A DIAMETER GAP FOR QUOTIENTS OF THE UNIT SPHERE

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ABSTRACT. We prove that for any isometric action of a group on a unit sphere of dimension larger than one, the quotient space has diameter zero or larger than a universal dimension-independent positive constant.

1. INTRODUCTION

1.1. **Main result.** We prove the following gap theorem, answering a question going back to Karsten Grove and investigated in [McG93] and [Gre00].

Main Theorem. *There exists some $\epsilon > 0$ such that for any $n \geq 2$ and any group G acting by isometries on the unit n-dimensional sphere $\mathbb{S}^n$ the quotient space $\mathbb{S}^n/G$ either has diameter 0 or at least ϵ.*

The diameter of $\mathbb{S}^n/G$ is 0 if and only if any orbit of G is dense in $\mathbb{S}^n$, thus if the closure $\bar{G}$ of G in $\mathrm{O}(n)$ acts transitively on $\mathbb{S}^n$.

The case $n = 1$ needs to be excluded, since quotients of $\mathbb{S}^1$ by the action of cyclic groups can have arbitrary small diameter.

The existence of a dimension-dependent bound has been proved in [Gre00]. There, earlier in [McG93] and later in [MS05] and [DGMS09] explicit lower bounds on the diameter have been found for some special classes of actions. Most notably, a lower bound $\alpha > \frac{1}{4}$ has been verified by explicit calculation for free actions, for finite Coxeter groups and for actions with quotients of dimension ≤ 2. Recently in an independent preprint the existence of such a lower bound was claimed for *unitary* actions of *connected* Lie groups [GR18]. However, according to the authors of [GR18], their preprint contains a gap. We also refer the reader to [GR18] for the relevance of this problem to Control Theory.

Unlike the existence of a dimension-dependent constant $\epsilon(n)$ from [Gre00], our dimension-independent bound cannot be derived by a limiting argument. A related fact is that no such lower bound exists for isometric actions on the unit sphere in an infinite-dimensional Hilbert space [Wea]. While we have not tried to determine our constant explicitly, the proof indeed provides some explicit bound on ϵ in Theorem A. In a future work, we hope to bring this explicit bound in a range comparable with the existing examples, see [DGMS09] for some conjectures about the optimal value of ϵ.

1.2. **Related questions.** We start with a generalization of Greenwald's dimension-dependent bound [Gre00] and characterize compact Riemannian manifolds for which such a bound exists. The proof relies on a limiting argument similar to the one employed by Greenwald.

Theorem 1. *Let M be a compact homogeneous Riemannian manifold. Then $\pi_1(M)$ is finite if and only if there exists $\epsilon > 0$ such that, for every subgroup $G \subset \mathrm{Isom}(M)$, either $\mathrm{diam}(M/G) = 0$ or $\mathrm{diam}(M/G) > \epsilon$.*

Unlike the Main Theorem, the bound in Theorem 1 above cannot be made independent of the space M, even after the metric is rescaled to have a fixed diameter. A counter-example is given by the groups $\mathrm{SO}(n)$, see Example 31. Nevertheless, the following problem is likely to have an affirmative answer:

Question 2. *Does there exist a lower bound for the diameter of quotients of simply-connected compact symmetric spaces depending only on the rank?*

Note that the rank one case follows from the Main Theorem and Theorem 1.

A special case of the Main Theorem is the existence of a universal lower bound on the diameter of Riemannian orbifolds with constant curvature one (in the case of manifolds this is the main result in [McG93]). Considering curvature negative one instead of one yields the following natural question, which seems to be open:

Question 3. *Is there a universal lower bound for the diameter of hyperbolic manifolds (resp. orbifolds)?*

Note that the existence of a *dimension-dependent* bound follows from Margulis' Lemma, see e.g. [Rat06, Cor. 1, §12.7].

While our proof of the Main Theorem uses several geometric arguments, it heavily relies on the structure and classification of compact Lie groups and their representations. Even in the connected case it seems to be a difficult task to remove the representation-theoretic arguments from the proof and obtain an affirmative answer to the following:

Question 4. *Does there exist a universal constant ϵ such that for any non-trivial singular Riemannian foliation $\mathcal{F}$ on a unit sphere $\mathbb{S}^n$, the quotient $\mathbb{S}^n/\mathcal{F}$ has diameter at least ϵ?*

We refer to [Mol88], [Rad] for the theory of singular Riemannian foliations, being a group-free generalization of isometric group actions and to [LR18], [MR16] for algebraic properties of singular Riemannian foliations on spheres. While in codimension one a positive answer to the above problem is a famous theorem of Münzner [Mün81], nothing is known in higher *codimensions*. Even the existence of a dimension-dependent bound $\epsilon(n)$ is presently not known. Nevertheless, we note that Problem 4 has an affirmative answer for all *currently known* examples, because these are all constructed starting from a homogeneous foliation, and repeatedly composing it with Clifford foliations, see [Rad14].

1.3. **The proof of the Main Theorem.** We are going to explain the main steps involved in the proof of the Main Theorem now.

First, we may replace G by the closure of its image in $O(n+1)$, thus we may assume G to be compact.

If the representation of G on $V = \mathbb{R}^{n+1}$ is reducible then the quotient $\mathbb{S}^n/G$ has diameter $\frac{\pi}{2}$ or π, see Lemma 6. Using a slightly more refined argument, we deduce that the existence of a normal subgroup N of G acting reducibly on V implies that N acts as (real, complex, or quaternionic) scalars or that the diameter of $\mathbb{S}^n/G$ is at least $\frac{\pi}{4}$, see Lemma 12.

Replacing G by a larger group can only decrease diameter. Combining this with the previous observation and ruling out 4 special classes of examples by hand, we reduce the task to the following two main cases.

I) The group G is a (uniformly) finite extension of $G_0 \times G_1$ where G_0 is the group of F-scalars (with $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$) and G_1 is a simple, simply connected compact group acting on V irreducibly.

II) The connected component G^0 of G acts as F-scalars on V, thus, G is finite up to scalars.

In case (I), we invoke the algebraic result proved in [GS], saying that the orbit $G \cdot p$ of G through the *highest weight vector* has a *focal radius* bounded from below by a universal constant. Then we combine this with a quotient version of the Klingenberg injectivity radius estimate (Proposition 25) to finish the proof.

In the technically more involved case (II) we only explain the main idea, neglecting all difficulties arising from the presence of scalars, which force us to work with projective representations rather than actual representations. Thus we assume that G is a finite group and that it is a maximal subgroup of $\mathrm{O}(n)$, in particular, the representation is of real type.

We compare the diameter and the volume of the quotient $\mathbb{S}^n/G$. The volume equals $\mathrm{vol}(\mathbb{S}^n)/|G|$. On the other hand, by the theorem of Bishop–Gromov, the volume of $\mathbb{S}^n/|G|$ is bounded from above by $c \cdot r^n \cdot \mathrm{vol}(\mathbb{S}^n)$, where c is a universal constant and r is the diameter of $\mathbb{S}^n/G$. Thus, in order to obtain the conclusion, we only need to verify that $\log(|G|)/n$ has a universal upper bound (for all representations that we cannot rule out by other means).

If the group G is a finite simple group, then the classification of such groups and existing lower bounds on the dimension of their representations provide us with the needed bound (with the only exception of the minimal representation of the alternating group, for which we already have the bound of $\frac{\pi}{4}$, [Gre00]). If the group G is not simple, we consider a minimal normal subgroup N of G, use the fact that this normal subgroup must act irreducibly and that G/N (again up to scalars) embeds into the group of outer automorphisms of N. Since N must be a power of a simple group, we again apply the classification of finite simple groups and obtain the required bound on $\log(|G|)/n$.

Remark 5. If one is interested in the case of connected Lie groups only, the proof can be considerably shortened. Indeed by passing to a maximal non-transitive connected closed group and using Theorem 9 to discard polar actions (see section 3 for this concept) we directly arrive at one of first three cases in Lemma 22. Subsections 5.2 and 5.3 cover these cases and we obtain as a lower bound for ϵ half the focal radius of the orbit through the special point, hence $\approx \frac{1}{30}$ according to [GS].

1.4. **Organization.** In Section 2 we recall some known facts about diameter of quotients which we will use later. Section 3 concerns normal subgroups and reduces the proof of the Main Theorem to two cases, according to whether the identity component G^0 of the given group G acts irreducibly, or as scalar multiplication. Section 4 finishes the proof when G^0 acts as scalar multiplication, that is, when G is essentially a finite group, while Section 5 deals with the case where G^0 acts irreducibly. Section 6 is devoted to the proof of Theorem 1.

Finally, there are three Appendices. Appendix A contains a couple of basic facts and definitions about real, complex and quaternionic representations, which are used throughout this article. Appendices B and C contain proofs of two technical but essentially known Lemmas needed in Sections 4 and 5.

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2. Preliminaries

Here we collect some basic facts about the diameter of quotients. We start with a well-known result, whose proof can be found, for instance, in [GL14, pages 75–76].

Lemma 6. *Let $G \subset \mathrm{O}(n)$. Then the diameter of $\mathbb{S}^{n-1}/G$ is equal to π if and only G fixes some non-zero vector. Otherwise the diameter is less than or equal to $\pi/2$, with equality precisely when the representation of G on $\mathbb{R}^n$ is reducible.*

Next we turn to the behaviour of the diameter of the quotient with respect to inclusion of groups $K \subset G \subset \mathrm{O}(n)$. A simple fact we will use frequently is that $\mathrm{diam}(\mathbb{S}^{n-1}/K) \geq \mathrm{diam}(\mathbb{S}^{n-1}/G)$. In the opposite direction, there is the following result, which appears as Lemma 3.13 in [Gre00], and allows one to replace a group with a finite index subgroup, as long as the index is controlled:

Lemma 7. *Let $K \subset G \subset \mathrm{O}(n)$ be closed subgroups, and assume G/K is finite, with k elements. Then*

$$\mathrm{diam}(\mathbb{S}^{n-1}/G) \geq \frac{\mathrm{diam}(\mathbb{S}^{n-1}/K)}{2(k-1)}.$$

The existence of a dimension-dependent lower bound on the diameter of the orbit space was established in [Gre00, Theorem 4.3]:

Theorem 8 (Greenwald)**.** *For each $n \geq 3$, there exists $\epsilon(n) > 0$ such that, for all $G < \mathrm{O}(n)$ compact and non-transitive,* $\mathrm{diam}(\mathbb{S}^{n-1}/G) \geq \epsilon(n)$.

See Theorem 1 for a generalization of Theorem 8, with similar proof.

