CURVATURE BOUNDS OF SUBSETS IN DIMENSION TWO

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Abstract. We show that closed subsets with vanishing first homology in two-dimensional spaces inherit the upper curvature bound from their ambient spaces and discuss topological applications.

1. Introduction

This paper concerns the intrinsic geometry of subsets in two-dimensional metric spaces with upper curvature bounds. The main geometric result is

Theorem 1. *Let X be a two-dimensional contractible CAT(κ) space. Let $A \subset X$ be a closed, Lipschitz connected subset with $H_1(A) = 0$. Then A is a CAT(κ) space with respect to the induced intrinsic metric.*

For $\kappa = 0$, this confirms a folklore conjecture, which appeared in print in [Ber02, Conjecture 1]. Related statements and conjectures about subsets of non-positively curved spaces can be found in [AKP19b, Chapter 4].

Some special cases of Theorem 1 are known. In [Bis08] and later in [Ric20], it was shown that Jordan domains in the euclidean plane are CAT(0). A more general version appeared in [LW18]. In [Ric19], Theorem 1 is proved for CAT(0) euclidean simplical complexes. Another special case plays a central role in [NSY21].

The contractibility assumption is redundant for $\kappa \leq 0$; for $\kappa > 0$ it is satisfied if the diameter of X is less than $\frac{\pi}{\sqrt{\kappa}}$. For $\kappa > 0$ the statement is wrong without the contractibility assumption: A closed metric ball of radius $\pi > r > \frac{\pi}{2}$ in the round sphere $\mathbb{S}^2$ is contractible but not CAT(1) in its intrinsic metric.

Localizing the above result we deduce the following:

Corollary 2. *Let Y be a metric space of curvature bounded above by κ and dimension two. Let $A \subset Y$ be closed, Lipschitz connected and locally simply connected. Then A has curvature bounded above by κ with respect to the intrinsic metric.*

If $\kappa \leq 0$, the theorem of Cartan–Hadamard implies that the universal covering of A is contractible. Hence A is *aspherical* in the sense that all higher homotopy groups of A vanish:

Corollary 3. *Let Y be a two-dimensional space of non-positive curvature and let $A \subset Y$ be a closed, Lipschitz connected and locally simply connected subset. Then A is aspherical with respect to the topology induced by the intrinsic metric.*

Somewhat surprisingly, no topological assumption is needed for our next conclusion. Indeed, we obtain the following topological statement about *all* subsets of non-positively curved metric spaces of dimension two.

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Theorem 4. *Let Y be non-positively curved and two-dimensional. Let $A \subset Y$ be an arbitrary subset. Then all higher Lipschitz homotopy groups of A vanish: Every Lipschitz n-sphere in A with $n \geq 2$ bounds a Lipschitz ball in A.*

This result is of geometric origin and is deduced from our main theorem. It has the following purely topological application:

Corollary 5. *Let Y be a two-dimensional space of non-positive curvature. Then any neighborhood retract $A \subset Y$ is aspherical.*

It is known, but surprisingly difficult to prove that *all* subsets of the euclidean plane are aspherical, [CCZ02]. This and the results above make the following generalization of the famous Whitehead Conjecture [Whi41] plausible:

Conjecture 1. *Let X be a two-dimensional aspherical space. Then any subset A of X is aspherical.*

Possibly, a combination of the geometric ideas of the present paper and the purely topological methods of [CCZ02] may lead to the resolution of this conjecture for non-positively curved spaces.

Since there is no assumption on local compactness in Theorem 1, we gain some information on coarse topology of high-dimensional spaces. Recall that a CAT(0) space has *asymptotic rank at most two*, if no asymptotic cone contains an isometric copy of euclidean 3-space [Gro93, CL10, Wen11].

Proposition 6. *Let X be a CAT(0) space of asymptotic rank at most two and let $A \subset X$ be arbitrary. Then for every $n \in \mathbb{N}$ and $\epsilon > 0$ there exists $L_0 > 0$ such that the following holds for all $L \geq L_0$. Any L-Lipschitz sphere $f : \mathbb{S}^n \to A$ bounds a Lipschitz ball in $N_{\epsilon L}(A)$.*

As the proof shows, the numbers ϵ, L_0 are independent of A, but only depend on X. Moreover, the map f extends to a (πL)-Lipschitz map F from the euclidean unit ball B^{n+1} into the (ϵL)-neighborhood of the image of f.

Proposition 6 applies, in particular, to arbitrary subgroups of rank two CAT(0) groups, finitely presented or not, compare the discussion on the Coarse Whitehead Conjecture in [Kap05, p. 26].

We expect our results to simplify the description of geodesically complete two-dimensional CAT(κ) spaces obtained and announced in [NSY21]. Moreover, we expect the results to facilitate a good understanding of two-dimensional CAT(κ) spaces beyond geodesic completeness. For instance, they might lead to a resolution of the following conjecture of potential relevance to geometric group theory:

Conjecture 2. *Any compact two-dimensional non-positively curved space is homotopy equivalent to a finite, two-dimensional, non-positively curved euclidean complex.*

We want to point out that all the results above trivially hold in dimension one, since any one-dimensional CAT(κ) space is covered by a tree. On the other hand, all results completely fail in dimension at least three: already the complement of an open ball in $\mathbb{R}^3$ is not non-positively curved and not aspherical.

Sketch of proof. In order to control the curvature bound of A we need to majorize arbitrary Jordan curves $\Gamma \subset A$ by some CAT(κ)-discs inside A. We use the homological assumption, to find a geodesic of X completely contained in A which subdivides Γ into two smaller Jordan curves. Iterating this process, we subdivide Γ into a collection of 2^k smaller Jordan curves. The technically most challenging part of the proof controls this cutting process and confirms that the arising new Jordan curves will have arbitrary small diameters after sufficiently many steps. Now we majorize the small Jordan curves inside the ambient space X and observe that these majorizations glue together to a majorization of Γ within a small neighborhood of A in X. A limiting argument provides the required majorization contained in A.

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2. Metric geometry

2.1. **Basics and notation.** We refer to [BBI01], [Bal04], [AKP19a] for background on metric geometry and CAT(κ) spaces. Let us summarize notation and basic facts. As usual $\mathbb{R}^n$ will denote the euclidean space and $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ the unit sphere.

The distance on a metric space X will be denoted by $|\cdot,\cdot|_X$ and if there is no risk for confusion by $|\cdot,\cdot|$. If $A \subset X$ is a subset, then we denote by $N_r(A)$ and $\bar{N}_r(A)$ its open respectively closed r-neighborhoods. If A is just a point x, then $N_r(A)$ is the open r-ball which we denote by $B_r(x)$. The closed r-ball will be denoted by $\bar{B}_r(x)$.

The length of a curve c in a metric space X will be denoted by $\ell(c) \in [0,\infty]$. The space X is called Lipschitz connected if any two points in X are joined by a curve of finite length. X is called an *intrinsic space*, if the distance between any two points is equal to the greatest lower bound for lengths of curves connecting those points.

Isometric embeddings of intervals will be called *geodesics* and in the case of compact intervals also *geodesic segments* or simply *segments*. If c is a geodesic segment, then we will denote its boundary points by ∂c. The space X itself will be called *geodesic* if any two points in X are joined by a geodesic.

Any Lipschitz connected metric space has a canonical *induced intrinsic metric*, [BBI01, 2.3.3]. The length of all curves for the given metric and for the induced intrinsic metric coincide.

A *triangle* in X consists of three points and three geodesics connecting them. The three geodesics are called the *sides* of the triangle. For $\kappa \in \mathbb{R}$, let $D_\kappa \in (0,\infty]$ be the diameter of the complete, simply connected model surface M^2_κ of constant curvature κ. For any triangle $\triangle$ with perimeter $< 2D_\kappa$ in a metric space X, we can find a *comparison triangle* $\tilde{\triangle} \subset M^2_\kappa$ such that corresponding sides have equal lengths.

