Алгебра и анализ
Том 16 (2004), вып. 6

DIFFERENTIATION IN METRIC SPACES

© ALEXANDER LYTCHAK

We discuss differentiation of Lipschitz maps between abstract metric spaces and study such issues as differentiability of isometries, first variation formula and theorems of Rademacher type.

§1. Introduction

1.1. The Aim. This paper is devoted to the study of the first order geometry of metric spaces. Our study was mainly motivated by the observation that whereas the advanced features of the theories of Alexandrov spaces with upper and lower curvature bounds are quite different, the beginnings are almost identical, at least as far as only first order derivatives are concerned (for example tangent spaces and the first variation formula). One is naturally led to the question on which spaces the first order geometry can be established. As it turns out the same first order geometry exists in many other spaces that we call geometric. The class of geometric spaces contains all Hölder continuous Riemannian manifolds, sufficiently convex and smooth Finsler manifolds ([LY]), a big class of subsets of Riemannian manifolds (for example sets of positive reach, see [Fed59] and [Lyta]), surfaces with an integral curvature bound ([Res93]) and extremal subsets of Alexandrov spaces with lower curvature bound ([PP94a]). The last case was discussed in [Pet94] and the proof of the first variation formula was a major step towards proving the deep gluing theorem ([Pet94]). Moreover the class of geometric spaces is stable under metric operations, even under such a difficult one as taking quotients. Finally the existence of the first order geometry is a good assumption for studying features of higher order, such as gradient flows of semi-concave functions ([PP94b] and [Lytc]). One of the main issues of this paper is the establishing of natural, easily verifiable axioms, that describe this first order geometry and their consequences.

Keywords and phrases: Alexandrov spaces, Rademacher theorem, variation formulas, tangent cones.

Another more direct motivation comes from the question whether a submetry (or more specially an isometry) between metric spaces must be differentiable in some suitable sense. This question was answered affirmatively for smooth Riemannian manifolds in [BG00]. On the other hand in [CH70] an example of a non-differentiable isometry between Riemannian manifolds with continuous Riemannian metrics is constructed. Even for asking this question a language is needed that allows us to speak about differentiability of Lipschitz mappings. Another main issue of this paper is the establishing of such a language.

Remark 1.1. For (special) doubling metric measure spaces Cheeger has developed in [Che99] a deep theory giving a Rademacher type theorem for such spaces. However this approach does not allow to speak about differentiability in a given (singular) point. Moreover it is essentially restricted to differentiation of functions and does not apply to maps into another singular space. Kirchheim developed in [Kir94] a very interesting theory of metric differentiation of Lipschitz mappings of the Euclidean space into arbitrary metric spaces. The disadvantage of this theory is that it completely neglects the (possibly existing) tangent structure of the image space. Whereas Kirchheims definition has a clear interpretation in our language, the connections to the theory of Cheeger are much less clear and will not be discussed here.

In this paper we discuss the basics of the theory. In [Lytc] we study connections between properties of the differentials and the map itself, in [Lytb] we apply these ideas to differentiability in Carnot–Caratheodory spaces.

1.2. The Problem. The notion of a tangent cone at a point of a proper metric space was defined by Gromov using Gromov–Hausdorff convergence of rescaled spaces, through the requirement that an infinitesimal portion of the space at x does not depend on the infinitesimal scale. In many situations this concept has been used to study the properties of the original space (for example [BGP92, Pet94, Mit85] and many others). Unfortunately this definition being perfect for the study of the infinitesimal portions of a given space is not very suitable for the study of differentials. The problem is that Gromov–Hausdorff convergence of abstract metric spaces is defined only on the set of isometry classes. For example the question about differentiability of isometries does not make any sense in this context. Dealing with differentials one would prefer to know what happens in a fixed direction in the tangent space. Mostow and Margulis encountered this problem as they were dealing with differentials between Carnot–Caratheodory spaces ([MM00]).

1.3. The Method. To circumvent this problem we give a slightly different definition of the tangent cone, working with ultra-convergence instead of the

Gromov–Hausdorff convergence, a notion widely used in the theory of non-positively curved spaces. Namely for each zero sequence $(o) = (\epsilon_i)$ which we call an (infinitesimal) scale one can consider the blow up $X_x^{(o)}$ of the pointed space (X, x) at the scale (o), given as the ultralimit $X_x^{(o)} = \lim_\omega(\frac{1}{\epsilon_i}X, x)$. Now we say that a tangent cone T_xX of X at the point x is a metric cone $(T, 0)$ together with a fixed choice of pointed isometries $i^{(o)} : T \to X_x^{(o)}$ for each scale (o), such that certain natural commutation relations (Definition 6.1) are satisfied.

If the tangent space exists in the sense of Gromov, our definition just makes the additional requirement of fixing a special choice of a metric space in the isometry class of the tangent space in the sense of Gromov (Remark 6.2). With this definition of the tangent space, the differential of a Lipschitz map is the blow up at the given point, if this blow up is unique.

If the tangent spaces in X and in Y exist, then they exist in a natural way in the product $X \times Y$ and in the Euclidean cone CX. Moreover there is a natural choice (up to the tangent cones in X) of the tangent cones to subsets of X.

In general no tangent space in our sense may exist or there may be no natural choice (we assumed in the definition, that the isometries $i^{(\epsilon_i)}$ are given somehow). However, tangent spaces exist in lots of important singular metric spaces. This existence is given by a (not necessarily continuous) map e from a small portion of a metric cone T to a small neighborhood of the given point x, that is an infinitesimal isometry at x (thus being a very singular equivalent of the exponential map, see Subsection 3.5 and Subsection 6.1 for the precise definition). All examples of tangent cones known to the author arise in this way. One problem closely related to the question whether the tangent cone is defined in a natural way is that the identification of the tangent space at x with the ultraproduct T^ω (this ultraproduct is equal to T if T is proper) via this map e depends not only on e and the metric of T but also on the particular metric cone structure on T, i.e. a particular choice of the dilations (see Section 4). This is the reason for the pathological example of [CH70], see Example 7.6.

Even though we use a choice of an ultrafilter ω in our definitions, the notions of differentiability and differential do not depend on ω if the tangent cones are given by a map e as above (see Subsection 7.2). For example for Lipschitz mappings between Banach spaces we get the usual definition of directional differentiability.

1.4. Geometric conditions. In order to obtain the tangent cones (the isometries $i^{(\epsilon_i)}$) in a natural way, we observe that each metric space defines in a natural way a cone C_x at each point x, being the set of germs of unparameterized geodesics starting at x. Moreover this cone C_x comes along with a natural family of 1-Lipschitz exponential mappings $\exp_x^{(\epsilon_i)} : C_x \to X_x^{(\epsilon_i)}$ to the different blow ups of X at x. We now define a generalized angle condition (A), that is satisfied by spaces with one-sided curvature bound, by strongly convex Banach spaces and many others (see below). It generalizes the usual condition of the equality of the upper and the lower angles (Example 5.3). We say that X has the property (A) at x if $\lim_{t\to 0} \frac{d(\gamma_1(t),\gamma_2(st))}{t}$ exists for all $s \in \mathbb{R}^+$ and all geodesics γ_1, γ_2 starting at x.

However even in proper geodesic spaces geodesics may see only a small part of the blow ups, as the example of Carnot–Caratheodory spaces shows. To guarantee the surjectivity of the exponential maps, we impose a uniformity condition (U). We say that a locally geodesic space X with the property (A) at x has the property (U) at x, if the geodesic cone C_x is proper and $d(\gamma_1(t), \gamma_2(t)) \leqslant O(t, d(\gamma_1^+, \gamma_2^+))t$ holds, where $d(\gamma_1^+, \gamma_2^+)$ is the distance between the starting directions γ_i^+ of γ_i in C_x and O is some function going to 0 if both arguments go to 0. Given this condition one can define a natural (however not continuous) exponential map $e : C_x \to X$ identifying C_x with the tangent cone T_xX. Hence in spaces with the property (U) the tangent space exist in a natural way. We call a locally geodesic space X infinitesimally cone-like, if it has the property (U) at each point and each tangent cone $T_xX = C_x$ is a Euclidean cone (Definition 6.3). In [Lytc] we prove that gradient flows of semi-concave functions exist in such spaces, generalizing the corresponding result of [PP94b].

Finally to be able to deal with distance functions, we need a further condition. We say that geodesics vary smoothly at x, if small long and thin quadrangles with a vertex at x essentially look like quadrangles in C_x (see Definition 9.2). This expresses the fact that geodesics converging pointwise to a given geodesic converge also in some better sense. For example it is true in a continuous Riemannian metric, if all geodesics are uniformly $C^{1,\alpha}$ for some $\alpha > 0$, for this reason the name.

Remark 1.2. This (local) condition is almost equivalent to the global statement that the first variation formula is valid in X, see Section 9 for details.

We call a proper geodesic space geometric if it has the property (U) at each point, each tangent cone $T_xX = C_x$ is a uniformly convex and smooth cone (for example a Euclidean cone or a Banach space with a strongly convex and smooth norm, see Definition 4.3 and Definition 4.4) and if geodesics vary smoothly at each point (Definition 10.1).

1.5. Results. As was already mentioned in the beginning the class of geometric spaces is very big. Alexandrov spaces (see Definition 2.2), surfaces with an integral curvature bound, manifolds with only Hölder continuous Riemannian metrics, sets of positive reach and some more general subsets of Riemannian manifolds are geometric and infinitesimally cone-like. A finite dimensional Banach space is geometric iff its norm is strongly convex and smooth. Finsler manifolds with Hölder continuous and pointwise smooth and sufficiently convex norms are geometric. Products and convex sets of and Euclidean cones over (infinitesimally cone-like) geometric spaces are (infinitesimally cone-like) geometric. Each open subset of an infinitesimally cone-like space is infinitesimally cone-like.

We can now state our results. A map $f : X \to Z$ of a space X with property (U) at x to another metric space Z is differentiable at x iff it is directionally differentiable at x, i.e. if $f \circ \gamma : [0, \epsilon) \to Z$ is differentiable at 0 for all geodesics γ starting at x. This implies:

Proposition 1.1. *Let $f : X \to Z$ be an isometric embedding. If X and Z are infinitesimally cone-like* (*or more general just have the property* (U)) *then f is differentiable at all points.*

In geometric spaces the first variation formula holds, i.e. the distance functions d_S to subsets S and the metric $d : X \times X \to \mathbb{R}$ itself are differentiable and the differential of d_S at x depends only on the set of directions in C_x of minimal geodesics, between x and S, see Proposition 9.3 for the precise formulation, where the usual angles are replaced by the corresponding Busemann functions. If the tangent spaces are Euclidean cones one gets the usual first variation formula:

Theorem 1.2. *Let X be an infinitesimally cone-like space, $x \neq z \in X$. Let γ be a geodesic between x and z with starting resp. ending directions $\gamma^+ \in T_xX$ and $\gamma^- \in T_zX$. Then the differential of the distance $d : X \times X \to \mathbb{R}$ can be estimated by $D_{(x,z)}d(v, w) \leqslant -\langle \gamma^+, v\rangle - \langle \gamma^-, w\rangle$. If X is in addition geometric, then $D_{(x,z)}d(v, w)$ exists and is equal to the above sum for some geodesic γ between x and z.*

Using the uniform convexity of the tangent spaces we see that the distance functions to points in a geometric space play the role of the coordinate functions in the Euclidean space, i.e. a Lipschitz map $f : X \to Z$ of a space X to a geometric space Z is differentiable at x if for a dense sequence of points z_n in a punctured neighborhood of $f(x)$ the composition functions $d_{z_n} \circ f$ are differentiable at x. Now the first statement of the next proposition is an easy application, whereas the second one requires some work. It shows that our notion of geometric spaces is stable enough to survive such a difficult operation as taking quotients.

Theorem 1.3. *Let $f : X \to Y$ be a submetry. If X and Y are geometric, then f is differentiable at each point. Moreover the assumption that X is geometric already implies that Y is geometric.*

Moreover it is possible to describe precisely the differential structure of a submetry, getting the usual vertical (tangent space to the fiber) and horizontal (tangent space to the union of horizontal geodesics) subspaces of the tangent space.

