IMPROVEMENTS OF UPPER CURVATURE BOUNDS

ALEXANDER LYTCHAK AND STEPHAN STADLER

Abstract. We prove that upper curvature bounds in the sense of Alexandrov can be improved locally by using appropriate conformal changes. As a new technical tool we derive a generalization to metric spaces and semiconvex functions of the classical differential geometric property that compositions of harmonic maps with convex functions are subharmonic.

1. Introduction

1.1. **Main results.** There is a significant difference between the existence of global *non-positive* upper curvature bounds and the existence of *some* upper curvature bound $\kappa \in \mathbb{R}$. For instance, any complete non-positively curved space is aspherical. On the other hand, *any* simplicial complex carries a metric of curvature bounded from above by 1 in the sense of Alexandrov, [Ber83].

The following result confirms the expectation that in *local* considerations the value of the upper curvature bound does not matter:

Theorem 1.1. *For $\kappa \in \mathbb{R}$ let the metric space (X, d) be CAT(κ). Let $O = B_r(x)$ be an open ball of radius r around x in X. If $\kappa > 0$ assume $r < \frac{\pi}{2\sqrt{\kappa}}$. Then there exists a complete CAT(-1) metric d' on O, such that the identity map $(O, d) \to (O, d')$ is locally bilipschitz.*

In particular, in many questions concerning only local topological properties of CAT(κ) spaces, like most of [Kle99], [LN19], [LN18], one may always assume κ to be -1.

Besides the theory of minimal discs, Theorem 1.1 relies on a generalization of the classical observation that the restriction of a convex function to a harmonic map is subharmonic, [Ish79], [KS93], [Che95], [Fug05]. Here we derive the following natural extension to semi-convex functions as a direct consequence of the contraction properties of gradient flows of such functions.

Theorem 1.2. *Let Ω be a domain in a Euclidean space $\mathbb{R}^n$ and let $u : \Omega \to X$ be a harmonic map into a CAT(κ) space X. Let $f : X \to \mathbb{R}$ be a Lipschitz continuous λ-convex function.*

2010 *Mathematics Subject Classification.* 53C20, 53C23, 58E20.

Key words and phrases. Non-positive curvature, conformal change, minimal disc, harmonic maps.

Then for the composition $f\circ u \in W^{1,2}_{loc}(\Omega)$ *the distributional Laplacian* $\Delta(f\circ u)$ *is a signed locally finite measure which satisfies*

$$\Delta(f\circ u) \geq \lambda \cdot e_u^2 \ ,$$

where $e_u^2 \in \mathcal{L}^1(\Omega)$ *is the energy density of* u.

Recall that a function $f : X \to \mathbb{R}$ is called λ-*convex* if the function

$$t \to f\circ\gamma(t) - \frac{\lambda}{2}\cdot t^2$$

is convex, for any unit speed geodesic γ in X.

Any Sobolev function with vanishing energy density has a constant representative. As an immediate consequence we obtain:

Corollary 1.3. *Let* Ω *be a domain in a Euclidean space* $\mathbb{R}^n$ *and let* $u : \Omega \to X$ *be a harmonic map into a CAT(*κ*) space* X. *Let* $f : X \to \mathbb{R}$ *be Lipschitz and* 1*-convex. If the composition* $f\circ u$ *is constant then* u *itself is a constant map.*

1.2. **Comments on Theorem 1.2.** The proof of Theorem 1.2 does not use any specific property of Ω and applies without changes to any domain in a Riemannian manifold and to domains in admissible Riemannian polyhedra in the sense of [Fug05], [DM10], [BFH$^+$16].

The proof of Theorem 1.2 does not use the upper curvature bound assumption in an essential way either. It only relies on the existence and the contracting behaviour of gradient flows of semi-convex functions, which is valid in much greater generality. For instance these features are true in spaces with lower curvature bounds, [AKP16], [Pet07] and in some spaces of probability measures, [AGS05], [Oht09].