Another result of Greenwald useful to us is [Gre00, Theorem 3.15]:

Theorem 9 (Greenwald)**.** *If $G < \mathrm{O}(n)$ is a finite group generated by reflections, then* $\mathrm{diam}(\mathbb{S}^{n-1}/G) \geq \frac{\pi}{8.1}$. *If G is of classical type, that is, types A, B, C or D, then* $\mathrm{diam}(\mathbb{S}^{n-1}/G) \geq \frac{\pi}{4}$.

3. Controlling normal subgroups via polar representations

We will need the following technical observation:

Lemma 10. *Let $N_0 \subset \mathrm{O}(l)$ (respectively $\mathrm{U}(l)$, $\mathrm{Sp}(l)$) be irreducible of real (respectively complex, quaternionic) type, and let $N = \Delta N_0 \subset \mathrm{O}(kl)$ (respectively $\mathrm{U}(kl)$, $\mathrm{Sp}(kl)$) be the diagonal group, seen as acting by left multiplication on the vector space V of $l \times k$ matrices with entries in $\mathbb{R}$ (respectively $\mathbb{C}$, $\mathbb{H}$). Let $\mathrm{O}(k)$ (respectively $\mathrm{U}(k)$, $\mathrm{Sp}(k)$) act by right multiplication, and denote by K the image in $\mathrm{O}(V)$*

of $\mathrm{O}(k) \times \mathrm{O}(l)$ *(respectively of the group generated by* $\mathrm{U}(k) \times \mathrm{U}(l)$ *and complex conjugation,* $\mathrm{Sp}(k) \times \mathrm{Sp}(l)$*). Then the normalizer* $N_{\mathrm{O}(V)}(N)$ *is contained in* K*.*

Proof. Let $g \in N_{\mathrm{O}(V)}(N)$. Since g normalizes N, it also normalizes the centralizer $C_{\mathrm{O}(V)}(N)$ of N, which, by Schur's lemma, equals $\mathrm{O}(k)$ (respectively $\mathrm{U}(k)$, $\mathrm{Sp}(k)$). Therefore g also normalizes $\mathrm{SO}(k)$ (respectively $\mathrm{SU}(k)$). But every automorphism of $\mathrm{SO}(k)$ is given by conjugation with some element of $\mathrm{O}(k)$, every automorphism of $\mathrm{SU}(k)$ is inner, or inner composed with complex conjugation, and every automorphism of $\mathrm{Sp}(k)$ is inner. Therefore there exists $g' \in \mathrm{O}(k)$ (respectively $\mathrm{SU}(k) \cup c\,\mathrm{SU}(k)$, $\mathrm{Sp}(k)$) such that conjugation by g and g' coincide on $\mathrm{SO}(k)$ (respectively $\mathrm{SU}(k)$, $\mathrm{Sp}(k)$). In other words, $g^{-1}g'$ centralizes $\mathrm{SO}(k)$ (respectively $\mathrm{SU}(k)$, $\mathrm{Sp}(k)$). By Schur's Lemma, $g^{-1}g'$ belongs to $\mathrm{O}(l)$ (respectively $\mathrm{U}(l)$, $\mathrm{Sp}(l)$), and therefore $g \in K$. □

The next lemma is analogous to Lemma 6 in that it provides algebraic information about a representation when the diameter of the quotient is assumed to be small. In the proof we use the concept of a *polar representation*, which is defined as a representation admitting a *section*, that is a vector subspace which meets all of the orbits orthogonally. The quotient space of the representation is isometric to the quotient of any section by its so-called *generalized Weyl group* (polar group). The latter is defined as the quotient of the subgroup which leaves the section invariant by the subgroup which fixes the section pointwise, and it is always finite. Moreover, the generalized Weyl group of a polar representation of a compact connected group is a finite reflection group. For a detailed account on polar representations and their generalized Weyl groups we refer to [PT87].

In order to make the statement of the lemma more convenient, we make the following definition, which corresponds to the case $l = 1$ in the notation of Lemma 10.

Definition 11. We will call a subgroup $N \subset \mathrm{O}(V)$ *super-reducible* if, as an N-representation, $V = W^k$, where W is irreducible with $\dim_F W = 1$, where $F = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ is the type of W.

In the following we denote the symmetric group on k letters by Σ_k.

Lemma 12. *Let* $G \subset \mathrm{O}(n)$ *be a closed subgroup, and assume* $\operatorname{diam} \mathbb{S}^{n-1}/G < \pi/4$*. Then, every normal subgroup* $N \subset G$ *is either irreducible or super-reducible.*

Proof. First we claim that, as an N-representation, $\mathbb{R}^n$ has one isotypical component. Let $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$ be the decomposition into isotypical components, and assume to the contrary that $k > 1$. Since N is normal in G, any $g \in G$ takes N-invariant subspaces to N-invariant subspaces, and hence N-irreducible subspaces to N-irreducible subspaces. But any N-irreducible subspace W must be contained in some V_i, by Schur's lemma. Thus, if $W \subset V_i$ is N-irreducible, and $g \in G$, there exists j such that $gW \subset V_j$. Moreover, the set of N-irreducible subspaces of V_i is connected, so that j does not depend on the choice of $W \subset V_i$. So $gV_i \subset V_j$, and, applying the same argument to g^{-1}, it follows that $gV_i = V_j$. Thus we obtain a group homomorphism $\phi : G \to \Sigma_k$ such that $gV_i = V_{\phi(g)(i)}$ for all i. Since G is irreducible, this action of G on $\{1, \ldots, k\}$ is transitive, and, in particular, all V_i have the same dimension d. Therefore $G \subset \Sigma_k \ltimes \mathrm{O}(d)^k$. The group $\Sigma_k \ltimes \mathrm{O}(d)^k$ is polar and the quotient $\mathbb{S}^{n-1}/(\Sigma_k \ltimes \mathrm{O}(d)^k)$ is isometric to the quotient of $\mathbb{S}^{k-1}$ by the Weyl group $\Sigma_k \ltimes \{\pm 1\}^k$, which, by Theorem 9, has diameter at least $\pi/4$.

Thus $\operatorname{diam}(\mathbb{S}^{n-1}/G) \geq \pi/4$, contradicting our hypothesis, and finishing the proof that $\mathbb{R}^n$ has only one N-isotypical component.

This puts us in the situation of Lemma 10, and, following the notation there, G is contained in K, which is the image in $\mathrm{O}(n)$ of $\mathrm{O}(k)\times\mathrm{O}(l)$ (respectively of the group generated by $\mathrm{U}(k)\times\mathrm{U}(l)$ and complex conjugation, $\mathrm{Sp}(k)\times\mathrm{Sp}(l)$). If both k, l are larger than one, the group K is polar, non-transitive, and by direct computations the associated generalized Weyl group is a finite reflection group of classical type. Thus Theorem 9 yields $\operatorname{diam}(\mathbb{S}^{n-1}/G) \geq \pi/4$, a contradiction. Therefore, either $k = 1$, that is, N is irreducible, or $l = 1$, that is, N is super-reducible. □

Let $G \subset \mathrm{O}(n)$ be a compact subgroup with identity component G^0, and assume $\operatorname{diam}(S^{n-1}/G) < \pi/4$. Since G^0 is a normal subgroup of G, we may apply Lemma 12 above to conclude that G^0 is either super-reducible, or irreducible. Thus the proof of the Main Theorem reduces to these two cases, which we will deal with separately in the next two sections.

4. Case where G^0 is super-reducible

4.1. **Finite simple groups and projective representations.** In this subsection we collect some facts about the projective representations and the automorphism groups of powers S^r of a finite simple group S. For more details on projective representations of finite groups we refer to [Kar85].

Recall that an n-dimensional (complex) *projective representation* of a group G is a group homomorphism $G \to \mathrm{PGL}(n,\mathbb{C})$. If this homomorphism can be lifted to a group homomorphism $\rho : G \to \mathrm{GL}(n,\mathbb{C})$, the representation is called *linear*. In general, it can be lifted to a map $\rho : G \to \mathrm{GL}(n,\mathbb{C})$ which is a group homomorphism only up to scalar multiplication. In other words, there is a map $\alpha : G \times G \to \mathbb{C}^\times$ such that $\rho(1) = 1$, and $\rho(xy) = \alpha(x,y)\rho(x)\rho(y)$ for all $x, y \in G$. Such a map ρ is called an α-representation. The group axioms imply that α is a *cocycle* (or *Schur multiplier*), that is, it satisfies $\alpha(x,1) = \alpha(1,x) = 1$ and $\alpha(x,y)\alpha(xy,z) = \alpha(y,z)\alpha(x,yz)$, for all $x,y,z \in G$. The set of all cocycles is called $Z^2(G,\mathbb{C}^\times)$, and it forms an Abelian group under pointwise multiplication. Moreover, one defines the subgroup $B^2(G,\mathbb{C}^\times)$ of *coboundaries*, and the cohomology group $H^2(G,\mathbb{C}^\times)$ as the quotient. Two lifts of the same projective representation have cohomologous Schur multipliers, and a projective representation is linear if and only if the associated cohomology class vanishes. Let $l(G)$ denote the smallest dimension of a faithful irreducible projective representation of G (if they exist, which is the case for a nonabelian, simple group G).

Lemma 13. *There exists a constant c such that, for every finite simple group S that is not cyclic or alternating, one has*

$$\frac{\log|S|}{l(S)} \leq c.$$

See Appendix B for the proof, which consist of a case-by-case verification following the classification of the finite simple groups and their representations.

Lemma 14. *Let A_n denote the alternating group in n letters. Then, for $n \geq 12$, the smallest dimension $l(A_n)$ of an irreducible faithful projective complex representation is $n-1$, uniquely achieved by the standard permutation representation on $\mathbb{C}^{n-1}$, and the second smallest is at least $n(n-3)/4$.*

Proof. Since A_n is simple, every non-trivial representation is faithful. A_n has exactly two cohomology classes of Schur multipliers [Sch11]. Denoting by α the non-trivial Schur multiplier, the smallest dimension of an irreducible α-representation is $2^{\lfloor (n-2)/2 \rfloor}$, see [KT12, page 1774]. Since $n \geq 12$, this is larger than $n(n-3)/4$, so it suffices to consider linear representations.