For maps into a geometric space the theorem of Rademacher is equivalent to the theorem of Rademacher for functions:

Proposition 1.4. *Let Z be a metric space with a Borel measure μ, and tangent spaces at almost each point, such that each Lipschitz function $f : Z \to \mathbb{R}$ is differentiable μ-almost every where. Then for each geometric space X each Lipschitz map $f : Z \to X$ is differentiable almost everywhere.*

Corollary 1.5. *If Z is a measurable subset of the Euclidean space $\mathbb{R}^n$ and $f : Z \to X$ is a locally Lipschitz map to a geometric space X, then for almost all $z \in Z$ the differential $D_z f$ exists, the image $D_z f(\mathbb{R}^n) \subset T_{f(z)}X$ is a Banach space and the restriction $D_z f : \mathbb{R}^n \to D_z f(\mathbb{R}^n)$ is linear.*

A final issue that we address in this paper is differentiability of maps into arbitrary spaces with a one-sided curvature bound. In this situation the tangent space in our sense may not exist, however one can use the same ideas and work with the geodesic cone C_x instead of the tangent cone. For semi-concave functions this is used in [Lytc], here we prove:

Theorem 1.6. *Let Z be either $CAT(\kappa)$ space or a space with curvature $\geqslant \kappa$. Let $S \subset \mathbb{R}^n$ be a measurable subset, $f : S \to Z$ a locally Lipschitz map. Then f has at almost each point a differential $D_x f : T_x S \to C_{f(x)} Z$.*

Remark 1.3. If Z is an Alexandrov space in the sense of Definition 2.2 then Theorem 1.6 is a special case of Proposition 1.4.

1.6. The Plan. After the preliminaries we recall some basic notions concerning ultra-convergence of spaces and maps, a major tool for this paper. In Section 4 we discuss basic issues about general metric cones. In Section 5 we start with differential issues and discuss geodesic cones and the exponential mappings. In Section 6 and Section 7 we give the definition of tangent cones differentials, give the main examples and discuss the condition (U) and some other related topics. In Section 8 we recall Kirchheim's notion of metric differentiability. In Section 9 we discuss the first variation formula. In Section 10 and Section 11 geometric spaces are studied. Finally in Section 12 we prove Theorem 1.6.

1.7. Acknowledgments. I would like to thank Werner Ballmann for encouragement, support, helpful discussions and comments. I am very grateful to Sergei Buyalo for many remarks and corrections. I am thankful to Juan Souto for useful comments and suggestions.

§2. Preliminaries and Notations

2.1. Notations. By $\mathbb{R}^+$ resp. $\mathbb{R}^n$ we will denote the positive real numbers resp. the Euclidean space. We shall denote by d the distance in metric spaces. For a subset A of a metric space X we denote by d_A the distance function to the set A. For a positive number r we denote by rX the set X with the metric scaled by r. By $B_r(x)$ we denote the closed ball of radius r around x. A pseudo metric d on a space X is a metric for which the distance between different points may be 0. Identifying in X points x, z with $d(x, z) = 0$ we get the corresponding metric space. A map $f : X \to Y$ between metric spaces is called L-Lipschitz if for all $x, z \in X$ one has $d(f(x), f(z)) \leqslant Ld(x, z)$.

Example 2.1. Each distance function d_A is 1-Lipschitz, whereas the metric $d : X \times X \to \mathbb{R}$ is a $\sqrt{2}$-Lipschitz function.

An s-dilation is a bijective map $f : X \to Y$ between metric spaces with $d(f(x), f(\bar{x})) = sd(x, \bar{x})$ for all $x, \bar{x} \in X$. An isometry is a 1-dilation.

Definition 2.1. By a scale we will denote a sequence $(o) = (\epsilon_i)$ of positive real numbers converging to 0.

2.2. Geodesics. For a curve γ in X we will denote its length by $L(\gamma)$. A geodesic resp. ray resp. line in X is an isometric embedding of an interval resp. half-line resp. the whole real line into X. For disjoint subsets $S, T \subset X$ we denote by $\Gamma_{S,T}$ the set of all geodesics of length $d(S, T)$ starting in S and ending in T. The space X is called geodesic if for all points $x \neq z$ in X the set $\Gamma_{x,z}$ is not empty. Finally we will denote by Γ_x the set of all geodesics starting at x.

A metric space X is called proper if its closed bounded subsets are compact. In a proper geodesic space X the set $\Gamma_{S,T}$ is compact and not empty if S is compact and T is closed.

2.3. Busemann functions. For a ray $h : [0,\infty) \to X$ in space X we denote by b_h its Busemann function $b_h(x) = \lim_{t\to\infty}(d(x,h(t)) - t)$. This limit always exists and b_h is a 1-Lipschitz function.

Example 2.2. If $f : X \to \mathbb{R}$ is a 1-Lipschitz map with $f(h(t)) = -t$, then $f \leqslant b_h$ holds. Especially for rays γ_j converging to a ray γ the inequality $\liminf b_{h_j} \leqslant b_h$ holds.

Example 2.3. Let γ be a line defining two rays γ^+ and γ^-. Then $-b_{\gamma^-}$ is a 1-Lipschitz function satisfying $-b_{\gamma^-} = b_{\gamma^+}$ on γ. Therefore we get $b_{\gamma^-} + b_{\gamma^+} \geqslant 0$ on X. We will call γ straight if $b_{\gamma^-} + b_{\gamma^+} = 0$ in X.

Example 2.4. For $i = 1,2$ let h_i be a ray in the space X_i. Then $h(t) = (h_1(\frac{t}{\sqrt{2}}), h_2(\frac{t}{\sqrt{2}}))$ is a ray in $X_1 \times X_2$. Let $f : X_1 \times X_2$ be a $\sqrt{2}$-Lipschitz function satisfying $f(h_1(t), h_2(t)) = -2t$. Then for $(v,w) \in X_1 \times X_2$ we get the inequality $f((v,w)) \leqslant \sqrt{2} b_h((v,w)) = b_{h_1}(v) + b_{h_2}(w)$.

2.4. Alexandrov spaces. We refer to [BBI01, BGP92, BH99] for the theory of spaces with one-sided curvature bound. A space X is called a $CAT(\kappa)$ space resp. a space with curvature $\geqslant \kappa$ if it is complete and geodesic and triangles in X are not thicker resp. not thinner than triangles in the two dimensional simply connected manifold M^2_κ of constant curvature κ.

Definition 2.2. We will call a space X an Alexandrov space, if X is a proper space, that either has curvature $\geqslant \kappa$ and finite Hausdorff dimension or that is geodesically complete (i.e. each geodesic is part of an infinite locally geodesic) and contains a $CAT(\kappa)$ neighborhood of each of its points, for some $\kappa \in \mathbb{R}$.

§3. Ultralimits

3.1. Ultraconvergence of spaces. A reader not used to ultrafilters and ultralimits should consult [BH99] or [KL97] for excellent accounts. Let ω denote an arbitrary non-principal ultrafilter on the set of natural numbers. It allows to choose for each sequence (x_i) in a compact Hausdorff space X a point $\lim_\omega(x_i)$ among the limit points of the sequence. It also allows us to construct a limit space of a sequence of spaces and limits of Lipschitz maps between them in the following manner.

For a sequence (X_i, x_i) of pointed metric spaces their ultralimit $(X,x) =: \lim_\omega(X_i,x_i)$ is defined to be the set of all sequences (z_i) of points $z_i \in X_i$ with $\sup\{d(z_i,x_i)\} < \infty$. On this set one considers the pseudo metric $d((z_i),(y_i)) := \lim_\omega(d(z_i,y_i))$. The ultralimit (X,x) is the metric space arising from this pseudo metric.

Example 3.1. Let (X_i, x_i) be a constant sequence (X, x). We call then $\lim_\omega(X_i, x_i)$ the ultraproduct of (X, x) and denote it by X^ω. This space contains (X, x) in a natural way $(z \to (z, z, z \dots))$ and does not depend on the base point x. It coincides with (X, x) iff X is a proper space.

3.2. Relation to the usual convergence. The following lemma allows us to replace ultralimits by limits, if the statement concerns all sequences:

Lemma 3.1. *Let (x_j) be a sequence in a complete metric space (X, x) with uniformly bounded distances to x. If for each subsequence (x_{k_i}) of this sequence the point $z = (x_{k_i})$ in the ultraproduct $X^\omega = \lim_\omega(X, x)$ does not depend on the subsequence, then z is in X and the sequence (x_j) converges to z.*

Proof. Assume that x_i is not a Cauchy sequence. Then replacing (x_i) by a subsequence, we may assume $d(x_i, x_{i+1}) > \epsilon$ for all i. Consider the subsequence y_i of x_i given by $y_i = x_{i+1}$. Then the points (y_i) and (x_i) have in X^ω distance at least ϵ from each other. Contradiction. •

Gromov–Hausdorff topology on the set of isometry classes of pointed proper metric spaces is closely related to ultralimits. If a sequence (X_i, x_i) of proper metric spaces converges to a proper space (X, x) in the Gromov-Hausdorff topology, then $\lim_\omega(X_i, x_i)$ is in the isometry class of (X, x) (see [KL97, p. 132]).

3.3. Ultralimits of maps. Each sequence of L-Lipschitz maps $f_j : (X_j, x_j) \to (Y_j, y_j)$ induces in a natural way an ultralimit $f = \lim_\omega f_j$ that is an L-Lipschitz map between the ultralimits (X, x) resp. (Y, y) of the sequences (X_j, x_j) resp. (Y_j, y_j), defined by $f((z_j)) := (f_j(z_j))$. These ultralimits of maps commute with compositions.

Example 3.2. If γ_j are L-Lipschitz curves in X_j starting at x_j, then $\gamma = \lim_\omega \gamma_j$ is an L-Lipschitz curve in $(X, x) = \lim_\omega(X_j, x_j)$ starting at x. If all curves γ_j are geodesics, then so is γ. In particular if all the spaces X_j are geodesic, then so is X. Actually X is geodesic if X_j are only length metric spaces.

Example 3.3. The ultralimit of products of spaces is the product of the corresponding ultralimits. If (S_j, x_j) are subsets of (X_j, x_j) then the ultralimit $\lim_\omega(S_j, x_j)$ is embedded into $\lim_\omega(X_j, x_j)$ in a natural way.

Example 3.4. Let X_j be a $CAT(\kappa_j)$ space resp. a space with curvature $\geqslant \kappa_j$, with $\kappa_j \to \kappa$. Then $\lim_\omega(X_j, x_j)$ is a $CAT(\kappa)$ space resp. a space with curvature $\geqslant \kappa$. For spaces with upper curvature bound this is proved in

[KL97]. For lower curvature bound the statement is not completely trivial, but it follows directly from [PP94b], Subsection 1.6.

Remark 3.5. The ultralimits of sequences of spaces and maps usually depend on the choice of the ultrafilter ω. In fact if for a sequence (X_i, x_i) of proper metric spaces the isometry class of $(X, x) = \lim_\omega (X_i, x_i)$ does not depend on the ultrafilter ω and if this space X is proper, then the sequence of the isometry classes of (X_i, x_i) is a convergent sequence with respect to the Gromov–Hausdorff topology.

3.4. Blow up. Let X be a metric space, $x \in X$. For each scale $(o) = (\epsilon_i)$ we get a blow up $X_x^{(o)} = \lim_\omega (\frac{1}{\epsilon_i} X, x)$ at the scale (o). It is a space with a distinguished point $0 = (x, x, \ldots)$. If $f : (X, x) \to (Y, y)$ is a locally Lipschitz map, we get a blown up map: $f_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$. For a subspace S of X containing x we get a subspace $S_x^{(o)}$ of $X_x^{(o)}$. In particular a geodesic γ starting at x defines a ray $\gamma_x^{(o)}$ starting at 0.

Remark 3.6. If X is a doubling metric space near x, i.e. if for some $C > 0$, each $r \leqslant \frac{1}{C}$ and each point $z \in B_{\frac{1}{C}}(x)$ the ball $B_r(z)$ can be covered by C balls of radius $\frac{r}{2}$, then each blow up $X_x^{(o)}$ is a proper metric space. For example this is the case if X is a doubling measure space (see [Che99]).

Example 3.7. If X is a Banach space, resp. has lower resp. upper curvature bound, then for each scale (o) the blow up $X_x^{(o)}$ is a Banach resp. a non-negatively curved resp. a $CAT(0)$ space.

Example 3.8. Let $(o) = (t_i)$ and $(\tilde{o}) = (r_i)$ be different scales. In general there is no possibility to compare the blow ups $X_x^{(o)}$ and $X_x^{(\tilde{o})}$. However if the scales are comparable, i.e. if $0 < \lim_\omega (\frac{t_i}{r_i}) := s < \infty$ holds, then the identity $id : (\frac{1}{t_i} X, x) \to (\frac{1}{r_i} X, x)$, being an $\frac{t_i}{r_i}$-dilation induces a natural s-dilation $id_{(\tilde{o})}^{(o)} : (X_x^{(o)}, 0) \to (X_x^{(\tilde{o})}, 0)$.

