mapped by f isometrically onto $y\bar{y}_1$. Choose $z_1 \in \mathrm{Ant}(\bar{z}_1)$ with $d(z_1, x) = \epsilon$. Then $\delta(z_1) = \delta(\bar{z}_1) = \delta_f(\bar{y}_1) = \delta_f(y_1)$. Hence z_1 is 1-*near* to x. Since H_x is 1-symmetric a starting part of the geodesic $x\bar{z}_1$ is *horizontal.*

Then the image of this part is δ_f-*horizontal.* Therefore $w \in H_y^f$ and $w \in D_x f(H_x)$. Hence (applying Corollary 6.4) H_y^f is 1-symmetric and $D_x f : H_x \to H_y^f$ is surjective.

Let on the other hand H_y^f be 1-symmetric, $v \in H_x$. Let $w = D_x f(v)$. Consider $y_1 = exp(\epsilon w)$ and an antipode $\bar{y}_1$ of y_1. A starting part of the geodesic $y\bar{y}_1$ is δ_f-*horizontal.* So for each preimage $\bar{z}$ of $\bar{y}_1$ a starting part of $x\bar{z}$ is *horizontal.* Since the starting direction of $y\bar{y}_1$ is antipodal to w, the starting direction of $x\bar{z}$ is antipodal to v. Thus each $v \in H_x$ has an antipode $\bar{v}$ in H_x.

Let $v_1 \in \mathrm{Ant}(v)$ be another *genuine* direction. Since $\exp(\epsilon v_1)$ and $\exp(\epsilon \bar{v})$ are both contained in $\mathrm{Ant}(\exp((\pi - \epsilon)v))$ we get $\delta(\exp(\epsilon v_1)) = \delta(\exp(\epsilon \bar{v}))$. Therefore $\exp(\epsilon v_1)$ is 1-*near* to x. Thus $v_1 \in H_x$ and using Corollary 6.4 we see that the set H_x is 1-symmetric. □

Consider now an *s.L.m.* $g : X \to S^n$. The set $g(A_x^1)$ is a dense subset in the fiber $g(B_x^1)$ of the submetry $\delta_g : S^n \to \Delta_X$. From the appendix or [Lyt01],p.48 we derive that for some point $a = g(z) \in g(A_x^1)$ the δ_g-*horizontal* space $H_a^g \subset S_a S^n$ is a sphere. From Lemma 8.5 we see that H_z is 1-symmetric. Taking together the assertions of Lemma 8.5 and Corollary 8.4 we get:

Proposition 8.6. *For each $x \in X$ the* horizontal *space H_x is a closed convex 1-symmetric subset of S_x. For each $z \in A_x^1$ the spaces H_x and H_z are isometric. If $f : X \to Y$ is an* s.L.m.*, then the δ_f-*horizontal *space $H_{f(x)}^f$ is closed convex and 1-symmetric too. The* horizontal *differential $D_x f : H_x \to H_y^f$ is an* s.L.m..

Finally we show

Lemma 8.7. *Each* horizontal *space H_x is a (maybe trivial) factor of S_x. Moreover H_x is a nice space. The same is valid for $H_{f(x)}^f$*

Proof. From Lemma 6.3 and the fact that H_x is convex and 1-symmetric we derive that H_x is geodesically complete. Moreover using Lemma 6.3 and the fact that the set of *genuine* directions is dense in S_x we can repeat the proof of Proposition 4.2 and get the splitting $S_x = H_x * \mathrm{Pol}(H_x)$.

To prove that H_x is nice take an arbitrary $v \in H_x$ and consider the point $z = \exp(\epsilon v)$. Consider the *s.L.m.* $f = f^z : X \to S^n$ as in Definition 7.1. The *horizontal* differential $D_x f^z : H_x \to H_{f(x)}^f$ is an *s.L.m.* of H_x onto a sphere. Moreover since for each $w \in H_x$ the points x and $x_w = \exp(\epsilon w)$ are *almost near*, the triangle zxx_w is mapped isometrically. This proves $d(D_x f^z(v), D_x f^z(w)) = d(v, w)$.

Since H_y^f is a closed convex 1-symmetric subset of the geodesically complete space H_y we can apply Proposition 4.2 and Lemma 7.1. □

9. Geodesics in horizontal directions

Let X be a nice space. Using the fact that we know, what happens in *horizontal* direction of a submetry of a Euclidean sphere, we are going to study what happens in *horizontal* directions of X.

Lemma 9.1. *For $x \in X$, $v \in H_x$, let γ be a geodesic of length π starting in the direction of v. Then for some numbers $0 = t_1 < t_2 < ... < t_k = \pi$, the points $\gamma(t_i)$ and $\gamma(t_{i+1})$ are* near.