A complete metric space X is CAT(κ) if points at distance $< D_\kappa$ in X are joined by geodesics and if all triangles in X with perimeter $< 2D_\kappa$ are *not thicker* than their comparison triangles. A metric space X is said to have *curvature bounded above* by κ, if X is locally CAT(κ).

Any CAT(0) space is contractible; the universal covering of any complete nonpositively curved space is CAT(0), [AKP19a, 8.13.1, Theorem of Cartan–Hadamard].

In a CAT(κ) space X any point y in $B_{D_\kappa}(x)$ is connected with x by a unique geodesic, denoted by xy and depending continuously on y. Angles between geodesics starting at x are well defined. There is a *space of directions* $\Sigma_x X$ which is CAT(1) with respect to the angle metric. The logarithm map

$$\log_x : B_{D_\kappa}(x) \setminus \{x\} \to \Sigma_x X\ ,$$

which assigns to y the starting direction of xy is a homotopy equivalence [Kra11].

In a CAT(κ) space Z, there are natural ways of straightening singular simplices using iterated geodesic coning or barycentric simplices, see [KL97, Section 6.1] or [Kle99, Lemma 5.1]. As a consequence, for any finite simplicial complex K, Lipschitz maps are dense in the space of continuous maps $K \to Z$ with respect to compact-open topology. Moreover, if a continuous map f is already Lipschitz continuous on a subcomplex K_0 of K then the Lipschitz continuous approximations $\tilde{f}$ of f can be chosen to coincide with f on K_0.

On a space X of curvature bounded above, there is a natural notion of dimension $\dim(X)$, introduced and investigated by Kleiner in [Kle99]. It is equal to the supremum of topological dimensions of compact subsets of X, satisfies $\dim(X) = \sup_{x\in X}\{\dim(\Sigma_x X) + 1\}$ and coincides with the homological dimension of X.

The homological dimension of a space X is the supremum of all n, such that for some open pair $V \subset U \subset X$ the relative homology $H_n(U, V)$ is non-zero. Here and below H_n denotes singular homology with $\mathbb{Z}$ coefficients.

3. Majorizations

A *Jordan curve* in a metric space X is a subset homeomorphic to a circle. We say that a metric space Y majorizes a rectifiable Jordan curve Γ in a metric space X if there exists a 1-Lipschitz map $f : Y \to X$ which sends a Jordan curve $\Gamma' \subset Y$ bijectively in an arc length preserving way onto Γ. By Reshetnyak's Majorization Theorem, [AKP19a, 8.12.4], any Jordan curve of length $< 2D_\kappa$ in a CAT(κ) space is majorized by a closed convex subset of M^2_κ. On the other hand, we have

Proposition 7. [LS20, Proposition 4.2] *Let X be a complete intrinsic space. If any Jordan curve Γ of length $< 2D_\kappa$ in X is majorized by some CAT(κ) space Y_Γ, then X is CAT(κ).*

Majorizations stay close to the curve:

Lemma 8. *Let X be a CAT(κ) space and let $\Gamma \subset X$ be a Jordan curve of length $< 2D_\kappa$. Suppose that the diameter of Γ is at most $\epsilon \leq \frac{D_\kappa}{2}$. Then Γ has a majorization by a convex subset of M^2_κ whose image in X is contained in $\bar{N}_\epsilon(\Gamma)$.*

Proof. For any $x \in \Gamma$, the closed ball $\bar{B}_\epsilon(x)$ is CAT(κ) and contains Γ. Hence, we can find a majorization of Γ within this ball. ☐

The following simple gluing/subdivision lemma will be repeatedly applied in the proof of the main theorem.

Lemma 9. *Let X be a CAT(κ) space and $\Gamma^\pm \subset X$ rectifiable Jordan curves. Suppose Γ^+ intersects Γ^- in a geodesic segment γ and denote by Γ the Jordan curve $(\Gamma^+ \cup \Gamma^-) \setminus \gamma$. Let $Z^\pm$ be CAT(κ) discs and let $w^\pm : Z^\pm \to X$ be majorizations of $\Gamma^\pm$ which restrict to arc length preserving homeomorphisms $\partial Z^\pm \to \Gamma^\pm$. Then there is a CAT($\kappa$) disc Z and a majorization $w : Z \to X$ of Γ which sends ∂Z in an arc length preserving way onto Γ and whose image is the union of the images of $w^\pm$.*

Proof. Since γ is a geodesic in X and $w^\pm$ is 1-Lipschitz, the subarc $c^\pm$ of $\partial Z^\pm$ which gets mapped to γ has to be geodesic in $Z^\pm$, cf. [AKP19a, 8.12.2]. Let $f : c^+ \to c^-$ be the canonical isometry. By Reshetnyak's gluing theorem [AKP19a, 8.9.1], the space $Z := Z^+ \cup_f Z^-$ is CAT(κ). Moreover, Z is homeomorphic to a closed disc and contains $Z^\pm$ as convex subspaces. We define $w : Z \to X$ such that it restricts to $w^\pm$ on $Z^\pm$. By construction, w is a well-defined majorization of Γ, as required. ☐

4. Support sets of cycles

Support sets of top-dimensional cycles provide a useful tool in the study of finite dimensional spaces [KL97, BKS16, Hua17, Ric19]. We will recall their definition here and prove basic properties required in later sections, in order to define our cutting procedure for Jordan curves. Support sets were also used by Ricks in his proof of Theorem 1 for simplicial complexes, although in a different vein.

Definition 10. *Let Y be a subset of a metric space X. The* support $\mathrm{spt}(\alpha)$ *of a homology class $\alpha \in H_n(X,Y)$ is the set of all points $x \in X \setminus Y$ such that the image of α is non-trivial under the inclusion homomorphism*

$$H_n(X,Y) \to H_n(X, X \setminus \{x\})$$

The support of α is the intersection of $X \setminus Y$ with all images of chains representing α. Thus, $\mathrm{spt}(\alpha)$ is closed in $X \setminus Y$ and its closure in X is compact.

For instance, the support of the fundamental class $[M] \in H_n(M)$ of a compact, oriented n-manifold M is the whole manifold M.

The following result has been verified in [Hua17, p. 2342]:

Lemma 11. *Let Y be a subset of a metric space X of homological dimension n. Let S be the support of a class $\alpha \in H_n(X,Y)$. Then, for any neighborhood U of $S \cup Y$, the class α can be represented in U; thus, α is in the image of $i_* : H_n(U,Y) \to H_n(X,Y)$.*

If X is contractible, then the boundary homomorphism $\partial : H_n(X,Y) \to H_{n-1}(Y)$ is an isomorphism and the support of $\alpha \in H_n(X,Y)$ is the set of all $x \in X \setminus Y$ such that the image $i_*(\partial\alpha) \in H_{n-1}(X \setminus \{x\})$ is non-zero for the inclusion $i : Y \to X \setminus \{x\}$.

Corollary 12. *Let X be a contractible metric space of homological dimension n. Let $Y \subset X$ and $\alpha \in H_n(X,Y)$ be given. Set $S = \mathrm{spt}(\alpha) \subset X \setminus Y$. Let $p \in S$ and $0 < r < |p,Y|$ be arbitrary. Then, for all neighborhoods W of $\partial B_r(p) \cap S$ in $X \setminus \{p\}$, the canonical map $i_* : H_{n-1}(W) \to H_{n-1}(X \setminus \{p\})$ is non-trivial.*

Proof. Fix a neighborhood W, which we may assume to be open. Assume that we find a larger neighborhood $V \supset W$ such that the non-zero image of $\alpha \in H_n(X,Y)$ in $H_n(X, X \setminus \{p\})$ can be represented by an element $\beta \in H_n(V,W)$. Then, using that the connecting homomorphism $\partial : H_n(X, X \setminus \{p\}) \to H_{n-1}(X \setminus \{p\})$ is an isomorphism, we would deduce that $i_*(\partial\beta)$ is non-zero in $H_{n-1}(X \setminus \{p\})$.