3.5. Infinitesimal isometries. The following definition is the metric analog of the notion of a Lebesgue point.

Definition 3.1. Let (S, x) be a subset of (X, x). We will say that S is infinitesimally dense at x if for each scale (o) the canonical isometric embedding $i^{(o)} : S_x^{(o)} \to X_x^{(o)}$ is onto (i.e. an isometry).

The above definition just says, that for each $\epsilon > 0$ and all sufficiently small δ the ball $B_\delta(x) \subset S$ is $\epsilon\delta$-dense in the ball $B_\delta(x) \subset X$.

Example 3.9. If S is dense in a neighborhood of x in X, then S is infinitesimally dense at x. If X is a doubling metric measure space ([Che99]) and S a measurable subset, then S is infinitesimally dense at each of its Lebesgue points.

Example 3.10. Let X be complete and geodesic. If a closed subset S of X is infinitesimally dense at each point $x \in S$, then $S = X$ ([Lytc]).

Definition 3.2. Let $e : (X, x) \to (Y, y)$ be a not necessarily continuous map. We will call e an infinitesimal isometric embedding (at x) if $|d(e(x_1), e(x_2)) - d(x_1, x_2)| \leqslant o(d(x_1, x) + d(x_2, x))$, for all $x_1, x_2 \in X$ and some function $o : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t\to 0} \frac{o(t)}{t} = 0$. We will say that e is an infinitesimal isometry (at x) if in addition the image $e(B_\delta(x))$ of each ball $B_\delta(x)$ around x is infinitesimally dense at y in Y.

Example 3.11. A Lipschitz map $e : (X, x) \to (Y, y)$ is an infinitesimal isometry iff for each scale (o) the blow up $e_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ is an isometry.

A composition of infinitesimal isometries is again an infinitesimal isometry. The importance of this notion is due to the following easy observation:

Lemma 3.2. *Let $e : (X, x) \to (Y, y)$ be an infinitesimal isometry. Then for each scale (o) the map $e_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ given by $e_x^{(o)}((x_i)) = (e(x_i))$ is well defined. Moreover it is an isometry.*

Example 3.12. Let (S, x) be a subset of (X, x). Define a map $e : (X, x) \to (S, x)$ by setting $e(z) = \bar{z}$, where $\bar{z}$ is an arbitrary point in S with $d(z, \bar{z}) \leqslant 2d(z, S)$. Then e is an infinitesimal isometry iff S is infinitesimally dense at x. In this case $e_x^{(o)} : X_x^{(o)} \to S_x^{(o)}$ is the canonical identification.

These (generalized) blow ups are again compatible with compositions (of infinitesimal isometries). From Example 3.12 one deduces, that for each infinitesimal isometry $e : (X, x) \to (Y, y)$ there is an infinitesimal isometry $\bar{e} : (Y, y) \to (X, x)$ such that for each scale (o) one has $e^{(o)} \circ \bar{e}^{(o)} = id$ and $\bar{e}^{(o)} \circ e^{(o)} = id$.

§4. Metric cones

4.1. Group of dilations. Let (X, x) be a pointed metric space. Consider the group $Dil_x(X)$ of all dilations of X leaving the point x invariant, equipped with the topology of pointwise convergence. The natural map $P : Dil_x(X) \to \mathbb{R}^+$ sending an s-dilation to the number s is a continuous homomorphism. The kernel of P is the group I_x of isometries of X fixing the point x.

Definition 4.1. A metric cone structure on the space (X, x) is a continuous section of the homomorphism P above, i.e. a continuous homomorphism $\rho : \mathbb{R}^+ \to Dil_x(X)$ that sends s to some s-dilation ρ_s. A metric cone is a space with a metric cone structure. We call x the origin of the metric cone X and denote it by 0. A map $f : X \to Y$ between metric cones is called homogeneous if it commutes with all dilations ρ_t.

A metric space (X, x) can admit several families of dilations making it to a metric cone. If X is a proper metric space, then the pointwise topology on $Dil_x(X)$ coincides with the compact-open topology and the group $Dil_x(X)$ resp. I_x is locally compact resp. compact. If a metric cone structure on (X, x) exists, the projection $P : Dil_x(X) \to \mathbb{R}^+$ is surjective. On the other hand if the map P is surjective and X is proper it is easy to see, that the group $Dil_x(X)$ splits as a direct product $I_x \times \mathbb{R}^+$, such that P becomes the projection onto the second factor (First reduce to the connected component of $Dil_x(X)$. Then use the fact the group of outer automorphisms of I_x is totally disconnected, see [HM98, p. 512]). In particular in this case a metric cone structure on X exists, such that all dilations ρ_s are in the center of Dil_x. Moreover different metric cone structures are in one-to-one correspondance with different continuous homomorphisms $p : \mathbb{R}^+ \to I_x$.

4.2. Cones. The products and ultralimits of metric cones are metric cones with naturally defined dilations ρ_s. For a metric cone $(X, 0)$ the metric $d : X \times X \to \mathbb{R}$ is a homogeneous function. By the norm $|\cdot|$ we will denote the homogeneous function d_0.

A ray $\gamma : [0, \infty) \to X$ starting at the origin of the cone X is called radial, if it is stable under the dilations, i.e. if it is a homogeneous map. If a ray γ is radial, then its Busemann function $b_\gamma : X \to \mathbb{R}$ is homogeneous.

Example 4.1. A Banach space B is a cone with dilations $\rho_t(v) = tv$. Radial rays are precisely the linear ones $\gamma(t) = tv$. The Busemann function b_γ of such a ray γ is linear iff v is a smooth point of the unit sphere (see [JL01, p. 30] for the definition), i.e. iff the affine line in the direction of v is straight in the sense of Example 2.3.

Example 4.2. The Euclidean cone CY over a metric space Y ([BBI01, p. 91]) is a metric cone. Each ray starting at 0 is radial and has the form $\gamma(t) = tv$ with $v \in Y$. Its Busemann function is given for $w \in Y$ by $b_\gamma(sw) = -\langle v, sw \rangle := -s\cos(d^Y(v, w))$.

4.3. Special metric cones. Cones can be arbitrary wild in general. We will use the following particularly nice classes of metric cones.

Definition 4.2. We will call a metric cone X radial, if for each $x \in X$ with $|x| = 1$ the map $t \to \rho_t(x)$ is a ray.

A cone is radial iff it is the union of its radial rays. Consider the unit sphere S in a radial cone X, i.e. the set of all points $v \in X$ with $|v| = 1$. Then it is easy to check using only the triangle inequality, that the natural homogeneous map $CS \to X$ of the Euclidean cone over S to X that sends the point $tv \in CS$ to $\rho_t(v)$ is biLipschitz.

Definition 4.3. We call a radial cone X uniformly convex if for each $\epsilon > 0$ there is some $\delta > 0$, such that for each radial ray $\gamma(t) = \rho_t(v_0)$ ($|v_0| = 1$) and each $v \in X$ with $|v| = 1$ and $d(v, v_0) \geqslant \epsilon$ one has $b_\gamma(v) \geqslant -1 + \delta$.

Definition 4.4. We call a metric cone X smooth if for each sequence of radial rays γ_j converging to a radial ray γ the Busemann functions b_{γ_j} converge pointwise to the Busemann functions b_γ.

A direct product or a subcone of radial resp. uniformly convex resp. smooth cones is radial resp. uniformly convex resp. smooth. A completion or an ultraproduct of radial resp. of uniformly convex cones is radial resp. uniformly convex. Euclidean cones are uniformly convex and smooth. Banach spaces are radial cones and the notion of uniform convexity resp. of smoothness is the usual uniform convexity resp. smoothness of the norm.

Example 4.3. Carnot groups are not radial, but it is possible to prove that they are smooth cones.

§5. Geodesic cones

5.1. Germs of geodesics. Let x be a point in a space X. Consider the set Γ_x of all geodesics starting at x and the direct product $\Gamma_x \times [0, \infty)$. Define a pseudo metric $\tilde{d}$ on $\Gamma_x \times [0, \infty)$ by setting $\tilde{d}((\gamma_1, s_1), (\gamma_2, s_2)) = \limsup_{t \to 0} \frac{d(\gamma_1(s_1 t), \gamma_2(s_2 t))}{t}$. Denote by $\widetilde{C}_x$ the metric space corresponding to the pseudo metric space $(\Gamma_x \times [0, \infty), \tilde{d})$ and by C_x its metric completion. The set $\Gamma_x \times \{0\}$ is identified to a point 0 in C_x. The dilations of $[0, \infty)$ define on the spaces $(\widetilde{C}_x, 0)$ and $(C_x, 0)$ structures of metric cones.

Each geodesic $\gamma \in \Gamma_x$ defines a radial ray $\bar{\gamma}(t) = (\gamma, t)$ in $\widetilde{C}_x \subset C_x$. Hence C_x is always a radial metric cone. We will denote by γ^+ the ray (γ, t) in C_x as well as the point $\gamma^+(1) = (\gamma, 1)$. By S_x we denote the unit sphere in C_x and call it the link at x. One can think of C_x as the space of unparameterized geodesic germs at x, however one should be cautious:

Example 5.1. Let X be a Banach space and $x = 0$ its origin. If the norm of X is strongly convex, then all geodesics are straight lines and C_x is naturally isometric to X. But if the norm is not strongly convex, the space C_x is much bigger than X, for instance C_x is never locally compact in this case.

5.2. Exponential mappings. For each scale $(o) = (t_i)$ there is a natural map $\exp_x^{(o)} : \Gamma_x \times [0, \infty) \to X_x^{(o)}$ defined by $\exp_x^{(o)}((\gamma, t)) = (\gamma(tt_i)) \in X_x^{(t_i)}$. The exponential map $\exp_x^{(o)}$ goes down and defines a 1-Lipschitz map $\exp_x^{(o)} : \widetilde{C}_x \to X_x^{(o)}$, due to the limit superior in the definition of the distance in $\widetilde{C}_x$. Also by $\exp_x^{(o)}$ we will denote the unique 1-Lipschitz extension $\exp_x^{(o)} : C_x \to X_x^{(o)}$.

Example 5.2. Let $(o) = (t_i), (\tilde{o}) = (r_i)$ be two scales with $0 < \lim_\omega(\frac{t_i}{r_i}) := s < \infty$. For the canonical s-dilation $id_{(\tilde{o})}^{(o)} : (X_x^{(o)}, 0) \to (X_x^{(\tilde{o})}, 0)$ we get: $id_{(\tilde{o})}^{(o)} \circ \exp_x^{(o)} = \exp_x^{(\tilde{o})} \circ \rho_s$, where ρ_s is the natural s-dilation on C_x.

For each $\gamma \in \Gamma_x$ the radial ray $\gamma^+ \subset C_x$ is mapped by $\exp_x : C_x \to X_x^{(o)}$ isometrically onto the ray $\gamma_x^{(o)}$. The exponential mappings $\exp_x^{(o)} : C_x \to X_x^{(o)}$ are isometric embeddings for all scales $(o) = (t_i)$ if and only if the limit superior in the definition of the distance in C_x is always a limit. This property being quite fundamental justify the following

Definition 5.1. We say that the space X has the property (A) at x if the limit superior in the definition of the distance in C_x is always a limit.

Example 5.3. The upper angle coincides with the lower angle between arbitrary geodesics starting at x (see [BBI01, p. 98] for the definition) iff the condition (A) holds at x and the geodesic cone is a Euclidean cone $C_x = CS_x$.

Remark 5.4. Even if X is a geodesic space and the property (A) holds at x, and so the mappings $\exp_x^{(o)}$ define isometric embeddings of C_x into geodesic spaces $X_x^{(o)}$, the geodesic cone C_x need not be a geodesic space. For example it is not the case in Carnot–Caratheodory spaces or in general spaces with lower curvature bound ([Hal00]).

Remark 5.5. If X has the property (A) at x and some blow up $X_x^{(o)}$ is a proper space (compare Remark 3.6), then C_x is a proper space too.

5.3. Geodesic cones under one-sided curvature bounds. In a general space X with one-sided curvature bound the lower and the upper angle between arbitrary geodesics coincide (see [BBI01]). Therefore the geodesic cone C_x at each point $x \in X$ is a Euclidean cone and for each scale (o) the exponential map $\exp_x^{(o)} : C_x \to X_x^{(o)}$ is an isometric embedding. If X is an Alexandrov space, we will see below, that $\exp_x^{(o)}$ is also onto. However in general spaces with one-sided curvature bound it is almost never the case.

