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CURVATURE EXPLOSION IN QUOTIENTS AND APPLICATIONS

Alexander Lytchak & Gudlaugur Thorbergsson

Abstract

We prove that the quotient space of a variationally complete group action is a good Riemannian orbifold. The result is generalized to singular Riemannian foliations without horizontal conjugate points.

1. Introduction

Let M be a Riemannian manifold and let K be a closed group of isometries of M. Usually, the quotient space $B = M/K$ is not a Riemannian manifold, but an Alexandrov space of curvature (locally) bounded below, stratified by Riemannian manifolds that consist of orbits of the same type. The set M_0 of all points in M with principal isotropy group is open and dense in M, and it is invariant under the action of K. The quotient $B_0 = M_0/K$ is the maximal stratum of B. In many cases, the sectional curvatures in B_0 explode as one approaches a singular point $y \in B_{sing} = B \setminus B_0$. However, sometimes it does not happen, like in the case of an exceptional orbit. An interesting class of examples is given by polar actions, where the quotient is a smooth Riemannian orbifold and, therefore, its sectional curvatures are uniformly bounded on compact subsets. Our first objective is a precise description of such (non-) explosions. For a point $z \in B_0$, we denote by $\bar{\kappa}(z)$ the maximum of the sectional curvatures of tangent planes at z. Then we have:

Theorem 1.1. *Let M be a Riemannian manifold and let K be a closed group of isometries of M. Let $B = M/K$ be the quotient. Let $x \in M$ be a point with isotropy group K_x acting on the normal space H_x of the orbit $Kx \subset M$. Set $y = Kx \in B$. Then the following are equivalent:*

1) $\limsup_{z \in B_0, z \to y} \bar{\kappa}(z) < \infty$;
2) $\limsup_{z \in B_0, z \to y} \bar{\kappa}(z) \cdot d^2(y,z) = 0$;
3) *The action of K_x on H_x is polar;*
4) *A neighborhood of y in B is a smooth Riemannian orbifold.*

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REMARK 1.1. With arguments similar to those used in the proof of Theorem 1.1, one sees that the explosion of curvatures in B_0 is at most quadratic in the distance to the singular strata; namely, for each compact subset U of B there is a constant $C > 0$ such that for all $z \in B_0 \cap U$,

$$\bar{\kappa}(z) \leq \frac{C}{d^2(z, B_{sing})}.$$

Since all orthogonal actions on Euclidean spaces of cohomogeneity at most 2 are polar, we deduce that the complement of the set of Riemannian orbifold points in the quotient has codimension at least 3:

Corollary 1.2. *Let M be a Riemannian manifold and let K be a closed group of isometries of M. Then each point y in the quotient $B = M/K$ contained in a stratum of codimension ≤ 2 has a neighborhood in B isometric to a smooth Riemannian orbifold.*

This result makes large parts of the quotient accessible with differential geometric tools and can be used to derive some geometric properties of the quotient, like the invariance of the Liouville measure under the (quasi-) geodesic flow; see Theorem 1.6 below.

Theorem 1.1 can be applied to study variationally complete actions. Variationally complete actions were introduced by Bott and used in [**Bot56**] and [**BS58**], as a mean for studying the topology of (loop spaces of) symmetric spaces. In these papers it is observed that the orbits of such actions are taut submanifolds of the ambient (symmetric) space, which establishes strong relations between the topology of the ambient manifold and the topology of the orbits. Conlon [**Con72**] showed that hyperpolar actions are variationally complete. The converse was proven in [**DO01**] for Euclidean spaces, in [**GT02**] for compact symmetric spaces and, more generally, in [**LT07**] for non-negatively curved Riemannian manifolds.

In general, a variationally complete action of a group K on a Riemannian manifold M does not need to be polar, nor does an action with taut orbits need to be variationally complete. We describe all variationally complete actions in terms of the quotient space and give a precise meaning to the following remark of Bott and Samelson [**BS58**, p. 965]: "Intuitively, we like to think of variational completeness as absence of conjugate points on the decomposition space M/K."

Theorem 1.3. *Let M be a complete Riemannian manifold and K a closed group of isometries of M. The action of K on M is variationally complete if and only if the quotient $B = M/K$ is a good Riemannian orbifold without conjugate points, i.e., if and only if B is isometric to a quotient $B = N/\Gamma$, where N is a smooth, complete, simply connected Riemannian manifold without conjugate points and Γ a discrete group of isometries of N.*

If the manifold M is non-negatively curved, then so is the quotient B and the universal orbifold covering N of B. The absence of conjugate points implies flatness of N in such a case. This flatness, on the other hand, implies that the action is hyperpolar. Thus Theorem 1.3 generalizes theorem A from [**LT07**].

As soon as one knows that the quotient $B = M/K$ is a Riemannian orbifold, the statement of Theorem 1.3 is a more or less direct consequence of the definition of variational completeness. Thus the main part of the work consists in proving that variational completeness implies that the quotient M/K is a Riemannian orbifold. In this part of the proof Theorem 1.1 plays an essential role.

As in [**LT07**], our results generalize to singular Riemannian foliations. Recall that a *transnormal system* $\mathcal{F}$ on a Riemannian manifold M is a decomposition of M into smooth injectively immersed connected submanifolds, called *leaves*, such that geodesics emanating perpendicularly from one leaf stay perpendicularly to all leaves. A transnormal system $\mathcal{F}$ is called a *singular Riemannian foliation* if there are smooth vector fields X_i on M such that for each point $p \in M$ the tangent space $T_pL(p)$ of the leaf $L(p)$ through p is given as the span of the vectors $X_i(p) \in T_pM$. We refer to [**Mol88**, pp. 185–216] and [**Wil07**] for more on singular Riemannian foliations. Examples of singular Riemannian foliations are (regular) Riemannian foliations and the orbit decomposition of an isometric group action. A singular Riemannian foliation $\mathcal{F}$ will be called *closed* if all of its leaves are closed in M, and it will be called *locally closed at a point* $x \in M$ if for some neighborhood U of x the restriction of $\mathcal{F}$ to U is closed (i.e., connected components of the intersection of the leaves of $\mathcal{F}$ with U are closed in U). If $\mathcal{F}$ is locally closed at x (if $\mathcal{F}$ is closed and M is complete, respectively) then the local quotient (the global quotient, respectively) $U/\mathcal{F}$ $(M/\mathcal{F})$ is a well defined Alexandrov space of curvature locally bounded from below [**BGP92**, **Bol07**, **Lyt01**]. Note, that closedness and completeness are only needed to assure that the quotient space is Hausdorff.

For a singular Riemannian foliation $\mathcal{F}$ on M, denote by M_0 the set of all points $z \in M$, called *regular points*, whose leaves have maximal dimension. The set M_0 is open and dense in M, and the restriction of $\mathcal{F}$ to M_0 is a (non-singular) Riemannian foliation. At each point $z \in M_0$, the local quotient of M modulo $\mathcal{F}$ is a smooth Riemannian manifold and we denote by $\bar{\kappa}(z)$ the maximum of all sectional curvatures at the image of z in this quotient. The next result describes (non-) explosion of these curvatures as one approaches a boundary point x of M_0. The isotropy representation at x used in Theorem 1.1 now has to be replaced by the infinitesimal singular Riemannian foliation $T_x\mathcal{F}$ on the tangent space T_xM (see [**Mol88**, pp. 202–205] and Subsection 2.1).

The following result generalizes Theorem 1.1:

Theorem 1.4. *Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. Let $x \in M$ be a point and let $T_x\mathcal{F}$ be the infinitesimal singular Riemannian foliation induced by $\mathcal{F}$ on the tangent space T_xM. Then the following are equivalent:*

1) $\limsup_{z\in M_0, z\to x} \bar\kappa(z) < \infty$*;*
2) $\limsup_{z\in M_0, z\to x} \bar\kappa(z) \cdot d^2(x,z) = 0$*;*
3) *The singular Riemannian foliation $T_x\mathcal{F}$ is polar;*
4) *$\mathcal{F}$ is locally closed at x and a local quotient $U/\mathcal{F}$ of a neighborhood U of x is a Riemannian orbifold.*

Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. We will call $\mathcal{F}$ *infinitesimally polar* at the point $x \in M$ if the equivalent conditions of Theorem 1.4 are fulfilled. We call $\mathcal{F}$ infinitesimally polar if it is infinitesimally polar at all points of M. The class of infinitesimally polar singular Riemannian foliation is a broad generalization of regular Riemannian foliations and singular Riemannian foliations with sections. On the one hand, such singular Riemannian foliation are well accessible by differential geometric methods, since their local quotients are finite(!) quotients of smooth Riemannian manifolds. On the other hand, each singular Riemannian foliation is infinitesimally polar on large parts of M (compare Proposition 3.1). We hope to discuss general features of infinitesimally polar singular Riemannian foliations somewhere else. Here we discuss a surprising characterization of infinitesimally polar singular Riemannian foliations and an application to general closed singular Riemannian foliations.