The importance of the result for spaces of curvature bounded above lies in a great variety of semi-convex functions. For instance, for every point x in a CAT(κ) space X the function $f(y) = d^2(x,y)$ is 1-convex if $\kappa \leq 0$. If $\kappa > 0$ then the function f is ϵ-convex on the ball $B_r(x)$ for any $r < \frac{\pi}{2\sqrt{\kappa}}$ and some $\epsilon = \epsilon(r,\kappa) > 0$. This in conjunction with Theorem 1.4 below leads to the proof of Theorem 1.1.

Moreover, on any CAT(0) space X the distance function $d : X\times X \to \mathbb{R}$ is convex. Similarly, for any closed ball B of radius less than $\frac{\pi}{2\sqrt{\kappa}}$ in any CAT(κ) space there exists a convex function $\psi : B\times B \to \mathbb{R}$, comparable (up to a bounded factor) with the distance $d(x,y)$, [Ken91], [Yok16]. This directly implies the uniqueness of solutions of the Dirichlet problem and the continuous dependence of harmonic maps on their traces. The existence of the harmonic maps with prescribed trace and their regularity involve some finer arguments but are heavily based on Theorem 1.2 for the functions d and ψ, respectively, [KS93], [Ser95].

1.3. **Comments and generalizations of Theorem 1.1.** The proof of Theorem 1.1 follows [LS17] and defines the metric d' via a conformal change of the original metric d with a sufficiently convex function.

We refer to [LS17] and Subsection 4.1 below for the definition and basic properties of conformally changed spaces. We just note here that the metric d' provided by the proof of Theorem 1.1 has the following additional properties. The distance to the central point x in (O, d') depends only on the distance to x in (O, d). Moreover, the tangent spaces at any point $y \in O$ with respect to both metrics d and d' are isometric. This condition implies that (O, d') is geodesically complete if (O, d) is locally geodesically complete, see [LN19].

The control of the upper curvature bound under conformal changes is obtained by an application of the theory of minimal discs as in [LS17] together with Theorem 1.2. The following more general statement provides the optimal analogue of the the formula expressing the curvature of a Riemannian manifold after a conformal change.

Theorem 1.4. *For $c, C, \kappa, \lambda \in \mathbb{R}$, let X be a CAT(κ) space and let $f : X \to [c, C]$ be a Lipschitz continuous λ-convex function. Further, let $Y = e^f \cdot X$ denote the conformally equivalent space.*

- *If $\kappa - 4\lambda \leq 0$ then Y is CAT($\bar\kappa$) with $\bar\kappa = e^{-2C} \cdot (\kappa - 4\lambda)$.*
- *If $\kappa - 4\lambda > 0$ and $\lambda > 0$ then Y is CAT($\bar\kappa$) with $\bar\kappa = e^{-2c} \cdot (\kappa - 4\lambda)$.*

The proof of Theorem 1.4 provides also a local statement in the case $\kappa - 4\lambda > 0$ and $\lambda \leq 0$, see Theorem 5.1. Moreover, as the proof of Theorem 1.4 shows, one can localize the statement by writing the formulas in Theorem 1.4 using only local bounds and local semi-convexity of f.

As another direct application of Theorem 1.4 we prove that any CAT(0) space (X, d) admits another CAT(0) metric which is locally *negatively* curved, see Theorem 5.2 below. This result also applies to everywhere branching Euclidean buildings, where some rigidity might have been expected, see [KL97], [Kra11]. However, in Theorem 5.2 the local negative curvature bound needs to tend to zero at infinity. Indeed, no conformal change of the Euclidean plane results in a complete Riemannian manifold of curvature ≤ -1, [MT02, Corollary 7.3]. Thus, the answer to the following question, very natural in view of [AB04] and our Theorems 1.1, 5.2, cannot be obtained by means of this paper:

Question 1.5. *Given a CAT(0) space X, does there exists a CAT(-1) metric on X defining the same topology?*

1.4. **Structure of the paper.** In the preliminaries we recall basics of Sobolev maps and variation of length under gradient flows of semi-convex functions. This variation is applied in Section 3 to obtain a proof of Theorem 1.2. In Section 4, we recall some structural results about spaces with upper curvature bounds, minimal discs and conformal changes used in Section 5 to prove the main results of the paper.