If X has an upper curvature bound, it was proved in [Nik95] that C_x is a geodesic space, hence it is a totally convex subset of each blow up $X_x^{(o)}$. The example of [Hal00] shows that C_x is not necessarily a geodesic space if X has a lower curvature bound. However, since C_x is an isometrically embedded Euclidean cone in $X_x^{(o)}$, the rigidity theorem of Toponogov gives us that for all radial rays η_1, η_2 in C_x and an arbitrary geodesic η in $X_x^{(o)}$ connecting arbitrary points on η_1 and η_2, the triangle $\eta_1 \eta \eta_2$ is Euclidean.

Remark 5.6. Let X be a space with lower curvature bound. It is not difficult to prove, that if a unit vector $v \in C_x$ has an antipode $w \in X_x^{(o)}$, i.e. a point w with $d(w,0) = 1$ and $d(w,v) = 2$, then w is contained in $\exp_x^{(o)}(C_x)$ and the line defined by v and w is a Euclidean factor not only of $X_x^{(o)}$ but also of C_x, i.e. v is connected with each other point $\bar{v} \in C_x$ by a geodesic in C_x.

§6. Tangent cones

6.1. Definition and the main example. The following main definition of this paper is motivated by Example 5.2

Definition 6.1. Let (X,x) be a metric space. We say that the tangent space of X at x exists, if for some metric cone (T,ρ_s) and each scale $(o) = (t_i)$ an isometry $i^{(o)} : (T,0) \to (X_x^{(o)}, 0)$ is chosen, such that for different scales $(o) = (t_i)$ and $(\tilde{o}) = (r_i)$ with $0 < \lim_\omega(\frac{t_i}{r_i}) := s < \infty$ the map $i^{(\tilde{o})} \circ \rho_s \circ (i^{(o)})^{-1} : X_x^{(o)} \to X_x^{(\tilde{o})}$ is the natural s-dilation from Subsection 3.4.

This metric cone T together with the fixed isometries $i^{(o)}$ will be called the tangent space at x and denoted by $T_x X$.

Example 6.1. If $(T,0)$ is a metric cone then the tangent space at 0 is naturally isometric to the ultraproduct T^ω. The isometries $i^{(t_i)} : T^\omega \to T_0^{(t_i)}$ are given by $i^{(t_i)}((x_i)) = (\rho_{t_i}(x_i))$.

If the tangent cone $T_x X$ exists, then the isometries $i^{(o)}$ allow us to identify in a unique way points from the blow up $X_x^{(o)}$ with points in $T_x X$. We will always use this particular identification below.

The following remark refers our definition to that of Gromov:

Remark 6.2. Let (X,x) be a proper space. Assume that for $t \to 0$ the set of all (isometry classes of) spaces $(\frac{1}{t}X, x)$ is relatively compact in the G.-H.-topology (compare Remark 3.6). If $T_x X$ exist in the sense of Definition 6.1 then for $t \to 0$ the isometry classes of $(\frac{1}{t}X, x)$ converge to the isometry class of $T_x X$ (Subsection 3.2). On the other hand if such a convergence takes place, we know that for each scale $(o) = (t_i)$ there is an isometry $i^{(o)} : T_x X \to X_x^{(o)}$.

The natural s-dilations between the blow ups define s-dilations on T_xX. It is possible to choose a metric cone structure on T_xX (see Subsection 4.1) and to change the isometries $i^{(o)}$, such that the commutation relations of Definition 6.1 are satisfied.

The most tangent spaces arise from Example 6.1 and the following:

Example 6.3. Let $e : (X, x) \to (Y, y)$ be a (not necessarily continuous) infinitesimal isometry. If T_xX exists, then T_yY exists too and is naturally isometric to T_yY. Namely the isometries $i^{(o)} : T_xX \to X_x^{(o)}$ define isometries $\tilde{i}^{(o)} : T_xX \to Y_y^{(o)}$ by $\tilde{i}^{(o)} = e_x^{(o)} \circ i^{(o)}$, where the isometries $= e_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ are given by Lemma 3.2.

Combining Example 6.1, Example 6.3 and Example 3.12 we obtain:

Lemma 6.1. *Let $(T, 0)$ be a metric cone, $(D, 0)$ a subset of T infinitesimally dense at 0 and let $e : (D, 0) \to (X, x)$ be an infinitesimal isometry. Then T_xX exists and is naturally isometric to the ultraproduct T^ω.*

Remark 6.4. The identification of T_xX with T^ω as above depends on the metric cone structure of T in an essential way (Example 6.1). Thus changing the cone structure on T (and letting e and the metric on T fixed) we get a different tangent cone structure at x.

Remark 6.5. If the cone T in Lemma 6.1 is proper then the tangent cone T_xX is isometric to T, in particulary it does not depend on the choice of the ultrafilter ω.

Example 6.6. Let $(M, |\cdot|)$ be a smooth manifold with a continuous Finsler metric. For $x \in M$ let $(T_xM, |\cdot|_x)$ be the usual tangent space at x. Then each chart $e : T_xM \to M$ with $e(0) = x$ whose differential at 0 is the identity, satisfies the assumptions of Lemma 6.1. Therefore the tangent space in the sense of Definition 6.1 coincides with the Banach space $(T_xM, |\cdot|_x)$. Remark that the identification (i.e. the maps $i^{(o)}$) does not depend on the choice of the chart e. Moreover the topology, the metric cone structure and the identification map e only depend on the manifold structure of X. Only the metric (norm) on T_xM depends on the Finsler metric $|\cdot|$.

Example 6.7. Belaiche constructed in [Bel96] for each Carnot–Caratheodory space M an almost isometry $e : G_x \to M$ from the Nilpotenization G_x of M at x to M, identifying T_xM with G_x.

6.2. Metric operation. Let X, Y be metric spaces. If T_xX and T_yY exist, then $T_{(x,y)}X \times Y$ exists and is naturally isometric to $T_xX \times T_yY$. The tangent space at x to the rescaled space tX is naturally isometric to $tT_xX = T_xX$. If

X and Y are geodesic spaces and $f : X \to \mathbb{R}^+$ a continuous function, then the warped product $X \times_f Y$ (compare [BBI01, p. 95]) has at (x, y) the tangent cone $T_xX \times T_yY$ (Use Example 6.3 for the identity map between $X \times_f Y$ and $X \times_{\tilde{f}} Y$ for the constant function $\tilde{f} = f(x)$). In particular if T_xX exists then for each $t \in \mathbb{R}^+$ the tangent space $T_{tx}CX$ to the Euclidean cone exists and is naturally isometric to $T_xX \times \mathbb{R}$.

Let S be a subset of X, $x \in S$ and let T_xX exist. If we say that S has a tangent cone at x it means, that the subset $S_x^{(o)} \subset X_x^{(o)} = T_xX$ does not depend on the scale (o). If S_1, S_2 are subsets of X both containing x with tangent cones $T_xS_1, T_xS_2 \subset T_xX$ then the union $S_1 \cup S_2$ has the tangent cone $T_xS_1 \cup T_xS_2$ at x.

Example 6.8. If the subset (S, x) of (X, x) is infinitesimally dense at x, then T_xS exists and is equal to T_xX. In particular if X is a doubling metric measure space such that T_xX exists for almost all $x \in X$, then for each measurable subset $S \subset X$ and almost each point (with respect to the induced measure) $x \in S$ the tangent space $T_xS \subset T_xX$ exists and coincides with T_xX.

Example 6.9. Let X be an Alexandrov space with curvature $\geqslant k$, S an extremal subset of X ([PP94a]). Then at each point $x \in S$ the tangent space $T_xS \subset T_xX$ exists and it is an extremal subset of T_xX.

6.3. Property (U)**.** The following condition seems to be very natural. It is a very rough generalization of the lower curvature bound condition:

Definition 6.2. Let X be a space, $x \in X$. Assume that the union of all geodesics starting in x contains a neighborhood of x, that the property (A) holds at x and that the geodesic cone C_x is proper. We say that X has the property (U) at x if for each $\epsilon > 0$ there is some $\rho > 0$, such that $d(\gamma(t), \eta(t)) \leqslant \epsilon t$ for all $t < \rho$ and all $\gamma, \eta \in \Gamma_x$ with $d(\gamma^+, \eta^+) < \rho$.

Example 6.10. If X has the property (U) at x, then so does each subset S of X that is a union of geodesics starting at x.

Example 6.11. A complete metric cone T has the property (U) at the origin iff it is proper and one can change the metric cone structure such that T becomes radial and the only geodesics starting at 0 are parts of radial rays. The if direction is clear and the only if implication follows from the fact (see below for a proof) that under the condition (U) the geodesic cone C_0 is isometric to the ultraproduct T^ω. In particular each proper Euclidean cone and each proper, uniformly convex Banach space have the property (U).

The property (U) allows us to compare distances in C_x and in X.

Proposition 6.2. *Let X be a space with the property (U) at x. Then for each $\epsilon > 0$ there is some $\rho > 0$, such that for all $r \leqslant t \leqslant \rho$ and all $\gamma, \eta \in \Gamma_x$ the inequality $|d(\gamma(r), \eta(t)) - d((\gamma, r), (\eta, t))| \leqslant \epsilon t$ holds.*

Proof. Assume that there are sequences $\gamma_i, \eta_i \in \Gamma_x$ and zero sequences $r_i \leqslant t_i \to 0$ violating the above inequality. Choosing a subsequence we may assume that γ_i^+ and η_i^+ are Cauchy sequences and $\frac{r_i}{t_i}$ converge to a number s with $0 \leqslant s \leqslant 1$. Moreover we may assume $r_i = st_i$ and that the sequence t_i is non-increasing.

For arbitrary small $\rho > 0$ we can choose i big enough such that for all $j \geqslant i$ we get $d(\eta_i^+, \eta_j^+) + d(\gamma_i^+, \gamma_j^+) < \rho < \frac{\epsilon}{5}$. Using the property (U), increasing i if necessary and having chosen ρ small enough we get $d(\gamma_i(t), \gamma_j(t)) + d(\eta_i(t), \eta_j(t)) \leqslant \frac{\epsilon t}{5}$ for all $t \leqslant t_i$. Hence we get $|d(\gamma_j(st_j), \eta_j(t_j)) - d(\gamma_i(st_j), \eta_i(t_j))| + |d((\gamma_j, st_j), (\eta_j, t_j)) - d((\gamma_i, st_j), (\eta_i, t_j))| \leqslant \frac{4\epsilon}{5}$. Therefore we obtain the inequality $|d(\gamma_i(st_j), \eta_i(t_j)) - d((\gamma_i, st_j), (\eta_i, t_j))| \geqslant \frac{\epsilon t_j}{5}$ for all $j \geqslant i$. This is a contradiction to the property (A). •

Corollary 6.3. *Let X have the property (U) at x and let γ_i be a sequence in Γ_x converging pointwise to a geodesic γ of positive length. Then γ_i^+ converge to γ^+ in C_x. For each $\epsilon > 0$ there is some $\rho > 0$, such that for each z with $d(z, x) < \rho$ the inequality $d(\gamma^+, \eta^+) < \epsilon$ holds for all geodesics $\gamma, \eta \in \Gamma_{x,z}$.*

Now we can use Proposition 6.2 and Lemma 6.1 to identify C_x with T_xX. Namely we consider the logarithmic map $h : X \to C_x$, that sends a point $z \in X$ to some pair $(\gamma, t) \in C_x$ with $t = d(x, z)$ and $\gamma \in \Gamma_{x,z} \subset \Gamma_x$. By assumption h is defined on the neighborhood $\cup_{\gamma \in \Gamma_x}$ of x. From Proposition 6.2 we conclude, that h is an infinitesimal isometry, hence each map $e : h(X) \subset C_x \to X$ satisfying $e \circ h = id$ has the properties used in Lemma 6.1 to identify C_x with T_xX. The identification between $X_x^{(o)}$ and C_x given by Lemma 6.1 is exactly the exponential map $\exp_x^{(o)}$.

Remark 6.12. The last construction is well known in many cases. In the case of Riemannian manifolds the logarithmic map h above is just the inversion of the usual exponential map. If X is an Alexandrov space with curvature $\leqslant 0$, then h is uniquely defined, surjective and 1-Lipschitz. If X is an Alexandrov space with curvature $\geqslant 0$, then one can define the almost inversion $e : C_x \to X$ to be surjective and 1-Lipschitz ([PP94b]).

Definition 6.3. A space X will be called infinitesimally cone-like, if it is locally geodesic, at each point $x \in X$ the property (U) holds and each tangent cone $T_xX = C_x$ is a Euclidean cone.

§7. Differentials

7.1. Generalities. Let $f : (X, x) \to (Y, y)$ be a locally Lipschitz map and assume that T_xX and T_yY exist. For each scale (o) the blow up $f_x^{(o)} : X_x^{(o)} \to Y_y^{(o)}$ gives us a map between the tangent spaces.