In order to state the characterization, recall that a geodesic on M is called *horizontal* if it meets the leaves of $\mathcal{F}$ perpendicularly. We will call such a geodesic $\gamma : [a,b] \to M$ *regular*, if $\gamma(a)$ and $\gamma(b)$ are regular points of $\mathcal{F}$. A regular horizontal geodesic intersects the singular strata of $\mathcal{F}$ only in finitely many points $a < t_1 < .... < t_k < b$ (see Corollary 4.6). We set $c(\gamma) = c_1 + ... + c_k$, where c_i is given by $\dim(L(\gamma(a))) - \dim(L(\gamma(t_i)))$, and call this number the *crossing number* of γ. Then we have:

Theorem 1.5. *Let M be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M. Let G denote the space of all regular horizontal geodesics with the topology of pointwise convergence. Let $c : G \to \mathbb{N}$ be the crossing number function. The function c is continuous if and only if $\mathcal{F}$ is infinitesimally polar.*

Remark 1.2. The crossing numbers $c(\gamma)$ were implicitly used in [**Bot56**] and [**BS58**] to calculate the indices of geodesics in compact symmetric spaces and to deduce consequences about homology groups of their loop spaces. The equality between the index and the crossing number holds true for *variationally complete* actions, or, more generally, for singular Riemannian foliations without horizontal conjugate points (see below). Generalizing [**BS58**], one can deduce from this fact that the leaves are taut submanifolds, and the crossing numbers can be used to compare the topology of the leaves with the topology of the ambient space (cf. [**Now08**]).

Now let $\mathcal{F}$ be an arbitrary singular Riemannian foliation on a complete Riemannian manifold M. Then on the quotient $M/\mathcal{F}$ one can define a canonical "quasi-geodesic flow." Restricted to the regular part B_0 of $M/\mathcal{F}$, this flow coincides with the geodesic flow. However, at some time instances the flow may leave the regular part, and then a local increase or decrease of volume could happen a priori. Using Theorem 1.4, we show that it does not happen. In case of a compact quotient, the next result can be used to obtain almost recurrent horizontal geodesics.

Theorem 1.6. *Let M be a complete Riemannian manifold and let $\mathcal{F}$ be a closed singular Riemannian foliation on M. Then the projection of the horizontal geodesic flow on M leaves the Liouville measure of $M/\mathcal{F}$ invariant.*

The notion of variational completeness generalizes to the setting of singular Riemannian foliation as the notion of absence of horizontal conjugate points (cf. [**LT07**]). Namely, let $\gamma : [a, b] \to M$ be a horizontal geodesic. An *$\mathcal{F}$-Jacobi field* along γ is a variational field through horizontal geodesics starting on the leaf $L(\gamma(a))$. An *$\mathcal{F}$-vertical Jacobi field* along γ is an $\mathcal{F}$-Jacobi field J with $J(t) \in T_{\gamma(t)}L(\gamma(t))$ for all t. We say that γ *has no horizontal conjugate points* if each $\mathcal{F}$-Jacobi field J with $J(t_0) \in T_{\gamma(t_0)}L(\gamma(t_0))$ for some $a < t_0 < b$ is $\mathcal{F}$-vertical. We say that $\mathcal{F}$ *has no horizontal conjugate points* if no horizontal geodesics in M have horizontal conjugate points.

The following result generalizes Theorem 1.3:

Theorem 1.7. *Let M be a complete Riemannian manifold with a singular Riemannian foliation $\mathcal{F}$. If $\mathcal{F}$ has no horizontal conjugate points then it is infinitesimally polar. If $\mathcal{F}$ is closed, then $\mathcal{F}$ has no horizontal conjugate points if and only if quotient $M/\mathcal{F}$ is a good Riemannian orbifold without conjugate points.*

The paper is structured as follows. In Section 2 we recall some basic facts about singular Riemannian foliations and Riemannian orbifolds. In Section 3 we prove Theorem 1.4 and its consequences Theorem 1.1 and Proposition 3.1, which generalize Corollary 1.2 to the case of singular Riemannian foliations. In Section 4 we discuss the horizontal geodesic flow and prove Theorem 1.6. The main technical observation of this section is the fact that the horizontal geodesic flow in the total space defines a flow in a quotient, i.e., that two projections of horizontal geodesics that coincide initially coincide for their life span. This result was proved in [**Lyt01**] and [**Bol07**] for the case of proper singular Riemannian foliations, and in [**LT07**] and [**Now08**] for the case of singular

Riemannian foliations without conjugate points. An independent proof of this fact recently appeared in [**AT08**]. Finally, in Section 5 we discuss various notions of conjugate points, prove stability of the absence of conjugate points and deduce Theorem 1.7 and Theorem 1.5.

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2. Preliminaries

Let M be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M.

2.1. Distinguished tubular neighborhoods and infinitesimal foliations. Let $x \in M$ be a point. Then there is a small open ball P around x in the leaf $L(x)$, a number $\epsilon > 0$ and a neighborhood O of P in M, called a *distinguished tubular neighborhood at x* such that the following holds true [**Mol88**, pp. 192–193 and pp. 202–205]:

1) The foot point projection $F : O \to P$ is well defined;
2) O is the image of the ϵ-tube $N^\epsilon(P)$ in the normal bundle $N(P)$ of P under the exponential map, and the map $\exp : N^\epsilon(P) \to O$ is a diffeomorphism;
3) For each real positive number $\lambda \leq 1$ the map $h_\lambda : O \to O$, given by $h_\lambda(\exp(v)) = \exp(\lambda v)$ for all $v \in N^\epsilon(P)$, preserves $\mathcal{F}$;
4) There is a diffeomorphism ϕ of O into the tangent space T_xM with $D_x\phi = \mathrm{Id}$ and a singular Riemannian foliation $T_x\mathcal{F}$ on T_xM that coincides with $\phi_*\mathcal{F}$ on $\phi(O)$ and such that $T_x\mathcal{F}$ is invariant under all rescalings $r_\lambda : T_xM \to T_xM, r_\lambda(v) = \lambda v$, for all $\lambda > 0$.

Remark 2.1. In Section 4 we will see that the restriction of $\mathcal{F}$ to O is invariant under the reflection $h_{-1} : O \to O$ at P.

The singular Riemannian foliation $T_x\mathcal{F}$ on the tangent space T_xM will be called the *infinitesimal singular Riemannian foliation of $\mathcal{F}$ at the point x*. The infinitesimal foliation $T_x\mathcal{F}$ can be considered as a blow up of $\mathcal{F}$ in the following sense. Let ϕ be as above. Identify O with $\phi(O)$. Set $O^\lambda = r_\lambda(O)$ and define the Riemannian metric g^λ on O^λ as $g^\lambda = \lambda^2 \cdot (r_\lambda)_*g$. We have $\cup O^\lambda = T_xM$. On compact subsets of T_xM, the blow up metrics g^λ smoothly converge to the flat metric g_x. By construction, the restriction of $T_x\mathcal{F}$ to O^λ is a singular Riemannian foliation with respect to g^λ.

2.2. Local quotients. We will continue to use the notations introduced above. Note that $\mathcal{F}$ is locally closed at the point x if and only if the infinitesimal foliation $T_x\mathcal{F}$ is closed. In such a case, the quotient $T_xM/T_x\mathcal{F}$ is a non-negatively curved Alexandrov space and $\phi(O)/T_x\mathcal{F}$ is a ball around the origin (the leaf through 0) in this space. The space $O/\mathcal{F}$ is an inner metric space of curvature bounded below in the sense of Alexandrov. Moreover, the space $T_xM/T_x\mathcal{F}$ is the tangent space to this Alexandrov space at the leaf $L(x) \in O/\mathcal{F}$.

Let us now assume that M is complete and that $\mathcal{F}$ is closed. Let $x \in M$ be given, and let O be a small distinguished tubular neighborhood of x, such that $O \cap L(x) = P$, in the notations of Subsection 2.1. Then $M/\mathcal{F}$ and $O/\mathcal{F}$ are spaces with curvature locally bounded below. The embedding $O \to M$ induces an open map $i : O/\mathcal{F} \to M/\mathcal{F}$. Since $\mathcal{F}$ is closed, the map is finite-to-one and, by construction, the leaf $L(x) \in M/\mathcal{F}$ has only one preimage in $O/\mathcal{F}$. The map i preserves the lengths of all curves.