1.5. **Acknowledgments.** We would like to thank Grigori Avramidi for posing a question which has lead to Theorem 1.1.

Both authors were partially supported by DFG grant SPP 2026.

2. Preliminaries

2.1. **Notations and spaces with upper curvature bounds.** We refer the reader to [BBI01], [BH99] and [AKP16] for basics on metric geometry and CAT(κ) spaces. Here we just agree on notation, some finer properties will be discussed in Subsection 4.3.

In this paper all CAT(κ) spaces will be complete length spaces by definition. By d we will denote distances in metric spaces. We will let $D \subset \mathbb{R}^2$ be the open Euclidean unit disc, $\bar{D} \subset \mathbb{R}^2$ the closed Euclidean unit disc and $S^1 = \partial \bar{D} \subset \mathbb{R}^2$ the unit circle.

For a Lipschitz function $f : X \to \mathbb{R}$ on a metric space X we denote by $|\nabla_p^- f| \in [0, \infty)$ the *descending slope* of f at $p \in X$ defined by

$$|\nabla_p^- f| = \max\{0, \limsup_{x \to p} \frac{f(p) - f(x)}{d(p, x)}\}.$$

2.2. **Semi-convex functions and their gradient flows.** Let X be a CAT(κ) space. For any Lipschitz continuous λ-convex function $f : X \to \mathbb{R}$ there exists the locally Lipschitz continuous *gradient flow* $\Phi : [0, \infty) \times X \to X$ of f, such that for any x the flow line $t \to \Phi_t(x)$ is the gradient curve of the function f starting at x.

As a reference one can use [OP17] or [Lyt05], see also [Pet07], [May98] and [AGS05] for a general theory of gradient flows in metric spaces.

From all properties of gradient flows we will only need the following distance estimate on the change of length under the gradient flow. In [Pet07, Lemma 2.2.1, Lemma 2.1.4] it is proven for Alexandrov spaces but the proof relies only on the first variation formula and is identical in our setting of CAT(κ) spaces.

Corollary 2.1. *Let X be CAT(κ) and let f a Lipschitz continuous λ-convex function on X with gradient flow Φ. Let $\gamma : [a, b] \to X$ be an absolutely continuous curve and let $\rho : [a, b] \to [0, \infty)$ be Lipschitz.*

Then $\eta(s) := \Phi_{\rho(s)}(\gamma(s))$ is an absolutely continuous curve and for almost all $s \in [a, b]$ its velocity is bounded by

$$|\eta'(s)|^2 \leq e^{-2\lambda \cdot \rho(s)}(|\gamma'(s)|^2 - 2(f \circ \gamma)'(s) \cdot \rho'(s) + |\nabla_{\gamma(s)}^- f|^2 \cdot (\rho'(s))^2).$$

2.3. **Sobolev maps and energy.** By now there exists a well established theory of Sobolev maps with values in metric spaces, [HKST15]. We will follow [LW17] and restrict our revision to the special case needed in this paper.

Let X be a complete metric space. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and denote by $L^2(\Omega, X)$ the set of measurable and essentially separably valued maps $u : \Omega \to X$ such that for some and thus every $x \in X$ the function $u_x(z) := d(x, u(z))$ belongs to $L^2(\Omega)$.

Definition 2.2. A map $u \in L^2(\Omega, X)$ belongs to the Sobolev space $W^{1,2}(\Omega, X)$ if there exists $h \in L^2(\Omega)$ such that for every $x \in X$ the composition u_x is contained in the classical Sobolev space $W^{1,2}(\Omega)$ and its weak gradient satisfies $|\nabla u_x| \leq h$ almost everywhere on Ω.