In order to find such V, consider $\hat{V} := B_r(p) \cup W \cup (X \setminus \bar{B}_r(p))$ and its subset $\hat{W} := W \cup (X \setminus \bar{B}_r(p))$. By Lemma 11, the class α can be represented in $H_n(\hat{V}, Y)$. Thus, the image of α in $H_n(X, X \setminus \{p\})$ can be represented by an element $\hat{\alpha} \in H_n(\hat{V}, \hat{W})$. Now we use excision and see that $\hat{\alpha}$ can be represented by an element in $H_n(V,W)$ where $V = \hat{V} \setminus (\hat{W} \setminus W) = B_r(p) \cup W$. This finishes the proof. □

The following observation is essentially contained in [Ric19, Lemma 2.4].

Lemma 13. *Let X be contractible metric space of homological dimension n. Let $M \subset X$ be a compact oriented $(n-1)$-manifold with fundamental class $[M] \in H_{n-1}(M)$. Consider the unique $\alpha \in H_n(X,M)$ with $\partial\alpha = [M]$ and let $S \subset X \setminus M$ be the support of α. Then $S \neq \emptyset$ and $\overline{S} = S \cup M$.*

Proof. Let $x \in M$ and $\delta > 0$ be arbitrary. We claim that there exists an open neighborhood O of M, such that $O \cup B_{2\delta}(x) = X$ and such that the image $\hat{\alpha}$ of α in $H_n(X,O)$ is non-zero. Once the claim is verified, the support $\hat{S}$ of $\hat{\alpha}$ would be a non-empty subset of $X\setminus O$, since otherwise an application of Lemma 11 with $U=O$ would provide a contradiction to $\hat\alpha\neq 0$. Any $p\in\hat S$ would also be contained in the support S of α. Since δ was arbitrary, this would imply $x\in\bar S$. But since x was arbitrary, this would show $M\subset\bar S$. On the other hand, S is closed in $X\setminus M$, thus, we would deduce $\bar S=S\cup M$. The statement $S\neq\emptyset$ would follow as well.

In order to verify the claim, we use that M is an absolute neighborhood retract, and find a retraction $r:V\to M$ of a neighborhood V of M, [Han51]. Restricting to a smaller neighborhood, if needed, we may assume that $T:=r^{-1}(x)$ has diameter smaller δ. Then the injectivity of the map $H_{n-1}(M)\to H_{n-1}(M,M\setminus\{x\})$ implies that the image of $[M]\in H_{n-1}(V)$ is non-zero in $H_{n-1}(V,V\setminus T)$.

Setting $O:=V\cup(X\setminus\bar B_\delta(x)))$ we deduce by excision

$$H_{n-1}(O,O\setminus T)=H_{n-1}(V,V\setminus T)\ .$$

Thus, $[M]$ is non-zero in $H_{n-1}(O,O\setminus T)$, hence also in $H_{n-1}(O)$. Therefore, $[M]$ is not in the image of $\partial:H_n(O,M)\to H_{n-1}(M)$. Now, the long exact sequence of the triple (M,O,X) shows that $\hat\alpha\neq 0$. ☐

If X is a CAT(κ) space of diameter $<D_\kappa$, the logarithm map $\log_x:X\setminus\{x\}\to\Sigma_xX$ is a homotopy equivalence. Thus, for a subset $Y\subset X$ and a class $\alpha\in H_n(X,Y)$, a point $x\in X\setminus Y$ is in the support of α if and only if $(\log_x)_*(\partial\alpha)\neq 0\in H_{n-1}(\Sigma_xX)$, compare [Ric19, Definition 1.3], [Hua17, p.2345].

Proposition 14. [BKS16, Lemma 3.1][Hua17, Lemma A-11] *Let X be an n-dimensional CAT(κ) space of diameter $<D_\kappa$, let $Y\subset X$ be a closed subset and α a class in $H_n(X,Y)$ with support $S:=\operatorname{spt}(\alpha)$. Then the following* geodesic extension property *holds. For any pair of points $x\in X$ and $p\in S$ the segment xp extends beyond p to a point $y\in Y$ such that the subsegment py lies in $S\cup\{y\}$.*

Proof. It is enough to find, for any $p\in S$, $x\in X\setminus\{p\}$ and any $0<r<|p,Y|$, a point $q\in\partial B_r(p)\cap S$ such that p lies on the segment xq. Then an iteration of this property, as at the end of [Hua17, Lemma A-11], finishes the proof.

If there is no such q, then there exists a neighborhood W of $\partial B_r(p)\cap S$ in X, such that p does not lie on geodesic from x to a point in W. Thus, the geodesics towards x provide a contraction of W inside $X\setminus\{p\}$, in contradiction to Corollary 12. ☐

For $Y\subset X$ as in Proposition 14 and a class α in $H_n(X,Y)$, consider again the support $S:=\operatorname{spt}(\alpha)$. For $p\in S$, we define the space of directions $\Sigma_pS\subset\Sigma_pX$ to be the set of starting directions of geodesics py contained in S.

Due to Proposition 14 any such segment extends within S until Y (in particular, to a uniformly positive length). By compactness of $\bar S$, this implies that Σ_pS is a compact subset of Σ_pX. Another application of Proposition 14 shows

$$\Sigma_pS=\bigcap_{r>0}\log_p(\dot B_r(p)\cap S)$$

where $\dot B_r(p)$ refers to the punctured ball $B_r(p)\setminus\{p\}$. We refer to [Hua17] for further properties of Σ_pS. Here we will only need:

Lemma 15. *In the notations above, assume that $\Sigma_pS=\mathcal V^+\cup\mathcal V^-$, with $\mathcal V^\pm$ being closed, non-empty and disjoint. Then we find $v^+\in\mathcal V^+$ and $v^-\in\mathcal V^-$ with*

$$|v^+,v^-|=\pi.$$

Proof. Assume the contrary and find a small ϵ, such that $\epsilon < |v^+, v^-| < \pi - \epsilon$ for all $v^\pm \in \mathcal{V}^\pm$. Choose arbitrary $v_0^\pm \in \mathcal{V}^\pm$ and $x_0^\pm \in S$ lying in the direction of $v_0^\pm$ from p.

We find a small positive $r > 0$, such that for all $q \in S \cap \partial B_r(p)$ we have $|\Sigma_p S, \log_p(q)| \leq \frac{\epsilon}{3}$. Thus, $S \cap \partial B_r(p)$ is a disjoint union of two compact subsets $K^\pm$ such that $|\mathcal{V}^\pm, \log_p(q)| \leq \frac{\epsilon}{3}$, for $q \in K^\pm$.

The geodesics towards $x_0^\pm$ provide a contraction of a neighborhood of $K^\mp$ in X inside $X \setminus \{p\}$. Thus, a neighborhood of $S \cap \partial B_r(p)$ is contractible inside $X \setminus \{p\}$, in contradiction to Corollary 12. □

5. Reduction to a cutting Lemma

Throughout this section X is a contractible CAT(κ) space of dimension two.

Definition 16. *Let Γ be a Jordan curve in X and $[\Gamma]$ a generator of $H_1(\Gamma)$. Denote by $\alpha \in H_2(X, \Gamma)$ the unique element with $\partial\alpha = [\Gamma]$. The support* $\operatorname{spt}(\alpha) \subset X \setminus \Gamma$ *will be called* interior *of Γ.*

We borrow the term *interior* in this context from [Ric19].

A closed convex subset X' of X of diameter less than D_κ is contractible. If Γ is contained in X', then, by the definition of support, the interior of Γ is contained in X' and so are all subsets resulting from the constructions performed in this section. Recall that any Jordan curve Γ of length $l < 2D_\kappa$ is contained in a closed ball X' of radius $\frac{l}{4}$, [Bal04, Prop. 3.20]. Since all subsequent considerations concern only such curves, we may always replace X by X' and assume that the diameter of X is at most $\frac{l}{2} < D_\kappa$ to begin with.

