Cyclic projections in Hadamard spaces

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Abstract

We prove that iterating projections onto convex subsets of Hadamard spaces can behave in a more complicated way than in Hilbert spaces, resolving a problem formulated by Miroslav Bačák.

1 Introduction

Let $C_1, \dots, C_k$ be closed convex subsets in a Hadamard space X. Denote by P_i the closest-point projection $X \to C_i$; it sends a point $x \in X$ to the (necessarily unique) point $P(x)$ in C_i that minimize the distance to x. Given a point $x \in X$ consider the sequence $x_n = P^n(x)$, where $P = P_1 \circ \dots \circ P_k$. Various properties of such compositions P and arising sequences x_n are classical topics of research, especially in Hilbert spaces, originated from convex optimization, see [5, 6, 8, 9, 12] and the bibliography therein.

If X is a Hilbert space, then the fundamental result of Heinz Bauschke [8, 13], states that the map P is *asymptotically regular*; that is, for any x, we have $|x_n - x_{n+1}| \to 0$ as $n \to \infty$. In this note we show that this statement does not hold in general Hadamard spaces, despite the fact that many related statements admit generalizations and, thus, depend only on the convexity properties of the distance functions and not on the linear structure. That answers the question of Miroslav Bačák [5, Problem 6.13].

We will denote by $|x - y|$ the distance between points x and y in any metric space, even without linear structure.

1.1. Theorem. *There exists a Hadamard space X and closed convex subsets $C_1, \dots, C_k$ in X such that the composition of the closest-point projections $P =$ $= P_1 \circ \dots \circ P_k$ is not asymptotically regular.*

If the sets $C_1, \dots, C_k$ have a common intersection, then such examples are impossible [4–6].

We provide two explicit examples proving the theorem for $k = 3$. Setting $C_3 = \dots = C_k$ defines examples for any $k \geqslant 3$.

In the first example (Section 2), the constructed space is a product of two tripods; it contains three convex flat quadrangles Q_1, Q_2, and Q_3 with pairs of opposite sides (C_1, C_2), (C_2, C_3), and (C_3, C_1) such that the composition of projections swaps the ends of C_1.

In the second example (Section 3), the convex subsets C_1, C_2, C_3 are isometric to the unit disc, and the composition $P \colon C_1 \to C_1$ rotates the disc by an arbitrary angle. If the angle is chosen irrational, then no power of P is asymptotically regular.

For $k = 2$, the result of Heinz Bauschke [8] admits a generalization:

1.2. Proposition. *Let C_1, C_2 be two closed convex subsets of a Hadamard space X. Then the composition $P = P_1 \circ P_2$ is asymptotically regular.*

Moreover, $|x_n - x_{n+1}| = o(\frac{1}{\sqrt{n}})$ for any $x \in X$ and $x_n = P^n(x)$.

Examples given by the real axes $C_1 \subset \mathbb{R}^2$ and the set

$$C_2 = \{\, (x, y) : x > 0, y \geqslant 1 + x^{-\varepsilon} \,\}$$

reveal that the convergence rate cannot be improved to $O(n^{-\frac{1}{2}-\varepsilon})$ for any $\varepsilon > 0$.

Further, we assume familiarity with the geometry of Hadamard spaces [2, 3, 7, 10, 11].

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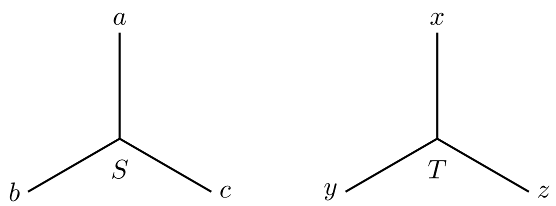
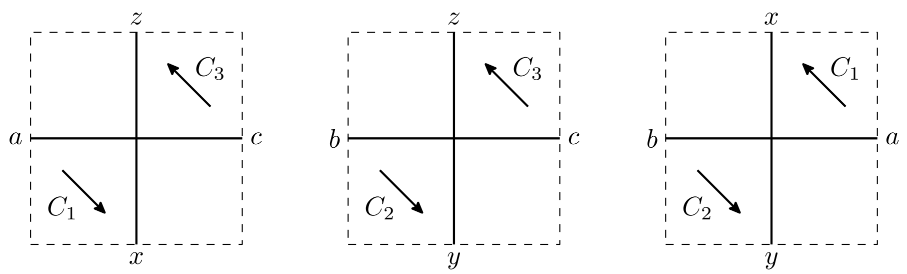
2 Product of tripods

A union of three unit segments that share one endpoint with the induced length metric will be called a *tripod*. Consider two tripods S and T. Denote by a, b, c and x, y, z the sides of S and T respectively.



By the Reshetnyak gluing theorem, S and T are CAT(0). Therefore, so is the product space $X = S \times T$. The following diagram shows 3 isometric copies of 2×2-square in X; they are obtained as the products of two pairs of sides in S and T as labeled:



Consider the segment C_1, C_2, and C_3 shown on the diagram; they all have slope -1 and project to each other isometrically. Note that each projection P_i reverses the shown orientation. It follows that the cyclic projection $P\colon C_1 \to C_1$ swaps the ends of C_1. In particular, P is *not* asymptotically regular.