Definition 7.1. Let $f : X \to Y$ be as above. We say that f is differentiable at x if the blow up $f_x^{(o)} : T_xX \to T_yY$ does not depend on the scale (o). In this case we denote this uniquely defined map by D_xf.

Example 7.1. If $f : (X, 0) \to (Y, 0)$ is a homogeneous map between cones, then f is differentiable at 0 and the differential D_0f is the ultraproduct $f^\omega = \lim_\omega f : X^\omega \to Y^\omega$ of f.

Example 7.2. Let $f : X \to Y$ be an isometry. If T_xX does not admit a non-trivial isometry fixing the origin 0, then f is differentiable at x, since for each scale (o) the map $f_x^{(o)} : T_xX \to T_yY$ is an origin preserving isometry.

Example 7.3. Let S be a subset of X, $x \in S$. If T_xX and $T_xS \subset T_xX$ exist as in Subsection 6.2 then the inclusion $I : S \to X$ is differentiable at x and the differential is the natural embedding $I_x : T_xS \to T_xX$.

Example 7.4. Let $f : X \to Y$ be a biLipschitz embedding. If f is differentiable at x, then $f(X)$ has a tangent cone at $f(x)$ given by $T_{f(x)}f(X) = D_xf(T_xX) \subset T_{f(x)}Y$. On the other hand if $f : X \to Y$ is a differentiable C-open map (see [Lytc]) and S a subset of Y that has a tangent cone at $f(x)$, then $f^{-1}(S)$ has a tangent cone at x given by $T_xf^{-1}(S) = (D_xf)^{-1}(T_{f(x)}S) \subset T_xX$.

Example 7.5. If T_xX and T_yY exist, then the projection $p : X \times Y \to X$ is differentiable and the differential is just the projection. If T_xX exist, then the metric $d : X \times X \to \mathbb{R}$ is differentiable at each point (x, x) on the diagonal and the differential is just the metric on T_xX. The distance function $d_x : X \to \mathbb{R}$ is differentiable at x with differential $D_xd_x(v) = |v|$. The differentiability of the metric at points outside the diagonal will be discussed in Section 9.

Example 7.6. Let $(T, 0)$ be a proper metric cone with dilations lying in the center of Dil_0. Let $(\widetilde{T}, 0)$ be the same space with a different metric cone structure given by a continuous homomorphism $p : \mathbb{R}^+ \to I_0$ (Subsection 4.1). Let $f : (T, 0) \to (\widetilde{T}, 0)$ be the identity. Then $f_0^{(t_i)}$ is exactly the isometry $\lim_\omega(p(t_i))$. Hence f is differentiable at 0 iff $\lim_{t \to 0} p(t)$ exists. However this can only happen if p is the trivial map. This suggests, that there is at most one natural tangent cone structure. Considering $T = \mathbb{R}^2$ we get essentially the counterexample of [CH70].

Since ultralimits commute with compositions we immediately see:

Lemma 7.1. *Let $f : X \to Y$ and $g : Y \to Z$ be Lipschitz maps, $f(x) = y, g(y) = z$. If f is differentiable at x and g differentiable at y then $g \circ f$ is differentiable at x with differential $D_x(g \circ f) = D_y g \circ D_x f$.*

Example 7.7. If $f : X \to Y$ is differentiable at x and S a subset of X such that $T_x S \subset T_x X$ exists, then $f : S \to Y$ is differentiable in x and the differential $D_x f : T_x S \to T_{f(x)} Y$ is the restriction of $D_x f : T_x X \to T_{f(x)} Y$. If on the other hand $T_x S = T_x X$ and the restriction $f : S \to X$ is differentiable at x, then $f : X \to Y$ is also differentiable at x.

7.2. Comparing with the usual differentiability. If the tangent spaces are given by Lemma 6.1, we get the usual definition of differentiability. Let namely $f : (X, x) \to (Y, y)$ be a Lipschitz map, $(T_1, 0)$ resp. $(T_2, 0)$ metric cones and $e_1 : T_1 \to X$ resp. $e_2 : T_2 \to Y$ be maps as in Subsection 6.1 (If e_i are defined only on infinitesimally dense subsets $(D_i, 0) \subset (T_i, 0)$ we may extend them by Example 3.12). If $A : T_1 \to T_2$ is a homogeneous Lipschitz map, such that for $v \in T_1$ one has $\lim_{|v| \to 0} \frac{d(f(e_1(v)), e_2(A(v)))}{|v|} = 0$ then the differential of f at x exists and is equal to the ultraproduct $A^\omega : T_1^\omega \to T_2^\omega$. On the other hand we can use Lemma 3.1 and see, that if the differential $D_x f$ exists, then the image of $D_x f(T_1) \subset T_2^\omega$ must be contained in $T_2 \subset T_2^\omega$, therefore the existence of a map A as above is also necessary in this case.

In particular the differentiability does not depend on the ultrafilter ω! Moreover if T_1 and T_2 are proper, then the differential does not depend on the ultrafilter too.

7.3. Separating maps. Let (Y, y) be a metric space, $\{f_j : Y \to Y_j\}$ a set of Lipschitz maps differentiable at y and separating the points in $T_y Y$, i.e. for $v_1 \neq v_2 \in T_y Y$ there is some j, such that $D_y f_j(v_1) \neq D_y f_j(v_2)$. Since ultralimits of maps commute with compositions we obtain, that a map $g : (X, x) \to (Y, y)$ is differentiable at x iff for each j the map $f_j \circ g$ is differentiable at x. For example a biLipischitz map $f_0 : Y \to Y_0$ differentiable at y satisfies the above conditions. In particular its inverse must be differentiable at $f_0(y)$.

7.4. Differentiating curves. Let $\gamma : [0, a] \to X$ be a Lipschitz curve, with $\gamma(0) = x$. If γ is differentiable at 0, then the differential is a homogeneous map h of the half-line $[0, \infty)$ to $T_x X$. Since this map is uniquely determined by $h(1)$, we will call the point $h(1)$ the right hand side differential of γ at 0 and denote it by γ^+. In the same way one defines γ^- if γ is differentiable at a. The differential exists at an inner point $t \in (0, a)$, iff γ^+ and γ^- exist in t.

7.5. Differentiating geodesics. The most natural and basic maps into a metric space are geodesics. One can only hope to get a rich theory of differentiation if many geodesics are differentiable. A geodesic $\gamma : [0,a] \to X$ starting at x is differentiable at 0 iff the ray $\gamma_x^{(o)} \subset X_x^{(o)} = T_xX$ does not depend on the scale (o). In this case we get a unique radial ray $\gamma_x \subset T_xX$. We see that all geodesics are differentiable at x iff the exponential mappings $\exp_x^{(o)} : C_x \to X_x^{(o)} = T_xX$ do not depend on the scale (o), that means iff C_x is naturally embedded in T_xX via the exponential mappings. In this case X has the property (A) at x. For example this is always true if X has the property (U) at x. In general however it does not need to be true even in quite tame spaces, see [CH70] or Example 7.6.

7.6. Directional derivatives. Let $f : (X,x) \to (Z,z)$ be a locally Lipschitz map and assume that T_zZ exists. We say that f has directional derivatives at x if the restriction $f \circ \gamma$ to each geodesic $\gamma \in \Gamma_x$ is differentiable at 0. In this case we obtain a well defined homogeneous map $D_xf : C_x \to T_zZ$ of the geodesic cone C_x into the tangent cone T_zZ. For each scale (o) we have $f_x^{(o)} \circ \exp_x^{(o)} = D_xf$, in particular D_xf inherits the Lipschitz constant of f.

Example 7.8. Each locally Lipschitz semi-concave function $f : X \to \mathbb{R}$ has directional derivatives at all points (see [Lytc] for more on this).

If X has a tangent cone at x and the geodesics are differentiable at x, then each Lipschitz map $f : (X,x) \to (Z,z)$ differentiable at x is also directionally differentiable and $D_xf : C_x \to T_zZ$ is just the restriction of $D_xf : T_xX \to T_zZ$ to C_x. On the other hand if all geodesics are differentiable at x and the map $f : X \to Z$ is directionally differentiable at x then the restriction of $f_x^{(o)} : T_xX \to T_zZ$ to the subset $C_x \subset T_xX$ is independent of the scale (o). This implies Proposition 1.1.

We can actually deduce a bit more smoothness of isometries:

Corollary 7.2. *Let X have the property (U) at x. Let $f_i : (X,x) \to (X,x)$ be isometries fixing x and converging pointwise to an isometry f. Then the isometries D_xf_i of T_xX converge to the isometry D_xf.*

Proof. Composing the isometries f_i with f^{-1} we may assume $f = Id$. Then for each geodesic γ the geodesics $\gamma_i = f_i(\gamma)$ converge to γ. By Corollary 6.3 for the starting direction γ^+ of γ the directions $D_xf_i(\gamma^+)$ converge to γ^+. •

7.7. Strong differentiability. Let $f : (X,x) \to (Z,z)$ be a locally Lipschitz map and assume that T_xX exists and Z has the property (A) at z. We will say that a strong differential $D_xf : T_xX \to C_z$ exists, if for each scale (o) one has $f_x^{(o)} = \exp_z^{(o)} \circ D_xf$. If T_zZ exists and geodesics are differentiable at

z, then a map f is strongly differentiable at x iff it is differentiable and the differential $D_xf : T_xX \to T_zZ$ satisfies $D_xf(T_xX) \subset C_z$. Remark that the strong differential (if it exists) is a homogeneous Lipschitz map.

§8. Metric Differentials

Example 7.5 gives rise to the following definition ([Kir94]):

Definition 8.1. Let $f : X \to Y$ be a Lipschitz map and let T_xX exist. We say that f has a metric differential at x if the composition $d \circ (f \times f) : X \times X \to \mathbb{R}$ is differentiable at (x, x). In this case the differential is a homogeneous pseudo metric on T_xX and we denote it by mD_xf. We will say that f has a weak metric differential at x if the map $d_{f(x)} \circ f : X \to \mathbb{R}$ is differentiable at x. Again by mD_xf we denote the differential of this map.

If f is differentiable at x then it also has a metric differential given by $mD_xf(v, w) = d(D_xf(v), D_xf(w))$. If f is metrically differentiable at x then it is also weakly metrically differentiable with weak metric differential $mD_x(v) = mD_x(0, v)$. Let on the other hand $f : X \to Y$ be a biLipschitz map, $f(x) = y$. If f is metrically differentiable at x then one can uniquely define a tangent space T_yY, such that f becomes differentiable at x. An isometric embedding $I : X \to Z$ is metrically differentiable at each point x where T_xX exists and the metric differential is just the metric $mD_xI = d : T_xX \times T_xX \to \mathbb{R}$.

Example 8.1. The space X has the property (A) at the point x, iff for each pair of geodesics $\gamma_1, \gamma_2 \in \Gamma_x$ the map $\gamma : (-\epsilon, \epsilon) \to X$ given by $\gamma(t) = \gamma_1(t)$ for $t \leqslant 0$ and $\gamma(t) = \gamma_2(t)$ for $t \geqslant 0$ is metrically differentiable at 0.

Using this example we immediately obtain:

Lemma 8.1. *Let $f : (X, x) \to (Z, z)$ be a Lipschitz map that is an infinitesimal isometric embbeding at x and assume that T_zZ exists. If the image $f \circ \gamma$ of each geodesic $\gamma \in \Gamma_x$ is differentiable at 0, then X has the property (A) at x, the map f is directionally differentiable at x and the differential $D_xf : C_x \to T_zZ$ is an isometric embedding.*

Example 8.2. Let M be a Finsler manifold. If each geodesic $\gamma \in \Gamma_x$ is differentiable at 0, then M has the property (A) at x.

The following deep theorem was proved in [Kir94]:

Theorem 8.2. *Let K be a measurable subset of $\mathbb{R}^n$, $f : K \to Y$ a Lipschitz map. Then f has a metric differential at almost each point, this metric differential is almost everywhere a semi-norm, and the map $x \to mD_x = |\cdot|_x$ is measurable.*

Example 8.3. Let $\gamma : [p, q) \to Y$ be a Lipschitz map. If the weak metric differential of γ at p exists, we will denote by mD_p^+ the number $mD_p\gamma(1)$, i.e. $mD_p^+ = \lim_{t\to 0} \frac{d(\gamma(p+t),\gamma(p))}{t}$. The fact that the metric differential exists and is a semi-norm amounts to the much stronger statement

$$\lim_{t\to 0} \frac{d(\gamma(p + s_1t), \gamma(p + s_2t))}{t} = |s_2 - s_1| mD_p^+$$

for all $s_1, s_2 > 0$.