Each Sobolev map u has an associated *trace* $\mathrm{tr}(u) \in L^2(\partial\Omega, X)$, see [KS93]. If u extends continuously to a map $\hat{u}$ on $\bar{\Omega}$, then $\mathrm{tr}(u)$ is represented by the restriction $\hat{u}|_{\partial\bar{\Omega}}$.

There are several natural definitions of energy for Sobolev maps, see [LW17, Section 4]. We will only use the *Korevaar–Schoen energy.* It can be defined in many different ways, for instance, using the approximate metric differentials [LW17, Proposition 4.6]. The expression we are going to use is the following one.

Any map $u \in W^{1,2}(\Omega, X)$ has a representative, also denoted by u, which is absolutely continuous on *almost all curves* in Ω, [HKST15]. Then, for any vector $v \in \mathbb{R}^n$ the restriction of u to almost any segment parallel to v is absolutely continuous, hence has a well defined finite velocity at almost all times. Thus the function $m_u(x, v)$ which measures the velocity of the curve $t \to u(x + tv)$ at $t = 0$ is well-defined almost everywhere on $\Omega \times \mathbb{R}^n$. We mention that, for almost all $x \in \Omega$, the function $v \to m_u(x, v)$ is a semi-norm on $\mathbb{R}^n$, the *approximate metric differential* of u at x, which in fact is Euclidean, if X is CAT(κ), [KS93], [LW17, Section 11].

The *energy* density of u is defined as

$$e_u^2(z) = \frac{1}{\omega_n} \int_{S^{n-1}} |m_u(x, v)|^2 \, dv \, ,$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. The *Korevaar–Schoen energy* of u is given by

$$E^2(u) := \int_\Omega e_u^2(z) \, dz \, .$$

The map $u \in W^{1,2}(\Omega, X)$ is called *harmonic* if for all $w \in W^{1,2}(\Omega, X)$ with the same trace as u one has

$$E^2(u) \leq E^2(w) \, .$$

If X is CAT(0) or a CAT(κ) space of sufficiently small diameter, then any harmonic map is locally Lipschitz continuous and uniquely determined by a prescribed trace [KS93], [Ser95], [Fug08], [BFH$^+$16].

3. First variation of energy

For Sobolev maps in CAT(κ) spaces we have the following analog of the classical first variation formula.

Lemma 3.1. *Let X be a CAT(κ) space and f a Lipschitz continuous λ-convex function on X with gradient flow $\Phi : [0, \infty) \times X \to X$.*

Let $u \in W^{1,2}(\Omega, X)$ be given. For any Lipschitz continuous test function $\rho : \Omega \to [0,\infty)$ with compact support in Ω, define a variation u_t of u by

$$u_t(x) = \Phi(t\rho(x), u(x)).$$

Then $u_t \in W^{1,2}(\Omega, X)$ and the following inequality holds

$$\frac{d}{dt}^{+}\Big|_{t=0} E^2(u_t) \leq -2 \int_\Omega \left(\lambda \cdot e_u^2(x) \cdot \rho(x) + \langle \nabla_x (f \circ u), \nabla_x \rho \rangle\right) \, dx \tag{3.1}$$

where the left-hand side is the upper Dini derivative.

Proof. We fix $t > 0$. As a composition of a Sobolev and a Lipschitz map, the map u_t is contained in $W^{1,2}(\Omega, X)$.

Consider the curves $\gamma(s) = u(x+sv)$ and $\eta(s) = \Phi(t\cdot\rho(x+sv), \gamma(s))$.