Proof. Take an *s.L.m.* $f : X \to S^n$ that maps γ isometrically onto its image. The starting direction of $f(\gamma)$ at $y = f(x)$ is the δ_f-*horizontal* direction $w = D_x f(v)$. By Lemma 8.7 all horizontal spaces H_y^f of the submetry $\delta_f : S^n \to \Delta_X$ are spheres. Due to Lemma 14.3 the image $f(\gamma)$ consists of finitely many δ_f-*horizontal* geodesics. Then the same is true for γ. □

Now we make the statement of the last lemma uniform. To do so remark that for $v \in S_x$, $w \in B_v^2$, $0 \leq t \leq \pi$ and arbitrary geodesics η resp. γ starting at x in the direction v resp. w one has $\eta(t) \in B^2_{\gamma(t)}$. (This holds in all geodesically complete $CAT(1)$ spaces).

Lemma 9.2. *Let $x \in X$, $v \in H_x$ be given. Then for some $0 = t_1 < t_2 < ... < t_k = \pi$, each direction $w \in B_v^2$ and each geodesic γ in the direction w the points $\gamma(t_i)$ and $\gamma(t_{i+1})$ are* 1-near.

Proof. For an arbitrary geodesic η starting in the direction v and each $t < \pi$ holds $\eta(t) \in B^2_{\gamma(t)}$. If $\eta(t)$ and $\eta(s)$ are 1-*near*, then the points $\gamma(t)$ and $\gamma(s)$ are 1-*near* too. □

We are going to describe branchings in *horizontal* directions:

Lemma 9.3. *Let γ be a geodesic starting in a* horizontal *direction $v \in H_x$. If γ branches in a point $y = \gamma(t)$, then the branching has a positive angle, i.e. the incoming direction $w \in H_y$ of γ has in H_y more than one antipode.*

Proof. Each antipode $\bar{w}$ of w is a *horizontal* direction, therefore the point $\exp(\epsilon \bar{w})$ is uniquely defined. If $\bar{w}$ was the only antipode of w, γ could not branch in y. □

For $v \in H_x$ we denote by $b(v)$ the maximal number $t \leq \pi$, such that x and $\exp(b(v)v)$ are *almost near*. By Lemma 6.2 this is the first branching time of the geodesic starting in the direction v.

Lemma 9.4. *The function $b : H_x \to R^+$ defined above is constant on B_v^2 for each $v \in H_x$.*

Proof. For $w \in Ant(v)$, $\bar{v} \in Ant(w)$, set $z_1 = \exp(tv)$, $z_2 = \exp(t\bar{v})$ and choose $\bar{z} = \exp((\pi - t)w)$. The isometry $I_{z_1\bar{z}} \circ I_{\bar{z}z_2} : H_{z_1} \to H_{z_2}$ defined in Lemma 8.3 sends the incoming direction of xz_1 to the incoming direction of xz_2. By Lemma 9.3 the geodesic xz_1 branches in z_1 iff xz_2 branches in z_2. This implies by induction over j that the function b is constant on $A_v^2 = \cup \mathrm{Ant}^{2j}(v)$.

If directions $w_n \in H_x$ converge to w and $b(w_n)$ converge to t, then $\exp(b(w_n)w_n)$ converge to $\exp(tw)$ (by Proposition 6.1). Since the set of points *almost near* to x is closed, we derive $t \leq b(w)$.

Therefore $b(w) \geq b(v)$ for each $w \in B_v^2$. Since for $w \in B_v^2$ we also have the reversed inclusion $v \in B_w^2$, we get the equality. □

10. The Proof for nice spaces

We assume that Theorem 5.1 is true for all spaces of dimension at most $n-1$ and prove it for nice spaces of dimension n. Before embarking on the proof we make some easy remarks. The next lemma is motivated by Remark 10.1 below and will be the final point in the proof.

Lemma 10.1. *Let $C \subset S^m$ be a compact convex subset, $q \in C$. For $w \in S_qC$ we denote by $l(w)$ the length of the maximal geodesic $q\bar{q}$ in C that starts in direction w. Then l is a continuous function on S_qC.*

Let $C \subset S^m$ be a hemisphere $\{\mathrm{pt}\} * S^{m-1}$, $q \in C$ an arbitrary point, $H \subset S_qC$ a totally geodesic sphere. Then the function l cannot be constant on H and smaller than $\frac{\pi}{2}$.