Every irreducible (linear) representation of A_n is either the restriction of an irreducible representation of Σ_n, or a summand, with half the dimension, of such a restriction — see [FH91, page 64, Prop. 5.1]. By [Ras77, Result 2], when $n \geq 9$, the third smallest dimension of an irreducible representation of Σ_n (after 1 and $n-1$) is $n(n-3)/2$, completing the proof. □

Lemma 15. *Let S be a finite simple group. If S is non-Abelian, then $l(S^r) = l(S)^r$. If S is Abelian, that is, $S \simeq \mathbb{Z}/p$ for a prime p, then $l(S^r) = p^{r/2}$ if r is even, and S^r has no complex projective faithful irreducible representations if r is odd.*

Proof. Assume S non-Abelian. Then, by [Kar85, page 132, Prop. 4.1.2], the Schur multiplier $M(S^r)$ (that is, the cohomology group $H^2(S^r, \mathbb{C}^\times)$) is equal to $M(S)^r$, because of the definition of tensor product of groups in [Kar85, page 58]. That is, every Schur multiplier of S^r is cohomologous to a product of r Schur multipliers of S, which, by [Kar85, page 198, Corollary 5.1.3], implies that every irreducible projective representation of S^r is an outer tensor product of irreducible projective representations of S. Moreover, such an outer tensor product is faithful if and only if each factor is faithful, thus concluding the proof that $l(S^r) = l(S)^r$.

Next assume S Abelian, that is, $S = \mathbb{Z}/p$, for a prime p, and that S^r has a (complex) faithful irreducible α-representation for some Schur multiplier $\alpha \in Z^2(S^r, \mathbb{C}^\times)$. Then, by [Kar85, page 578, Lemma 10.4.3], the identity is the only element of S^r that is α-regular. Recall that, since S^r is Abelian, an element g is α-regular if and only if $\alpha(g, x) = \alpha(x, g)$ for all $x \in S^r$ (see [Kar85, page 107], or [Hig01, Definition 1.2] for the general definition of α-regularity). Therefore, by [Hig01, Lemma 2.2(1)], S^r is of symmetric type, which, in our case, simply means that r is even; and moreover every irreducible projective α-representation of S^r has degree $\sqrt{|S^r|} = p^{r/2}$. □

The outer automorphism groups of the finite simple groups are small. For our purposes, the very rough estimate below will suffice:

Lemma 16. *Let S be a finite simple group, and $r \geq 1$. If S is non-Abelian, then $|\operatorname{Out}(S)| \leq |S|$, $|\operatorname{Aut}(S)| \leq |S|^2$, and $|\operatorname{Aut}(S^r)| \leq r!|S|^{2r}$. If S is Abelian, $S = \mathbb{Z}/p$ for a prime p, then $|\operatorname{Aut}(S^r)| \leq p^{r^2}$.*

Proof. Assume S non-Abelian. Using the classification of finite simple groups, it has been proved in [Qui04, Lemma 2.2] that $|\operatorname{Out}(S)| \leq |S|/30$. Since $\operatorname{Inn}(S) \simeq S$, it follows that $|\operatorname{Aut}(S)| \leq |S|^2$. Moreover, $\operatorname{Aut}(S^r)$ is isomorphic to the semi-direct product between the permutation group in r letters, and $\operatorname{Aut}(S)^r$. Indeed, any group homomorphism $\phi : S^r \to S^r$ can be written as

$$\phi(g_1, \ldots, g_r) = \left(\prod_j \phi_{1j}(g_j), \prod_j \phi_{2j}(g_j), \ldots, \prod_j \phi_{rj}(g_j) \right)$$

where $\phi_{ij} : S \to S$ are group homomorphisms such that, for all i, and all $(g_1, \ldots, g_r) \in S^r$, $\{\phi_{ij}(g_j)\}_{j=1}^r$ commute. Since S is simple non-Abelian, this implies that for each

i, there is at most one value of j, such that ϕ_{ij} is non-trivial (and hence an automorphism). Assuming further that ϕ is an automorphism, there must in fact be a permutation $\sigma \in \Sigma_r$ such that ϕ_{ij} is non-trivial if and only if $\sigma(i) = j$.

In the Abelian case, an automorphism of S^r is represented by an $r \times r$ matrix with entries in $\mathbb{Z}/p$, and thus $|\operatorname{Aut}(S^r)| \leq p^{r^2}$. □

4.2. **Volume, diameter and dimension.** We will need a rough estimate for the volume of the compact rank one-symmetric spaces (which is actually known explicitly). Denote by B^n the unit Euclidean ball and by $\mathbb{C}P^n = \mathbb{S}^{2n+1}/\operatorname{U}(1)$ and $\mathbb{H}P^n = \mathbb{S}^{4n+3}/\operatorname{Sp}(1)$ the complex and the quaternionic projective spaces, respectively. Note that $\mathbb{C}P^n$ and $\mathbb{H}P^n$ have sectional curvatures bounded above by 4.

Lemma 17. *Let M be an n-dimensional compact, simply connected rank one symmetric space with curvature bounded above by 4. Then*

$$\operatorname{vol}(M^n) > \frac{1}{2^n} \operatorname{vol}(B^n) .$$

Proof. The injectivity radius of the symmetric space M is at least equal to the injectivity radius of the sphere $\frac{1}{2}\mathbb{S}^n$ of constant curvature 4. By the Bishop–Gromov volume comparison, we have

$$\operatorname{vol}(M) \geq \operatorname{vol}(\frac{1}{2}\mathbb{S}^n) .$$

Considering the orthogonal projection, the volume of $\frac{1}{2}\mathbb{S}^n$ is larger than the volume of the unit n-dimensional Euclidean ball of radius $\frac{1}{2}$. This implies the claim. □

As an application we deduce:

Lemma 18. *Let M be an n-dimensional compact, simply connected rank one symmetric space with curvature bounded above by 4. Let G be a finite group acting by isometries on M. Then*

$$\operatorname{diam}(M/G) \geq \frac{1}{2} \sqrt[n]{1/|G|} .$$

Proof. Let d denote the diameter of the quotient. Then a fundamental domain for the action is contained in a ball in M of radius d. Since M is positively curved, by the Bishop–Gromov Theorem, the volume of the quotient satisfies $d^n \operatorname{vol}(B^n) \geq \operatorname{vol}(M^n/G)$. On the other hand, $\operatorname{vol}(M/G) \geq \operatorname{vol}(M)/|G|$ (with equality if the action is effective). The result now follows from the previous Lemma. □

4.3. **Proof of Main Theorem — super-reducibly case.** We can now prove the Main Theorem under the assumption that the connected component G^0 of G acts as scalars, i.e. *super-reducibly* in terms of Definition 11).

Theorem 19. *There exists $\epsilon > 0$ such that $\operatorname{diam}(\mathbb{S}^{n-1}/G) > \epsilon$ for every group $G \subset \operatorname{O}(n)$, for which the connected component $G^0 \subset \operatorname{O}(n)$ is super-reducible.*

We start with a few reductions:

Lemma 20. *To prove Theorem 19, one may assume that $\operatorname{diam}(\mathbb{S}^{n-1}/G) < \pi/4$ and $n > 16$. Moreover, we may assume that one of the following three cases occurs:*

1. *The group G is finite, $\operatorname{O}(1) = \{\pm 1\} \subset G$ and $\{\pm 1\}$ is maximal among super-reducible normal subgroups of G.*

(2) *We have* $G = H \cdot \mathrm{Sp}(1)$. *The group* $\mathrm{Sp}(1) = G^0$ *is maximal among super-reducible normal subgroups of* G. *The group* H *is finite and contains the center* Z *of* $\mathrm{Sp}(n/4)$.
(3) *We have* $G = H \cdot \mathrm{Sp}(1)$. *The group* $\mathrm{U}(1) = G^0$ *is maximal among super-reducible normal subgroups of* G. *The group* H *is finite and contains the center* Z *of* $\mathrm{SU}(n/2)$.

Proof. The first statement is clear. By Theorem 8 we may assume $n > 16$.

Consider a subgroup L of $\mathrm{O}(n)$ which is maximal among subgroups that are super-reducible, contain G^0 and are normalized by G. Replacing G by $G \cdot L$ and observing that the connected component of $G \cdot L$ is the super-reducible group L^0, we may assume that $G = G \cdot L$, hence $L \subset G$.

Clearly, $\pm 1 \in L$. If $L = \{\pm 1\}$ we are in case (1).

Otherwise, L is of complex or quaternionic type. Assume that L is of quaternionic type, hence $L \subset \mathrm{Sp}(1)$. By Lemma 10, the group G is contained in $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n/4)$. Hence $\mathrm{Sp}(1)$ is normalized by G, thus $L = \mathrm{Sp}(1)$, by maximality of L. We can now take H to be the intersection of G with $Sp(n/4)$.

Similarly, if L is of complex type then applying Lemma 10 and the maximality of L we obtain $L = \mathrm{U}(1)$ and G is contained in the extension of $\mathrm{U}(n/2)$ by the complex-conjugation. Replacing G by an index two subgroup, which is possible by Lemma 7, we may assume that $G \subset \mathrm{U}(n)$. Again, we obtain H as the intersection of G with $\mathrm{SU}(n/2)$. □

Proof of Theorem 19. We make the assumptions listed in Lemma 20. If G is finite, we set $H = G$ to make the notation more uniform. We denote by Z the center of $\mathrm{O}(n)$ (resp. $\mathrm{SU}(n/2)$, $\mathrm{Sp}(n/4)$), so Z is cyclic of order 2 (resp. $n/2$, 2). Let $\bar{N}$ be a minimal normal subgroup of H/Z. Since $\bar{N}$ is minimal normal, it is characteristically simple, hence isomorphic to S^r, for some finite simple group S (see [Wil09, Lemmas 2.7 and 2.8]).

We will show that $\log(|H/Z|)/n$ is uniformly bounded from above by providing appropriate bounds on n and on $|H/Z|$. This will conclude the proof via an application of Lemma 18, because $\mathbb{S}^{n-1}/G = \mathbb{C}P^{(n-2)/2}/(H/Z)$ (resp. $\mathbb{H}P^{(n-4)/4}/(H/Z)$).

Let N be the inverse image of $\bar{N}$ in H. We claim that $N \subset \mathrm{O}(n)$ is irreducible. Indeed, $N.\{\pm 1\}$ (resp. $N.\mathrm{U}(1)$, $N.\mathrm{Sp}(1)$) must be irreducible, because it is normal in G, and strictly contains $\{\pm 1\}$ (resp. $\mathrm{U}(1)$, $\mathrm{Sp}(1)$), which is maximal super-reducible by assumption. This implies that, as an N-representation, $\mathbb{R}^n$ breaks into at most 1 (resp. 2, 4) irreducible factors. Since $n > 16$, N cannot be super-reducible, and since it is normal in G, it must be irreducible.