By definition, for almost all $x, v \in \Omega \times S^{n-1}$, the velocities of γ respectively η at $s = 0$ are exactly $m_u(x,v)$ and $m_{u_t}(x,v)$, the values of the corresponding approximate metric differentials. Applying Corollary 2.1 we get the estimate, valid at all such x, v:

$$m_{u_t}^2(x,v) \leq e^{-2\lambda \cdot t \cdot \rho(x)}(m_u^2(x,v) - 2t \langle \nabla_x (f \circ u), v\rangle \cdot \langle \nabla_x \rho, v\rangle + t^2 |\nabla_{u(x)} f|^2 \cdot \langle \nabla_x \rho, v\rangle^2)$$

Averaging over S^{n-1} and using the equality

$$\frac{1}{\omega_n} \int_{S^{n-1}} \langle w_1, v\rangle \cdot \langle w_2, v\rangle \, dv = \langle w_1, w_2\rangle \ ,$$

valid for all $w_1, w_2 \in \mathbb{R}^n$, we obtain the estimate of the energy densities, valid pointwise almost everywhere on Ω:

$$e_{u_t}^2(x) \leq e^{-2\lambda \cdot t \cdot \rho(x)}(e_u^2(x) - 2t \langle \nabla_x \rho, \nabla_x (f \circ u)\rangle + t^2 |\nabla_{u(x)} f|^2 \cdot |\nabla_x \rho|^2) \leq$$

$$\leq (1 - 2\lambda\rho(x) \cdot t)(e_u^2(x) - 2t \langle \nabla_x \rho, \nabla_x (f \circ u)\rangle) + Ct^2 \ ,$$

for some constant C depending on λ, ρ, f and u.

The claim follows now directly by integration over Ω. $\square$

Now we can easily derive:

Proof of Theorem 1.2. Appyling Lemma 3.1 and the definition of harmonicity, we see that the right hand side of (3.1) must be non-negative for any non-negative Lipschitz continuous function ρ with compact support in Ω. Thus the distributional Laplacian $\Delta(f \circ u)$ satisfies

$$\Delta(f \circ u)(\rho) = -\int_\Omega \langle \nabla_x (f \circ u), \nabla_x \rho\rangle \, dx \geq \lambda \int_\Omega e_u^2 \cdot \rho \ .$$

By the representation theorem of Riesz in distirbution theory, this is sufficient to draw the conclusion. $\square$

4. Preparations

4.1. **Length spaces and their conformal changes.** The length of a rectifiable curve γ in a metric space X is denoted by $\ell(\gamma)$. A metric space X is a *length space* if the distance between any two points is equal to the greatest lower bound for lengths of curves connecting the respective points. A curve $c : [a,b] \to X$ will be called *geodesic* if it is an isometric embedding. The space X itself will be called *geodesic* if any two points in X are joined by a geodesic.

We refer to [LS17] for more details on what follows here. Let X be a length space and $f : X \to (0,\infty)$ be a continuous function.

We define the f-length of a rectifiable curve $\gamma : [a,b] \to X$ by

$$\ell_f(\gamma) = \int_a^b f(\gamma(t)) \cdot |\dot\gamma(t)|\, dt\,, \tag{4.1}$$

where $|\dot\gamma(t)|$ denotes the velocity of the curve γ at time t. The conformally changed metric d_f on the space X is defined by

$$d_f(x,y) = \inf_\gamma \left\{ \ell_f(\gamma)\ ;\ \gamma \text{ Lipschitz curve from } x \text{ to } y \right\}\ . \tag{4.2}$$

The space $f \cdot X := (X, d_f)$ is a length space called the *metric space conformally equivalent to X with conformal factor f.*

The identity map $\mathrm{id}_f : X \to f \cdot X$ is a locally bilipschitz homeomorphism. If f is bounded from below by a positive constant and X is complete then $f \cdot X$ is complete as well.