Remark 10.1. If Z is a spherical join $Z = Z_1 * ... * Z_k$ of irreducible factors for which Theorem 5.1 is true, then Δ_Z^2 is a π-convex subset of an m-dimensional sphere for some natural number m. For $x \in Z, v \in H_x$ let $q = \delta_Z^2(x)$ and $w = D_x\delta_Z^2(v) \in S_q\Delta^2$. Then $l(w)$ is the maximum of all t, such that the points $\exp(tv)$ and x are 2-*near*. From the knowledge of Theorem 5.1 it is not difficult to derive, that this quantity $l(w)$ is equal to $b(v)$, for the function b defined in Lemma 9.4.

The next lemma follows directly from Example 5.1

Lemma 10.2. *Let Y be a space with $\Delta_Y^2 = \{\mathrm{pt}\}$, $Z = Y * S^k$, $z \in Z$. For a point $z \in (Y \cup S^k)$ holds that each geodesic starting in a direction $v \in H_z$ does not branch before $\frac{\pi}{2}$. If z does not lie in $Y \cup S^k$, then the* horizontal *space H_z is a k-dimensional sphere and this sphere is mapped by the* horizontal *differential of $\delta_Z^2 : Z \to \Delta_Z^2 = \{\mathrm{pt}\} * S^k$ isometrically onto its image.*

Let now $z \in Z = Z_1 * Z_2 * ... * Z_k$ be a point with more than one antipode in Z and assume that the spaces Z_k are nice and Theorem 5.1 is true for them. For some $i = 1, ..., k$ the inequality $d(z, Z_i) < \frac{\pi}{2}$ must hold and the projection of z on this Z_i must have more than one antipode. For each $j \neq i$ we map the space Z_j by an *s.L.m.* to a Euclidean sphere S^{n_j}, and so we get an *s.L.m.* $f : Z \to S^m * Z_i$, such that the point $f(z)$ has more than one antipode.

Let finally G be a thick irreducible non-discrete building, $z \in G$. We can find a wall L of codimension 1 and a point $x \in L$, such that the geodesic zx branches in x. Then the image of z under $p_x : G \to S^0 * S_x$ has more than one antipode and $\Delta^2_{S_x} = \{\mathrm{pt}\} * S^l$. We just have proved:

Lemma 10.3. *Assume that for irreducible factors of a nice space Z the conclusion of Theorem 5.1 holds. Let $z \in Z$ a point with more than one antipode. Then there is an* s.L.m. $f : Z \to S^m * Y$ *with* $\Delta^2_Y = \{\mathrm{pt}\}$, *such that $f(z)$ has in $S^m * Y$ more than one antipode.*

Now we come to the core of the proof. We assume that Theorem 5.1 is true in dimensions smaller than n and consider a nice irreducible n-dimensional space X with a proper closed symmetric subset. Since Δ^2_X is not trivial, each *horizontal* subspace H_x is not empty.

First we assume that each *horizontal* subspace $H_z \subset S_z$ is a building. Let $x \in X$ be an arbitrary point, $z \neq x$ a point 2-*near* to x. Assume that the set of antipodes of x is not discrete and let $\bar{x}_j \to \bar{x}$ be a non-constant convergent sequence in $\mathrm{Ant}(x)$. For each j we have $d(z, \bar{x}_j) = \pi - d(z, x)$ and geodesics $z\bar{x}_j$ converge to $z\bar{x}$. Since the geodesic $z\bar{x}$ branches only in finitely many points, we can replace $\bar{x}_j$ by a subsequence and assume that the intersection of each geodesic $z\bar{x}_j$ with $z\bar{x}$ is the same geodesic zz_0. Then the starting directions $v_j \in S_{z_0}$ of $z_0\bar{x}_j$ are different from the starting direction v of $z_0\bar{x}$ (Lemma 9.3). Since the *horizontal* directions v_j and v are antipodes of the incoming direction of zz_0 and v_j converge to v, we get a contradiction to the assumption that H_{z_0} is a building. Therefore $\mathrm{Ant}(x)$ is discrete and by Proposition 4.5 the space X is a building.

Assume now that for some $x \in X$ the space H_x is not a building. By induction H_x has a non-discrete factor F with $\Delta^2_F = \{\mathrm{pt}\}$, i.e. for each $v \in F$ holds $B^2_v = F$. By Lemma 9.4 there is a number $t = b(v)$ such that the first branching of each geodesic starting in a direction $w \in F$ occurs exactly at the time t.