Next, we claim that the centralizer $C_H(N)$ of N in H is Z, so that, in particular, Z is the center of N. Indeed, since N is irreducible, $C_H(N)$ is super-reducible. This implies that $C_H(N)$ (resp. $C_H(N).\mathrm{U}(1)$, $C_H(N).\mathrm{Sp}(1)$) is not irreducible, because $n > 16$. Thus, being normal in G, it must be super-reducible. By maximality of $\{\pm 1\}$ (resp. $\mathrm{U}(1)$, $\mathrm{Sp}(1)$) among super-reducible normal subgroups of G, we must have $C_H(N) = \{\pm 1\}$ (resp. $C_H(N).\mathrm{U}(1) = \mathrm{U}(1)$, $C_H(N).\mathrm{Sp}(1) = \mathrm{Sp}(1)$), which implies $C_H(N) = Z$.

Bounding $|H/Z|$ from above. H acts by conjugation on N, so we have a group homomorphism $\eta : H \to \mathrm{Aut}(N)$, whose kernel is $C_H(N) = Z$. Thus $|H| \leq |Z|.|\,\mathrm{image}(\eta)|$. But

$$\mathrm{image}(\eta) \subset \mathrm{Aut}_Z(N) = \{\phi \in \mathrm{Aut}(N) \mid \phi(z) = z \;\; \forall z \in Z\}$$

and each element of $\mathrm{Aut}_Z(N)$ induces an automorphism of $\bar{N} = N/Z$. Thus, denoting $\mathrm{Aut}_0(N) = \{\phi \in \mathrm{Aut}_Z(N) \mid \phi \text{ induces the trivial automorphism of } \bar{N}\}$, we have a short exact sequence

$$1 \to \mathrm{Aut}_0(N) \to \mathrm{Aut}_Z(N) \to \mathrm{Aut}(\bar{N}) \to 1.$$

Moreover, the map that sends $\phi \in \mathrm{Aut}_0(N)$ to $\alpha : \bar{N} \to Z$ defined by $\alpha(x) = \phi(x)x^{-1}$ establishes an isomorphism $\mathrm{Aut}_0(N) \simeq \mathrm{Hom}(\bar{N}, Z)$. If S is non-Abelian, $\mathrm{Hom}(\bar{N}, Z)$ is trivial, and if S is Abelian, we have $|\mathrm{Hom}(\bar{N}, Z)| \leq |Z|^r$. Therefore, we may use Lemma 16 to obtain the bound

$$|H/Z| \leq \begin{cases} r!|S|^{2r} & \text{if } S \text{ is non-Abelian} \\ n^r p^{r^2} & \text{if } S = \mathbb{Z}/p \end{cases} \tag{1}$$

Bounding n from below. Consider the representation of N on $\mathbb{R}^n$. It is faithful and irreducible. If it is of complex or quaternionic type (that is, if it commutes with some complex structure) then $U = \mathbb{R}^n = \mathbb{C}^{n/2}$ is a faithful irreducible complex representation of N. Otherwise, the representation of N on $\mathbb{R}^n$ is of real type, so that its complexification $U = \mathbb{C}^n$ is a faithful irreducible complex N-representation.

The projectivization of U has kernel which must be equal to Z, because Z is the center of N. Thus we have obtained a projective faithful irreducible representation of $\bar{N} = S^r$ of dimension n or $n/2$, and thus, via Lemma 15, the bound

$$n \geq \begin{cases} l(S^r) = l(S)^r & \text{if } S \text{ is non-Abelian} \\ p^{r/2} & \text{if } S = \mathbb{Z}/p. \end{cases} \tag{2}$$

To show that $\log |H/Z|/n$ is uniformly bounded and conclude the proof, we divide into three cases: S Abelian, S non-Abelian and non-alternating, and S alternating and non-Abelian.

If S is Abelian, isomorphic to $\mathbb{Z}/p$, then from (1) and (2) we obtain

$$\frac{\log |H/Z|}{n} \leq \frac{r \log n + r^2 \log p}{n} \leq \frac{r}{p^{r/4}} . \frac{\log(n)}{\sqrt{n}} + \frac{r^2}{p^{r/4}} . \frac{\log p}{p^{r/4}} \tag{3}$$

which is bounded from above.

If S is non-Abelian, then (1) and (2) yield

$$\frac{\log |H/Z|}{n} \leq \frac{\log(r!) + 2r \log |S|}{n} \leq \frac{\log(r!)}{l(S)^r} + \frac{2r \log |S|}{l(S)^r} \tag{4}$$

Since $l(S) \geq 2$, the term $\frac{\log(r!)}{l(S)^r}$ is bounded. When S is non-alternating, the last term $\frac{2r \log |S|}{l(S)^r}$ is bounded, because, by Lemma 13, the quantity $\frac{\log |S|}{l(S)}$ is bounded. From now on assume S is the alternating group A_d. If $r \geq 2$ and $d \geq 12$, then using Lemma 14 we see that the last term in (4) is again bounded:

$$\frac{2r \log |S|}{l(S)^r} \leq \frac{2r \log(d!/2)}{(d-1)^r} \leq \frac{2r}{(d-1)^{r-3/2}} . \frac{d \log d}{(d-1)^{3/2}} . \tag{5}$$

If $r \geq 2$ and $5 \leq d < 12$ then

$$\frac{2r \log |S|}{l(S)^r} \leq \frac{2r \log(12!/2)}{2^r} \tag{6}$$

which is bounded. Thus we may assume $r = 1$. If the faithful irreducible projective representation of $S = A_d$ constructed above is not the standard permutation representation, then by Lemma 14 its dimension is at least $d(d-3)/4$, so that $\frac{2r \log |S|}{n}$ is again bounded.

Therefore we have reduced to the case where $S = A_d$ and the projective representation U of $\bar{N} = S = A_d$ constructed above is the standard representation on $\mathbb{C}^{d-1}$. Since this projective representation $A_d = \bar{N} \to \mathrm{PGL}(U)$ lifts to the linear representation $A_d \to \mathrm{GL}(U)$, the short exact sequence $1 \to Z \to N \to \bar{N} \to 1$ splits, which implies that $N \simeq A_d \times Z$ (because Z is the center of N). Thus A_d is a normal subgroup of G, and it acts in the standard way on $U = \mathbb{C}^{d-1}$. Recall that $U = \mathbb{C}^{d-1}$ was isomorphic to either $\mathbb{R}^n$, or its complexification. The first case is precluded by our hypotheses, since then the restriction of the G-representation $\mathbb{R}^n$ to A_d would be neither irreducible nor super-reducible. Therefore the subgroup A_d acts on $\mathbb{R}^n = \mathbb{R}^{d-1}$ in the standard way. Since this representation is of real type, that is, it does not leave any complex structure invariant, we are in the case where $G = H$ is finite, so, in particular, $Z = \pm 1$. Since the automorphism group of A_d is isomorphic to Σ_d (because $d \geq 7$, see [Wil09, Theorem 2.3]), the index of A_d in G is at most four, and the desired diameter bound follows from Theorem 9 and Lemma 7. □

5. Case where G^0 is irreducible

As noted at the end of Section 3, Lemma 12 reduces the proof of the Main Theorem to two cases, according to whether the identity component G^0 acts as scalar multiplication or irreducibly. We have dealt with the former in Section 4, and this section is devoted to the latter.

For convenience in this section we will consider *almost* faithful representations $\rho : G \to \mathrm{O}(V)$. We first lift ρ to a representation of a semidirect product. By [Wil99, Lemma 7.5], there is a finite subgroup Γ meeting all connected components of G. Since the identity component G^0 is a normal subgroup of G, we can write $G = G^0 \cdot \Gamma$. Now there is a finite covering $G^0 \rtimes \Gamma \to G$ and we can lift ρ to the semidirect product. Therefore from now on we assume G splits as $G^0 \rtimes \Gamma$.

Furthermore, by passing to a finite cover we may also assume that G^0 is a product of simply connected simple Lie groups, and a torus. Altogether, we have reduced to proving the following:

Theorem 21. *There exists $\epsilon > 0$ with the following property. Let G^0 be a product of a torus with finitely many simply-connected compact connected simple Lie groups, let Γ be a finite group acting on G^0 by automorphisms, and set $G = G^0 \rtimes \Gamma$. Let $\rho : G \to \mathrm{O}(V)$ be an almost faithful representation whose restriction to G^0 is irreducible but not transitive. Then* $\mathrm{diam}(\mathbb{S}(V)/G) > \epsilon$.

Our strategy to prove Theorem 21 is the following reduction:

Lemma 22. *To prove Theorem 21, it suffices to find a common lower diameter bound for all non-transitive representations of the following types:*

- *(i) $G = G'$ is a simply-connected compact connected simple Lie group, $V = V'$ is of real type.*
- *(ii) $G = \mathrm{U}(1) \times G'$, $V = \mathbb{C} \otimes_{\mathbb{C}} V'$, G' is a simply-connected compact connected simple Lie group and V' is of complex type.*
- *(iii) $G = \mathrm{Sp}(1) \times G'$, $V = \mathbb{H} \otimes_{\mathbb{H}} V'$, G' is a simply-connected compact connected simple Lie group and V' is of quaternionic type.*
- *(iv) $G = \Sigma_k \ltimes \mathrm{SO}(n)^k$, $V = \otimes^k \mathbb{R}^n$, where $n \geq 3$ and $k > 2$.*
- *(v) $G = \mathrm{U}(1) \times \Sigma_k \ltimes \mathrm{SU}(n)^k$, $V = \mathbb{C} \otimes_{\mathbb{C}} \otimes^k \mathbb{C}^n$, where $n \geq 3$ and $k > 2$.*
- *(vi) $G = \Sigma_k \ltimes \mathrm{Sp}(n)^k$, $V = \otimes^k \mathbb{H}^n$, where $n \geq 1$ and $k \geq 4$ is even.*

(vii) $G = \mathrm{Sp}(1) \times \Sigma_k \ltimes \mathrm{Sp}(n)^k$, $V = \mathbb{H} \otimes_{\mathbb{H}} \otimes^k \mathbb{H}^n$, *where* $n \geq 1$ *and* $k \geq 3$ *is odd.*

(in the last four cases, the permutation group Σ_k *acts by permuting the factors of the tensor product)*

Remark 23. Some explanation about quaternionic tensor products is in order for cases (vi) and (vii) above. The complex representation of $\mathrm{Sp}(n)$ on $\mathbb{H}^n = \mathbb{C}^{2n}$ is of quaternionic type, and thus $W = \otimes^k_{\mathbb{C}}(\mathbb{H}^n)$ is a complex irreducible representation of $\mathrm{Sp}(n)^k$, which is of real type when k is even, and of quaternionic type when k is odd. Denoting by $\epsilon_i : \mathbb{H}^n \to \mathbb{H}^n$ the standard quaternionic structure on the ith factor of this tensor product, $\epsilon = \epsilon_1 \otimes \cdots \otimes \epsilon_k$ is a real (if k even) or quaternionic (if k odd) structure on W. In case (vi) we take V to be real form of W, that is, the fixed point set of $\epsilon : W \to W$. In case (vii) we take V to be the real form of $\mathbb{C}^2 \otimes_{\mathbb{C}} W$ relative to $\epsilon_0 \otimes \epsilon$, where ϵ_0 is the standard quaternionic structure on $\mathbb{H} = \mathbb{C}^2$. In both cases the permutation group Σ_k acts on W by permuting the factors, and this action commutes with ϵ, so that it induces an action on V.