We will need the following observation:

Lemma 4.1. *Let X be a length space and assume $X = B_r(x)$ for some $x \in X$ and $r > 0$. Let $\xi : [0,r) \to (0,\infty)$ be continuous and consider the function $f(y) := \xi(d(x,y))$ on X. Then, in the conformally changed space $f \cdot X$, the distance function to x can be computed as:*

$$d_f(x,y) = \int_0^{d(x,y)} \xi(t)\ dt\ .$$

Proof. Consider concentric metric spheres $S_s(x)$ of radii $s \le d(x,y)$ around x. Then for any $0 \le s_0 < s_1 \le d(x,y)$ and any point $z \in S_{s_1}(x)$ we can estimate the d_f-distance from z to $S_{s_0}(x)$ as

$$|s_1 - s_0| \cdot \min_{s_0 \le s \le s_1} \xi(s) \le d_f(z, S_{s_0}(x)) \le |s_1 - s_0| \cdot \max_{s_0 \le s \le s_1} \xi(s)\,.$$

The proof of the lemma follows by writing the integral $\int_0^{d(x,y)} \xi(t)\ dt$ as a limit of Riemann sums as on the right and left hand sides of the above inequality.

∎

4.2. **Recognizing CAT(κ) spaces.** For us it will be important that CAT(κ) spaces can be recognized without referring to geodesic triangles. By a *Jordan curve* in a metric space X we denote a subset homeomorphic to a circle.

We say that a metric space Y majorizes a rectifiable Jordan curve Γ in a metric space X if there exists a 1-Lipschitz map $P : Y \to X$ which sends a Jordan curve $\Gamma' \subset Y$ bijectively in an arc length preserving way onto Γ. The following is proved in [LS17, Proposition 2] for CAT(0) spaces. Along the same lines we deduce:

Proposition 4.2. *Let $\kappa \in \mathbb{R}$ and $\Lambda > 0$ be such that $\Lambda \leq \frac{\pi}{\sqrt{k}}$ if $\kappa > 0$. Let X be a complete length metric space.*

If any Jordan curve Γ in X of length $< 2\Lambda$ is majorized by some CAT(κ) space Y_Γ, then any closed ball B in X of any radius $r < \frac{\Lambda}{2}$ is convex in X and CAT(κ) .

Moreover, if $\kappa > 0$ and $\Lambda = \frac{\pi}{\sqrt{k}}$ then X is CAT(κ).

Proof. The argument in [LS17, Proposition 2] shows that any pair of points in X at distance less than Λ is connected by a unique geodesic in X. Moreover, such geodesics depend continuously on their endpoints.

As in [LS17, Proposition 2] the assumption implies that any triangle in X of perimeter less than 2Λ is not thicker than its comparison triangle in the constant curvature surface.

For $\kappa > 0$ and $\Lambda = \frac{\pi}{\sqrt{k}}$ this implies by definition, that X is CAT(κ).

For general Λ, the condition implies that X is locally CAT(κ) and the statement follows from [Bal04, 6.10], see Lemma 4.3 below. □

4.3. **Local-to-global in CAT(κ) spaces.** The basic local-to-global theorem about CAT(κ) spaces is the theorem of Cartan–Hadamard, saying that a complete length space, which is locally CAT(κ) with $\kappa \leq 0$, is a CAT(κ) space if and only if it is simply connected.

In order to describe related local-to-global statements for all κ, we recall that the injectivity radius of a local CAT(κ) space X is the supremum of all $r > 0$, such that any pair of points x and y in X at distance less than r are connected by a unique geodesic and that this geodesic depends continuously on the endpoints.

Combining [Bal04, 6.10] and [AKP16, 8.11.3] we obtain the following.

Lemma 4.3. *Let X be a complete length space which is locally CAT(κ). Let $r > 0$ be such that $r < \frac{\pi}{2\sqrt{\kappa}}$ for $\kappa > 0$.*

If the injectivity radius of X is larger than r then any ball $\bar{B}_r(x)$ is a convex CAT(κ) subset of X.

The next local-to-global result is also well-known.

Lemma 4.4. *Let X be a complete length space which is locally CAT(κ). Let $\Lambda > 0$ be such that $\Lambda \leq \frac{\pi}{\sqrt{\kappa}}$ if $\kappa > 0$.*

Assume that for any closed curve Γ of length less than 2Λ there exists a homotopy $\Gamma_t, t \in [0,1]$ from $\Gamma = \Gamma_0$ to a constant curve Γ_1 such that the length of Γ_t is less than 2Λ for all t.