Assume $t \geq \frac{\pi}{2}$. Then for each $w \in F$ the point $z_w = \exp(\frac{\pi}{2}w)$ is well defined and *almost near* to x by Lemma 6.2. Denote by Z the set of all points of the form z_w, $w \in F$. For $w_1, w_2 \in F$ the triangle $xz_{w_1}z_{w_2}$ is spherical, so the set Z is convex and closed. For $z \in Z$ and $\bar{z} \in \mathrm{Ant}(z)$

holds $d(\bar{z}, x) = \pi - d(z, x) = \frac{\pi}{2}$ and since F is a 1-symmetric subset of H_x, we see that $\bar{z}$ lies in Z. Therefore Z is a proper closed convex 1-symmetric subset of X. By Proposition 4.2 we see that Z is a factor of X and X is reducible. Contradiction.

The rest is devoted to the proof that the case $t < \frac{\pi}{2}$ cannot occur.

Since F is not discrete we can choose a non-isolated point $v \in F$ and a sequence of distinct points $v_n \in F$ converging to v. Set $z = \exp(tv)$, $z_j = \exp(tv_j)$ and let $w \in H_z$ be the incoming direction of xz. By Lemma 9.3 the direction w has more than one antipode in $H_z \subset S_z$. Since H_z is nice by Lemma 8.7 and has dimension less than n, we can use Lemma 10.3 and find an *s.L.m.* $g : H_z \to S^m * Y$ with $\Delta^2_Y = \{\mathrm{pt}\}$, such that $g(w)$ has in $S^m * Y$ several antipodes.

Consider now the map $f^z : X \to H_z * S^l$ constructed in Proposition 7.3 and the composition $f = (g * id) \circ f^z : X \to H_z * S^l \to Y * S^{m+l+1}$.

The images $\bar{y}_j = f(z_j)$ and $\bar{y} = f(z)$ are different by construction. Moreover for $y = f(x)$ the geodesic $y\bar{y}$ branches in $\bar{y}$. By Proposition 8.6 we know that $D_x f : H_x \to H^f_y$ is an *s.L.m.* and by Proposition 4.3 the factor F of H_x is mapped by $D_x f$ onto a factor $\bar{F}$ of H^f_y. Since H^f_y is a factor of H_y, the image $\bar{F}$ is also a factor of H_y. Moreover since $\bar{y}_j \neq \bar{y}$ (and since z_j is *almost near* to x), the sequence $D_x f(v_j)$ converging to $D_x f(v)$ is not constant. Thus $\bar{F}$ is not discrete!

So far we have constructed an *s.L.m.* $f : X \to Z = Y * S^a$ with the following properties. The space $\Delta^2_Z = \Delta^2_{S^a} * \Delta^2_Y$ is a hemisphere $S^a * \{\mathrm{pt}\}$. At the point $y = f(x)$ there is a non-discrete factor $\bar{F} = D_x f(F)$ of H_y, such that for a direction $\bar{v} = D_x f(v) \in \bar{F}$ the geodesic γ starting in the direction of $\bar{v}$ branches at the time $t < \frac{\pi}{2}$. Moreover there are real numbers $0 = t_1 < t_2 < ... < t_k = \pi$, such that for each $w \in \bar{F}$ and each geodesic η in the direction w, the points $\eta(t_i)$ and $\eta(t_{i+1})$ are 2-*near* (Lemma 9.2).

Set $q = \delta^2_Z(y)$. Since the geodesic in the direction of $\bar{v}$ branches before $\frac{\pi}{2}$, the point y is not contained in $Y \cup S^a \subset Y * S^a$. Due to Lemma 10.2 the space $\bar{F}$ is mapped by the differential of δ^2_Z isometrically onto a non-discrete totally geodesic sphere T in $S_q(\Delta^2_Z)$. The function l defined in Lemma 10.1 (the distance to the boundary of Δ^2_Z) can assume on T only the values $t_1, ..., t_k$. Since T is connected and l continuous, it must be constant t_j on T. Since the geodesic in the direction $\bar{v}$ branches at the time $t < \frac{\pi}{2}$, the value l_0 must be smaller than $\frac{\pi}{2}$. Contradiction to Lemma 10.1.

This finishes the proof of Theorem 5.1 in the case of nice spaces.

11. Ultratrick

The reader will find more detailed information about ultralimits for instance in [BH99],p.78 and [KL97],pp.131-133.

11.1. **Ultralimits.** Let ω be a fixed non-principal ultrafilter on the set of natural numbers. It allows us to choose for a sequence (x_n) in a compact space X a point $\lim_\omega x_n \in X$ among the limit points of the sequence. Moreover it allows to build ultralimits of metric spaces and Lipschitz maps. Let namely (X_j, x_j) be a sequence of pointed metric spaces. Then $(X, x) = \lim_\omega(X_j, x_j)$ consists of all sequences (z_j) with uniformely bounded $d(z_j, x_j)$, where z_j is a point in the space X_j. The metric on this set X is defined by the pseudo-metric $d((z_j), (\bar{z}_j)) = \lim_\omega(d(z_j, \bar{z}_j))$.