The proof of Lemma 22 is obtained by analysing the action by G^0 using Lemmas 10 and 12, and is relegated to Appendix C (alternatively, one may also note that every maximal closed *non-transitive* subgroup of $\mathrm{O}(n)$ (up to taking subgroup of small index) is either a maximal closed subgroup of $\mathrm{O}(n)$, or $\mathrm{U}(1)$ times a maximal closed subgroup of $\mathrm{SU}(n)$, or $\mathrm{Sp}(1)$ times a maximal closed subgroup of $\mathrm{Sp}(n)$, and then use the classification of infinite, non-simple maximal closed subgroups of the classical groups obtained in [AFG12]). In the remaining of this section, we run through the cases of Lemma 22.

5.1. **The tensor power representations.** The goal of this subsection is to show the existence of a universal lower bound on $\operatorname{diam} \mathbb{S}(V)/G$, where (V, G) is one of the representations listed in cases (iv)-(vii) of Lemma 22.

For an arbitrary metric space X, define the *radius* at $x \in X$ to be $r_x = \inf\{r > 0 : X \subset B(x, r)\}$. It is immediate from the triangle inequality that it compares to the diameter of X as follows:

$$r_x \leq \operatorname{diam} X \leq 2r_x. \tag{7}$$

Lemma 24. *Let G be a locally compact topological group acting continuously, properly and isometrically on a metric space X. Assume the fixed point set of G on X is non-empty. Then* $\operatorname{diam} X/G \geq \frac{1}{2} \operatorname{diam} X$.

Proof. Let $x_0 \in X$ be a fixed point of G and denote by $\pi : X \to X/G$ the natural projection. For every $x \in X$, the distance from x_0 to Gx is constant. It follows that the distances $d(x, x_0) = d(\pi(x), \pi(x_0))$ and hence the radii $r_{x_0} = r_{\pi(x_0)}$. The desired result now follows from (7). □

Next consider a representation $\rho : G \to \mathrm{O}(V)$ as in the last four cases of Lemma 22. Then G and G^0 share a common orbit in V, namely, that one consisting of "pure tensors". Indeed, following the notation in Remark 23, it is the orbit through $p = v_1 \otimes \cdots \otimes v_k \in \otimes^k \mathbb{R}^n$ in case (iv), $p = v_1 \otimes \cdots \otimes v_k \in \otimes^k \mathbb{C}^n$ in case (v), $p = v_1 \otimes \cdots \otimes v_k + \epsilon_1 v_1 \otimes \cdots \otimes \epsilon_k v_k \in \otimes^k \mathbb{C}^{2n}$ in case (vi) and $p = v_0 \otimes \cdots \otimes v_k + \epsilon_0 v_0 \otimes \cdots \otimes \epsilon_k v_k \in \mathbb{C}^2 \otimes_{\mathbb{C}} \otimes^k \mathbb{C}^{2n}$ in case (vii).

Denoting $X = \mathbb{S}(V)/G$ and $X^0 = \mathbb{S}(V)/G^0$, we conclude that G/G^0 acts on X^0 with a fixed point and $X = X^0/\Gamma$, so we can apply Lemma 24 and Lemma 12 to deduce that $\operatorname{diam} X \geq \frac{1}{2} \operatorname{diam} X^0 \geq \pi/8$.

5.2. **Normal injectivity radius and focal radius.** This subsection is devoted to proving a version of the injectivity radius estimate of Klingenberg for quotients $\mathbb{S}(V)/G$, that is, to give a lower bound for the normal injectivity radius of a G-orbit in terms of the focal radius. In the next section this will be combined with a universal lower bound (found in [GS]) for the focal radius for a special G-orbit to finish the proof of Theorem 21.

Let N be a properly embedded submanifold of a complete Riemannian manifold M. Consider the normal bundle νN in M and the normal exponential map $\exp^\perp : \nu N \to M$. Denote the open ball bundle of radius r in νN by $\nu^r N$. The *normal injectivity radius* ι_N of N is the the supremum of the numbers r such that $\exp^\perp$ is an embedding on $\nu^r N$, and the image of $\nu^{\iota_N} N$ is called the *maximal tubular neighborhood* of N. If N is compact, $\iota_N > 0$. On the other hand, a *focal point* of N relative to $p \in N$ is a critical value of $\exp^\perp : \nu N \to M$ such that $\exp^\perp(v) = q$ for some $v \in \nu_p N$. In this case, the *focal distance* associated to q is the length $|v|$ of the normal geodesic from p to q. The *focal radius* f_N of N is the infimum of all focal distances to N along normal geodesics. It is clear that $\iota_N \leq f_N$.

Proposition 25. *Let G be a compact Lie group acting isometrically on a compact Riemannian manifold M. Let $p \in M$ and consider the orbit $N = Gp$. Assume the fixed point set of the identity component $(G_p)^0$ of the isotropy group at p in the closure of the maximal tubular neighborhood of N is contained in N. Then $f_N/2 \leq \iota_N$. In particular, the diameter of M/G is bounded below by $f_N/2$.*

Proof. If $\iota_N < f_N$, we argue as in Klingenberg's Lemma [dC92, Chap. 13 Proposition 2.12] (see also [CE08, Lemma 5.6]) to deduce the existence of a horizontal geodesic segment γ of length $2\iota_N$, entirely contained in closure of the maximal tubular neighborhood of N, that starts at p and ends at a point $q \in N$. By assumption, $\mathfrak{g}_p$ is not contained in $\mathfrak{g}_{\gamma(t)}$ for all small $t > 0$. Therefore there is a non-trivial variation of γ through horizontal geodesics fixing p, and ending on N, and hence p is a focal point of N. It follows that the length of γ is at least f_N, as desired. Finally, any point in M outside the maximal tubular neighborhood of N has distance at least i_N to N, which proves the last statement. □

5.3. **The case of simple Lie groups and their extensions by scalars.** Having dealt with cases (iv)–(vii) of Lemma 22 in Subsection 5.1, it remains to treat cases (i)–(iii) to finish the proof of Theorem 21, and hence of the Main Theorem. The strategy to prove the existence of a lower diameter bound for $\mathbb{S}(V)/G$ for the representations listed in cases (i)–(iii) of Lemma 22 is to use the universal lower bound for the focal radius of a special orbit [GS] in combination with the Klingenberg-type Proposition 25. In fact, this subsection is devoted to showing that, in these cases, the hypothesis in Proposition 25 concerning the fixed point set is satisfied.

Fix a maximal torus of G, consider the corresponding root system and fix an ordering of the roots. Since the representation of G is irreducible and non-transitive on the unit sphere, we may apply the main result of [GS] to deduce there exists $\delta > 0$ such that the focal radius f_N of the orbit $N = Gp$ is bigger than δ, where $p = v_\lambda$ or $p = \frac{1}{\sqrt{2}}(v_\lambda + \epsilon(v_\lambda))$, and v_λ is a unit highest weight vector of V or its complexification V^c, according to whether ρ admits an invariant complex structure or not; in the latter case it admits a real structure ϵ. Note that ρ admits an invariant complex structure in case (ii) and it does not in cases (i) and (iii).

In case (ii), we have $v_\lambda = v_{\lambda'}$ is also a unit highest weight vector of V'. In case (iii), V' admits an invariant complex structure and it is easier to do the computations in V'; let $v_{\lambda'}$ be a unit highest weight vector. The G'-action on V' admits an extension to a G-action. We have $V = \mathbb{H} \otimes_{\mathbb{H}} V'$ is a real form of $\mathbb{C}^2 \otimes_{\mathbb{C}} V'$ and there is a G-equivariant isometry $V' \to V$ mapping $v_{\lambda'}$ to $\frac{1}{\sqrt{2}}(v_\lambda + v_{-\lambda})$, where v_λ is the highest weight vector of V and $v_{-\lambda} = \epsilon(v_\lambda)$, where ϵ is the real structure on $\mathbb{C}^2 \otimes_{\mathbb{C}} V'$.

5.3.1. *The complex and quaternionic cases.* In this section, we check the hypothesis of Proposition 25 in cases (ii) and (iii).

Lemma 26. *Let $p = v_{\lambda'}$ in cases (ii) and (iii). Then the fixed point set of G_p^0 in $\mathbb{S}(V')$ is contained in Gp.*

Proof. The proof is the same in both cases. The Lie algebra of the maximal torus of G has the form $\mathfrak{t}' \oplus \mathfrak{u}(1)$ where $\mathfrak{t}'$ is the Lie algebra of the maximal torus of G'. The isotropy algebra $\mathfrak{g}_p$ contains the kernel of λ in $\mathfrak{t}'$ and an element of the form $h_1 - h_0$, where $h_1 \in \mathfrak{t}'$ satisfies $\lambda(h_1) = i$ and $h_0 \in \mathfrak{u}(1)$ acts as multiplication by i on V. Write an arbitrary element of $\mathbb{S}(V')$ as $v = \sum_\mu c_\mu v_\mu$, where the sum runs through the different weights of V', $c_\mu \in \mathbb{C}$ and v_μ is a weight vector of weight μ. Then $\ker \lambda|_{\mathfrak{t}'} \cdot v = 0$ implies $c_\mu = 0$ unless μ is a multiple of λ'. Moreover, if $\mu = c\lambda'$ then $(h_1 - h_0) \cdot v_\mu = (ci - i)v_\mu$ can be zero only if $c = 1$. It follows that $\mathfrak{g}_p \cdot v = 0$ implies $v = c_{\lambda'} v_{\lambda'}$ with $|c_{\lambda'}| = 1$, so $v \in Gp$. □

5.3.2. *The real case.* It remains to tackle case (i) from Lemma 22. In view of Theorem 9, we may assume that the representation is not polar. We claim that we may also assume that $\mathfrak{g}_p$ is a maximal isotropy algebra, up to conjugation. Indeed, let $q \in \mathbb{S}(V) \setminus \{-p\}$ be arbitrary, consider the minimal geodesic segment γ in $\mathbb{S}(V)$ from p to the orbit Gq and let $q_1 \in Gq$ be its endpoint. Of course, $\mathfrak{g}_{q_1}$ and $\mathfrak{g}_q$ are Ad_G-conjugate. If $\mathfrak{g}_{q_1}$ is not contained in $\mathfrak{g}_p$, an element in $\mathfrak{g}_{q_1} \setminus \mathfrak{g}_p$ produces a non-trivial variation of γ through horizontal geodesics fixing q_1, which implies that q_1 is a focal point of Gp. We deduce that $\operatorname{diam} X \geq \ell \geq f_N > \delta$, where ℓ is the length of γ and $N = Gp$.