3 Three discs

Fix an angle α and a small $\varepsilon > 0$. Consider the closed ε-neighborhood A of a closed geodesic in the unit sphere $\mathbb{S}^3$. By the result of Stephanie Alexander, David Berg, and Richard Bishop [1], the space A is locally CAT(1). The universal cover $\tilde{A}$ of A with its induced metric is locally CAT(1) as well.

Denote by E the preimage of the geodesic in $\tilde{A}$. The isometry group of $\tilde{A}$ contains the group of translations along E and the rotations that fix E. Let T be the composition of translation along E of length $2\pi + 10{\cdot}\varepsilon$ and the rotation by angle α. The element T generates a discrete subgroup Γ in the group of isometries of $\tilde{A}$ that acts freely.

Set $Y = \tilde{A}/\Gamma$. Since ε is small, every point of $\tilde{A}$ is moved by any element of Γ more than $2{\cdot}\pi$. Therefore, Y is a compact locally CAT(1) space that does not contain closed geodesics of length less than 2π. Hence, by the generalized Hadamard–Cartan theorem [2], Y is CAT(1). By construction, Y is locally isometric to $\mathbb{S}^3$ outside its boundary B. The projection of E to Y is a closed geodesic G of length $2\pi + 10{\cdot}\varepsilon$.

Denote by X the Euclidean cone over Y; since Y is CAT(1), we get that X is CAT(0); see [2]. Moreover, X is locally Euclidean outside its *boundary* — the cone over B.

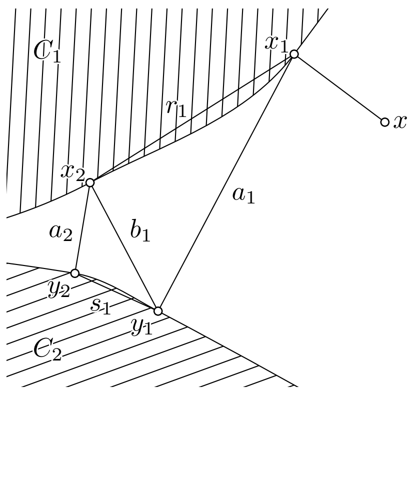
The cone Z over the closed geodesic G is the Euclidean cone over a circle of length $2\pi + 10{\cdot}\varepsilon$. By construction, Z is locally convex, hence a convex subset of X [3, 2.2.12]. Let us consider a geodesic triangle $[q_1q_2q_3]$ in Z that surrounds the origin o of the cone Z.

By construction, the sides of triangle $[q_1q_2q_3]$ lie in the flat part of X. Thus, we can find a small $r > 0$ such that the $2{\cdot}r$-neighborhood U_1 of the geodesic $[q_1q_2]$ isometric to a convex subset of the Euclidean space. We can assume that $2{\cdot}r$-neighborhoods U_2 of $[q_2q_3]$ and U_3 of $[q_3q_1]$ have the same property.

Denote by C_i the disc of radius r centered at q_i and being orthogonal to Z. By construction, C_i and C_{i+1}, for $i = 1, 2, 3 \pmod 3$ are contained in U_i. Since Z is convex, C_i and C_{i+1} are *parallel* inside U_i, thus their convex hull Q_i is isometric to a right cylinder with bases C_i and C_{i+1}. In particular, the projection P_i defines an isometry $C_{i+1} \to C_i$.

By construction, the composition $P = P_1 \circ P_2 \circ P_3 \colon C_1 \to C_1$ rotates C_1 by angle α. If $\frac{\alpha}{\pi}$ is irrational, then P, as well as all its powers, are *not* asymptotically regular.

4 Two sets



Proof of 1.2. By definition, $x_n \in C_1$ for all n. Set $y_n = P_2(x_n)$, so y_n is the closest-point projection of x_n to C_2. Further set

$$r_n := |x_n - x_{n+1}|,$$
$$s_n := |y_n - y_{n+1}|.$$

Since the closest-point projection is nonexpanding, we get

$$r_1 \geqslant s_1 \geqslant r_2 \geqslant s_2 \geqslant \dots \qquad ❶$$

Set

$$a_n := |x_n - y_n| = \operatorname{dist}_{C_2} x_n,$$
$$b_n := |y_n - x_{n+1}| = \operatorname{dist}_{C_1} y_n.$$

Note that

$$a_1 \geqslant b_1 \geqslant a_2 \geqslant b_2 \geqslant \dots \qquad ❷$$

Since C_1 is convex and $x_{n+1} \in C_1$ lies at the minimal distance from y_n, we have $\measuredangle[x_{n+1}\,{}^{x_n}_{y_n}] \geqslant \frac{\pi}{2}$. By CAT(0) comparison,

$$r_n^2 \leqslant a_n^2 - b_n^2.$$

Therefore, ❷ implies that

$$\sum_n r_n^2 \leqslant a_1^2.$$

By ❶, r_n is non-increasing. Therefore, $r_n = o(\frac{1}{\sqrt{n}})$. □

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