Then any closed ball of radius less than $\frac{\Lambda}{2}$ in X is convex and CAT(κ).

Proof. Let Γ be a closed curve of length less than $2\,\Lambda$. Our assumptions allow to apply [AKP16, 8.13.4], to conclude that Γ is majorized by a CAT(κ) space. The claim now follows from Proposition 4.2. $\square$

Corollary 4.6 below will slightly strengthen the next result.

Proposition 4.5. *Let Z be a compact geodesic space homeomorphic to $\bar{D}$. If Z is locally CAT(1) and has Hausdorff area $\mathcal{H}^2(Z)$ less than 2π, then Z is CAT(1).*

Proof. Otherwise Z contains an isometric embedding Γ of a round circle S^1_{2l} of length $2l < 2\pi$ in Z, [Bal04, 6.9]. The closed Jordan domain Z_1 cut out of Z by Γ is convex, hence locally CAT(1) and its Hausdorff area is also less than 2π.

Doubling Z_1, thus gluing two copies of it along Γ, we obtain a space homeomorphic to the 2-dimensional sphere, which has Hausdorff area less than 4π and which is locally CAT(1), by Reshetnyak's gluing theorem, [AKP16, Theorem 8.9.1].

But this contradicts the Gauss-Bonnet formula, [Res93, (8.15)]. $\square$

As a consequence we deduce the following analog of [LW18, Proposition 12.1] for $\kappa \neq 0$.

Corollary 4.6. *Let $\hat{Z}$ be a length space homeomorphic to the open disc D. For $\kappa \in \mathbb{R}$ let $\hat{Z}$ be locally CAT(κ) and assume that for $\kappa > 0$ the area $\mathcal{H}^2(\hat{Z})$ is at most $\frac{2\pi}{\kappa}$. Then the completion Z of $\hat{Z}$ is CAT(κ).*

Proof. We exhaust the space $\hat{Z}$ by compact closed discs Z_n with boundary being a geodesic polygon as in the proof of [LW18, Proposition 12.1]. As in [LW18, Section 11.2], we readily see that these subsets Z_n are locally CAT(κ) in their intrinsic metrics.

Moreover, a limiting argument as in [LW18, Proposition 12.1] shows that it suffices to verify that the closed discs Z_n are globally CAT(κ).

For $\kappa \leq 0$, this statement follows directly by the theorem of Cartan–Hadamard. For $\kappa > 0$, we may rescale the space and assume $\kappa = 1$. Any open non-empty subset of Z_0 has positive $\mathcal{H}^2$-area, [LN19, Theorem 1.2], thus $\mathcal{H}^2(Z_n) < 2\pi$, for any n. The global CAT(1) property of Z_n is exactly Proposition 4.5. $\square$

4.4. **Surfaces.** In the case of flat domains the curvature of conformally changed metrics has been investigated in detail by Yuri Reshetnyak, see [Res93] and the references therein. In this case it is even possible to relax the continuity and positivity assumptions on conformal factors.

We say that a function $f : U \to [0,\infty)$ on a domain $U \subset \mathbb{R}^2$ is *κ-log-subharmonic*, if f is upper semi-continuous, contained in L^1_{loc} and satisfies weakly

$$\Delta \log f + \frac{\kappa}{2} f^2 \geq 0 \; .$$

For a κ-log-subharmonic function f one can use formulas (4.1) and (4.2) to define the conformally changed metric on U. Indeed, we have the following result due to Reshetnyak, see Theorem 7.1.1 in [Res93], see also [Mes01, Theorem 6.1] and [LW18, Theorem 8.1, Section 17].

Theorem 4.7. *Let $U \subset \mathbb{R}^2$ be a domain and f a κ-log-subharmonic function on U. Then $f \cdot U$ is locally CAT(κ) and $\mathrm{id}_f : U \to f \cdot U$ is a homeomorphism.*

The next computational lemma will provide control on double conformal changes.