Let U be an ϵ-tube around the leaf L with the same ϵ as in the definition of O. Then U is a union of leaves of $\mathcal{F}$ and $U/\mathcal{F}$ is a neighborhood of $L(x)$ in $M/\mathcal{F}$; see [**Lyt01**] or Section 4. The global quotient $U/\mathcal{F}$ is mapped by i onto the local quotient $O/\mathcal{F}$. To understand the map $i : O/\mathcal{F} \to U/\mathcal{F}$, consider the universal covering $\tilde{U}$ of U with the lifted singular Riemannian foliation $\tilde{\mathcal{F}}$ and the group of deck transformations Γ. We have $\tilde{U}/\tilde{\mathcal{F}} = O/\mathcal{F}$. The action of Γ on $\tilde{U}$ induces an isometric action on $\tilde{U}/\tilde{\mathcal{F}}$ whose quotient is precisely $U/\mathcal{F}$. Thus we deduce that a neighborhood of $L(x)$ in $M/\mathcal{F}$ is the quotient of the local quotient $O/\mathcal{F}$ by a finite group of isometries of $O/\mathcal{F}$.

2.3. Stratification. Let $\mathcal{F}$ again be a singular Riemannian foliation on the Riemannian manifold M. By the dimension, $\dim(\mathcal{F})$, and the codimension of $\mathcal{F}$, $\operatorname{codim}(\mathcal{F}, M)$, we denote the maximal dimension, respectively the minimal codimension of its leaves. For $s \leq \dim(\mathcal{F})$ denote by Σ_s the subset of all points $x \in M$ with $\dim(L(x)) = s$. Then Σ_s is an embedded submanifold of M and the restriction of $\mathcal{F}$ to Σ_s is a Riemannian foliation [**Mol88**, pp. 194–198]. For a point $x \in M$, we denote by Σ^x the connected component of Σ_s through x, where $s = \dim(L(x))$. We call the decomposition of M into the manifolds Σ^x the *canonical stratification* of M.

The subset $\Sigma_{\dim(\mathcal{F})}$ is open, dense, and connected in M. It is the regular stratum M_0 of M. All other singular strata Σ^x have codimension at least 2 in M. The *quotient codimension of a stratum* Σ^x is defined to be $\operatorname{codim}(\mathcal{F}, M) - \operatorname{codim}(\mathcal{F}, \Sigma^x)$.

If the singular Riemannian foliation $\mathcal{F}$ is locally closed at x, then a local quotient $O/\mathcal{F}$ is a space stratified by smooth Riemannian orbifolds $\Sigma^z/\mathcal{F}$ and the quotient codimension of the stratum Σ^x is just the codimension of the quotient $\Sigma^x/\mathcal{F}$ in the whole quotient $O/\mathcal{F}$.

Let a point $x \in M$ be fixed. Then the tangent space T_xM decomposes as $T_xM = T_x\Sigma^x \oplus N_x\Sigma^x$, where $N_x\Sigma^x$ is the normal space in M to

Σ_x. The infinitesimal Riemannian foliation $T_x\mathcal{F}$ on T_xM is the direct product of the foliation of $T_x\Sigma^x$ by affine subspaces parallel to $T_xL(x)$ and a singular Riemannian foliation $\widetilde{T_x\mathcal{F}}$ on $N_x\Sigma^x$. This last (the main) part $\widetilde{T_x\mathcal{F}}$ is invariant under positive homotheties of $N_x\Sigma^x$, and the only 0-dimensional leaf of $\widetilde{T_x\mathcal{F}}$ is the origin $\{0\}$.

The quotient codimension of the stratum Σ^x is the codimension of the singular Riemannian foliation $\widetilde{T_x\mathcal{F}}$ on the Euclidean space $N_x\Sigma^x$.

2.4. Riemannian orbifolds. We refer to [**BH99**, pp. 584–619] for a more advanced and refined study of Riemannian orbifolds. A metric space X is called a *good Riemannian orbifold* if X is isometric to M/Γ, where M is a smooth Riemannian manifold and Γ a discrete group of isometries.

A point x in a metric space X is called an *orbifold point* if x has a neighborhood U that is a good Riemannian orbifold. The set O of all orbifold points in X is open. We call X a *Riemannian orbifold* if $X = O$ holds. Note that the quotient of a Riemannian orbifold by a finite group of isometries is again a Riemannian orbifold.

Let B be a Riemannian orbifold. Then locally B is a finite isometric quotient of a smooth Riemannian manifold. Since geodesics, tangent spaces, and the Liouville measure on the unit tangent bundle of Riemannian manifolds are invariant under isometries, one gets corresponding notions on B. Namely, the "unit tangent bundle" UB being a disjoint union $UB = \cup_{y\in B} S_yB$ of spaces of directions S_yB, locally being a finite quotient of the unit tangent bundle of a "covering" Riemannian manifold. This unit tangent bundle comes along with the foot point projection $p : UB \to B$, a locally compact (quotient) topology, a local geodesic flow ϕ, and the Liouville measure μ, that is a Borel measure on UB. The local flow ϕ_t preserves the Liouville measure, whenever it is defined, since this is the case for Riemannian manifolds and since the Liouville measure and the local geodesic flow are preserved under isometries. For $v \in UB$, we set $\eta_v(t) = p(\phi_t(v))$, i.e., the curve η_v is locally the image of a geodesic in a Riemannian manifold under the quotient map. We call η_v the *orbifold-geodesic* in the direction of v.

A Riemannian orbifold B is stratified by Riemannian manifolds with a unique maximal stratum B_0 that is open and dense in B. The unit tangent bundle UB_0 is an open and dense subset of UB that has full measure with respect to μ. Moreover, the set U' of all vectors $v \in UB_0$ such that the orbifold-geodesic η_v does not cross strata of codimension ≥ 2 is of full measure in UB.

For each orbifold-geodesic η_v, the curvature endomorphism along η_v is well defined. Therefore, the notions of Jacobi fields and conjugate points are also well-defined. Let us now assume that B is complete as a metric space. Then each orbifold-geodesic is defined on $\mathbb{R}$ and the local geodesic flow is a global flow. Take a regular point $x \in B_0 \subset B$. Consider the orbifold exponential map $\exp : T_xB \to B$ given by $\exp(tv) = \eta_v(t)$, for a unit vector $v \in T_x$. This map (since defined in metric terms) factors over local branched covers of B, i.e., for each $w \in T_xB$ there is a finite quotient $N/\Gamma_w = O \subset B$, with $\exp(w) \in O$, such that exp lifts on a neighborhood of w to a smooth map to N. The vector $w = tv$ is a conjugate vector along the geodesic η_v, if and only if this lift has a non-injective differential at w.

If no vector in T_xB is a conjugate vector of x, i.e., if x has no conjugate points, then one can pull back the metric from B (in fact from the local covers N) to a Riemannian metric $\tilde{g}$ on T_xB. The space T_xB with this Riemannian metric $\tilde{g}$ is a complete Riemannian manifold and the map $\exp : T_xB \to B$ becomes an arclength preserving orbifold covering (as in the Theorem of Cartan-Hadamard). Thus T_xB is the universal orbifold covering of B and we get $B = (T_xB, \tilde{g})/\Gamma$ for some group Γ of isometries of T_xB. Hence B is a good orbifold in this case. Thus we have the following observation that is implicitly contained in the proof of the developability results stated in [**BH99**, p. 603].

Lemma 2.1. *If B is a complete Riemannian orbifold without conjugate points then B is a good orbifold. More precisely, there is a complete Riemannian manifold N without conjugate points and a discrete group Γ of isometries of N such that $B = N/\Gamma$.*

3. Infinitesimal polarity

3.1. Horizontal sections. Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. A *global (local) horizontal section through* x is a smooth immersed submanifold N in M through x that intersects all leaves of $\mathcal{F}$ (all leaves in a neighborhood of x), such that all intersections are orthogonal. We say that $\mathcal{F}$ is *polar* if there are global horizontal sections through every point $x \in M$. It is called *hyperpolar* if all these sections are flat. Recall that all local sections are totally geodesic; thus each polar foliation of $\mathbb{R}^n$ is hyperpolar. Finally, each hyperpolar foliation of $\mathbb{R}^n$ is closed. We refer to [**Bou95**], [**Ale04**], and [**Ale06**] for more on singular Riemannian foliation with sections.

Remark 3.1. In many important cases, like in Euclidean or symmetric spaces, polar singular Riemannian foliation have been objects of an extensive study, where they are better known as isoparametric foliations. See, for instance, [**PT88**].

3.2. Non-explosion of curvature. Now we can start with the

Proof of Theorem 1.4. The implications (4) $\Longrightarrow$ (1) $\Longrightarrow$ (2) are clear.

Assume (2). We are going to use the notations introduced in Subsection 2.1. Let $z \in T_xM$ be a $T_x\mathcal{F}$-regular point. For all t with $z \in O^t$ we denote by $\bar{\kappa}^t(z)$ the supremum of all sectional curvatures at the local projection of z in $(O^t, g^t)/T_x\mathcal{F}$. Since g^t smoothly converge to g_x for $t \to \infty$, the horizontal curvatures satisfy $\lim_{t\to\infty}(\bar{\kappa}^t(z)) = \bar{\kappa}_x(z)$. On the other hand, the assumption (2) implies $\lim_{t\to\infty}(\bar{\kappa}^t(z)) = 0$. Thus we deduce that the local quotient of (T_xM, g_x) modulo $T_x\mathcal{F}$ is flat at all regular points. Due to the flatness of (T_xM, g_x), this implies the vanishing of the O'Neill tensor of the Riemannian foliations $T_x\mathcal{F}$ on the regular part. Thus the horizontal distribution on the regular part of $T_x\mathcal{F}$ is integrable and we deduce that $T_x\mathcal{F}$ is hyperpolar [**Ale06**].