Let $\Gamma \subset X$ be a Jordan curve with interior S. We define a *cut* of Γ to be a geodesic c with $c \subset \bar{S}$ and $c \cap \Gamma = \partial c$. A cut c divides Γ into two arcs Γ^+ and Γ^- whose boundaries coincide with ∂c. When performing a cut we obtain two new Jordan curves $\Gamma_c^\pm := \Gamma^\pm \cup c$.

A *k-fold (iterated) cut* is defined inductively, where a 1-fold iterated cut is just a cut and a k-fold iterated cut are 2^{k-1} cuts performed at each of the Jordan curves resulting from a $(k-1)$-fold iterated cut.

Iterated cuts stay in a controlled neighborhood of the original curve:

Lemma 17. *Let $\Gamma \subset X$ be a Jordan curve of diameter at most $\epsilon < \frac{D_\kappa}{2}$. Denote by G_k the union of Γ with all the geodesics from a k-fold iterated cut. Then G_k is contained in the ϵ-neighborhood of Γ.*

Proof. Γ is contained in a convex ball $X' = \bar{B}_\epsilon(x)$, for any $x \in \Gamma$. Then, by induction, all iterated cuts of Γ are contained in X'. This implies the claim. □

For any cut c of a Jordan curve Γ of length $< 2D_\kappa$ in X, both arising Jordan curves $\Gamma_c^\pm$ have length strictly smaller than the length of Γ. The same is true for any Jordan curve obtained as a result of an iterated cut. The following cutting lemma is a uniform version of this statement. We are going to postpone its proof to the next section. In this section, we derive from it the main results of the present paper.

Lemma 18. *For any Jordan curve $\Gamma \subset X$ of length $< 2D_\kappa$ and $\epsilon > 0$, there exists a k-fold cut of Γ, such that all resulting Jordan curves have diameter at most ϵ.*

Using this lemma we can now provide:

Proof of Theorem 1. Thus, let $A \subset X$ be a Lipschitz connected subset of a contractible CAT(κ) space X such that $H_1(A) = 0$.

Since $H_1(A) = 0$, for any Jordan curve Γ contained in A, the interior of Γ is contained in A (by the very definition of support). By induction on the number of cuts, all iterated cuts of Γ are contained in A.

Denote by $\hat{A}$ the set A with its intrinsic metric. By assumption $\hat{A}$ is a intrinsic space. Since A is closed, $\hat{A}$ is complete [Pet20, Exercise 1.19].

The identity map $I : \hat{A} \to A$ is 1-Lipschitz. For any Lipschitz curve $\gamma : [a, b] \to A$, the curve $I^{-1} \circ \gamma$ has the same length in A and in $\hat{A}$. Therefore, for any 1-Lipschitz map $f : Z \to A$ from an intrinsic space Z, the composition $I^{-1} \circ f$ is 1-Lipschitz as well. Thus, by Proposition 7, we only need to find for any Jordan curve Γ in A of length $< 2D_\kappa$ a majorization of Γ in A by some CAT(κ) disc Z.

We fix such a curve Γ. By Lemma 18, we find an infinite sequence of iterated cuts of Γ such that the diameters of all occurring Jordan curves go to zero uniformly. Namely, for each $\epsilon > 0$ there exists $k_\epsilon \in \mathbb{N}$ such that all the resulting Jordan curves after $k \geq k_\epsilon$ iterations have diameter at most ϵ. Denote by G_k the union of Γ with all cuts after k iterations. Then $(G_k)_{k\in\mathbb{N}}$ forms an increasing sequence of compact sets in A, $G_k \subset G_{k+1}$. By Lemma 17, we have $G_k \subset N_\epsilon(G_{k_\epsilon})$. This implies that the closure $\overline{\bigcup_{k\in\mathbb{N}} G_k}$ is compact. Indeed, for all $\epsilon > 0$ we can cover G_{k_ϵ} by finitely many ϵ-balls. Hence $\overline{\bigcup_{k\in\mathbb{N}} G_k} \subset \bar{N}_\epsilon(G_{k_\epsilon})$ is covered by finitely many 2ϵ-balls.

Since all Jordan curves which arose from our cutting process have diameter at most ϵ, Lemma 8 ensures that they can be majorized within their closed ϵ-neighborhoods. Using Lemma 9, we can then inductively glue these majorizations to obtain a majorization $f_k : Z_k \to X$ of Γ with image contained in $\bar{N}_\epsilon(G_k)$. Precomposing with a majorization of $\partial Z_k \subset Z_k$, we may assume that Z_k is a convex region in M^2_κ. The sequence $(Z_k)_{k\in\mathbb{N}}$ subconverges with respect to Hausdorff distance to a convex region Z_∞. Thus we obtain a partial limit $f_\infty : Z_\infty \to X$ which is 1-Lipschitz and has image in $\overline{\bigcup_{k\in\mathbb{N}} G_k} \subset A$. To see that it is a majorization of Γ, we note

$$\ell(\Gamma) \leq \ell(\partial Z_\infty) \leq \liminf \ell(\partial Z_k) = \ell(\Gamma).$$

Hence the boundary ∂Z_∞ is mapped in an arc length preserving way by f_∞. □

From the above proof we now derive

Proof of Corollary 2. As before, denote by $\hat{A}$ the set A with the induced intrinsic metric. Let $x \in A$ be an arbitrary point. Since Y has curvature bounded above by κ, we find a small ball $\bar{B}_r(x)$ around x in Y which is CAT(κ) and such that $r < \frac{D_\kappa}{2}$.

Since A is locally simply connected at x, we find some $s < r$, such that the inclusion $\bar{B}_s(x) \cap A \to B_r(x) \cap A$ induces a trivial map on π_1, hence also on H_1.

Denote by P the set of all points in $\bar{B}_s(x) \cap A$ which are connected to x by a Lipschitz curve inside $\bar{B}_s(x) \cap A$. Let $\hat{P}$ be the set P equipped with the induced intrinsic metric. Note that the $\frac{s}{4}$-ball in $\hat{P}$ around x coincides with the $\frac{s}{4}$-ball around x in $\hat{A}$. Thus, it suffices to prove that $\hat{P}$ is CAT(κ). The space $\hat{P}$ is a complete intrinsic space, by the same argument as in the solution of [Pet20, Exercise 1.19]. Assume that Γ is a Jordan curve in $\hat{P}$ of length $< 2D_\kappa$. Then Γ defines a trivial element in $H_1(A \cap B_r(x))$, thus, the interior of Γ within $\bar{B}_r(x)$ is contained in $A \cap \bar{B}_r(x)$. So, any cut of Γ (considered within the CAT(κ)-space $\bar{B}_r(x)$) is contained in $A \cap \bar{B}_r(x)$. Since a cut is a geodesic, it is also contained in the convex ball $\bar{B}_s(x)$.

Repeating the argument, any iterated cut of Γ is contained in $A \cap \bar{B}_s(x)$. As in the proof of Theorem 1, we now find a majorization of Γ within $A \cap \bar{B}_s(x)$ by a convex subset of M^2_κ.

The image of this majorization lies in P and provides a majorization of Γ within $\hat{P}$. By Proposition 7, this implies that $\hat{P}$ is CAT(κ) and finishes the proof. $\square$

In the case $\kappa = 0$, the theorem of Cartan–Hadamard yields Corollary 3.

6. Proof of the cutting Lemma

6.1. **Reduction to existence of essential cuts.** For $\delta > 0$, a k-fold cut of Γ will be called δ*-essential*, if all 2^k resulting Jordan curves Γ_i satisfy $\ell(\Gamma_i) \leq (1-\delta) \cdot \ell(\Gamma)$.

Lemma 18 is a direct consequence of the following:

Proposition 19. *For every $\epsilon_0 > 0$, $\kappa \in \mathbb{R}$ and $2D_\kappa > l_0 > 0$ there exists a positive constant $\delta = \delta(\epsilon_0, \kappa, l_0)$ such that the following holds:*

Let Γ be a Jordan curve of length $l \leq l_0$ in a CAT(κ) space X. If the diameter of Γ is at least ϵ_0, then Γ admits a δ-essential 2-fold cut.