Remark 11.1. If all the spaces X_j have uniformly bounded diameters, then the choice of the base point $x_j \in X_j$ does not play a role in the construction.

An ultralimit (X, x) of $CAT(\kappa)$ spaces (X_j, x_j) is a $CAT(\kappa)$ space, that is geodesically complete if so are all the spaces X_j.

A sequence $f_j : (X_j, x_j) \to (Y_j, y_j)$ of L-Lipschitz maps induces an L-Lipschitz map $f = \lim_\omega f_j : \lim_\omega X_j \to \lim_\omega Y_j$ defined by $f((x_j)) = (f_j(x_j))$. If the diameters of the spaces are uniformly bounded and the maps $f_j : X_j \to Y_j$ have dense images, then the ultralimit $f = \lim_\omega f_j$ is surjective.

11.2. **Ultraproducts.** For a space X we denote by X^ω its ultraproduct $X^\omega = \lim_\omega X_i$ with the constant sequence $X_i = (X, x)$, where x is an arbitrary point. We remark, that the ultraproduct X^ω does not depend on the choice of the point $x \in X$. The space X has a natural isometric embedding in X^ω given by $z \to (z, z, ...) \in X^\omega$. This embedding is onto iff X is proper. For a subset $A \subset X$ we get a closed subset $A^\omega \subset X^\omega$ whose intersection with X is the closure of A in X.

11.3. **Remarks on geometric dimension.** An important result of [Kle99] is that the geometric dimension is the greatest number n, such that for some points $x_0, x_1, ..., x_n$ in X the barycentric simplex $S(x_0, x_1, .., x_n)$ defined in [Kle99] is non-degenerate.

Lemma 11.1. *Let (X_j, x_j) be a sequence of $CAT(1)$ spaces of dimension $\leq r$. Then the ultralimit $(X, x) = \lim_\omega(X_j, x_j)$ has also dimension at most r.*

Proof. Let $x_0, x_1, ..., x_n$ be points in X such that the barycentric simplex $S(x_0, x_1, ...x_n)$ is not degenerate. Let x_i be given by a sequence

$z_i^j \in X_j$. Consider for each j the simplex $S(z_0^j, z_1^j, ..., z_n^j)$. The definition of barycentric simplices directly implies that $\lim_\omega S(z_0^j, z_1^j, ..., z_n^j)$ is exactly the simplex $S(x_0, ...x_n)$. If all the simplices $S(z_0^j, z_1^j, ..., z_n^j)$ were degenerate, then the simplex $S(x_0, x_1, ...x_n)$ would be degenerate too. □

Since X is isometrically embedded in X^ω we obtain

Corollary 11.2. *Let X be a $CAT(1)$ space. Then $dim(X) = dim(X^\omega)$.*

Remark 11.2. By rescaling the spaces we see, that the last two results are true for arbitrary $CAT(\kappa)$ spaces.

11.4. **Splittings and buildings.** We show now that if the conclusion of Theorem 5.1 is true for X^ω then it is true for X.

Lemma 11.3. *Let X, Y, Z be $CAT(1)$ spaces with $\dim(X) = \dim(Y * Z) = n < \infty$. Let an isometric embedding $i : X \to Y * Z$ be given. If X is geodesically complete then $X \cap Y$ and $X \cap Z$ are not empty.*

Proof. Assume that $X \cap Y$ is empty. Then for each $y \in Y$ and $z \in Z$ the geodesic yz may contain at most one point of X, otherwise the geodesic completness of X would imply $y \in X$. Let K be a compact subset of X with topological dimension n. Then the projection of K onto Z is well defined and injective. Its image in Z is a compact set of topological dimension n, therefore $Y * Z$ has dimension at least $n + 1$. □

As a consequence we get:

Corollary 11.4. *Let X be a geodesically complete finite dimensional $CAT(1)$ space. If X_ω splits as $X_\omega = Y * Z$ then X is reducible too.*

Proof. By Lemma 11.3 we know that $X \cap Y$ and $X \cap Z$ are not empty. Thus $X \cap Y$ is a proper closed convex 1-symmetric subset of X. By Proposition 4.2 the subset $X \cap Y$ is a factor of X. □

Lemma 11.5. *Let X be a geodesically complete $CAT(1)$ space. Then X is a building iff X^ω is a building.*

Proof. Assume that X^ω is a building. Then X is a convex subset of X^ω with $\dim(X) = \dim(X^\omega) = n$. Thus X contains a point x whose link S_xX contains a sphere S^{n-1}. If $\bar{x}$ is an antipode of x in X then X must contain the sphere S^n defined by $x, \bar{x}$ and $S^{n-1} \in S_x$. By [KL97],p.145 we know that X is a building.