The main implication (3) $\Longrightarrow$ (4) is more subtle. Let $T_x\mathcal{F}$ be hyperpolar and let N be a horizontal section through 0. There is a finite group Γ of isometries of N, called the *Weyl group* of N (cf. [**PT88**]), with $\Gamma(0) = 0$ such that N/Γ is isometric to $T_xM/T_x\mathcal{F}$. Let O be again as in Subsection 2.1 and let us again identify it with $\phi(O)$. Let N_0 be a small ball in N around 0 that is contained in O.

In general, one cannot expect that N is a g-horizontal section of $\mathcal{F}$ nor must $\mathcal{F}$ have any horizontal local sections. The idea is to define a new "horizontal" metric on N_0 that is invariant under Γ and such that N_0/Γ becomes isometric to a neighborhood of x in $O/\mathcal{F}$.

For a point $z \in N_0$, we denote by $\tilde{V}(z)$ the orthogonal complement of T_zN with respect to the (constant) metric g_x. Let $\tilde{H}(z)$ denote the orthogonal complement of $\tilde{V}(z)$ with respect to the original metric g. Then $\tilde{H}(z)$ depends smoothly on z. Moreover, each space $\tilde{H}(z)$ is contained in the g-orthogonal complement of $T_zL(z)$. By dimensional reasons, $\tilde{H}(z)$ is the orthogonal complement of $T_zL(z)$ at all regular points.

Define a Riemannian metric $\tilde{g}$ on N_0 by $\tilde{g}_z(v,w) = g_z(P^z(v), P^z(w))$, where P^z is the orthogonal projection to $\tilde{H}(z)$ with respect to g_z. Denote by $\tilde{N}$ the manifold N_0 with the inner metric defined by the Riemannian metric $\tilde{g}$. The projection $p : \tilde{N} \to O/\mathcal{F}$ preserves lengths of all curves contained in the set of regular points of N. On the other hand, p is invariant under the group of diffeomorphisms Γ of $\tilde{N}$. Thus each $k \in \Gamma$ preserves lengths of all curves contained in the regular part of N. By continuity, each $k \in \Gamma$ is an isometry of $\tilde{N}$. Moreover, the induced map $\tilde{N}/\Gamma \to O/\mathcal{F}$ is an isometric embedding. This proves (4) and finishes the proof of Theorem 1.4. q.e.d.

Let us now assume that M is complete and that $\mathcal{F}$ is closed. Let $x \in M$ be given and let O be a small distinguished tubular neighborhood around x. If $\mathcal{F}$ is infinitesimally polar, then $O/\mathcal{F}$ is a Riemannian orbifold. Therefore, the image of O in $M/\mathcal{F}$, which is a finite quotient of $O/\mathcal{F}$, is a Riemannian orbifold. On the other hand, if $L(x) \in M/\mathcal{F}$ is an orbifold point, then the regular part of $T_xM/T_x\mathcal{F}$ is flat and we get that $T_x\mathcal{F}$ is hyperpolar. Thus $L(x)$ is an orbifold point of the global quotient

$M/\mathcal{F}$ if and only if $\mathcal{F}$ is infinitesimally polar at x. Thus Theorem 1.4 implies Theorem 1.1.

3.3. Small codimensions. Let $\mathcal{F}$ be a singular Riemannian foliation on the Euclidean space $\mathbb{R}^n$ that is invariant under positive rescalings and satisfies $L(0) = \{0\}$. Then all leaves of $\mathcal{F}$ are contained in concentric spheres around 0, and $\mathcal{F}$ is the cone over the restriction of $\mathcal{F}$ to the unit sphere $\mathbb{S}^{n-1}$. Note that $\mathcal{F}$ is polar on $\mathbb{R}^n$ if and only if its restriction to $\mathbb{S}^{n-1}$ is polar. We have $\operatorname{codim}(\mathcal{F}, \mathbb{R}^n) = \operatorname{codim}(\mathcal{F}, \mathbb{S}^{n-1}) + 1$. Finally, each singular Riemannian foliation of codimension 1 in a complete Riemannian manifold is polar. Thus each scaling invariant singular Riemannian foliation on $\mathbb{R}^n$ of codimension ≤ 2 is polar. This proves the following result generalizing Corollary 1.2:

Proposition 3.1. *Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M. Let $x \in M$ be a point with stratum Σ^x of quotient codimension ≤ 2. Then $\mathcal{F}$ is infinitesimally polar at x.*

4. Horizontal exponential map

4.1. Horizontal vectors. Consider the subset D of the unit tangent bundle UM that consists of all starting vectors v of horizontal geodesics γ_v. The set D is closed and invariant under the local geodesic flow ϕ_t, whenever it is defined. By $p : D \to M$ we denote the foot point projection.

We now discuss a preliminary stratification of D; later we will derive a more natural stratification adapted to the geodesic flow. For each stratum Σ^x of M, the preimage $D^x = p^{-1}(\Sigma^x) \subset D$ is a smooth submanifold of UM (and in fact a sphere bundle over Σ^x) of dimension $\dim(M) - 1 + \dim(\Sigma^x) - \dim(L(x))$. Thus D is stratified by smooth submanifolds D^x of UM. The main stratum D_0 is open and dense in D and the codimension of the stratum D^x (i.e., $\dim(D_0) - \dim(D^x)$) coincides with the quotient codimension of the stratum Σ^x of M.

Let M_i be the open subset of all points $x \in M$ such that the quotient codimension of Σ^x is at most i. Set $D_i = p^{-1}(M_i) \subset D$. Then D_i is the union of all strata in the above stratification of D that have codimension at most i. Since the geodesic flow ϕ_t is locally Lipschitz, the shadow of $D \setminus D_i$ under ϕ (i.e., the set of all directions $v \in D$, such that γ_v intersects $M \setminus M_i$) has Hausdorff dimension at most $\dim(D) - i$. Moreover, for each fixed t, the Hausdorff dimension of $\phi_t(D \setminus D_i)$ is at most $\dim(D) - i - 1$. In particular, we deduce:

Lemma 4.1. *There is a subset D' of full measure in the manifold D_0, such that for all $v \in D'$ the whole geodesic γ_v is contained in M_1.*

4.2. Horizontal geodesics in the nice part. Let x be a regular point ($x \in M_0$). Then a local quotient of $O/\mathcal{F}$ around x is a smooth

Riemannian manifold, and horizontal geodesics in O are projected to geodesics in $O/\mathcal{F}$. In particular, two such projections coincide, if they coincide initially.

Now let x be a point, such that Σ^x has quotient codimension 1. Then the infinitesimal quotient $T_xM/T_x\mathcal{F}$ is isometric to $\mathbb{R}^{q-1} \times [0, \infty)$ and (due to Theorem 1.4) the local quotient $O/\mathcal{F}$ is a Riemannian orbifold of the form $N/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts as an isometric reflection at a totally geodesic hypersurface of a smooth Riemannian manifold N. A horizontal geodesic γ in O is either completely contained in the regular part of O, or it is completely contained in the singular stratum Σ^x, or it intersects Σ^x in precisely one point. In the first two cases the image of γ is a geodesic in $O/\mathcal{F}$. In the last case the image is the concatenation of two geodesics that meet at the boundary of $O/\mathcal{F} = N/\mathbb{Z}_2$ and satisfy the reflection law. In any case, such a projection is an orbifold-geodesic in $N/\mathbb{Z}_2$. In particular, two projections of horizontal geodesics coincide if they coincide initially. Thus we have shown:

Lemma 4.2. *If M does not have any strata of quotient codimension ≥ 2, then in each local quotient $O/\mathcal{F}$ projections of horizontal geodesics coincide if they coincide initially.*

4.3. Equivalence relation. We are going to define a natural equivalence relation $\mathcal{R}$ on D that identifies two directions $v, w \in D$ if the corresponding geodesics have equal images in $M/\mathcal{F}$.

To be more precise, let first $\mathcal{L}$ be a leaf of $\mathcal{F}$. There is a small (not necessarily tubular) neighborhood U of the zero section of the normal bundle $N(\mathcal{L})$ of $\mathcal{L}$ with the following properties (see Subsection 2.1). The exponential map restricted to U is a local diffeomorphism. The set U is pointwise star-shaped, i.e., it is invariant under the maps $h_\lambda(v) = \lambda v$, $0 < \lambda \leq 1$. Finally, the pull-back $\exp^*(\mathcal{F})$ is invariant under all h_λ. Thus there is a unique singular foliation on $N(\mathcal{L})$ invariant under all h_λ, $0 < \lambda < \infty$, that coincides with $\exp^*(\mathcal{F})$ on U.