Assuming Proposition 19, we now provide

Proof of Lemma 18. Let l be the length of Γ. Choose $\delta = \delta(\epsilon, \kappa, l)$ in Proposition 19. If Γ has diameter $\leq \epsilon$ there is nothing to prove. Otherwise, we apply Proposition 19 and obtain a δ-essential 2-fold cut of Γ resulting in four new Jordan curves.

We proceed inductively as follows. At each step we consider the Jordan curves produced by the previous step and split them into two groups depending on their diameter. If a Jordan curve has diameter at most ϵ, we perform an arbitrary 2-fold cut, thereby keeping the diameter bound. If, on the other hand, a Jordan curve has diameter larger than ϵ, then, since its length is less than l, we can apply Proposition 19 to make a δ-essential 2-fold cut.

Thus, after k steps, the diameters of the arising Jordan curves are at most $\max\{\epsilon, (1-\delta)^k \cdot l\}$. For sufficiently large k, this number is at most ϵ as required. $\square$

6.2. **Setting for finding essential cuts.** In order to prove Proposition 19, we fix κ, $l_0 < 2D_\kappa$ and ϵ_0. By scaling, we may assume $\kappa = 1$. Making ϵ_0 smaller, we may assume $\epsilon_0 < \frac{1}{8}$.

Henceforth we fix a CAT(1) space X and a Jordan curve Γ in X of length $l \in (\epsilon_0, l_0]$. The curve Γ is contained in a ball of radius at most $\frac{l}{4}$ and replacing X by this ball, we may assume that X has diameter at most $\frac{l}{2}$. Let S denote the interior of Γ. We are going to construct a 2-fold δ-essential cut of Γ for some δ depending only on l_0 and ϵ_0.

The proof will be divided into two cases, depending on the existence of long cuts: For $\epsilon > 0$, we say that Γ is ϵ*-degenerated* if all cuts of Γ are shorter than $2\epsilon \cdot l$.

6.3. **The non-degenerated case.** In this case we proceed as follows. First we are going to verify the following simple claim about spherical triangles:

There exists a positive constant $\rho = \rho(l_0)$ such that for any spherical triangle $\triangle(x, y, z) \subset \mathbb{S}^2$ with $|y, z| \leq \pi/2$, $|x, y| \leq \frac{l_0}{2}$, $\angle_y(x, z) \geq \frac{\pi}{2}$ we have

$$|x, y| \leq |x, z| + |z, y| - \rho \cdot |z, y|.$$

If $|x, y| \leq \frac{\pi}{2}$, we can take $\rho = 1$.

For $\frac{l_0}{2} \geq |x,y| \geq \frac{\pi}{2}$, we may assume $\angle_y(x,z) = \frac{\pi}{2}$ and $|x,y| = \frac{l_0}{2}$. The claim follows by compactness and the fact that for sufficiently small $|x,y|$ the statement is true by the first variation formula.

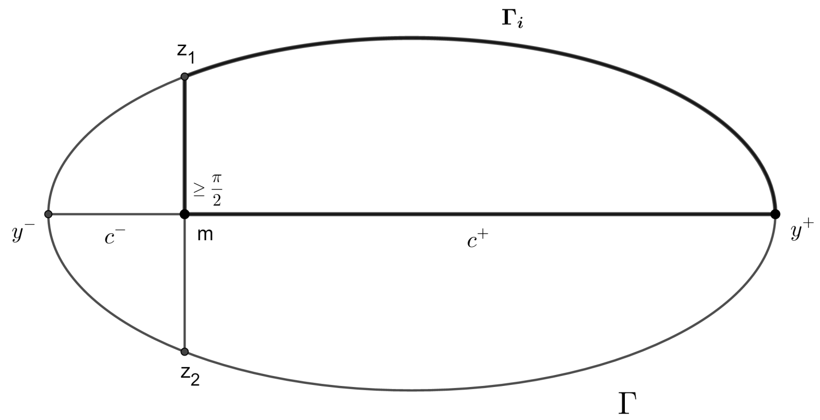
Using this $\rho = \rho(l_0)$ we can now state:

Lemma 20. *If, for some $\epsilon \in (0, \frac{1}{8})$, the curve Γ is not ϵ-degenerated then there is a $(\rho\epsilon)$-essential 2-fold cut of Γ.*

Proof. Let c be any cut of length at least $2\epsilon \cdot l$, which exists by assumption. Denote by $z_1, z_2 \in \Gamma$ the endpoints of c. Denote by Γ_c^+ and Γ_c^- the arising Jordan curves with interiors $S^\pm$. Let m denote the midpoint of c.

In S^+ we consider a sequence of points x_k converging to $m \in c \subset \Gamma^+$, which is possible by Lemma 13. Denote by $\bar{x}_k \in \Gamma^+$ the closest point to x_k. Then $\bar{x}_k$ converges to m, and taking k large enough, we may assume that $\bar{x}_k \in c$. Using the geodesic extension property, Proposition 14, we can extend the segments $\bar{x}_k x_k$ to a segment $c_k = \bar{x}_k y_k$ such that $x_k y_k$ is contained in the closure $\bar{S}^+$. Since $\bar{S}^+$ is compact, we can pass to a partial limit c' of c_k, which is a geodesic segment my starting in m and contained in $\bar{S}^+$. By the upper semi-continuity of angles, both angles enclosed between c and c' are at least $\frac{\pi}{2}$, since the same is true for c_k and c at $\bar{x}_k$. Restricting c' to the first intersection point $y^+ \neq m$ of my with Γ^+ we obtain a cut $c^+ = my^+$ of Γ^+ starting at m and enclosing angles at least $\frac{\pi}{2}$ with c.

In the same way, we construct a cut $c^- = my^-$ of Γ^- starting at m and enclosing angles at least $\frac{\pi}{2}$ with c. The geodesics c, c^+, c^- provide a 2-fold cut of Γ. We claim that this 2-fold cut is $(\rho\epsilon)$-essential.



In order to see this, consider one of the four Jordan curves Γ_i. Without loss of generality, we may assume that Γ_i consists of the geodesics $z_1 m$, my^+ and the part of Γ between y^+ and z_1. Thus, Γ_i arose from the cut c^+ of Γ^+. The difference of lengths

$$\ell(\Gamma^+) - \ell(\Gamma_i)$$

is at least as large as the triangular defect

$$|y^+, z_2| + |z_2, m| - |m, y^+|$$

which by construction of ρ is at least

$$\rho \cdot |z_2, m| \geq \rho \cdot \epsilon \cdot l \ .$$

Therefore, we deduce

$$\ell(\Gamma) - \ell(\Gamma_i) \geq \ell(\Gamma^+) - \ell(\Gamma_i) \geq \rho \cdot \epsilon \cdot l \ .$$

This finishes the proof. □

6.4. **The degenerated case.** This case is technically more complicated and requires several steps. In the rest of this subsection, we fix $\delta := \frac{\epsilon_0}{1000 \cdot \pi}$ and assume that Γ is δ-degenerated. Since $l < 2\pi$, we have $5 \cdot \delta \cdot l < \frac{\epsilon_0}{100}$.

We note that the closure $\bar{S}$ of S is Lipschitz connected: Γ has finite length and any point in S is connected to some point on Γ by an (extrinsic) geodesic completely contained in $\bar{S}$, due to Proposition 14.

Denote by $\hat{S}$ the set $\bar{S}$ equipped with the induced intrinsic metric. If we consider Γ as a subset of $\hat{S}$ we will denote it by $\hat{\Gamma}$. The identification $\hat{\Gamma} \to \Gamma$ is a length preserving, 1-Lipschitz homeomorphism.

We observe: $\hat{\Gamma}$ is $(\delta \cdot l)$-dense in $\hat{S}$.

Otherwise, we could find a point $x \in \hat{S}$ at distance at least $\delta \cdot l$ from $\hat{\Gamma}$. By Proposition 14, we find a cut c of Γ through the point x. By assumption on x, the cut c has length at least $2\delta \cdot l$, in contradiction to our degeneracy assumption.