So in the sequel we may assume $\mathfrak{g}_p$ is a maximal isotropy algebra, up to conjugation. We will show that this implies that $\operatorname{rk} \mathfrak{g} \geq 2$ and V^c is a *minuscule* representation, that is, all weights comprise a single Weyl orbit.

Lemma 27. $\operatorname{rk} \mathfrak{g}_p = \operatorname{rk} \mathfrak{g} - 1$.

Proof. Denote the Lie algebra of the maximal torus of G by $\mathfrak{t}$, and the corresponding system of roots by Δ, where we have already chosen an ordering of the roots. Consider the root space decomposition

$$\mathfrak{g} = \mathfrak{t} + \mathfrak{t}^\perp, \qquad \mathfrak{t}^\perp = \sum_{\alpha \in \Delta^+} (\mathfrak{g}_\alpha^{\mathbb{C}} + \mathfrak{g}_{-\alpha}^{\mathbb{C}}) \cap \mathfrak{g}$$

It is clear that $\mathfrak{g}_p = \mathfrak{g}_p \cap \mathfrak{t} + \mathfrak{g}_p \cap \mathfrak{t}^\perp$, where $\mathfrak{g}_p \cap \mathfrak{t} = \ker \lambda$. Suppose, to the contrary, that $\operatorname{rk} \mathfrak{g}_p = \operatorname{rk} \mathfrak{g}$. Then $\ker \lambda$ can be enlarged to a Cartan subalgebra of $\mathfrak{g}_p$ by adding an element u of $\mathfrak{t}^\perp$. It follows from $[\ker \lambda, u] = 0$ that $u = x_\alpha + \epsilon x_\alpha$ for some $\alpha \in \Delta^+$, where $x_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}$, and λ is a multiple of α. Since $0 = u \cdot (v_\lambda + v_{-\lambda})$, we deduce that $-\lambda + \alpha = \lambda - \alpha$ and thus $\lambda = \alpha$. The only dominant roots are the highest root or the highest short root. In first case, our representation is the adjoint representation and hence polar. The remaining cases that need to be analyzed occur only for simple groups of type B_n, C_n, F_4, G_2, in which our representation is respectively the isotropy representation of the symmetric space $\mathbb{S}^{2n+1}$, $\mathrm{SU}(2n)/\mathrm{Sp}(n)$, $\mathrm{E}_6/\mathrm{F}_4$ or the 7-dimensional representation of G_2, again all polar. In any case, we reach a contradiction to our previous assumption. □

Lemma 28. $\mathrm{rk}\,\mathfrak{g} \geq 2$ *and* V^c *is* minuscule.

Proof. It follows from Lemma 27 that zero cannot be a weight, because the isotropy algebra of a real zero-weight vector would have full rank in $\mathfrak{g}$, bigger than $\mathrm{rk}\,\mathfrak{g}_p$. This already rules out representations of real type of a rank one group, since the odd dimensional representations of SO(3) always have zero as a weight.

Take $q = \frac{1}{\sqrt{2}}(v_\mu + v_{-\mu})$ where μ is an arbitrary nonzero weight μ of V^c. Then $\ker\mu \subset \mathfrak{g}_q$ and $\mathrm{rk}\,\mathfrak{g}_q \leq \mathrm{rk}\,\mathfrak{g}_p$. Again Lemma 27 implies that $\mathrm{rk}\,\mathfrak{g}_q = \mathrm{rk}\,\mathfrak{g}_p$ and $\ker\mu$, $\ker\lambda$, viewed as subspaces of $\mathfrak{t}$, are Ad-conjugate. Since two maximal tori of a compact connected Lie group are Ad-conjugate by a transformation that fixes pointwise their intersection, we deduce that $\ker\mu$, $\ker\lambda$ are conjugate under the Weyl group W. Now μ is W-conjugate to a multiple of λ, say $c\cdot\lambda$ with $0 < c \leq 1$.

Since $\mathrm{rk}\,\mathfrak{g} \geq 2$, we can find a simple root α of $\mathfrak{g}$ which is neither proportional nor orthogonal to λ. Then $s_\alpha\lambda := \lambda - 2\frac{\langle\lambda,\alpha\rangle}{||\alpha||^2}\alpha$ is a weight of V^c and so are λ, $\lambda-\alpha,\dots,\lambda-q\alpha$ where $q = 2\frac{\langle\lambda,\alpha\rangle}{||\alpha||^2}$ is a positive integer. These weights are all W-conjugate to a multiple of λ by what we have seen above, and they all lie in the union of two closed chambers because α is simple. Since W acts transitively on the set of chambers, we deduce that $q = 1$. In particular, there can be no weights of V^c of the form $c\cdot\lambda$, $0 < c < 1$. We have proved that all non-zero weights of V^c are W-conjugate. Therefore, V^c is minuscule. □

We finally check the hypothesis of Proposition 25.

Lemma 29. *Let* $p = \frac{1}{\sqrt{2}}(v_\lambda + v_{-\lambda})$. *Then the fixed point set of* G_p^0 *in* $\mathbb{S}(V)$ *is* $\{\pm p\}$.

Proof. Recall zero is not a weight of V^c. Write an arbitrary element of $\mathbb{S}(V)$ as $v = \sum_\mu c_\mu(v_\mu + v_{-\mu})$, where $c_\mu \in \mathbb{R}$, $v_{\pm\mu}$ are weight vectors and μ runs through the "positive" weights of V^c. If v is killed by $\ker\lambda$, then $c_\mu = 0$ unless μ is a positive multiple of λ. Since V^c is minuscule, we deduce that $\mathfrak{g}_p\cdot v = 0$ implies $v = c_\lambda(v_\lambda + v_{-\lambda}) = \pm p$. □

6. Non-spherical quotients

We start with the characterization of compact Riemannian manifolds M admitting a positive lower bound on the diameter of quotients by isometric actions. The first obvious observation is that if M is non-homogeneous, then such a bound exists, namely $\mathrm{diam}(M/\mathrm{Iso}(M))$. The homogeneous case is Theorem 1 from the Introduction. To prove it we need the following lemma:

Lemma 30. *Let G be a compact Lie group acting transitively on a compact connected smooth manifold M. Let $G' = [G^0, G^0]$ be the semi-simple part of G. Then $\pi_1(M)$ is finite if and only if G' acts transitively on M.*

Proof. If $\pi_1(M)$ is finite, then G' acts transitively by [Oni94, Proposition 4.9, page 94]. Conversely, if G' acts transitively, we choose any $x \in M$, and the long exact sequence of homotopy groups associated to $G'_x \to G' \to M$ implies that $\pi_1(M)$ is finite, because $\pi_1(G')$ is finite and G'_x has finitely many connected components. □

Proof of Theorem 1. Recall $\mathrm{Isom}(M)$ is a compact Lie group and assume first $\pi_1(M)$ is finite. Suppose to the contrary that no such ϵ exists. Then there exists a sequence of compact non-transitive subgroups G_i of the isometry group of M such that $\lim \mathrm{diam}(M/G_i) = 0$. By compactness of the Hausdorff metric, we may assume, after passing to a subsequence, that G_i converges to a compact subset $G_\infty \subset \mathrm{Isom}(M)$. Then G_∞ is a group, and $\mathrm{diam}(M/G_\infty) = 0$, that is, G_∞ acts transitively on M. By [MZ42], the groups G_i are eventually conjugate to subgroups of G_∞, so we may assume that $G_i \subset G_\infty$ for all i.

Since M has finite fundamental group, we may apply Lemma 30 to conclude that the semi-simple part $G'_\infty = [G^0_\infty, G^0_\infty]$ also acts transitively on M. The semi-simple parts G'_i, being subgroups of G_i, also act non-transitively, and thus form a sequence of proper subgroups of G'_∞ that converges to G'_∞. This contradicts [tD87, Chapter IV, Proposition 3.7], which says that a compact Lie group is a limit of proper subgroups if and only if it is not semi-simple. Therefore an $\epsilon > 0$ satisfying the statement of the theorem must exist.

For the converse, assume that $\pi_1(M)$ is infinite. Let G be a finite cover of $\mathrm{Isom}(M)$ of the form $G' \times T^k$, where G' is semi-simple. By Lemma 30, G' does not act transitively on M. Therefore neither does any group of the form $G' \times \Gamma$, for Γ a finite subgroup of the torus T^k. Taking a sequence of finite subgroups $\Gamma_i \subset T^k$ converging to T^k, we obtain a sequence of non-transitive subgroups $G_i = G' \times \Gamma_i$ of G such that $\lim_{i\to\infty} \mathrm{diam}(M/G_i) = 0$. □

In light of Theorem 1, one might suspect that the Main Theorem also generalizes to the class of all compact homogeneous spaces with finite fundamental group (normalized to have a fixed diameter). This turns out to be false, as the next example shows:

Example 31. Endow $\mathrm{SO}(n+1) \subset \mathbb{R}^{n^2}$ with the Riemannian metric g induced by the inner product $\langle A, B\rangle = \mathrm{tr}(AB^t)/2$ on $\mathbb{R}^{n^2}$. A straight-forward computation shows that the natural quotient map $\mathrm{SO}(n+1) \to \mathrm{SO}(n+1)/\mathrm{SO}(n) = \mathbb{S}^n$ is a Riemannian submersion, where $\mathbb{S}^n$ is endowed with the standard metric. The diameter of $(\mathrm{SO}(n+1), g)$ goes to infinity as $n \to \infty$, because it is bounded from below by the extrinsic diameter as a subset of $\mathbb{R}^{n^2}$. Indeed,

$$\mathrm{diam}(\mathrm{SO}(n+1), g) \ge d_g(I, -I) \ge d_{\mathbb{R}^{n^2}}(I, -I) = \sqrt{2(n+1)}$$

when n is odd, and similarly for n even.