We will call two vectors $v, w \in D$ equivalent if v and w are normal vectors to the same leaf $\mathcal{L}$ of $\mathcal{F}$, and if they are in the same leaf of the singular foliation on the normal bundle $N(\mathcal{L})$ described above. Equivalently, v and w are in the same equivalence class if and only if there is a smooth (or piecewise smooth) curve η connecting v and w in D and a small positive number ϵ, such that the leaf through $\gamma_{\eta(s)}(t)$ does not depend on s, for all $0 \leq t < \epsilon$. The last condition just means, that for all $0 \leq t < \epsilon$ the curve $\eta_t(s) = \phi_t(\eta(s))$ is contained in the normal bundle to some leaf $\mathcal{L}_t$ of $\mathcal{F}$.

We will denote the equivalence relation by $\mathcal{R}$. By $\mathcal{R}(v)$ we will denote the equivalence class of v. Note that the restriction of $\mathcal{R}$ to the manifold D_0 is given by leaves of a smooth foliation.

We are going to prove the invariance of $\mathcal{R}$ under the geodesic flow (cf. [**AT08**] for an alternative proof and [**Lyt01**], [**Bol07**], and [**Now08**] for some special cases).

Proposition 4.3. *Let $\eta : [a,b] \to D$ be a curve in an equivalence class $\mathcal{R}(v)$. If $\phi_t(\eta)$ is defined for some $t > 0$, then $\phi_t(\eta)$ is contained in an equivalence class of $\mathcal{R}$. Moreover, $\mathcal{R}$ is invariant under the reversion $-Id : D \to D$, given by $-Id(v) = -v$.*

Proof. The equivalence classes of $\mathcal{R}$ are smooth injectively immersed submanifolds of the unit tangent bundle of M. Denote by $\tilde{\mathcal{R}}(v)$ the tangent space to the vector $v \in D$ of its equivalence class $\mathcal{R}(v)$. The claim can now be restated as follows: The local flow ϕ_t and the reversion $-\mathrm{Id}$ leave the "distribution" $\tilde{\mathcal{R}}$ invariant, i.e., for all $v \in D$ we have $\tilde{\mathcal{R}}(-v) = (-\mathrm{Id})_*(\tilde{\mathcal{R}}(v))$ and $(\phi_t)_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(\phi_t(v))$, for all t such that $\phi_t(v)$ is defined.

Given an open subset V of M, the restriction of D, $\tilde{\mathcal{R}}$, and the flow ϕ to the unit tangent bundle of V coincides with the corresponding objects for the restriction of $\mathcal{F}$ to V. Thus our claim is local on M.

Due to Lemma 4.2, the claim is true if in M there are no strata of quotient codimension ≥ 2. Thus for all v, such that the geodesic $\gamma_v : [0,t] \to M$ is contained in M_1, we have $(\phi_t)_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(\phi_t(v))$.

In particular, this is true for all $v \in D' \subset D_0$ from Lemma 4.1 and all t, such that $\phi_t(v)$ is defined. Notice that $\tilde{\mathcal{R}}$ is a smooth foliation on D_0, ϕ is smooth, and D' is dense in D_0. Therefore, for all $v \in D_0$ and all t with $\phi_t(v) \in D_0$, we must have $(\phi_t)_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(\phi_t(v))$.

Remark 4.1. Continuity arguments could be used to finish the proof at this point if all leaves were assumed to be closed.

Let $x \in M$ be a point and let the plaque $P \subset L(x)$, the number $\epsilon > 0$, and a distinguished tubular neighborhood O at x be chosen as in Subsection 2.1. For a unit normal vector v to the plaque P, we get from the definition of O and $\mathcal{R}$ that $(\phi_t)_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(\phi_t(v))$ and $(\phi_t)_*(\tilde{\mathcal{R}}(-v)) = \tilde{\mathcal{R}}(\phi_t(-v))$, for all $0 \leq t < \epsilon$. Moreover, $(\phi_t)_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(\phi_t(v))$ for all $-\epsilon < t < 0$ if and only if $(-\mathrm{Id})_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(-v)$.

As we have seen, ϕ_t leaves $\tilde{\mathcal{R}}$ on the regular part D_0 invariant, and therefore $(-\mathrm{Id})_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(-v)$, for all normal v to P, such that $\exp(tv)$ and $\exp(-tv)$ are in M_0, for some (and hence all) $0 < t < \epsilon$.

Consider the diffeomorphism $I : O \to O$ (reflection at P; in terms of Subsection 2.1 it is just h_{-1}), defined by $I(\exp(tv)) = \exp(-tv)$ for unit normal vectors v to P and $0 \leq t < \epsilon$. By definition, $(-\mathrm{Id})_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(-v)$ for a unit normal vector v to P if and only if I preserves $\mathcal{F}$ at $\exp(\frac{\epsilon}{2}v)$. By the observation above, I preserves $\mathcal{F}$ on the open dense subset $M_0 \cap O \cap I(M_0 \cap O)$ of O. But a singular Riemannian foliation is uniquely defined by its restriction to an open dense subset; see

Lemma 4.4 below. We deduce $I_*(\mathcal{F}) = \mathcal{F}$. Thus we have shown the invariance of $\tilde{\mathcal{R}}$ under the reversion $-\mathrm{Id}$.

For each vector $v \in D$, we can now take its foot point x and a distinguished tubular neighborhood of x and deduce that $(\phi_t)_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(\phi_t(v))$, for all $-\epsilon < t < \epsilon$, where $\epsilon = \epsilon(x)$ is chosen as in Subsection 2.1. Covering an arbitrary geodesic $\gamma_v : [0, t] \to M$ by finitely many distinguished tubular neighborhoods, we deduce $(\phi_t)_*(\tilde{\mathcal{R}}(v)) = \tilde{\mathcal{R}}(\phi_t(v))$.

This finishes the proof of Proposition 4.3. q.e.d.

In the proof above we used the following:

Lemma 4.4. *Let M be a manifold. For $i = 1, 2$, let g_i be a Riemannian metric on M and let $\mathcal{F}_i$ be a singular Riemannian foliation on M with respect to g_i. If $\mathcal{F}_i$ coincide on an open and dense subset U of M, then they coincide on all of M.*

Proof. Choose an arbitrary point $x \in M$. The claim is local; thus restricting to a small relatively compact neighborhood O of x, we may assume that the leaves $L_1(x)$ and $L_2(x)$ are closed. Then for each sequence $x_n \to x$ the leaves $L_i(x_n)$ (or, equivalently, their closures) converge in the Gromov-Hausdorff topology to the leaf $L_i(x)$.

Thus it is enough to prove that $L_1(x) = L_2(x)$ if x is a regular point of $\mathcal{F}_1$. Hence we may assume that $\mathcal{F}_1$ is a regular foliation. By continuity the leaves of $\mathcal{F}_2$ are contained in the leaves of $\mathcal{F}_1$ in such a case. Thus for each $\mathcal{F}_2$-regular point y we get $L_1(y) = L_2(y)$. Then the above limiting argument shows that leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ through all points coincide. q.e.d.

4.4. Natural stratification of the space of horizontal geodesics. Let v be a horizontal vector and let $\gamma = \gamma_v$ be the horizontal geodesic with $\gamma_v'(0) = v$. For t in the interval of definition of γ, we let $l(t)$ be the dimension of the leaf $L(\gamma(t))$. Due to the semi-continuity of the leaf dimension, we have $\liminf_{t_i \to t} l(t_i) \geq l(t)$.

Let t be fixed and consider a small distinguished neighborhood O of $\gamma(t)$ as in Subsection 2.1. Then $h_s(\gamma(t+\rho)) = \gamma(t+s\rho)$, for $-1 \leq s \leq 1$. For $0 < s \leq 1$, h_s preserves $\mathcal{F}$. On the other hand, in the course of the proof of Proposition 4.3, we have seen that h_{-1} preserves $\mathcal{F}$ as well. Thus $h_s : O \to O$ preserves $\mathcal{F}$ for all $s \in [-1, 1] \setminus \{0\}$. In particular, $l(t + \rho)$ does not depend on ρ for $\rho \in [-\epsilon, \epsilon] \setminus \{0\}$.

This shows that $l(t)$ is equal to the constant $d(\gamma) := \max(l(t))$ for all but discretely many t. We summarize our observations:

Lemma 4.5. *Let γ be a horizontal geodesic in M. Let $d(\gamma)$ denote the maximal dimension of the leaves $L(\gamma(t))$. Then for all but discretely many t, the leaf $L(\gamma(t))$ has dimension $d(\gamma)$.*

In the case of maximal dimension we get:

Corollary 4.6. *A compact horizontal geodesic that contains a regular point is contained in the set of regular points, with exception of at most finitely many points.*

Remark 4.2. In fact, we have shown a slightly more general statement than Lemma 4.5. Namely, for each horizontal geodesic γ and for all but discretely many times t, the infinitesimal foliation $T_{\gamma(t)}\mathcal{F}$ does not depend on t.