Using this observation, we are going to control the *absolute filling radius* of $\hat{\Gamma}$. Recall that the absolute filling radius of $\hat{\Gamma}$ is the greatest lower bound of numbers r such that $\hat{\Gamma}$ embeds isometrically as an r-dense subset into a metric space Y, $\iota : \hat{\Gamma} \hookrightarrow Y$, and $\iota_*[\hat{\Gamma}] = 0 \in H_1(Y)$.

We will use the following basic observations about absolute filling radii:

- The absolute filling radius of $\mathbb{S}^1$ with its intrinsic metric is $\frac{\pi}{3}$, [Kat83].
- If $f : \hat{\Gamma} \to \tilde{\Gamma}$ is an L-Lipschitz map of degree one, then the absolute filling radius of $\tilde{\Gamma}$ is at most L times the filling radius of $\hat{\Gamma}$, [Gro83, p. 8].
- If $|\cdot,\cdot|_k$ is a sequence of metrics on $\hat{\Gamma}$ converging uniformly to $|\cdot,\cdot|_{\hat{\Gamma}}$ then the absolute filling radii of $(\hat{\Gamma}, |\cdot,\cdot|_k)$ converge to the absolute filling radius of $(\hat{\Gamma}, |\cdot,\cdot|_{\hat{\Gamma}})$, [LMO20, Proposition 9.34].

Under our degeneracy assumption we can now show:

Lemma 21. *The Jordan curve $\hat{\Gamma} \subset \hat{S}$ has absolute filling radius at most $\delta \cdot l$.*

Proof. For every $k \in \mathbb{N}$, consider the neighborhood $W_k = N_{\frac{1}{k}}(\bar{S})$ of $\bar{S}$. Due to Lemma 13, the curve Γ bounds a chain in W_k. This relative cycle can be represented by a continuous map $f : \Sigma \to W_k$, where Σ is a smooth Riemannian surface with one boundary curve which is mapped by f onto Γ in a length preserving way, [Hat02, p. 109]. Due to the straightening of simplices mentioned above, [KL97, Section 6.1], we may assume that f is Lipschitz continuous after perturbation.

The set W_k is also Lipschitz connected. We denote by $\hat{W}_k$ the set W_k with its intrinsic metric and by Γ_k, the curve Γ as a subset of $\hat{W}_k$.

Since f is Lipschitz continuous, it remains continuous as a map $f : \Sigma \to \hat{W}_k$. Thus, Γ_k induces a trivial 1-cycle in $H_1(\hat{W}_k)$. Since any point of $\bar{S}$ is connected to Γ with a curve of length at most $\delta \cdot l$, the curve Γ_k is $(\delta \cdot l + \frac{1}{k})$-dense in $\hat{W}_k$. Therefore, the filling radius of Γ_k is at most $(\delta \cdot l + \frac{1}{k})$.

In order to reach the conclusion, we only need to show that $|\cdot,\cdot|_{\Gamma_k}$ converge uniformly to $|\cdot,\cdot|_{\hat\Gamma}$. By the uniform compactness of Γ_k it suffices to prove that for any $p,q\in\Gamma$ the distances $|p,q|_{\Gamma_k}=|p,q|_{W_k}$ converge to $|p,q|_{\hat\Gamma}=|p,q|_{\hat S}$.

Clearly, $|p,q|_{\hat W_k}\leq |p,q|_{\hat S}$.

On the other hand, consider a sequence of curves η_k in W_k connecting p,q, parametrized by arclength and such that $\lim\limits_{k\to\infty}\ell(\eta_k)=\lim\limits_{k\to\infty}|p,q|_{\hat W_k}$. By compactness of $\bar S$, we find a subsequence of η_k which converges pointwise to a curve η in $\bar S$. Then $\ell(\eta)\leq \lim\limits_{k\to\infty}\ell(\eta_k)$, hence

$$\lim_{k\to\infty}|p,q|_{\hat W_k}\geq |p,q|_{\hat S}\,.$$

This finishes the proof of the lemma. □

We are going to show that the δ-degenerated $\hat\Gamma$ "comes close to itself".

Consider points $p,q\in\hat\Gamma$ realizing the diameter d of $\hat\Gamma$. Since the diameter of Γ does not exceed the diameter of $\hat\Gamma$, by our assumption $d>\epsilon_0$.

Denote by $\Gamma^\pm$ the arcs of $\hat\Gamma$ defined by the points p and q. Denote by $\Gamma_0^\pm$ the set of all points in $\Gamma^\pm$ that have $\hat S$-distance at least $\frac{d}{3}$ to p and q. We claim

Lemma 22. *There exist points $x^+\in\Gamma_0^+$ and $x^-\in\Gamma_0^-$ with $|x^+,x^-|_{\hat S}\leq 4\cdot\delta\cdot l$.*

Proof. Assume the contrary. The function $\hat f(x):=|p,x|_{\hat S}$ defines 1-Lipschitz maps of both arcs $\Gamma^\pm$ onto the interval $[0,d]$. Moreover, $\hat f$ sends the points p,q onto the ends of the interval and is a degree one map of $\Gamma^\pm$ onto $[0,d]$ modulo endpoints.

Hence also the maps $\tilde f:\Gamma^\pm\to[\frac d3,\frac{2d}{3}]$, defined as composition of $\hat f$ and the closest-point projection from $[0,d]$ to $[\frac d3,\frac{2d}{3}]$, are 1-Lipschitz and have degree one.

Consider the constant speed parametrizations $\eta^\pm:[\frac d3,\frac{2d}{3}]\to\mathbb S^1$ of the upper and lower hemi-circle, respectively. Let $f:\hat\Gamma\to\mathbb S^1$ be defined on $\Gamma^\pm$ as $\eta^\pm\circ\tilde f$. By construction, the map f has degree one and its restrictions to Γ^+ and to Γ^- are both $(\frac3d\cdot\pi)$-Lipschitz continuous.

For all $x^+\in\Gamma^+$ and $x^-\in\Gamma^-$, either one of the points is not in $\Gamma_0^\pm$ and then

$$\frac{|f(x^+),f(x^-)|}{|x^+,x^-|_{\hat S}}\leq\frac3d\cdot\pi\,,$$

by the 1-Lipschitz property of $\tilde f$. Or, otherwise, the distance between x^+ and x^- is at least $4\cdot\delta\cdot l$ and then

$$\frac{|f(x^+),f(x^-)|}{|x^+,x^-|_{\hat S}}\leq\frac{\pi}{4\cdot\delta\cdot l}\,.$$

By assumption on δ we have $\frac3d\leq\frac{1}{4\cdot\delta\cdot l}$. Thus, the map f is $\frac{\pi}{4\cdot\delta\cdot l}$-Lipschitz.

Due to Lemma 21 and the properties of the filling radius listed above, we deduce

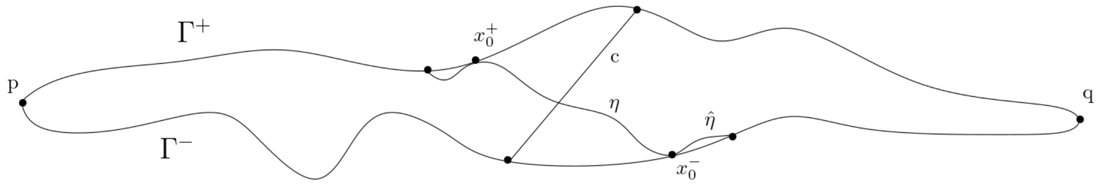
$$\frac{\pi}{4\cdot\delta\cdot l}\cdot\delta\cdot l\geq\frac\pi3\,.$$

This is a contradiction, finishing the proof. □

Now we will find a "good" cut near the points provided by the above lemma:

Corollary 23. *The curve Γ admits a δ-essential cut.*

Proof. We continue to use notations introduced prior to Lemma 22. Consider points $x^\pm \in \Gamma_0^\pm$ provided by Lemma 22. We find a curve $\hat\eta$ in S connecting x^+ with x^- of length $< 5 \cdot \delta \cdot l$. Let x_0^+ be the last intersection point of $\hat\eta$ with Γ^+ and let x_0^- be the first intersection point of $\hat\eta$ with Γ^-. Denote by η the part of $\hat\eta$ between x_0^+ and x_0^-.