Assume now that X is a building. The submetry $\theta : X \to \Delta$ induces a submetry $\theta^\omega : X^\omega \to \Delta$. For each sequence of isometrically embedded S^n in X we get an isometrically embedded S^n in X^ω. We see that each two points of X^ω lie in an isometrically embedded S^n and from [KL97],p.139 we derive that X^ω is a building. □

The next lemma shows that we do not lose the assumptions under taking ultraproducts.

Lemma 11.6. *Let X be a geodesically complete $CAT(1)$. If $A \subset X$ is 1-symmetric, then $A^\omega \subset X^\omega$ is 1-symmetric.*

Proof. For $x = (x_j) \in A^\omega$ consider a point $z = (z_j) \in \mathrm{Ant}(x)$. By definition $\lim_\omega d(x_j, z_j) \geq \pi$. If for some j the point z_j is not in A then $d(x_j, z_j) < \pi$. For such j choose a point $\bar{z}_j \in \mathrm{Ant}(x_j) \subset A$ with $d(z_j, \bar{z}_j) = \pi - d(x_j, z_j)$. For other j set $\bar{z}_j = z_j$. The point $(\bar{z}_j) \in X^\omega$ is contained in A^ω and coincides with (z_j) in X^ω. □

11.5. **Ultraproducts are nice.**

Lemma 11.7. *Let X be a $CAT(1)$ space of dimension k, that contains an isometrically embedded S^k. Then X admits a 1-Lipschitz retraction $p : X \to S^k$.*

Proof. For $k = 0$ the statement is clear. For $k > 0$ we choose a point x in the given $S^k \subset X$. Then $S_x X$ contains $S^{k-1} = S_x S^k$ and therefore $\dim(S_x X) = k - 1$. By induction we find a 1-Lipschitz retraction $r : S_x \to S^{k-1} = S_x X$. Now take the composition $(r * id) \circ p_x : X \to S_x X * S^0 \to (S_x S^k) * S^0 = S^k$. □

Lemma 11.8. *Let X be a finite dimensional geodesically complete $CAT(1)$ space. Then its ultraproduct X^ω is a nice space.*

Proof. Let the point x be given by a sequence (x_j) in X. Replacing x_j by a point $\bar{x}_j$ with $d(x_j, \bar{x}_j) \to 0$ we may assume (see Subsection 2.8) that each link S_{x_j} contains a sphere S^{m_j} with $m_j = \dim(S_{x_j})$. Consider an *s.L.m.* $q_j : S_{x_j} \to S^{m_j}$ and the compositions $f_j = (q_j * id) \circ p_{x_j} : X \to S_{x_j} * S^0 \to S^{m_j} * S^0$. The maps f_j are by construction 1-Lipschitz and have dense images. Moreover $d(f(x_j), f(z)) = d(x_j, z)$ for each $z \in X$. The map $f^x = f = \lim_\omega f_j : X^\omega \to \lim_\omega S^{m_j+1} = S^{\lim_\omega m_j + 1}$ has all the needed properties. □

11.6. **Happy ending.** Let now X be an n-dimensional geodesically complete $CAT(1)$ space with a proper closed 1-symmetric subset A. Then A^ω is a proper closed i-symmetric subset of the nice n-dimensional space X^ω. In the last section we proved that X^ω is a building or a spherical join. Lemma 11.5 and Corollary 11.4 imply now that X itself is a building or a spherical join. This concludes the proof of Theorem 5.1.

12. Conclusions

Let X be a finite dimensional geodesically complete $CAT(1)$ space. Let $X = S^l * G_1 * ... * G_k * X_1 * ... * X_m$ be a decomposition of X where G_j are thick irreducible buildings and X_j are irreducible but not buildings. Then $\Delta_X^2 = S^l * \Delta(G_1) * ... * \Delta(G_k) * \{\mathrm{pt}\} ... * \{\mathrm{pt}\}$ where $\Delta(G_i)$ is the Weyl simplex of the building G_i.

We see that a decomposition of X as above determines a decomposition of the space Δ_X^2 in a spherical join of irreducible factors. Since such a decomposition of a convex subset of a sphere is unique, we just have finished the proof of Corollary 1.2. Moreover from the description of the space Δ_X^2 we see that the canonical weak submetries $\delta^2 : X \to \Delta_X^2$ are indeed submetries.