§9. Differentiation of distance functions

9.1. Generalities. We start with the following paradigmatic example:

Example 9.1. Let T be a proper metric cone, h a radial ray, $x = h(1)$. Then the distance function d_x is differentiable at the origin 0 and the differential is given by $D_0 d_x(v) = b_h(v)$, since $D_0 d_x(v) = \lim_{t\to 0} \frac{d(x,\rho_t(v))-d(x,0)}{t} = \lim_{t\to\infty}(d(\rho_t(x), v) - t) = b_h(v)$.

To state our results we need the following extension of Definition 6.1. Let $f : X \to Y$ be a Lipschitz map and let T_xX and T_yY exist. We say that D_xf has some property (even if it does not exist) if each blow up $f_x^{(o)} : T_xX \to T_yY$ has this property.

Let now X be a space, $x \neq z$ points in X such that T_xX and T_zX exist. Assume that each geodesic $\gamma \in \Gamma_{x,z}$ is differentiable at x and at z thus defining radial rays $\gamma^+ \subset T_xX$ and $\gamma^- \subset T_zX$. Then we show:

Lemma 9.1. *In the above notations the differential $D_{(x,z)}d : T_xX \times T_zX \to \mathbb{R}$ of the metric $d : X \times X \to \mathbb{R}^+$ can be estimated by $D_{(x,z)}d(v, w) \leqslant \inf_{\gamma\in\Gamma_{x,z}}(b_{\gamma^+}(v) + b_{\gamma^-}(w))$.*

Proof. Choose some geodesic $\gamma : [a_1, a_2] \to X$ connecting x and z. Then $d(\gamma(a_1+s), \gamma(a_2-s)) = d(x, z)-2s$. Hence each blow up $d^{(o)}_{(x,z)} : T_xX \times T_zX \to \mathbb{R}$ satisfies $d^{(o)}_{(x,z)}(\gamma^+(s), \gamma^-(s)) = -2s$. We are done by Example 2.4. •

In the same way using Example 2.2 instead of Example 2.4 we see:

Lemma 9.2. *Let X be a space, S a closed subset of X, $x \in X \setminus S$. Assume that T_xX exists and each geodesic $\gamma \in \Gamma_{x,S}$ connecting x and S is differentiable at 0. Then the differential of the distance function d_S is bounded from above by $D_x d_S(v) \leqslant \inf_{\gamma\in\Gamma_{x,S}} b_{\gamma^+}(v)$.*

Remark 9.2. Even if T_xX and T_zX do not exist, one can work with the directional differentials $D_{(x,z)} : C_x \times C_z \to \mathbb{R}$ and get (for the same reason) the same estimations as in Lemma 9.1.

9.2. First variation formula. As in the Riemannian geometry one would like to have equalities in the last two lemmas.

Definition 9.1. We say that the first variation formula holds for $S \subset X$ and $x \in X \setminus S$ if in the statement of Lemma 9.2 equality holds.

Example 9.3. Let $\gamma \in \Gamma_{z,S}$ be a geodesic, $x = \gamma(t)$ an inner point of γ. If γ is differentiable at x, it defines a homogeneous line $\widetilde{\gamma}$ in T_xX. Moreover $D_t(d_S \circ \gamma) = -id$. If $\widetilde{\gamma}$ is straight in the sense of Example 2.3, then the first variation formula holds for S and x.

The validity of the first variation formula is closely related to the question whether geodesics vary smoothly in X.

Definition 9.2. Let X be a space with the property (U) at x. We say that geodesics vary smoothly at x if for all geodesics γ and η in Γ_x and each sequence of geodesics γ_n with $\gamma_n(0) = \eta(t_n)$ converging to γ, the following condition holds. For each $\epsilon > 0$ there is some $n > 0$ and $\rho_n > 0$, such that $\rho_n - d(x, \gamma_n(\rho_n)) \geqslant (b_{\gamma^+}(v) - \epsilon)t_n$, where $v = \eta^+(1)$ is the starting direction of η.

If this condition holds for all x and all $\gamma \in \Gamma_x$, we will say that geodesics vary smoothly in X.

Remark 9.4. First of all we see, that since we require the above condition to hold for all convergent sequences of geodesics, the number ρ_n as above exists for all sufficiently large n. Moreover the numbers ρ_n can be chosen, such that $\rho_n \to 0$. Finally if the inequality in Definition 9.2 holds for some ρ_n, it also holds for all $\rho \geqslant \rho_n$. Hence we may also choose all ρ_n to be equal to a small constant ρ depending on γ and ϵ.

The connection between Definition 9.2 and the first variation formula is provided by the next three results.

Proposition 9.3. *Let X be a proper geodesic space, $x, z \in X$ points at which X has the property (U). Assume that geodesics vary smoothly at x and at z. Then in Lemma* 9.1 *equality holds.*

Proof. Since $X \times X$ has the property (U) at (x, z) it is enough to prove, that the equality holds for the starting direction of each geodesic $\tilde{\eta}$ in $X \times X$ starting at (x, z). Hence it is enough to prove, that for arbitrary geodesics $\eta_1 \in \Gamma_x$ and $\eta_2 \in \Gamma_z$ with starting directions $v \in T_xX$ resp. $w \in T_zX$ one has $\liminf_{t \to 0} \frac{d(\eta_1(t), \eta_2(st))}{t} \geqslant \inf_{\gamma \in \Gamma_{x,z}}(b_{\gamma^+}(v) + b_{\gamma^-}(sw))$, for all $s \geqslant 0$. Assume the contrary and choose a zero sequence (t_n) violating the above inequality. Choose a geodesic γ_n from $\eta_1(t_n)$ to $\eta_2(st_n)$. Going to a subsequence we may assume, that γ_n converge to a geodesic $\gamma \in \Gamma_{x,z}$. For given $\epsilon > 0$ and all n big enough we can find numbers ρ^+ and ρ^-, such that $d(x,\gamma_n(\rho^+)) \leqslant \rho^+ - (b_{\gamma^+}(v) - \epsilon)t_n$ and $d(x,\gamma_n(L(\gamma_n) - \rho^-)) \leqslant \rho^- - (b_{\gamma^-}(sw) - \epsilon)t_n$. But since γ is a geodesic we get $L(\gamma) \leqslant d(x,\gamma_n(\rho^+)) + (L(\gamma_n) - \rho^+ - \rho^-) + d(z,\gamma_n(L(\gamma_n) - \rho^-))$. Hence $L(\gamma) - L(\gamma_n) \leqslant (2\epsilon + b_{\gamma^+}(v) + b_{\gamma^-}(sw))t_n$. This proves the result. •

In the same way we see:

Proposition 9.4. *Let X be a proper geodesic space with property (U) at x, $S \subset X$ a closed subset not containing x. Assume that geodesics vary smoothly at x. Then the first variation formula holds for S and x.*

Example 9.5. Under the assumptions of Proposition 9.3 assume in addition, that the tangent cones T_xX and T_zX are smooth (Definition 4.4). For $v \in T_xX, w \in T_zX$ choose a sequence $\gamma_i \in \Gamma_{x,z}$ such that $D_{(x,z)}d(v,w) = b_{\gamma_i^+}(v) + b_{\gamma_i^-}(w)$. Let $\gamma \in \Gamma_{x,z}$ be a pointwise limit of γ_i. Then γ_i^+ resp. γ_i^- converge to γ^+ resp. γ^- by Corollary 6.3 and from the smoothness of the tangent cones we obtain $D_{(x,z)}d(v,w) = b_{\gamma^+}(v) + b_{\gamma^-}(w)$. This finishes the proof of Theorem 1.2.

Lemma 9.5. *Let X be a proper geodesic space with the property (U) at x. Assume that the first variation formula is valid at x for each closed subset S not containing x. If the tangent cone T_xX is smooth then geodesics vary smoothly at x.*

Proof. Let $\gamma, \eta, \gamma_n \to \gamma$ be as in Definition 9.2 and set $v = \eta^+$. For sufficiently small δ and each geodesic $\tilde{\gamma} \in \Gamma_{x,\gamma(\delta)}$ we obtain from Corollary 6.3 that $d(\gamma^+,\tilde{\gamma}^+) \leqslant \epsilon_1$ with $\epsilon_1 \to 0$ if $\delta \to 0$. By the smoothness of the tangent cone we get $|b_{\gamma^+}(v) - b_{\tilde{\gamma}^+}(v)| \leqslant \epsilon$ with $\epsilon \to 0$ if $\delta \to 0$.

Let now ϵ be given. Choose δ small enough and consider δ_n such that $d(\gamma_n(\delta_n),x) = \delta + t_n^2$. Let S be the closed subset of X that consists of the sequence $\gamma_n(\delta_n)$ and the point $z = \gamma(\delta)$. The first variation formula gives us $D_xd_S(v) = D_xd_z(v) = \inf_{\tilde{\gamma} \in \Gamma_{x,z}} b_{\tilde{\gamma}^+}(v)$. This and the above estimate of $b_{\tilde{\gamma}^+}(v)$ directly imply the inequality of Definition 9.2. •

At the beginning of this section we have seen that in Banach spaces the first variation formula always holds for distance functions to points. The situation for the distance functions to subsets is more complicated, namely:

Lemma 9.6. *Let B be a finite dimensional uniformly convex Banach space. Then the first variation formula holds in B for all distance functions iff the norm of B is smooth.*

Proof. Assume that the norm is smooth. Let γ_n be a sequence of geodesics converging to a geodesic γ. We may assume that $\gamma(s) = sh$, $\gamma_n(s) = t_nv + sh_n$ with $t_n \to 0$ and some unit vectors v, h_n and h, where h_n converge to h. Fix some positive ϵ. By the smoothness of the norm, for large positive C we get $|Ch+v|+|Ch-v|-2C<\epsilon$. But $d(0,\gamma_n(Ct_n))=|t_nv+Ct_nh_n|=t_n|v+Ch_n|$. Choosing n such that Ch_n is very close to Ch, we get $Ct_n-d(0,\gamma_n(Ct_n))\geqslant t_n(C-|v+Ch_n|)\geqslant t_n(|Ch-v|-C-\frac{\epsilon}{2})\geqslant t_n(b_h(v)-\epsilon)$. Therefore geodesics vary smoothly in X and we are done by Proposition 9.4.

If the norm is not smooth, one can consider a non-smooth point x of the unit sphere in B. Let H be a supporting hyperplane at x. It is easy to see that for the distance function d_H the first variation formula does not hold at the origin. We leave the details to the reader. •

§10. The class of geometric spaces

We recall from the introduction:

Definition 10.1. A proper geodesic space X is called geometric if it has property (U) at each point, each tangent space $T_xX=C_x$ is uniformly convex and smooth and if geodesics vary smoothly in X.

We are going to show now that many important spaces are geometric.

10.1. Alexandrov spaces. Let X be an Alexandrov space. The upper angle coincides with the lower angle for each pair of geodesics starting at the same point. Hence X has the property (A) at each point and each geodesic cone is a Euclidean cone. If X has a lower curvature bound, then the geodesic cone C_x is proper by [BGP92] and the property (U) holds by the very definition of lower curvature bound. For Alexandrov spaces with an upper curvature bound the property (U) easily follows from the geodesically completeness (see [OT]). Hence Alexandrov spaces are infinitesimally cone-like.

In order to prove that they are geometric, consider geodesics γ and η starting at x at the angle α and a sequence of geodesics γ_i converging to γ with $\gamma_i(0)=\eta(t_i)$. If X has a lower curvature bound, then by the semi-continuity of angles the angle between γ_i and η^+ is $\geqslant \alpha-\epsilon$ for arbitrary small ϵ and sufficiently big i. Hence the angle between γ_i and η^- is at most $\pi-\alpha+\epsilon$. Now using the comparison triangle for $x\eta(t_i)\gamma_i(\rho)$ we get the needed upper bound for $d(x,\gamma_i(\rho))$. If X has an upper curvature bound, then the angle between η and the geodesic connecting x with $\gamma_i(\rho)$ is at least $\alpha-\epsilon$. Again the comparison triangle to $x\eta(t_i)\gamma_i(\rho)$ gives us the needed upper bound for $d(x,\gamma_i(\rho))$.

10.2. Extremal subsets. Petrunin has proved in [Pet94] that an extremal subset of an Alexandrov space with a lower curvature bound is infinitesimally cone-like and geometric with respect to the inner metric.