By definition, the function $d : D \to \mathbb{N}$, given by $d(v) = d(\gamma_v)$, is invariant under the local geodesic flow ϕ under the multiplication by -1 and under the equivalence relation $\mathcal{R}$. From the above we deduce, that $d(v)$ is the dimension of the leaf $L(\gamma_v(\epsilon))$ for small positive ϵ.

Set $D^i := d^{-1}(i) \subset D$. Then D is decomposed into the disjoint union of the sets D^i. Due to the semi-continuity of leaf dimensions, the closure of D^i is contained in the union of D^j, $j \leq i$. We claim that D^i is a smooth submanifold of the unit tangent bundle UM of M. To see this, let $v \in D^i$ be given. Then $\phi_\epsilon(v)$ is a horizontal vector of the restricted Riemannian foliation $\mathcal{F}$ on the submanifold $\Sigma^{\gamma_v(\epsilon)}$ of M. The space $\mathcal{H}$ of unit horizontal vectors of the restriction of $\mathcal{F}$ to the manifold $\Sigma^{\gamma_v(\epsilon)}$ is a smooth submanifold of the unit tangent bundle of $\Sigma^{\gamma_v(\epsilon)}$. By definition, the diffeomorphism $\phi_{-\epsilon}$ sends a neighborhood of $\phi_\epsilon(v)$ in $\mathcal{H}$ to a neighborhood of v in D^i. This shows that D^i are submanifolds of UM.

If $\mathcal{F}$ is a Riemannian foliation, then the function $d : D \to \mathbb{N}$ defined above is constant. For each vector $v \in D$, we have $d(v) = \dim(L(p(v))) = \dim(\mathcal{R}(v))$. Moreover, the equivalence classes of $\mathcal{R}$ are leaves of a foliation on the manifold D in this case. Finally, the equivalence classes of $\mathcal{R}$ are closed if $\mathcal{F}$ is a closed Riemannian foliation.

For a general singular Riemannian foliation, the observation above shows that each small open subset of D^i, for each i, can be moved by the geodesic flow to an open part of the set of horizontal vectors of a smooth Riemannian foliation (restriction of $\mathcal{F}$ to a stratum). Thus we arrive at the following:

Proposition 4.7. *For each $i \in D$, the subset D^i of all vectors $v \in V$ with dim$(\mathcal{R}(v)) = i$ is a submanifold of the unit tangent bundle, which is invariant under the local geodesic flow ϕ. The equivalence classes of $\mathcal{R}$ on D^i are leaves of a smooth foliation. This foliation has closed leaves, if the leaves of $\mathcal{F}$ are closed. For each $v \in D^i$, we have $d(\gamma_v) = i$.*

4.5. Vertical Jacobi fields. Let v be a horizontal vector and $\gamma = \gamma_v$ the geodesic in the direction v. Consider the leaf $\mathcal{R}(v)$ and a small neighborhood V of v in $\mathcal{R}(v)$. For each $\bar{v}$ in V consider the horizontal geodesic $\gamma_{\bar{v}}$. Due to Proposition 4.3, for each compact interval I of definition of γ, we may choose V so small that for all $t \in I$ and all $\bar{v} \in V$ we have $\gamma_{\bar{v}}(t) \in L(\gamma(t))$.

The space W^γ of variational fields through geodesics $\gamma_{\bar v}$ is a vector space of Jacobi fields along γ of dimension $\dim(\mathcal{R}(v)) = d(v)$. Due to Proposition 4.3, W^γ does not depend on the starting point of γ. Moreover, we have $W^\gamma(t) := \{J(t)|J \in W^\gamma\} = T_{\gamma(t)}L(\gamma(t))$, for all t in the interval of definition of γ.

4.6. Invariance of the Liouville measure. Now let M be complete and let $\mathcal{F}$ be closed. Consider the space of horizontal vectors D and its decomposition $D = \cup D^i$ discussed in Subsection 4.4. We have $v \in D^i$ if and only if $\dim L(\gamma_v(t)) = i$, for all but discretely many times t. The relation $v \in \mathcal{R}(w)$ is equivalent to $L(\gamma_v(t)) = L(\gamma_w(t))$, for all $t \in \mathbb{R}$.

Denote by G the space of equivalence classes $D/\mathcal{R}$. We consider G with the induced quotient topology (which is Hausdorff and locally compact in our case). The decomposition of D induces a decomposition of G as $G = \cup G^i$. The flow ϕ descends to a flow on G. Denote by D^+ and G^+ the maximal stratum of D and its projection to G, i.e., the set of all horizontal geodesics that contain at least one regular point.

The subspace $D_0 \subset D^+$ of all horizontal vectors with regular starting point is of full measure in the manifold D^+ (since the complement of D_0 is a countable union of submanifolds of positive codimension). The subspace $D' \subset D_0$ defined in Lemma 4.1 is of full measure in D^+, invariant under ϕ, and saturated under $\mathcal{R}$.

For each $v \in D'$, the singular Riemannian foliation $\mathcal{F}$ is infinitesimally polar at all points $\gamma_v(t)$. Moreover, due to Subsection 4.2, γ_v projects to an orbifold geodesic in each local quotient $O/\mathcal{F}$. Thus the image of γ_v in $M/\mathcal{F}$ is contained in the set B of orbifold points of $M/\mathcal{F}$, and this image is an orbifold geodesic in the orbifold B.

We set $G' = D'/\mathcal{R}$. Identify G' with a subset of the unit tangent bundle UB of the orbifold B. The argument above shows that on G' the flow ϕ coincides with the orbifold-geodesic flow of the orbifold B.

Define the measure μ on G by setting $\mu(G \setminus UB) = 0$ and by letting μ be the (usual) Liouville measure on the unit tangent bundle UB. Thus we deduce that ϕ preserves this Liouville measure μ. This proves Theorem 1.6.

Note that by construction the Liouville measure μ is positive on non-empty open subsets of G and that the total mass of μ is proportional to the volume of B. The last one is finite if $M/\mathcal{F}$ is compact [**BGP92**].

5. Conjugate points

5.1. Jacobi equation and Jacobi fields. We recall here some basic facts about the Jacobi equation, Jacobi fields, and focal points. We refer to [**Lyt09**] for extended explanations.

Let M be a Riemannian manifold, let $\gamma : I = [a, b] \to M$ be a geodesic, and let $\mathcal{N}$ be the normal bundle of γ. Let Jac denote the space of all normal Jacobi fields along γ, i.e., solutions of the equation

$J'' + R(J) = 0$, where R denotes the curvature endomorphism. By ω we denote the canonical symplectic form on Jac, defined by $\omega(J_1, J_2) = \langle J_1', J_2\rangle + \langle J_1, J_2'\rangle$. For subspaces W of Jac, we denote by $W^\perp$ the orthogonal complement with respect to ω. A subspace W of Jac is called *isotropic* if $W \subset W^\perp$, and it is called *Lagrangian* if $W = W^\perp$. For an isotropic subspace W and $t \in I$, we define the W*-focal index* of t to be $f^W(t) = \dim(W) - \dim(W(t))$, where $W(t) = \{J(t) | J \in W\}$. The set of points with non-zero focal index is discrete [**Lyt09**] and such points are called W*-focal.* The W*-index of* γ is defined by $\mathrm{ind}_W(\gamma) = \Sigma_{t\in I}(f^W(t))$. We have the following semi-continuity property [**Lyt09**]:

Lemma 5.1. *Let g_n be a sequence of Riemannian metrics that smoothly converges to g. Let $\gamma_n : [a_n, b_n] \to M$ be a sequence of g_n-geodesics converging to γ. Let $W_n \subset Jac(\gamma_n)$ be isotropic subspaces of normal Jacobi fields along γ_n that converge to an isotropic subspace $W \subset Jac(\gamma)$. If $f^{W_n}(a_n) = f^W(a)$ and $f^{W_n}(b_n) = f^W(b)$, then $ind_{W_n}(\gamma_n) \leq ind_{W_n}(\gamma)$ for all n large enough. If, in addition, W_n are Lagrangians, then $ind_{W_n}(\gamma_n) = ind_{W_n}(\gamma)$ for all n large enough.*

The following example is the main source of Lagrangians.