Let $X = S^l * G_1 * ... * G_k * X_1 * ... * X_m$ be as in Corollary 1.2. Consider the building $\hat{X}$ defined as $\hat{X} = S^l * G_1 * ... * G_k * \hat{X}_1 * ... * \hat{X}_m$ where $\hat{X}_j$ is the set X_j with a discrete metric. This is a building satisfying $\dim(\hat{X}) \leq \dim(X)$. Moreover the underlying sets of X and $\hat{X}$ are the same and the identity $id : \hat{X} \to X$ is a bijective 1-Lipschitz map.

From Example 5.1 we see, that points are 2-*near* in X if and only if they are 2-*near* in $\hat{X}$. Since each geodesic in the building $\hat{X}$ consists of finitely many *horizontal* parts, we see that the metric $\hat{d}$ on $\hat{X}$ can be defined in the following natural way: $\hat{d}(x,z) = \inf(d(x_0,x_1) + d(x_1,x_2) + ... + d(x_{n-1},x_n))$ where $x = x_0, x_1, ..., x_n = z$ run through finite sequences of points in X such that x_j and x_{j+1} are 2-*near*.

For an *s.L.m.* $f : X \to Y$ pairs of *i-near* points in X are mapped isometrically to *i-near* points in Y. This implies that f considered as a map $\hat{f} : \hat{X} \to \hat{Y}$ is still 1-Lipschitz, providing a proof of Corollary 1.3.

13. Local structure of $CAT(\kappa)$ spaces

We start with a quite general construction valid in arbitrary $CAT(\kappa)$ spaces. Let namely (X_i, x_i) be a sequence of pointed $CAT(\kappa)$ spaces and let $(X,x) = \lim_\omega(X_i, x_i)$ be their ultralimit. Consider arbitrary points $y, z \neq x$ in X. Choose $y_i, z_i \in X_i$ such that $(y_i) = y$ and $(z_i) = z$ holds. Let v_i resp. w_i be the starting direction in S_{x_i} of the geodesic $x_i y_i$ resp. $x_i z_i$. Let finally v resp. w be the starting direction in $S_x X$ of xy resp. of xz.

From $\lim_\omega(x_i z_i) = xz$ and $\lim_\omega(x_i y_i) = xy$ and the $CAT(\kappa)$ property we obtain $d(v,w) \geq \lim_\omega d(v_i, w_i)$ ([BH99],p.185). Therefore sending the direction v to the sequence $(v_i) \in \lim_\omega(S_{x_i})$ we obtain a well defined 1-Lipschitz map F to the space $\lim_\omega(S_{x_i})$ defined on the set of all genuine directions in S_x. Moreover this map does not depend on the choice of the point z and the sequence (z_i). Extending the map to the completion S_x of the set of all genuine directions we see:

Lemma 13.1. *Let* $(X, x) = \lim_\omega(X_i, x_i)$ *be an ultralimit of* $CAT(\kappa)$ *spaces. Then there is a natural (up to* ω*)* 1*-Lipschitz map* $F : S_xX \to \lim_\omega(S_{x_i}X_i)$*.*

Assume now in addition that each space X_i is geodesically complete. Then for each sequence $v_i \in S_{x_i}X_i$ of genuine directions we can choose a point $z_i = \exp(rv_i)$ with a fixed positive r. Considering the point $z = (z_i) \in \lim_\omega(X_i, x_i) = (X, x)$ we see, that $(v_i) \in \lim_\omega(S_{x_i})$ is contained in the image of the map F. Since the set of genuine directions is dense in S_{x_i} we obtain:

Lemma 13.2. *If in Lemma 13.1 all the spaces* (X_i, x_i) *are geodesically complete, the map* $F : S_x \to \lim_\omega(S_{x_i})$ *defined in Lemma 13.1 is surjective, hence* F *is an* s.L.m.

The most interesting consequences of the above lemma arise in the locally compact case:

Proposition 13.3. *Let* (X, x) *be a locally compact geodesically complete* $CAT(\kappa)$ *space,* (x_i) *a sequence in* X *converging to* x*. Then there is a well defined surjective* 1*-Lipschitz map* $p : S_x \to \lim_\omega(S_{x_i}X)$*.*

Proof. Apply Lemma 13.2 to the ultralimit $(X, x) = \lim_\omega(X, x_i)$. □

Under the assumptions of Proposition 13.3 we see, that $\lim_\omega(S_{x_i}X)$ is compact and therefore the spaces $S_{x_i}X$ subconverge to $\lim_\omega(S_{x_i}X)$ in the Gromov-Hausdorff topology. Now Corollary 1.5 follows directly from Corollary 1.4.