10.3. Surfaces with an integral curvature bound. We will assume that the reader is familiar with the notion of a two-dimensional surface with an integral curvature bound, see [Res93] for the definition and an excellent survey. Let M be a surface with an integral curvature bound. By Theorem 8.2.3 of [Res93] the upper and lower angle between each pair of geodesics coincide, hence M has at each point the property (A) and the geodesic cone is a Euclidean cone. We will denote by Ω^+ resp. Ω^- the Borel measures that describe the positive resp. the negative part of the curvature. We will use Theorem 8.2.2 of [Res93] saying the following: let T be a triangle in M such that the concatenation of its sides is a simple closed curve and its inner part T^0 is homeomorphic to a ball. Let α be the angle between two sides of T and let $\tilde{\alpha}$ be the corresponding angle in the comparison triangle in the Euclidean plane. Then $\alpha - \widetilde{\alpha} \leqslant \Omega^+(T^0)$. Now let $x \in M$ be an arbitrary point. Since the intersection of the punctured balls $B^0_r(x) := B_r(x) \setminus \{x\}$ is empty, for each $\epsilon \geqslant 0$ we can find a $r > 0$, such that $\Omega^+(B^0_r(x)) + \Omega^-(B^0_r(x)) \leqslant \epsilon$. Hence for each triangle T as above with a vertex in x and sidelengthes $\leqslant r$, we obtain, that each angle of T differs from the corresponding angle of the comparison triangle by at most 3ϵ.

Consider now two geodesics γ_1, γ_2 of length $t \leqslant r$ starting at x at an angle $\leqslant \epsilon$. In order to verify the property (U) we have to estimate $\frac{d(\gamma_1(t),\gamma_2(t))}{t}$ from above. If γ_1 and γ_2 intersect at $\gamma_1(t_0) = \gamma_2(t_0)$, then the angle between γ_1^+ and γ_2^+ at $\gamma_1(t_0)$ is at most ϵ. Hence we may assume that γ_1 and γ_2 do not intersect. Now it is easy to see, that for each geodesic η between $\gamma_1(t)$ and $\gamma_2(t)$ does not intersect $\gamma_1[0,t) \cup \gamma_2[0,t)$. Hence we may apply the above remark to the triangle $\gamma_1\eta\gamma_2$ and get the needed estimate for the length of η. Thus M is infinitesimally cone-like.

In order to prove that geodesics vary smoothly at x consider two geodesics $\gamma, \eta \in \Gamma_x$ enclosing a positive angle α at x. Let γ_n be a sequence of geodesics converging to γ with $\gamma_n(0) = \eta(t_n)$. Consider a geodesic ν_n between x and $\gamma_n(r)$. Applying the above consideration we see that the angle between ν_n and γ is at most 2ϵ for big n. Hence the angle between η and ν_n is at least $\alpha - 2\epsilon$. Using the triangle $\eta\nu_n\gamma_n$ we get the needed upper bound for the length of ν_n.

10.4. Metric operations. If X and Y are geometric, then so is the product $X \times Y$. If X is geometric and C a closed convex subset of X then C is geometric. Moreover the Euclidean cone CX is geometric. The proofs are straightforward and left to the reader.

10.5. A class of interesting subsets of manifolds. Let M be a smooth manifold with a continuous Finsler metric. Let $K \subset M$ be a closed subset such that the inner metric on K is biLipschitz equivalent to the induced one, i.e. each two points $x, z \in K$ are connected in K by a curve of length at most

$Ld(x,z)$. Assume further that all geodesics in K with respect to the inner metric have uniformly bounded $C^{1,\alpha}$ norms for a fixed $0 < \alpha \leqslant 1$.

Remark 10.1. In [Lyta] it is shown, that the above conditions are satisfied by sets of positive reach ($\alpha = 1$) and similar big classes of subsets in smooth Riemannian manifolds. Moreover they are satisfied if $K = M$ and the Finsler metric on M is Hölder continuous and sufficiently convex ([LY]).

We are going to prove now that K with its inner metric has the property (U) at each point and that it has continuously varying geodesics if all norms $|\cdot|_x$ are strongly convex and smooth. We will denote by d^K resp. by d the inner resp. the induced metric on K. The question is local, so we may assume that M is a chart $U \subset \mathbb{R}^n$ and the Finsler structure is uniformly continuous. We denote by $||\cdot||$ the Euclidean norm on $\mathbb{R}^n$ and by $|\cdot|_x$ the norm defined by the Finsler structure at x. For each K-geodesic γ in U we have $||\gamma'(t) - \gamma'(0)|| \leqslant Lt^\alpha$ for some fixed constant L. Moreover $||\gamma(0) - \gamma(t)|_{\gamma(0)} - t| \leqslant o(t)$ where the function $o(t)$ depends only on U and satisfy $\lim_{t\to 0} \frac{o(t)}{t} \to 0$. This implies the inequality $d^K(x,z) \leqslant d(x,z) + o(d(x,z))$ (compare [LY]). By Lemma 8.1 the space K has the property (A) at each point.

If γ_1, γ_2 are two geodesics starting at x, then $|d^K(\gamma_1(t), \gamma_2(t)) - |\gamma_1(t) - \gamma_2(t)|_x| \leqslant o(t)$. Since $|\gamma_i(t) - t\gamma_i^+(0)|_x \leqslant o(t)$ we conclude that K has the property (U) at x.

Let finally γ and η be geodesics starting at x and let γ_n be a sequence of geodesics converging to γ with $\gamma_n(0) = x_n = \eta(t_n)$. Let v be the starting direction of η and let h resp. h_n be the starting directions of γ resp. of γ_n. From the uniform $\mathcal{C}^{1,\alpha}$ bound of γ_n we see, that h_n converge to h. Fix some $\epsilon > 0$ and choose a sufficiently big $C = C(\epsilon) > 0$. Consider the triangle $x\gamma_n(0)\gamma_n(Ct_n)$.

We have $d(x, \gamma_n(Ct_n)) \leqslant |\gamma_n(Ct_n) - x|_x + o(t_n)$. On the other hand we have $|\gamma_n(Ct_n) - x|_x \leqslant t_n|v + Ch_n|_x + o(t_n)$. Hence geodesics in X vary continuously at x if this is true in the Banach space T_xM, i.e. if the norm of T_xM is smooth and uniformly convex (Lemma 9.6).

Finally remark, that if each norm $|\cdot|_x$ is a Euclidean norm, then K is infinitesimally cone-like.

§11. Differentiating in geometric spaces

11.1. Basics. Let X be a geometric space, F a closed subset of X and $x \in X \setminus F$. The uniform convexity of T_xX and the first variation formula show, that $D_xd_F(v) \leqslant -1 + \delta$ for a vector $v \in S_x \subset C_x$ implies $d(v, \gamma^+) \leqslant \epsilon$ for some $\gamma \in \Gamma_{x,F}$ and $\epsilon = \epsilon(\delta) = \epsilon(x, \delta)$ with $\lim_{\delta\to 0} \epsilon(\delta) = 0$. In particular $D_xd_F(v) = -1$ iff v is the starting direction γ^+ of some $\gamma \in \Gamma_{x,F}$.

Choose now a dense countable subset S of a punctured neighborhood of x. For each $z \in S$ the function d_z is differentiable at x with differential given by the first variation formula. For each unit vector $v \in T_x X$ and each $\epsilon > 0$ we can find a point $z \in S$ such that $D_x d_z = \inf_{\gamma \in \Gamma_{x,z}} b_{\gamma^+}$ where γ^+ runs over some radial rays h with $d(v, h(1)) < \epsilon$, i.e. z lies almost in the direction v from x. The uniform convexity of $T_x X$ shows:

Lemma 11.1. *The differentials $\{D_x d_z | z \in S\}$ of distance functions d_z separate the points in $T_x X$, i.e. functions d_z satisfy the conditions of Subsection* 7.3.

Now Subsection 7.3 gives us:

Corollary 11.2. *Let $f : Z \to X$ be a Lipschitz map. Assume that $T_z Z$ exists and that X is geometric. The map f is differentiable at z iff the compositions $d_{x_n} \circ f : Z \to \mathbb{R}$ are differentiable at z, for all points x_n in dense countable subset D of X.*

This implies Proposition 1.4 and from Theorem 8.2 we deduce Corollary 1.5.

11.2. Differentiating submetries. We recall some facts about submetries, a notion invented in [Ber87], see also [BG00].

Definition 11.1. A map $f : X \to Y$ is a submetry if $f(B_r(x)) = B_r(f(x))$ holds for all $x \in X$ and $r \in \mathbb{R}^+$.

If $f : X \to Y$ is a submetry, and X is proper resp. geodesic then so is Y. For each closed subset $A \subset Y$ we have $d_A \circ f = d_{f^{-1}(A)}$. Two points $x, \bar{x}$ in X are called near with respect to f, if $d(x, \bar{x}) = d(f(x), f(\bar{x}))$ holds. By N_x we denote the set of all points near to x. The restriction $f : N_x \to Y$ is a surjective map. Each geodesic γ between near points (called a horizontal geodesic) is mapped isometrically onto its image, that is itself a geodesic. If X is geodesic, then the set N_x is the union of horizontal geodesics starting at x and each geodesic in $\Gamma_{f(x)}$ has a horizontal lift in Γ_x.

Proposition 11.3. *Let $f : X \to Y$ be a submetry between geometric spaces. Then f is differentiable at each point and the differential $D_x f : T_x X \to T_{f(x)} Y$ is a homogeneous submetry.*

Proof. Consider a point $x \in X$ and $y = f(x)$. For each $\bar{y} \neq y$ the function $d_{\bar{y}} \circ f$ is the distance function $d_{F_{\bar{y}}}$ to the fiber $F_{\bar{y}} = f^{-1}(\bar{y})$ and therefore differentiable at x. By Corollary 11.2 the map f is differentiable at x. Being an ultralimit of submetries the differential $D_x f$ is a submetry. •

Under a convergence of submetries fibers converge to fibers, hence the tangent space to each fiber exists and is given by $T_x(f^{-1}(f(x))) = (D_x f)^{-1}(0) := V_x$ (compare Example 7.4).

The subset N_x of all points near to x is the union of all horizontal geodesics starting at x. Therefore we know by Example 6.10 that the space N_x has the property (U) at the point x. Hence the tangent space to N_x at x exists and is given by the closure of the union of radial rays in the tangent cone $T_x X$ corresponding to horizontal geodesics. In particular $T_x N_x$ is contained in the horizontal subcone $H_x = \{h \in T_x X | |h| = |D_x f(h)|\}$.

Take now an arbitrary unit direction $h \in H_x$ and consider $w = D_x f(h)$. Choose a sequence y_j converging to y from the direction w. Then $D_y d_{y_j}(w)$ goes to -1. Therefore $D_x d_{F_j}(h)$ goes to -1 too, where F_j is the fiber $f^{-1}(y_j)$. Thus the vector h is the limit of initial directions h_j corresponding to some geodesics $\gamma_j \in \Gamma_{x,F_j}$. But each geodesic γ in Γ_{x,F_j} is horizontal. Thus we have proved $T_x N_x = H_x = \{h \in T_x X | |h| = |D_x f(h)|\}$. Moreover the proof shows, that for each $h \in H_x$ and each geodesic γ in Y starting at y in the direction $D_x f(h)$ there is a horizontal lift $\bar{\gamma}$ of γ starting at x in the direction h.

11.3. More on submetries. The aim of this subsection is to sketch the proof of the following

Proposition 11.4. *Let X be a geometric space, $f : X \to Y$ a submetry. Then Y is geometric.*

Proof. Choose $x \in X$ and set $y = f(x)$. The set N_x of points near to x still has the property (U). Denote by $H_x = C_x(N_x) = T_x N_x \subset C_x = T_x X$ the tangent space so N_x. Each geodesic in N_x starting at x is mapped isometrically onto a geodesic in Y. Hence we get a natural surjective map $D_x f : H_x \to C_y$, that is 1-Lipschitz and maps radial rays isometrically. In particular the geodesic cone C_y must be proper. For each radial ray h and each point v in H_x we get the following inequality for the Busemann functions: $b_{D_x f(h)}(D_x f(v)) \leqslant b_h(v)$ (Example 2.2).

In order to prove the property (A) at y, consider two geodesics γ_1 and γ_2 starting at y and let $\bar{\gamma}_1, \bar{\gamma}_2 \in \Gamma_x$ be their horizontal lifts. Denote by F_r resp. G_r the fiber $f^{-1}(\gamma_1(r))$ through $\bar{\gamma}_1(r)$ resp. the fiber $f^{-1}(\gamma_2(r))$ through $\bar{\gamma}_2(r)$. We get $\lim_\omega \frac{d(\gamma_1(t_i),\gamma_2(st_i))}{t_i} = \lim_\omega \frac{d(F_{t_i},G_{st_i})}{t_i}$. Hence it is enough to prove, that the equidistant decomposition of $T_x X$ defined by the submetry $f_x^{(t_i)} : X_x^{(t_i)} \to Y_y^{(t_i)}$ is independent of the scale (t_i). However using the first variation formula and the uniform convexity of $T_x X$, it is possible to show, that $v, w \in T_x X$ are in the same fiber of $f_x^{(t_i)}$ iff $D_x d_F(v) = D_x d_F(w)$ holds for each fiber $F = f^{-1}(\bar{y})$ with $\bar{y} \neq y$. In fact this shows, that f is metrically differentiable at x.