Appendix A. Representations of real, complex, and quaternionic types

In this section we briefly collect a few definitions and basic facts about representations over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, of real, complex and quaternionic types that are used throughout the present article. A thorough treatment can be found in [BtD85, Section 2.6].

Let G be a compact group. A real representation of G is a group homomorphism $G \to \mathrm{GL}(U)$, where U is a real vector space. It is called irreducible when the only G-invariant real subspaces are $\{0\}$, U. In this case, Schur's lemma implies that the algebra of all G-equivariant endomorphisms of U is a real associative division algebra, which, by Frobenius' Theorem, must be isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. This representation is then called of real, complex, or quaternionic type, respectively.

A complex representation of G is a group homomorphism $G \to \mathrm{GL}(V)$, where V is a complex vector space, and it is called irreducible when the only G-invariant complex subspaces of V are $\{0\}$, V. In this case, it is called of real type (resp. quaternionic type) when it admits a real structure (resp. quaternionic structure), that is, a G-equivariant conjugate-linear map $\epsilon : V \to V$ with $\epsilon^2 = 1$ (resp. $\epsilon^2 = -1$). V is called of complex type when it admits neither a real nor a quaternionic structure, or equivalently, when V is not isomorphic to the complex-conjugate representation $\bar{V}$.

If U is a real irreducible representation of real type, then its complexification $V = \mathbb{C} \otimes_{\mathbb{R}} U$ (that is, the G-module obtained by extension of scalars) is a complex irreducible representation of real type. Conversely, given a complex irreducible representation of real type V, with real structure ϵ, then the fixed-point set U of ϵ is a real irreducible representation of real type, called the real form of V.

On the other hand, if U is a real irreducible representation of complex (resp. quaternionic) type, then it is the realification of an irreducible complex representation V of complex (resp. quaternionic) type, that is, it is obtained from V by restriction of scalars (from $\mathbb{C}$ to $\mathbb{R}$).

Appendix B. Proof of Lemma 13

Here we prove Lemma 13, which states: There exists a constant c such that, for every finite simple group S that is not cyclic or alternating, one has

$$\frac{\log |S|}{l(S)} \leq C.$$

We use the classification of finite simple groups, see e.g. [GLS94, Wil09]. We may discard the sporadic groups, as there are only finitely many of them. The remaining groups are the finite simple groups of Lie type, and come in 16 families, each parametrized by a prime power q, and possibly a natural number n. In [LS74], one finds lower bounds for $l(S, q)$ for all S of Lie type, where $l(S, q)$ is defined as the smallest dimension of a projective representation of G over a field of characteristic not dividing q. In each family there is a finite number of exceptions to this bound (listed in the third column of the table in [LS74, page 419]), which we may and will ignore. Since $l(S) \geq l(S, q)$, it suffices to show that, in each family, the quotient of $\log |S|$ by the bound provided in [LS74] is bounded from above. We proceed case by case, following Table 1 from [GLS94, page 8], and giving first the name as in [GLS94], followed by the name used in [LS74] (if different). In each case we find an upper bound for the order $|S|$ (whose exact value can be found in [GLS94, Table 1, page 8]), and a lower bound for the lower bound for $l(S, q)$ found in the table in [LS74, page 419].

(1) $A_n(q) = \mathrm{PSL}(n+1, q)$, $n \geq 1$. Then $|S| \leq q^{n^2+n-1}$ and $l(S) \geq (q^{n-1} - 1)/2 \geq q^{n-1}/4$, so that

$$\frac{\log |S|}{l(S)} \leq \frac{4(n+1)^2 \log(q)}{q^{n-1}}$$

goes to zero.

(2) ${}^2A_n(q) = \mathrm{PSU}(n+1,q)$, $n \geq 2$. Then $|S| \leq 2q^{n^2+n-1}$ and $l(S) \geq (q^n - q)/(q+1) \geq q^n/4$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{4(n+1)^2\log(q)}{q^n}$$
goes to zero.

(3) $B_n(q) = \mathrm{PSO}(2n+1,q)$, $n \geq 3$. Then $|S| \leq q^{2n^2+n}$ and $l(S) \geq q^{2(n-1)} - q^{(n-1)} \geq q^{2(n-1)}/4$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{4(2n^2+n)\log(q)}{q^{2(n-1)}}$$
goes to zero.

(4) ${}^2B_2(q) = \mathrm{Sz}(q)$. Then $|S| \leq q^5$ and $l(S) \geq \sqrt{q/2}(q-1) \geq q/4$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{20\log(q)}{q}$$
goes to zero.

(5) $C_n(q) = \mathrm{PSp}(2n,q)$, $n \geq 2$.Then $|S| \leq q^{2n^2+n}$ and $l(S) \geq \min\{q^n - 1, q^{n-1}(q^{n-1}-1)(q-1)\}/2 \geq q^n/4$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{4(2n^2+n)\log(q)}{q^n}$$
goes to zero.

(6) $D_n(q) = \mathrm{PSO}^+(2n,q)$, $n \geq 4$. Then $|S| \leq q^{(2n^2-n)}$ and $l(S) \geq q^{2n-3}/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{2(2n)^2\log(q)}{q^{2n-3}}$$
goes to zero.

(7) ${}^2D_n(q) = \mathrm{PSO}^-(2n,q)$, $n \geq 4$. Then $|S| \leq 2q^{2n^2-n}$ and $l(S) \geq q^{2n-3}/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{2((2n^2-n)\log(q)+\log(2))}{q^{2n-3}}$$
goes to zero.

(8) ${}^3D_4(q)$. Then $|S| \leq 2q^{28}$ and $l(S) \geq q^3(q^2-1) \geq q^5/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{2(28\log(q)+\log(2))}{q^5}$$
goes to zero.

(9) $G_2(q)$. Then $|S| \leq q^{14}$ and $l(S) \geq q(q^2-1) \geq q^3/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{28\log(q)}{q^3}$$
goes to zero.

(10) ${}^2G_2(q)$. Then $|S| \leq q^7$ and $l(S) \geq q(q-1) \geq q^2/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{14\log(q)}{q^2}$$
goes to zero.

(11) $F_4(q)$. Then $|S| \leq q^{52}$ and $l(S) \geq q^{10}/4$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{208\log(q)}{q^{10}}$$
goes to zero.

(12) ${}^2F_4(q)$. Then $|S| \leq q^{26}$ and $l(S) \geq q^5/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{52\log(q)}{q^{10}}$$
goes to zero.

(13) $E_6(q)$. Then $|S| \leq q^{78}$ and $l(S) \geq q^{11}/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{156\log(q)}{q^{11}}$$
goes to zero.

(14) ${}^2E_6(q)$. Then $|S| \leq q^{78}$ and $l(S) \geq q^{15}$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{78\log(q)}{q^{15}}$$
goes to zero.

(15) $E_7(q)$. Then $|S| \leq q^{133}$ and $l(S) \geq q^{17}/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{266\log(q)}{q^{17}}$$
goes to zero.

(16) $E_8(q)$. Then $|S| \leq q^{248}$ and $l(S) \geq q^{29}/2$, so that
$$\frac{\log|S|}{l(S)} \leq \frac{496\log(q)}{q^{29}}$$
goes to zero.

Appendix C. Proof of Lemma 22 – Analysing the representation of G^0

As in the statement of Theorem 21, let G^0 be a product of a torus with finitely many simply-connected compact connected simple Lie groups, Γ a finite group acting on G^0 by automorphisms, and $G = G^0 \rtimes \Gamma$. Let $\rho : G \to \mathrm{O}(V)$ be an almost faithful representation whose restriction $\rho^0 := \rho|_{G^0}$ to G^0 is irreducible, and such that $\operatorname{diam} \mathbb{S}(V)/G$ is small (but positive). The strategy to prove Lemma 22 is to show that, up to taking a subgroup of index at most 12, G is contained in one of the groups listed in Lemma 22, so that the statement will follow from Lemma 7. To achieve this, we will use Lemmas 10 and 12 to show that the normalizer of G^0 in $\mathrm{O}(V)$ is, up to small index, one of the groups listed in Lemma 22.

The proof will consist of a case-by-case analysis, with the division into cases as follows. First, G^0 is either semisimple, of the form $G^0 = G_1 \times \cdots \times G_k$ where the G_i are simply-connected compact connected simple Lie groups and $k \geq 1$; or G^0 not semisimple, of the form $G^0 = \mathrm{U}(1) \times G_1 \times \cdots \times G_k$ where the G_i are simply-connected compact connected simple Lie groups and $k \geq 1$. In the latter case, the torus is one-dimensional because the irreducibility of ρ^0 implies that the center of G is one-dimensional. As we will see below, $k = 1$ will lead to cases (i)–(iii) in the statement of Lemma 22, while $k \geq 2$ will lead to cases (iv)–(vii).

Second, the action of Γ on the simple factors may be transitive or not. And third, there is a complex irreducible representation $\pi : G^0 \to \mathrm{U}(W)$ such that either one of two cases happen: (i) ρ^0 is the real form of π; or (ii) ρ^0 is the realification of π. Thus there are in principle 8 cases, but as we will see below, only 4 may actually occur.

C.1. G^0 **semisimple, Γ-action transitive.** We can write $W = W_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} W_k$. where $\pi_i : G_i \to \mathrm{U}(W_i)$ is a complex irreducible representation. Since the action of Γ on the set of factors of G^0 is transitive, all factors are isomorphic. Fix isomorphisms once and for all. Now any two π_i, π_j differ by an automorphism of $G_i = G_j$. By composing π_i with an automorphism of G_i, we change ρ^0 to an orbit-equivalent representation and may assume all π_i equivalent representations.

Type (i): ρ^0 is a real form of π. If $k = 1$, then G^0 is simple, and hence its outer automorphism group has order at most 6. Since ρ^0 is of real type, its centralizer in $\mathrm{O}(V)$ is $\{\pm 1\}$. Together these imply that the index of G^0 in its normalizer in $\mathrm{O}(V)$ is at most 12. Now G^0 is as in case (i) of Lemma 22, and the desired lower bound on $\operatorname{diam} \mathbb{S}(V)/G$ is obtained from Lemma 7.