Remark 13.1. Easy examples show that Proposition 13.3 need not hold in the non locally compact case.

Finally we want to give an extension of Proposition 13.3, that will be used in a forthcoming paper to study singular points in $CAT(\kappa)$ spaces.

Proposition 13.4. *Under the assumptions of Proposition 13.3 assume in addition that* $x_i \neq x$ *and that the starting directions* $v_i \in S_x$ *of the geodesics* xx_i *converge to a direction* $v \in S_x$*. Then there is a natural* s.L.m. $F^v : S_v(S_xX) * S^0 \to \lim_\omega(S_{x_i}X)$*.*

Proof. Set $t_i = d(x, x_i)$. Then the rescaled spaces $(\frac{1}{t_i}X, x)$ converge in the Gromov-Hausdorff topology to the tangent cone $T_xX = CS_x$ ([BH99]), that is the Euclidean cone over S_x. Under this convergence the points x_i converge to the point $v \in S_x \subset C(S_x)$. The result follows now from Lemma 13.2 and the fact $S_v(CS_x) = S_v(S_x) * S^0$. □

14. **Appendix**

We recall some results about submetries of spheres from [Lyt01],pp.45-50 and sketch their proofs.

If $\delta : S^n \to \Delta$ is a submetry, then Δ is an Alexandrov space of Hausdorff dimension $\leq n$ and curvature bounded below by 1 (this can be found in [BGP92], p.16). One shows first

Lemma 14.1. *For each $x \in S^n$ the set N_x of points* near *to x is a convex set.*

To see this denote by F the fiber of δ through x. By definition we have $N_x = \cap_{\bar{x} \in F} N_{x,\bar{x}}$, where $N_{x,\bar{x}} = \{z \in S^n | d(x,z) \leq d(\bar{x},z)\}$. But in the sphere S^n each set $N_{x,\bar{x}}$ is a hemisphere, in particular it is convex.

Playing with convex sets one deduces that each fiber of δ is a set of positive reach in the sense of [Fed59], i.e. for each $y \in \Delta$ there is an $\epsilon > 0$, such that for each direction $v \in S_y\Delta$ there is a geodesic γ of length ϵ in Δ starting in y in the direction of v. In other words for each $x \in F = \delta^{-1}(y)$, and each δ-*horizontal* direction $w \in S_x$ the geodesic in the direction of w remains δ-*horizontal* at least for the time ϵ.

For another direct proof of this fact we refer to [Lyt04],p.11.

Each set of positive reach F contains a $C^{1,1}$ submanifold U that is open and dense in F ([Fed59] or [Lyt04],p.3). We get:

Lemma 14.2. *Each fiber F of δ contains an open dense subset U, such that for each $x \in U$ the set of δ-*horizontal *directions is a sphere.*

Let finally $\delta : S^n \to \Delta$ be a submetry, such that all *horizontal* spaces are spheres. For $x \in S^n$ and a δ-*horizontal* direction $v \in S_x S^n$ let γ be the geodesic in the direction of v. For some $t_1 > 0$ the points $\gamma(0) = x$ and $\gamma(t_1)$ are δ-*near*. Let t_1 maximal with this property. Since the set of *horizontal* directions at $\gamma(t_1)$ is a sphere, we find some $t_2 > t_1$ such that $\gamma(t_2)$ and $\gamma(t_1)$ are δ-*near*. Choose t_2 maximal. Continuing in this fashion we get a sequence $0 < t_1 < t_2 < ....$ It is possible to prove that t_i cannot converge to a finite number. To see this assume the contrary, i.e. assume $t_i \to t$. Then one easiely sees from the definition of quasi-geodesics (we refer to [PP94]), that the image $\delta \circ \gamma : [0,t] \to \Delta$ is a quasi-geodesic in Δ. This quasi-geodesic $\bar{\gamma}$ has an incoming direction in $\bar{\gamma}(t)$. Since in this direction a geodesic starts (here we use that the fiber through $\gamma(t)$ has positive reach), our quasi-geodesic must coincide with this geodesic ([PP94],p.8). This implies that $\gamma[t-\epsilon, t]$ is a δ-horizontal geodesic for some $\epsilon > 0$, in contradiction to the maximality of the choices of t_i. We have proved:

Lemma 14.3. *Let $\delta : S^n \to \Delta$ be a submetry, such that all horizontal spaces are spheres. Then for each geodesic γ starting in a horizontal direction, the segment $\gamma([0,\pi])$ consists of finitely many* horizontal *geodesics.*

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