Let now $\gamma \in \Gamma_y$ be a geodesic and $\bar{\gamma} \in \Gamma_x$ a horizontal lift of γ. Let $\bar{y} \neq y$ be an arbitrary point and set $F = f^{-1}(\bar{y})$. Then $d_{\bar{y}} \circ \gamma = d_F \circ \bar{\gamma}$. Therefore the differentials of these two maps at 0 coincide. If we denote by v the unit vector $\bar{\gamma}^+ \in H_x$ we get by the first variation formula $D_0(d_F \circ \gamma) = b_{\eta^+}(v)$, for some geodesic $\eta \in \Gamma_{x,F}$.

Geodesics from $\Gamma_{x,F}$ are mapped by f isometrically onto geodesics in $\Gamma_{y,\bar{y}}$. Set $w = D_x f(v)$. Then as in Lemma 9.2, we get $D_0(d_{\bar{y}} \circ \gamma) \leqslant b_{D_x f(\eta^+)}(w)$. But $b_{D_x f(\eta^+)} \circ D_x f \leqslant b_{\eta^+}$. Thus we obtain $b_{D_x f(\eta^+)}(w) = b_{\eta^+}(v)$.

Using once again the uniform convexity of C_x, the above equality and the property (U) in X we get the following: if $d(\gamma_1^+, \gamma_2^+) < \rho$ for some $\gamma_1, \gamma_2 \in \Gamma_y$, then $d(\bar{\gamma}_1(t), \bar{\gamma}_2(t)) < \epsilon t$ for all $t < \rho$, each horizontal lift $\bar{\gamma}_1$ of γ_1 and some horizontal lift $\bar{\gamma}_2$ of γ_2 starting at x. This verifies the property (U) at y. Moreover the above equality for Busemann functions implies, that for each $v \in H_x$ and each radial ray $\eta \subset C_y$ there is at least one radial ray $\bar{\eta} \subset H_x$ with $D_x f(\bar{\eta}) = \eta$ and $b_{\bar{\eta}}(v) = b_\eta(D_x f(v))$.

If both H_x and C_y were Euclidean cones, this would give us that the differential $D_x f : H_x \to C_y$ is a submetry. In general we do not know if this must be true. However the uniform convexity and the smoothness of H_x imply that the cone C_y is uniformly convex and smooth.

Finally the above equality of Busemann functions in C_x and in H_y shows that the first variation formula is valid at y. From Lemma 9.5 we deduce that Y is geometric. •

§12. Theorem of Rademacher

Now we are going to prove Theorem 1.6.

Proof. The question is local and we may assume that S is compact. Assume that we already know the result in the case $n = 1$. Then we can deduce it for arbitrary n by standard reasoning ([Kir94] and [MM00]). Let namely v be a unit vector in $\mathbb{R}^n$. For each $x \in \mathbb{R}^n$ consider the line γ_x through x in the direction v. The restriction of f to $\gamma_x \cap S$ is Lipschitz and by assumption this restriction is differentiable a.e. on $\gamma_x \cap S$. Denote by G^v the set of all $x \in S$, such that the restriction of f to $\gamma_x \cap S$ is differentiable at x. Then G^v has full measure in S, by the theorem of Fubini ([MM00]). Put $G = \cap_{v \in D} G^v$ where v runs over a countable dense subset of the unit sphere. The set G has also full measure in S and f is differentiable at each point of G.

So let $S \subset I$ be a compact subset of an interval and $f : S \to Z$ a Lipschitz map. Since Z is geodesic, we may extend f to a Lipschitz curve $\gamma : I \to Z$. Reparametrizing γ we may assume, that it is parameterized by the arclength. Then γ is 1-Lipschitz and by Theorem 8.2 there is a subset $\widetilde{G} \subset I$ of full measure in I such that for all $s \in \widetilde{G}$ the metric differential $mD_s\gamma$ exists and is the canonical metric d on $\mathbb{R} = T_sI$.

Set $x_t = \gamma(t)$ and let $h_t : I \to \mathbb{R}$ be the non-negative 1-Lipschitz function $h_t(s) = d_{x_t}(\gamma(s)) = d(x_t, x_s)$. Let T be a dense countable subset in I. By the usual theorem of Rademacher the set G of all points $x \in \widetilde{G}$, where $h_t'(x)$ exists and is linear for all $t \in T$, has full measure in I. Denote by N_ϵ^+ resp. N_ϵ^- the set of all point $x \in G$ such that $h_t'(x) < 1 - \epsilon$ for all $t \in T$ with $t < s$ resp. $h_r'(x) > -1 + \epsilon$ for all $r \in T$ with $r > s$. The set N_ϵ^+ is measurable. Assume that it has positive measure and take a Lebesgue point s of N_ϵ^+.

Choose ρ such that for all $t \in G$ with $s - \rho < t < s$ the inequalities $d(x_t, x_s) \geqslant (1 - \epsilon^2)|t - s|$ and $\mu(N_\epsilon^+ \cap [t, s]) \geqslant (1 - \epsilon^2)|s - t|$ hold.

But T is dense in I by assumption. Hence we can choose some $t \in T$ with $s - \rho < t < s$ and get $h_t(s) - h_t(t) \geqslant (s - t)(1 - \epsilon^2)$. On the other hand the differential of 1-Lipschitz function h_t on the subset $N_\epsilon^+ \cap [t, s]$ is bounded above by $1 - \epsilon$. Since this subset has measure at least $(s - t)(1 - \epsilon^2)$ we see $h_t(s) - h_t(t) \leqslant (s - t)(\epsilon^2) + (s - t)(1 - \epsilon^2)(1 - \epsilon) = (s - t)(1 - \epsilon + \epsilon^3)$. For small ϵ we get a contradiction to $h_t(s) - h_t(t) \geqslant |s - t|(1 - \epsilon^2)$.

In the same way we see, that N_ϵ^- has measure 0 in G. Hence also $N_\epsilon = N_\epsilon^+ \cup N_\epsilon^-$ and $N = \cup_{\epsilon > 0} N_\epsilon$ have measure 0 in I. Thus the complement $G^0 = G \setminus N$ has full measure in I. Until now we have not used the curvature assumptions and they will imply the result now.

Let $s \in G^0$ be arbitrary and set $z = \gamma(s)$. Choose sequences t_n and r_n with $t_n < s < r_n$, such that $h_{t_n}'(s) \to 1$ and $h_{r_n}'(s) \to -1$. Let v_n resp. w_n be the starting vectors of some geodesics from z to $\gamma(t_n)$ resp. from z to $\gamma(r_n)$.

We are going to prove that v_n and w_n converge in C_x to γ^+ resp. to γ^-. In order to prove this consider an arbitrary sequence $\epsilon_j \to 0$ and the point $w = (\gamma(s + \epsilon_j)) \in Z_z^{(\epsilon_j)}$.

Assume first that Z is a $CAT(\kappa)$ space. Then from the comparison triangle to $x_{r_n} x_s x_{s+\epsilon_j}$ with $\epsilon_j << r_n - s$ we see that $h_{r_n}^-(s) \to -1$ implies that the distance between w_n and w goes to 0 with $n \to \infty$. This finishes the proof in the case of upper curvature bound.

Let now Z be a space with curvature $\geqslant \kappa$. Then the comparison triangle to $x_{t_n} x_s x_{s+\epsilon_j}$ with $\epsilon_j << s - t_n$ shows that the distance between v_n and w goes to 2 as n goes to ∞. But $Z_z^{(\epsilon_j)}$ is a non-negatively curved space. This shows that v_n converge to a unique point $\bar{v} \in C_z \subset Z_z^{(\epsilon_j)}$ with $|\bar{v}| = 1$ and $d(\bar{v}, w) = 2$. But since the metric differential at s of γ is the usual metric on $\mathbb{R} = T_sI$, we see that the point $v = (\gamma(s - \epsilon_j)) \in Z_z^{(\epsilon_j)}$ also satisfies $d(v, 0) = 1$ and $d(v, w) = 2$. In the non-negatively curved space $Z_z^{(\epsilon_j)}$ geodesics cannot branch, hence v and $\bar{v}$ coincides. This finishes the proof in the case of lower curvature bound. ●

References

[BBI01] Burago D., Burago Yu., Ivanov S., *A course in metric geometry*, Grad. Stud. Math., vol. 33, Amer. Math. Soc., Providence, RI, 2001.

[Bel96] Bellaiche A., *The tangent space in sub-Riemannian geometry*, Sub-Riemannian Geometry, Progr. Math., vol. 144, Birkhauser, Basel, 1996, pp. 1–78.

[Ber87] Берестовский В. Н., *Субметрии пространственных форм неотрицательной кривизны*, Сиб. мат. ж. **28** (1987), №4, 44–56.

[BG00] Berestovskii V., Guijarro L., *A metric characterization of Riemannian submersions*, Ann. Global Anal. Geom. **18** (2000), no. 6, 577–588.

[BGP92] Бураго Ю., Громов М., Перельман Г., *Пространства А. Д. Александрова с ограниченными снизу кривизнами*, Успехи мат. наук **47** (1992), №2, 3–51.

[BH99] Bridson M., Haefliger A., *Metric spaces of non-positive curvature*, Grundlehren Math. Wiss., vol. 319, Springer-Verlag, Berlin, 1999.

[CH70] Calabi E., Hartman Ph., *On the smoothness of isometries*, Duke Math. J. **37** (1970), 741–750.

[Che99] Cheeger J., *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal. **9** (1999), 428–517.

[Fed59] Federer H., *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[Hal00] Halbeisen S., *On tangent cones of Alexandrov spaces with curvature bounded below*, Manuscripta Math. **103** (2000), no. 2, 169–182.

[HM98] Hoffmann K., Morris S., *The structure of compact groups*, Walter de Gruyter and Co., Berlin, 1998.

[JL01] Johnson W., Lindenstrauss L., *Basic concepts in the geometry of Banach spaces*, Handbook of the Geometry of Banach Spaces, Vol. 1, North-Holland, Amsterdam, 2001, pp. 1–84.

[Kir94] Kirchheim B., *Rectifiable metric spaces: local structure and regularity of the Hausdorff measure*, Proc. Amer. Math. Soc. **121** (1994), 113–123.

[KL97] Kleiner B., Leeb B., *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Inst. Hautes Etudes Sci. Publ. Math. №86 (1997), 115–197 (1998).

[Lyta] Lytchak A., *Almost convex subsets* (готовится к печати).

[Lytb] Lytchak A., *Differentiation in Carnot-Caratheodory spaces* (готовится к печати).

[Lytc] Lytchak A., *Open map theorem in metric spaces*, Preprint.

[LY] Lytchak A., Yaman A., *On Hölder continuous Riemannian and Finsler manifolds*, Preprint.

[Mit85] Mitchell J., *On Carnot-Caratheodory metrics*, J. Differential Geom. **21** (1985), no. 1, 35–45.

[MM00] Margulis G. A., Mostow G. D., *Some remarks on the definition of tangent cones in a Carnot-Caratheodory space*, J. Anal. Math **80** (2000), 299–317.

[Nik95] Nikolaev I., *The tangent cone of an Aleksandrov space of curvature* $\leqslant k$, Manuscripta Math. **86** (1995), 137–147.

[OT] Otsu Y., Tanoue H., *The Riemannian structure of Alexandrov spaces with curvature bounded above*, Preprint.

[Pet94] Petrunin A., *Applications of quasigeodesics and gradient curves*, Comparison Geometry (Berkeley, CA, 1993-94), Math. Sci. Res. Inst. Publ., vol. 30, Cambridge Univ. Press, Cambridge, 1997, pp. 203–219.

[PP94a] Перельман Г. Я., Петрунин А. М., *Экстремальные подмножества в пространствах Александрова и обобщенная теорема Либермана*, Алгебра и анализ **5** (1993), №1, 242–256.

[PP94b] Perel′man G., Getrunin A., *Quasigeodesics and gradient curves in Alexandrov spaces*, Preprint, 1994.

[Res93] Решетняк Ю. Г., *Двумерные многообразия ограниченной кривизны*, Геометрия-4, Итоги науки и техн., т. 70, ВИНИТИ, М., 1989, с. 7–189.

Mathematisches Institut
Universität Bonn
Beringstr. 1
53115 Bonn, Germany
E-mail: lytchak@math.uni-bonn.de

Поступило 12 мая 2004 г.