Example 5.1. If N is a submanifold of M through $\gamma(a)$ orthogonal to γ, then the space Λ^N of normal N-Jacobi fields is a Lagrangian. In this case the Λ^N-focal index of a is equal to $\dim(M)-1-\dim(N)$ and a point $t \neq a$ is Λ^N-focal if $\gamma(t)$ is a focal point of N along γ in the usual sense of Riemannian geometry. In particular, the space $\Lambda = \Lambda^{L(\gamma(a))}$ of all $\mathcal{F}$-Jacobi fields along a horizontal geodesic γ of a singular Riemannian foliation $\mathcal{F}$ is a Lagrangian. Thus the space W of all vertical $\mathcal{F}$-Jacobi fields is isotropic.

We recall now what we are going to use from Wilkings construction [**Wil07**] of a transversal Jacobi equation. Let W be an isotropic space of Jacobi fields along γ. Then there is a smooth Riemannian vector bundle $\mathcal{H}$ with a Riemannian connection $'$ and with a Riemannian projection $P : \mathcal{N} \to \mathcal{H}$, such that $\mathcal{H}(t) = \mathcal{N}(t)/W(t)$ for all t, that are not W-focal. There is a smooth symmetric operator $R^{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ such that solutions of the Jacobi equation $Y'' + R^{\mathcal{H}}(Y) = 0$ are precisely the projections (by the map P) of Jacobi fields $J \in W^\perp \subset \mathrm{Jac}(\mathcal{N})$ to $\mathcal{H}$. Lagrangians in $\mathrm{Jac}(\mathcal{H})$ are precisely the projections of Lagrangians in $\mathrm{Jac}(\mathcal{N})$ that contain W. Moreover (cf. [**Lyt09**]):

Lemma 5.2. *For each Lagrangian $\Lambda \subset Jac(\mathcal{N})$ that contains W, we have $ind_W(\gamma) + ind_{\Lambda/W}(\gamma) = ind_\Lambda(\gamma)$.*

Example 5.2. In the special case, where γ is a horizontal geodesic with respect to a Riemannian submersion $f : M \to B$, let W be the space of f-vertical Jacobi fields, i.e., variational fields through variations of horizontal lifts of $f(\gamma)$. Then $W(t)$ for each t is the vertical space of the submersion through $\gamma(t)$, $\mathcal{H}$ is canonically identified with the normal bundle of the projected geodesic $\bar{\gamma} = f(\gamma)$ in B, and the transversal operator $R^{\mathcal{H}}$ coincides with the curvature endomorphism in the base space B.

5.2. Vertical Jacobi fields. Let M be a Riemannian manifold and let $\mathcal{F}$ be a singular Riemannian foliation on M. Let $\gamma : [a,b] \to M$ be a horizontal geodesic. Then the space $\Lambda = \Lambda^{L(\gamma(a))}$ of all normal $\mathcal{F}$-Jacobi fields along γ is a Lagrangian space of Jacobi fields. Note that the space Λ depends not only on the maximal geodesic containing γ but also on the starting point $\gamma(a)$.

In the introduction, the space of $\mathcal{F}$-vertical Jacobi fields was defined as the space of all Jacobi fields $J \in \Lambda$ with $J(t) \in T_{\gamma(t)}L(\gamma(t))$ for all $t \in [a,b]$. Recall now that in Subsection 4.5 we have defined a space W^γ of Jacobi fields along γ that are defined (independently of the starting point) as variational fields through horizontal geodesics γ_s with $\gamma_s(t) \in L(\gamma(t))$, for all t. We have seen that $W^\gamma(t) := \{J(t) | J \in W^\gamma\}$ coincides with $T_{\gamma(t)}L(\gamma(t))$, for all t. By definition $W^\gamma \subset \Lambda$. Therefore, W^γ is precisely the space of all $\mathcal{F}$-vertical Jacobi fields along γ. In particular, the latter does not depend on the starting point, in contrast to Λ.

The number $d(\gamma)$, defined in Subsection 4.4, is the maximal dimension of $L(\gamma(t))$. We get $d(\gamma) = \dim W^\gamma$. The W-focal points along γ are precisely the points t_i with $\dim L(\gamma(t_i)) < d(\gamma)$ and the W-focal index of such points is $d(\gamma) - \dim L(\gamma(t_i))$. In particular, for a regular horizontal geodesic γ, its crossing number $c(\gamma)$ defined in the introduction coincides with the *vertical index* $\mathrm{ind}_W(\gamma)$.

Remark 5.3. In the above terminology it is possible to describe the space $W^\perp$ geometrically. Namely, it is possible to see that $W^\perp$ consists of normal *horizontal* Jacobi fields, where we call a Jacobi field $\mathcal{F}$-horizontal if it is the variational field of a variation of γ through horizontal geodesics. This observation together with Lemma 5.3 below proves the equivalence between two a priori slightly different definitions of variational completeness used in [**Bot56**] and [**BS58**]. In our terminology this equivalence reads as follows. The singular Riemannian foliation $\mathcal{F}$ does not have horizontal conjugate points if and only if any $\mathcal{F}$-horizontal Jacobi field along any horizontal geodesic that is tangent to the leaves at two points is tangent to the leaves at all points.

5.3. Horizontal conjugate points. Let γ be a curve and let T be a Riemannian bundle over γ with a Riemannian connection and a symmetric field of endomorphisms $R : T \to T$ and the corresponding symplectic vector space $\mathrm{Jac}(T)$ of Jacobi fields. (In this paper we are only interested in the cases $T = \mathcal{N}$ and $T = \mathcal{H}$). Points $c < d$ in the interval of definition I of γ are called *conjugate* if there is a non-zero Jacobi field

$J \in \mathrm{Jac}$ with $J(c) = J(d) = 0$. Note that if $c < d$ are conjugate, then for each $\bar{c} \leq c$ there is some $\bar{d} \in [c, d]$ that is conjugate to $\bar{c}$ [**Lyt09**].

Now let $M, \mathcal{F}, g$ be as always and let $\gamma : [a, b] \to M$ be a horizontal geodesic. Let $\Lambda = \Lambda^{L(\gamma(a))}$ and $W = W^{\gamma}$ be defined as above. Let $\mathcal{H}$ be the W-transversal bundle as defined in Subsection 5.1. The following result was independently obtained in [**Now08**]:

Lemma 5.3. *There are no horizontal conjugate points along γ if and only if $ind_{\Lambda}(\gamma_0) = ind_W(\gamma_0)$, where γ_0 denotes the subgeodesic $\gamma_0 : (a, b) \to M$ of γ. This condition is equivalent to the statement that the point a does not have conjugate points for the transversal Jacobi equation on $\mathcal{H}$.*

Proof. Assume that $\mathrm{ind}_{\Lambda}(\gamma) = \mathrm{ind}_W(\gamma)$ and let some $J \in \Lambda$ and $a < t_0 < b$ with $J(t_0) \in T_{\gamma(t_0)} L(\gamma(t_0))$ be given. We find some $\tilde{J} \in W$, with $\tilde{J}(t_0) = J(t_0)$. From $\mathrm{ind}_{\Lambda}(\gamma) = \mathrm{ind}_W(\gamma)$, we deduce that $\tilde{J} - J \in W$ and therefore $J \in W$. The other implication is a direct consequence of the definition.

To see the equivalence of $\mathrm{ind}_W(\gamma_0) = \mathrm{ind}_{\Lambda}(\gamma_0)$ to the absence of conjugate points in the quotient bundle $\mathcal{H}$, we use Lemma 5.2 to see that $\mathrm{ind}_W(\gamma_0) = \mathrm{ind}_{\Lambda}(\gamma_0)$ is equivalent to the absence of focal points of Λ/W on the open interval (a, b). But Λ/W is by definition the Lagrangian $\tilde{\Lambda}^a$ in $\mathrm{Jac}(\mathcal{H})$ of all Jacobi fields Y with $Y(a) = 0$. Thus the statement that $\mathrm{ind}_{\Lambda/W}(\gamma_0) = 0$ is equivalent to the fact that a does not have conjugate points with respect to the transversal Jacobi equation. q.e.d.

Note that if the point $\gamma(a)$ is regular, then the condition $\mathrm{ind}_{\Lambda}(\gamma_0) = \mathrm{ind}_W(\gamma_0)$ is equivalent to $\mathrm{ind}_{\Lambda}(\gamma) = \mathrm{ind}_W(\gamma) + \mathrm{codim}(\mathcal{F}, M) - 1$.

5.4. Horizontal conjugate points in the infinitesimally polar case. Let $(M, g, \mathcal{F})$ be as above and assume that $\mathcal{F}$ is infinitesimally polar. Let γ be a horizontal geodesic in M. Then one can cover γ by small distinguished neighborhoods O_i. The restriction of $\mathcal{F}$ to each O_i has a Riemannian orbifold $O_i/\mathcal{F}$ as quotient and $\gamma \cap O_i$ is projected to an orbifold geodesic $\bar{\gamma}$ in this quotient. We get a well-defined development along the projection $\bar{\gamma}$ of γ. We can consider it to be a smooth Riemannian manifold B containing our geodesic γ that we will denote by $\bar{\gamma}$, if we consider it as part of B.