Thus, the length of η is $< 5 \cdot \delta \cdot l$ and, by construction, the distances from $x_0^\pm$ to p and q are at least

$$\frac{d}{3} - 5 \cdot \delta \cdot l \geq \frac{d}{3} - \frac{d}{100} \geq \frac{d}{4}\,.$$

Consider the set $\mathcal{C}$ of all cuts c of Γ which contain a point on η. Let $\bar{\mathcal{C}}$ be the set of all geodesics which can be obtained as a limit of a sequence in $\mathcal{C}$.

Any geodesic $\hat c \in \bar{\mathcal{C}}$ connects two points on Γ and intersects η. By the degeneracy assumption, $\hat c$ has length at most $2 \cdot \delta \cdot l$. Thus, any point on $\hat c$ has distance from p and q at least equal to

$$\frac{d}{4} - 7 \cdot \delta \cdot l \geq \frac{d}{5}\,.$$

Assume that there exists an element $\hat c \in \bar{\mathcal{C}}$ which contains points from Γ^+ and Γ^- simultaneously. Then a subsegment c of $\hat c$ is a cut of Γ and has one endpoint on Γ^+ and the other endpoint on Γ^-. Then c subdivides Γ into two arcs, one of which contains p and the other contains q, hence both of them are at least $\frac{2d}{5}$ long.

Thus, both Jordan curves $\Gamma_c^\pm$ arising from the cut c have lengths at most

$$l - \frac{2d}{5} + 2 \cdot \delta \cdot l \leq l - \delta \cdot l\,.$$

Therefore, c is δ-essential and the proof would be complete.

Therefore, assuming that the corollary does not hold, we infer that no $\hat c \in \bar{\mathcal{C}}$ simultaneously intersects Γ^+ and Γ^-. Denote by $\bar{\mathcal{C}}^\pm$ the subsets of elements of $\bar{\mathcal{C}}$ which contain points in Γ^+, respectively in Γ^-. As we have observed, our assumption implies that the sets $\bar{\mathcal{C}}^\pm$ are disjoint. By definition, both these sets are closed under convergence, and we have $\bar{\mathcal{C}} = \bar{\mathcal{C}}^+ \cup \bar{\mathcal{C}}^-$.

By Proposition 14, there exists an element $c \in \mathcal{C}$ through each point in $\eta \setminus \{x_0^\pm\}$. Denote by $K^\pm$ the set of points on $\eta \setminus \{x_0^\pm\}$, which lie on some segment in $\bar{\mathcal{C}}^+$ and $\bar{\mathcal{C}}^-$, respectively. Hence, $\eta \setminus \{x_0^\pm\} = K^+ \cup K^-$. Since $\bar{\mathcal{C}}^\pm$ is closed, the sets $K^\pm$ are closed in the connected set $\eta \setminus \{x_0^\pm\}$.

We claim that the sets $K^\pm$ are non-empty. Indeed, consider points $y_m \neq x_0^+$ on η converging to x_0^+. Choose cuts c_m of Γ through y_m. If $c_m \in \bar{\mathcal{C}}^-$ for all m, then any partial limit $\hat c$ of c_m is contained in $\bar{\mathcal{C}}^-$. But $\hat c$ contains x_0^+, in contradiction to the disjointness of $\hat c \in \bar{\mathcal{C}}^-$ and Γ^+. Thus, $c_m \in \bar{\mathcal{C}}^+$, for large m. Therefore, K^+ is not empty. Similarly, K^- is non-empty as well.

The connectedness of $\eta \setminus \{x_0^{\pm}\}$ implies that $K^+ \cap K^-$ is not empty. Consider an arbitrary point $z \in K^+ \cap K^-$.

Denote by $\mathcal{V}^{\pm}$ the set of all vectors $v \in \Sigma_z X$ tangent to an element in $\bar{\mathcal{C}}^{\pm}$. By construction, $\mathcal{V}^{\pm}$ are both non-empty and their union $\mathcal{V}^+ \cup \mathcal{V}^-$ is the space of directions $\Sigma_z S$. Since the sets $\bar{\mathcal{C}}^{\pm}$ are closed, the sets $\mathcal{V}^{\pm}$ are closed in $\Sigma_z X$. Since $\bar{\mathcal{C}}^+ \cap \bar{\mathcal{C}}^-$ is empty, the intersection of $\mathcal{V}^+$ and $\mathcal{V}^-$ is empty and no direction $v \in \mathcal{V}^+$ has an antipodal direction in $\mathcal{V}^-$. This contradicts Lemma 15 and finishes the proof. $\square$

6.5. **Conclusion.** Now we finish the proof of the existence of essential iterated cuts.

Proof of Proposition 19. Set $\delta = \frac{\epsilon_0}{1000 \cdot \pi}$. If Γ is δ-degenerated we obtain a δ-essential cut from Corollary 23. If Γ is not δ-degenerated, we apply Lemma 20 and obtain a $(\rho \cdot \delta)$-essential 2-fold cut, where ρ depends only on l_0. $\square$

7. Lipschitz homotopy groups

In this final section we provide proofs for the applications of the main result.

Proof of Theorem 4. Let A be a subset of a two-dimensional non-positively curved space Y, let $n \geq 2$ be fixed and let $f : \mathbb{S}^n \to A$ be a Lipschitz map. In order to prove that f is contractible in A, we may replace A by the compact image $K = f(\mathbb{S}^n)$ and assume that A is compact. Rescaling the metric on Y by a factor, we may assume that f is 1-Lipschitz if $\mathbb{S}^n$ is considered with respect to its intrinsic metric.

Using that $\mathbb{S}^n$ is simply connected we can lift f to a map $\tilde{f} : \mathbb{S}^n \to X$ into the universal covering X of Y. Note that $\tilde{f}$ is still 1-Lipschitz and X is CAT(0), by the theorem of Cartan–Hadamard. Once we can contract $\tilde{f}$ in its image, we can also contract f. Thus we may assume that $Y = X$ is CAT(0), that A is compact and that f is 1-Lipschitz with respect to the intrinsic metric on $\mathbb{S}^n$.

For $\epsilon > 0$ we can cover the compact set A by finitely many closed ϵ-balls. Denote by A_ϵ the union of these balls. Then A_ϵ is a closed and Lipschitz connected subset of X. Moreover, A_ϵ is locally contractible since the union of those balls which contain a fixed point is star shaped. Denote by $\hat{A}_\epsilon$ the set A_ϵ with its the intrinsic metric. It follows from Corollary 2 that $\hat{A}_\epsilon$ is non-positively curved. Note that f is still 1-Lipschitz as a map from the intrinsic space $\mathbb{S}^n$ to $\hat{A}_\epsilon$. Denote by $\pi_\epsilon : Y_\epsilon \to \hat{A}_\epsilon$ the universal cover. We lift f to a 1-Lipschitz map $\tilde{f} : \mathbb{S}^n \to Y_\epsilon$. Then $\tilde{f}$ is $\frac{\pi}{2}$-Lipschitz when considering $\mathbb{S}^n$ as a subset of $\mathbb{R}^{n+1}$.

Since Y_ϵ is CAT(0), Kirszbraun's theorem [LS97, AKP11] implies that $\tilde{f}$ has a $\frac{\pi}{2}$-Lipschitz extension $\tilde{F} : B^{n+1} \to Y_\epsilon$ where B^{n+1} denotes the closed unit ball in $\mathbb{R}^{n+1}$. Projecting to A_ϵ, we extend f to a $\frac{\pi}{2}$-Lipschitz map $F := \pi_\epsilon \circ \tilde{F} : B^{n+1} \to \hat{A}_\epsilon$.

Hence F is also a $\frac{\pi}{2}$-Lipschitz extension of f as a map $B^{n+1} \to A_\epsilon$, where A_ϵ is equipped with the induced metric from Y (and not with its intrinsic metric).