Assume $k \geq 2$. Let $\gamma \in N_{\mathrm{O}(V)}(G^0)$. Then $\gamma(G_i) = G_{\sigma(i)}$ for all i and a permutation $\sigma \in \Sigma_k$. View $\gamma \in N_{\mathrm{U}(W)}(G^0)$ such that γ centralizes the real structure ϵ, which we take $\epsilon = \epsilon_1 \otimes \cdots \otimes \epsilon_k$, where ϵ_i are "the same". Define the complex endomorphism γ_0 of W by

$$\gamma_0(w_1 \otimes \cdots \otimes w_k) = w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}.$$

Then γ_0 centralizes ϵ and normalizes G^0 by a simple calculation. It follows that $\tilde{\gamma} := \gamma\gamma_0^{-1}$ defines a real endomorphism of V that normalizes G_i for all i.

Next we distinguish two cases:

(a) The π_i are of real type. Here $V = V_1 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V_k$ where $\rho_i : G_i \to \mathrm{O}(V_i)$ is a real form of π_i.

We have $\rho|_{G_1} = (\dim_{\mathbb{R}} V_1)^{k-1}\rho_1$. Lemma 10 (real case) now implies that $\tilde{\gamma} \in \mathrm{O}(V_1) \times \mathrm{O}(V')$, where $V' = V_2 \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} V_k$. Proceeding by induction, we see that $\tilde{\gamma} \in \mathrm{O}(V_1) \times \mathrm{O}(V_2) \times \cdots \times \mathrm{O}(V_k)$. Therefore, up to an index 2 subgroup, we have $N_{\mathrm{O}(V)}(G^0) \subset \Sigma_k \ltimes \mathrm{SO}(n)^k$, which appears as case (iv) in Lemma 22. Here we may assume $k > 2$, because in case $k = 2$ the representation is polar and a lower bound on the diameter of the quotient follows from Theorem 9.

(b) The π_i are of quaternionic type and k is even. Here $\pi_1 \otimes \pi_2$ and $\pi_3 \otimes \cdots \otimes \pi_k$ are of real type.

We have $\rho|_{G_1 \times G_2} = (\dim_{\mathbb{H}} W_1 \otimes_{\mathbb{H}} W_2)^{\frac{k}{2}-1}[\pi_1 \otimes \pi_2]_{\mathbb{R}}$ where $[]_{\mathbb{R}}$ denotes a real form. Lemma 10 (real case) now implies that $\tilde{\gamma} \in \mathrm{O}(W_1 \otimes_{\mathbb{H}} W_2) \times \mathrm{O}(V')$, where V' is a real form of $W_3 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} W_k$, with components $\tilde{\gamma}_{12}$ and $\tilde{\gamma}_{3\cdots k}$. Applying Lemma 10 (quaternionic case) to $\tilde{\gamma}_{12}$ and proceeding by induction with $\tilde{\gamma}_{3\cdots k}$, we see that $\tilde{\gamma} \in \mathrm{Sp}(W_1) \times \cdots \times \mathrm{Sp}(W_k)$. Therefore, $N_{\mathrm{O}(V)}(G^0)$ is contained in the group listed in part (vi) of Lemma 22. We may assume $k \geq 4$ because in case $k = 2$ the representation is polar.

Type (ii): ρ^0 is the realification of π.

If $k = 1$, then the identity component of the normalizer of G^0 falls into case (ii) or (iii) of Lemma 22, according to whether ρ^0 is of complex or quaternionic type. Moreover, the outer automorphism group of the simple group G^0 has order at most 6. Therefore the index of G^0 in G is bounded by 6 and the desired lower bound on $\operatorname{diam} \mathbb{S}(V)/G$ is obtained from Lemma 7.

Assume $k \geq 2$. Let $\gamma \in N_{\mathrm{O}(V)}(G^0)$. We have $\gamma(G_i) = G_{\sigma(i)}$ for all i and a permutation $\sigma \in \Sigma_k$. Define the complex endomorphism γ_0 of W by

$$\gamma_0(w_1 \otimes \cdots \otimes w_k) = w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}.$$

Then γ_0 normalizes G_0. Next we distinguish two cases:

(a) The π_i are of complex type. Here π is of complex type. The element γ normalizes the centralizer of G^0, which is to say that γ is a complex linear or conjugate linear endomophism of W. By composing with complex conjugation, we may assume γ is complex linear. It follows that $\tilde{\gamma} := \gamma\gamma_0^{-1}$ is a complex endomorphism of W that normalizes G_i for all i.

We have $\pi|_{G_1} = (\dim_{\mathbb{C}} W_1)^{k-1}\pi_1$. Lemma 10 (complex case) says $\tilde{\gamma} \in \mathrm{U}(W_1) \times \mathrm{U}(W')$, where $W' = W_2 \otimes_{\mathbb{C}} \otimes \cdots \otimes_{\mathbb{C}} W_k$ (recall $\tilde{\gamma}$ is complex linear). Proceeding by induction, we see that $\tilde{\gamma} \in \mathrm{U}(W_1) \times \cdots \times \mathrm{U}(W_k)$. Therefore, up to a subgroup of index 2, $N_{\mathrm{O}(V)}(G^0)$ is contained in the group listed in part (v) of Lemma 22. We may assume $k > 2$, since in case $k = 2$ the representation is polar.

(b) The π_i are of quaternionic type and k is odd.

Here π is of quaternionic type and $\mathrm{Sp}(1) = Z_{\mathrm{O}(V)}(G^0)$. The element γ normalizes $\mathrm{Sp}(1)$, and since this group has no outer automorphisms, we may assume γ centralizes it, which is to say that γ is quaternionic linear. Also γ_0 is quaternionic linear, so $\tilde{\gamma} := \gamma\gamma_0^{-1}$ defines a quaternionic endomorphism of V that normalizes G_i for all i.

Write $V = W_1^r \otimes_{\mathbb{R}} V'$, where W_1^r denotes the realification of W_1, and V' is a real form of $W_2 \otimes_{\mathbb{C}} \otimes \cdots \otimes_{\mathbb{C}} W_k$. Then $\rho|_{G_1} = (\dim_{\mathbb{R}} V')\pi_1^r$ and Lemma 10 (real case) says that $\tilde{\gamma} \in \mathrm{O}(W_1^r) \times \mathrm{O}(V')$, with components $\tilde{\gamma}_1$ and $\tilde{\gamma}'$. Now we recall that $\tilde{\gamma}$ is quaternionic and $\tilde{\gamma}'$ is real to note that indeed $\tilde{\gamma}_1 = \tilde{\gamma}\tilde{\gamma}'^{-1} \in \mathrm{Sp}(W_1)$, and we apply case (i)(b) to $\tilde{\gamma}'$ to deduce that $\tilde{\gamma} \in \mathrm{Sp}(W_1) \times \cdots \times \mathrm{Sp}(W_k)$. Therefore we are in case (vii) of Lemma 22, and we note that in case $k = 1$ the representation is polar.

C.2. G^0 **semi-simple, Γ-action non-transitive.** As above, we can write $W = W_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} W_k$, where $\pi_i : G_i \to \mathrm{U}(W_i)$ is a complex irreducible representation. If the action of Γ on the set of factors of G^0 is non-transitive, then $k \geq 2$ and each orbit produces a connected normal subgroup of G. We shall shortly see that here we are dealing with Type (i), that is, V is a real form of W.

Write $G^0 = G_a \times G_b$, where G_a and G_b are non-trivial Γ-invariant subgroups, and write $V = V_a \otimes_{\mathbb{F}} V_b$ accordingly. Since we are assuming $\operatorname{diam} \mathbb{S}(V)/G$ small, Lemma 12 says that the action of any normal subgroup of G is either irreducible or super-reducible. It follows that G_a acts by scalars and $V|_{G_b}$ is irreducible, up to interchanging a and b. Since G^0 is connected and semisimple, this says that $G_a = \mathrm{Sp}(1)$, $V_a = \mathbb{H}$, $\mathbb{F} = \mathbb{H}$ and V_b is of quaternionic type. Note that Γ must act transitively on the factors of G_b, for otherwise $G_b = \mathrm{Sp}(1) \times G_c$ is a non-trivial Γ-invariant decomposition and $\mathrm{Sp}(1)\,\mathrm{Sp}(1) = \mathrm{SO}(4)$ neither acts by quaternionic scalars nor is irreducible on V.

Now we can write $G_1 = \mathrm{Sp}(1)$ and $W_1 = \mathbb{C}^2$, $G_2 = \cdots = G_k$, $\pi_2 = \cdots = \pi_k$ are of quaternionic type with respective quaternionic structures $\epsilon_1 = \cdots = \epsilon_k$ and k is even.

Let $\gamma \in N_{\mathrm{O}(V)}(G^0)$. Then $\gamma \in N_{\mathrm{O}(V')}(G')$ where $G' = G_2 \times \cdots \times G_k$ and V' is the realification of $W_2 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} W_k$. By multiplying by an element of $\mathrm{Sp}(1)$, we may assume γ centralizes $\mathrm{Sp}(1)$. Now we apply case (ii)(b) to deduce that $\gamma \in \Sigma_{k-1} \ltimes \mathrm{Sp}(W_2) \times \cdots \times \mathrm{Sp}(W_k)$. Thus G is a subgroup of the group in case (vii) of Lemma 22 (note the different meanings of k here and there). Note also that the case $G^0 = \mathrm{Sp}(1) \times \mathrm{Sp}(n)$ is transitive on the unit sphere.

C.3. G^0 **non-semisimple.** As discussed above, $G^0 = \mathrm{U}(1) \times G_1 \times \cdots \times G_k$. The representation ρ^0 is necessarily of complex type, so it is the realification of $\pi : G^0 \to \mathrm{U}(W)$. Write $W = \mathbb{C} \otimes_{\mathbb{C}} \otimes W_1 \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} W_k$. where $\mathrm{U}(1)$ acts on $\mathbb{C}$ by complex scalar multiplication and $\pi_i : G_i \to \mathrm{U}(W_i)$ is a complex irreducible representation.

An argument using Lemma 12 similar to that in subsection C.2 shows that the action of Γ on set of the factors of $G^0/\,\mathrm{U}(1)$ is transitive. Let $\gamma \in N_{\mathrm{O}(V)}(G^0)$. Then $\gamma \in N_{\mathrm{O}(V')}(G/\,\mathrm{U}(1))$ and we apply case (ii)(a) to see that

$$\gamma \in \mathbb{Z}_2 \cdot \Sigma_k \ltimes \mathrm{U}(W_1) \times \cdots \times \mathrm{U}(W_k)$$

where $\mathbb{Z}_2$ acts on V as complex conjugation. Therefore up to taking a subgroup of index 2, G is a subgroup of the group in case (v) of Lemma 22. Note that case $k = 1$ is transitive on the unit sphere and case $k = 2$ is polar.

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