Remark 5.4. If $\mathcal{F}$ is infinitesimally polar and closed and if M is complete, then γ is projected to an orbifold-geodesic $\bar{\gamma}$ in the Riemannian orbifold $M/\mathcal{F}$ and the manifold B considered above is the local development of $M/\mathcal{F}$ along $\bar{\gamma}$.

For regular geodesics, the following lemma is a direct consequence of the basic Example 5.2.

Lemma 5.4. *There are no horizontal conjugate points along γ if and only if there are no conjugate points along $\bar{\gamma}$ in the local development B.*

Proof. Let $\gamma : [0, a] \to M$ be given. First let us assume that b is horizontally conjugate to 0 along γ. Then there is a normal $L(\gamma(0))$-Jacobi field $J \in \Lambda \setminus W$ with $J(b) = 0$. Adding some element of the vertical space W to J, we get another $L(\gamma(0))$-Jacobi field J_1 with $J_1(0) = 0$ and $J_1(b) \in T_{\gamma(b)} L(\gamma(b))$. Then J_1 is the variational field of a variation γ_s through horizontal geodesics with $\gamma_0 = \gamma$ and $\gamma_s(0) = 0$, for all s. Then in each local quotient γ_s is projected to a variation through orbifold geodesics. Thus we obtain a lift $\bar{\gamma}_s$ to B that gives us a variation of $\bar{\gamma}$ through geodesics in B with $\bar{\gamma}_s(0) = 0$. Moreover, the assumption $J_1(b) = T_{\gamma(b)} L(\gamma(b))$ implies for the variational field $Y := \frac{d}{ds}\bar{\gamma}_s$ that $Y(b) = 0$. If b is non-conjugate to a along $\bar{\gamma}$, then Y is constant 0. But this is equivalent to $J \in W$.

On the other hand, let us assume that there is some Jacobi field Y along $\bar{\gamma}$ with $Y(0) = 0$ and $Y(b) = 0$. Find a variation of geodesics $\bar{\gamma}_s$ corresponding to this Y, with $\bar{\gamma}_s(0) = \gamma(0)$. Since $T_{\gamma(0)}\mathcal{F}$ is polar, we may choose a horizontal section Z of $T_{\gamma(0)}\mathcal{F}$. Now we find a unique smooth lift η of the curve $\bar{\eta}(s) := \bar{\gamma}'_s(0)$ to Z with $\eta(0) = \gamma'(0)$. Then $\gamma_s(t) = \exp(t\eta(s))$ is a variation of γ through horizontal geodesics, whose Jacobi field J satisfies $J(0) = 0$ and $J(b) \in T_{\gamma(b)} L(\gamma(b))$. Moreover, J is not in W, since $Y \neq 0$. q.e.d.

Remark 5.5. Since regular horizontal geodesics are dense in the space of all geodesics and since the absence of conjugate points is an open condition for Riemannian manifolds, we deduce from Lemma 5.4 and the proof of Proposition 5.6 below that a singular Riemannian foliation $\mathcal{F}$ does not have horizontal conjugate points if and only if any *regular* horizontal geodesic does not have horizontal conjugate points.

5.5. Stability of absence of conjugate points. Let $(M, g, \mathcal{F})$ be as above. Let $\gamma : [a, b] \to M$ be a regular horizontal geodesic. Let Λ and W be defined as in Subsection 5.2. We assume that b is not Λ-focal. (The last condition can be achieved by slightly increasing b).

Now let g_n be a sequence of Riemannian metrics on M that smoothly converge to g and that are adapted to the singular Riemannian foliation $\mathcal{F}$. Let $\gamma_n : [a, b] \to M$ be a sequence of g_n-horizontal geodesics that converge to γ. Let Λ_n and W_n be the spaces of $\mathcal{F}$-Jacobi fields and $\mathcal{F}$-vertical Jacobi fields along γ_n (with respect to the metric g_n).

Since we are in the regular part of $\mathcal{F}$, the spaces Λ_n converge to Λ and W_n converge to W. Moreover, $f^{W_n} = f^W(a) = f^{W_n}(b) = f^W(b) = 0$ and $f^{\Lambda_n}(a) = f^\Lambda(a) = \dim(M) - 1 - \dim(\mathcal{F})$ and $f^{\Lambda_n}(b) = f^\Lambda(b) = 0$. From Lemma 5.1 we get $\mathrm{ind}_\Lambda(\gamma) - \mathrm{ind}_W(\gamma) \leq \mathrm{ind}_{\Lambda_n}(\gamma_n) - \mathrm{ind}_{W_n}(\gamma_n)$, for all n large enough. Using Lemma 5.3 we derive:

Lemma 5.5. *In the above situation, assume that γ_n has no horizontal conjugate points. Then γ has no horizontal conjugate points as well.*

5.6. Conclusions. Now we can finish the proofs of all results announced in the introduction. We start with another characterization of infinitesimal polarity.

Proposition 5.6. *Let $\mathcal{F}$ be a singular Riemannian foliation on a Riemannian manifold M and let x be a point in M. Then $\mathcal{F}$ is infinitesimally polar at x if and only if there is a neighborhood U of x, such that the horizontal geodesics contained in U do not have horizontal conjugate points.*

Proof. If $\mathcal{F}$ is infinitesimally polar at x, then we can find a small distinguished tubular neighborhood O of x such that $O/\mathcal{F}$ is a smooth Riemannian orbifold with bounded curvature. Then there is some $\epsilon > 0$ such that each orbifold-geodesic of length $\leq \epsilon$ in $O/\mathcal{F}$ has no conjugate points. Then taking $U \subset O$ to be an open ball of radius ϵ around x, we deduce from Lemma 5.4 that no horizontal geodesic in U has horizontal conjugate points.

On the other hand, let us assume that $T_x\mathcal{F}$ is not polar. Then there is at least one regular horizontal geodesic γ in T_xM with horizontal conjugate points (see [**LT07**]). Using the convergence of the rescaled metrics on a small tubular neighborhood O of x to the flat metric on T_xM from Subsection 2.1, we deduce from Lemma 5.5 that arbitrary small neighborhoods of x contain regular geodesics γ_n with horizontal conjugate points. q.e.d.

Now we can finish the proof of Theorem 1.7. If there are no horizontal conjugate points, then $\mathcal{F}$ is infinitesimally polar by Proposition 5.6. If, in addition, $\mathcal{F}$ is closed, then $B = M/\mathcal{F}$ is a complete Riemannian orbifold (Subsection 2.4). From Lemma 5.4 we deduce that this Riemannian orbifold does not have conjugate points if and only if in M there are no horizontal conjugate points. Now the result follows from Lemma 2.1.

5.7. Continuity of the crossing counting function. Now we are going to prove Theorem 1.5.

Proof. Recall that the crossing number $c(\gamma)$ of a regular geodesic is equal to its vertical index $\mathrm{ind}_W(\gamma)$. The claim of Theorem 1.5 is local, i.e., c is continuous if and only if each point $x \in M$ has a neighborhood U such that c is continuous for the restricted singular Riemannian foliation $(U, \mathcal{F})$.

If $\mathcal{F}$ is infinitesimally polar at x, we may choose U as in Proposition 5.6. Then for each regular geodesic γ in U we get $\mathrm{ind}_W(\gamma) = \mathrm{ind}_\Lambda(\gamma) - (\mathrm{codim}(\mathcal{F}, M) - 1)$. The result now follows from the continuity of indices for Lagrangians, Lemma 5.1.

Now let us assume that c is continuous and that $T_x\mathcal{F}$ is not polar. The proof of Corollary 5.6 shows that there are regular geodesics $\gamma_n : [0, \epsilon_n] \to M$, with $\epsilon_n \to 0$ and $\gamma_n(0) \to x$, that have horizontal conjugate points and such that the starting directions v_n of γ_n converge to a regular direction v in T_xM. Let $\gamma = \gamma_v$ be the geodesic in M in the direction v. For sufficiently small $\epsilon = \epsilon(\gamma)$, the geodesic $\gamma : [-\epsilon, \epsilon] \to M$ has no horizontal conjugate points. But the extended geodesics $\gamma_n : [-\epsilon, \epsilon] \to M$ still have horizontal conjugate points. Thus, for n large enough, $c(\gamma_n) = \mathrm{ind}_{W_n}(\gamma_n) < \mathrm{ind}_{\Lambda_n}(\gamma_n) - (\mathrm{codim}(\mathcal{F}, M) - 1)$ and $c(\gamma)) = \mathrm{ind}_W(\gamma) = \mathrm{ind}_\Lambda(\gamma) - (\mathrm{codim}(\mathcal{F}, M) - 1)$. Since γ_n converges to γ, this contradicts the continuity of c. q.e.d.

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Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn, Germany
E-mail address: lytchak@math.uni-bonn.de

Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln, Germany
E-mail address: gthorbergsson@mi.uni-koeln.de