Now we choose a sequence $\epsilon_k \to 0$. We obtain a nested sequence A_{ϵ_k} of closed sets with $A = \bigcap_{k \in \mathbb{N}} A_{\epsilon_k}$ and $\frac{\pi}{2}$-Lipschitz maps $F_k : B^{n+1} \to \hat{A}_{\epsilon_k}$ filling f. All the maps F_k are $\frac{\pi}{2}$-Lipschitz as maps to Y. Since K is compact, we obtain a partial limit $F_\infty : B^{n+1} \to Y$ which is a $\frac{\pi}{2}$-Lipschitz map extending f and has image in A. $\square$

Proof of Corollary 5. Let Y be a two-dimensional space of non-positive curvature, let $A \subset Y$ be a neighborhood retract and let $f : \mathbb{S}^n \to A$ be continuous for some $n \geq 2$. By assumption, we find a retraction $r : U \to A$ of an open neighborhood U of A and it suffices to find a continuous filling $F : B^{n+1} \to U$ of f.

Thus, we may assume that $A = U$ is open in Y. By staightening simplices, we see that f is homotopic to a Lipschitz map $\hat{f} : \mathbb{S}^n \to U$, [KL97, Section 6.1]. By Theorem 4 the map $\hat{f}$ is contractible in its image, hence f is contractible in U. $\square$

In the proof of Proposition 6 we will use the concept of ultralimits of metric spaces with respect to a non-principal ultrafilter ω. For precise definitions and properties we refer the reader to [KL97], [BH99, Chapter 1.5], [DK18, Chapter 10].

Proof of Proposition 6. Suppose for contradiction that the claim is wrong for some n and ϵ. Then we find a sequence $L_k \to \infty$ and a sequence of L_k-Lipschitz maps $f_k : \mathbb{S}^n \to X$ with images in A which do not bound Lipschitz balls in $N_{\epsilon L_k}(A)$.

Let p be a base point in $\mathbb{S}^n$ and set $x_k := f_k(p)$. Then we rescale X by $\frac{1}{L_k}$ and denote the resulting space by X_k. Note that f_k is 1-Lipschitz as a map to X_k. We pass to ultralimits $(X_\omega, x_\omega) = \omega\text{-lim}(X_k, p_k)$ and $f_\omega : \mathbb{S}^n \to X_\omega$. Then f_ω is 1-Lipschitz and by assumption, X_ω is a CAT(0) space of dimension at most two. By the proof of Theorem 4, f_ω bounds a $\frac{\pi}{2}$-Lipschitz ball F_ω in its image. Choose a finite $\frac{\epsilon}{2\pi}$-dense set T in the open unit ball B^{n+1} in $\mathbb{R}^{n+1}$. Define an extension $\tilde{F}_k : \mathbb{S}^n \cup T \to X_k$ of f_k such that $\omega\text{-lim}\,\tilde{F}_k = F_\omega|_{S^n \cup T}$. Then for k large enough, all $\tilde{F}_k$ are π-Lipschitz (where $\mathbb{S}^n \cup T$ carries the induced metric from $\mathbb{R}^{n+1}$). By Kirszbraun's theorem [LS97, AKP11], we can extend $\tilde{F}_k$ to a π-Lipschitz map $F_k : \bar{B}^{n+1} \to X_k$. Again, for k large enough, $F_k(T)$ lies at distance $< \frac{\epsilon}{2}$ from A and therefore the image of F_k lies at distance $< \epsilon$ from A. Contradiction. $\square$

References

[AKP11] S. Alexander, V. Kapovitch, and A. Petrunin. Alexandrov meets Kirszbraun. In *Proceedings of the Gökova Geometry-Topology Conference 2010*, pages 88–109, 2011.

[AKP19a] S. Alexander, V. Kapovitch, and A. Petrunin. Alexandrov geometry: preliminary version no. 1. *arXiv:1903.08539*, 2019.

[AKP19b] S. Alexander, V. Kapovitch, and A. Petrunin. *An invitation to Alexandrov geometry.* SpringerBriefs in Mathematics. 2019.

[Bal04] W. Ballmann. On the geometry of metric spaces. *Preprint, lecture notes, http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/sin40827.pdf*, 2004.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[Ber02] V. N. Berestovskiĭ. Pathologies in Alexandrov spaces with curvature bounded above. *Siberian Adv. Math.*, 12(4):1–18 (2003), 2002.

[BH99] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[Bis08] R. L. Bishop. The intrinsic geometry of a Jordan domain. *Int. Electron. J. Geom.*, 1(2):33–39, 2008.

[BKS16] M. Bestvina, B. Kleiner, and M. Sageev. Quasiflats in CAT(0) 2-complexes. *Algebr. Geom. Topol.*, 16(5):2663–2676, 2016.

[CCZ02] J. W. Cannon, G. R. Conner, and Andreas Zastrow. One-dimensional sets and planar sets are aspherical. *Topology Appl.*, 120(1-2):23–45, 2002.

[CL10] P.-E. Caprace and A. Lytchak. At infinity of finite-dimensional CAT(0) spaces. *Math. Ann.*, 346(1):1–21, 2010.

[DK18] C. Druţu and M. Kapovich. *Geometric group theory*, volume 63 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica.

[Gro83] M. Gromov. Filling Riemannian manifolds. *J. Differential Geom.*, 18(1):1–147, 1983.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Han51] O. Hanner. Some theorems on absolute neighborhood retracts. *Ark. Mat.*, 1:389–408, 1951.

[Hat02] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Hua17] J. Huang. Top-dimensional quasiflats in CAT(0) cube complexes. *Geom. Topol.*, 21(4):2281–2352, 2017.

[Kap05] M. Kapovich. Problems on boundaries of groups and Kleinian groups. *AIM, https://aimath.org/WWN/groupboundaries/groupboundaries.pdf*, 2005.

[Kat83] M. Katz. The filling radius of two-point homogeneous spaces. *J. Differential Geom.*, 18(3):505–511, 1983.

[KL97] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, 86:115–197, 1997.

[Kle99] B. Kleiner. The local structure of spaces with curvature bounded above. *Math. Z.*, 231:409–456, 1999.

[Kra11] L. Kramer. On the local structure and the homology of CAT(κ) spaces and Euclidean buildings. *Adv. Geom.*, 11(2):347–369, 2011.

[LMO20] S. Lim, F. Memoli, and O. B. Okutan. Vietoris-Rips persistent homology, injective metric spaces, and the filling radius. *arXiv:2001.07588*, 2020.

[LS97] U. Lang and V. Schroeder. Kirszbraun's theorem and metric spaces of bounded curvature. *Geom. Funct. Anal.*, 7(3):535–560, 1997.

[LS20] A. Lytchak and S. Stadler. Improvements of upper curvature bounds. *Trans. Amer. Math. Soc.*, 373(10):7153–7166, 2020.

[LW18] A. Lytchak and S. Wenger. Isoperimetric characterization of upper curvature bounds. *Acta Math.*, 221(1):159–202, 2018.

[NSY21] K. Nagano, T. Shioya, and T. Yamaguchi. Two-dimensional Alexandrov spaces with curvature bounded above I. *arXiv: 2102.00623*, 2021.

[Pet20] A. Petrunin. Pure metric geometry: introductory lectures. *arXiv:2007.09846*, 2020.

[Ric19] R. Ricks. Closed subsets of a CAT(0) 2-complex are intrinsically CAT(0). *arXiv:1909.00048*, 2019.

[Ric20] R. Ricks. Planar subspaces are intrinsically CAT(0). *arXiv:1001.2299 to appear in Tsukuba Journal of Mathematics*, 2020.

[Wen11] S. Wenger. The asymptotic rank of metric spaces. *Comment. Math. Helv.*, 86(2):247–275, 2011.

[Whi41] J. H. C. Whitehead. On adding relations to homotopy groups. *Ann. of Math. (2)*, 42:409–428, 1941.

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