ON CONTRACTIBLE ORBIFOLDS

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ABSTRACT. We prove that a contractible orbifold is a manifold.

1. Introduction

Following [Dav10], we call an orbifold X *contractible* if all of its orbifold homotopy groups $\pi_i^{orb}(X), i \geq 1$ vanish. We refer the reader to [Dav10] and the literature therein for basics about orbifolds. Davis has asked in [Dav10], whether any contractible orbifold X must be developable. In this note we answer this question affirmatively.

THEOREM 1.1. *Let X be a smooth contractible orbifold. Then it is a manifold.*

Proof. Since X is contractible, it is orientable. Let n be the dimension of X. Define on X a Riemannian metric and let the smooth manifold M be the bundle of oriented orthonormal frames on X (cf. [Hae84]). Then $G = SO(n)$ acts effectively and almost freely on M with $X = M/G$.

Let E denote a contractible CW complex on which G act freely, with quotient $E/G = BG$, the classifying space of G. Then $\hat{X} = (M \times E)/G$ is a model for the *classifying space of* X (cf. [Hae84]). By definition, the orbifold homotopy groups of X are the usual homotopy groups of $\hat{X}$. Thus, by our assumption, the topological space $\hat{X}$ is contractible.

The projection $M \times E \to \hat{X}$ is a homotopy fibration. Thus the contractibility of $\hat{X}$ implies that the embedding of any orbit of G into $M \times E$ is a homotopy equivalence between G and $M \times E$. Since E is contractible, the projection $M \times E \to M$ is a homotopy equivalence as well. Therefore, for any $p \in M$, the composition $o_p : G \to G \cdot p \to M$ given by orbit map $o_p(g) := g \cdot p$ is a homotopy equivalence.

Assume now that X is not a manifold. Then G does not act freely on M. Thus, for some $p \in M$, the stabilizer G_p of p is a finite non-trivial group. Then the orbit map $o_p : G \to M$ factors through the quotient

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map $\pi_p : G \to G/G_p$. Since the orbit map is a homotopy equivalence, there must exist some map $i : G/G_p \to G$ such that $i \circ \pi : G \to G$ is a homotopy equivalence. However, the manifolds G and G/G_p are orientable and the map $G \to G/G_p$ is a covering of degree $|G_p|$. Thus, for $m = \dim(G) = n(n-1)/2$, the image of π_p^* in $H^m(G, \mathbb{Z}) = \mathbb{Z}$ is a subgroup of $H^m(G, \mathbb{Z})$ of index $|G_p|$. In particular, $(i \circ \pi)^*$ cannot be surjective. Contradiction. $\square$

A small observation on the proof above: If we do not assume X to be contractible, but merely k-connected, then the orbit map $o_p : G \to M$ is k-connected as well. Thus, for any $l < k$, the map $H^l(M, \mathbb{Z}) \to H^l(G, \mathbb{Z})$ is surjective. Since the orbit map $o_p : G \to M$ factorizes through $\pi_p : G \to G/G_p$, we deduce as above that $H^l(G/G_p, \mathbb{Z}) \to H^l(G, \mathbb{Z})$ is surjective, for all $l < k$. If X is not a manifold, i.e., if some G_p is non-trivial, the above contradiction shows that $k \leq n(n-1)/2$. However, recall that the free part of $H^*(G)$ is generated by elements of degree at most $2n-3$ ([Hat02], p.300). Hence, if G_p is non-trivial, the map $H^l(G/G_p, \mathbb{Z}) \to H^l(G, \mathbb{Z})$ cannot be surjective for all $l \leq 2n-3$. We deduce, that X is a manifold if it is $(2n-2)$-connected.

I believe that this observation is far from optimal in high dimensions. In fact, I do not know a single example of a 4-connected non-developable orbifold. I would like to finish the note by formulating two problems.

PROBLEM 1.2. *Do highly connected non-developable orbifolds exist?*

PROBLEM 1.3. *Does an analogue of Theorem 1.1 hold true for non-smooth orbifolds? Does it hold true for etale groupoids of isometries?*

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References

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