Lemma 4.8. *Let $c, C, \kappa, \lambda \in \mathbb{R}$. Let $U \subset \mathbb{R}^2$ be a domain and φ a κ-log-subharmonic function on U. Suppose that $\psi : U \to [c, C]$ is a continuous function which safisfies $\Delta\psi \geq \mu \cdot \varphi^2$ weakly. Then the product $e^\psi \cdot \varphi$ is $\bar\kappa$-log-subharmonic, with*

- $\bar\kappa = e^{-2C} \cdot (\kappa - 2\mu)$ *if* $\kappa - 2\mu \leq 0$*;*
- $\bar\kappa = e^{-2c} \cdot (\kappa - 2\mu)$ *if* $\kappa - 2\mu \geq 0$*.*

Proof. Suppose $\kappa - 2\mu \leq 0$. Then $-(\kappa - 2\mu) \geq -\bar\kappa \cdot (e^\psi)^2$ and the claim follows from

$$\Delta \log(e^\psi \varphi) = \Delta\psi + \Delta \log \varphi \geq \mu \cdot \varphi^2 - \frac{\kappa}{2}\varphi^2 = -\frac{1}{2}(\kappa - 2\mu)\varphi^2$$

The case $\kappa - 2\mu \geq 0$ is analogous. ☐

Combining Theorem 4.7 and Lemma 4.8 leads to:

Lemma 4.9. *Let $c, C, \kappa, \mu \in \mathbb{R}$ and set*

- $\bar\kappa = e^{-2C} \cdot (\kappa - 2\mu)$ *if* $\kappa - 2\mu \leq 0$*;*
- $\bar\kappa = e^{-2c} \cdot (\kappa - 2\mu)$ *if* $\kappa - 2\mu \geq 0$*.*

Suppose that φ is a κ-log-subharmonic function on D and let $\varphi \cdot D$ be the conformally changed disc. Let Z denote the completion of $\varphi \cdot D$. If $\bar\kappa > 0$, assume in addition $\mathcal{H}^2(\varphi \cdot D) \leq e^{-2C} \cdot \frac{2\pi}{\bar\kappa}$. Finally, let $\psi : Z \to [c, C]$ be a continuous function on Z such that the restriction of ψ to D satisfies $\Delta\psi \geq \mu \cdot \varphi^2$ weakly. Then $e^\psi \cdot Z$ is CAT($\bar\kappa$).

Proof. The proof of Lemma 4 of [LS17] implies $e^\psi \cdot Z$ is the completion of the length space $e^\psi \cdot (\varphi \cdot D)$. Moreover, it shows that $(e^\psi \cdot \varphi) \cdot D$ is isometric to $e^\psi \cdot (\varphi \cdot D)$. Hence Lemma 4.8 together with Theorem 4.7 imply that $(e^\psi \cdot \varphi) \cdot D$ is locally CAT($\bar\kappa$).

The claim then follows from Corollary 4.6. ☐

4.5. **Minimal discs.** A general solution of the classical Plateau's problem has been provided in [LW17] for proper metric spaces and in [GW17] for complete CAT(0) spaces. We need to discuss an appropriate extension to non-proper CAT(κ) spaces with $\kappa > 0$.

Lemma 4.10. *Let X be a CAT(κ) space and let Γ be a Jordan curve in X of finite length l. If $\kappa > 0$ assume in addition, that $l < \frac{2\pi}{\sqrt{\kappa}}$. Then there exists a closed ball $B = \bar{B}_r(x)$ which contains Γ. Moreover, if $\kappa > 0$ we can choose $r < \frac{\pi}{2\sqrt{\kappa}}$.*

The space $\Lambda(\Gamma, B)$ of all maps $v \in W^{1,2}(D, B)$ such that $\mathrm{tr}(v)$ is a weakly monotone parametrization of Γ is non-empty and contains a map u_0 of smallest energy $E^2(u_0) < \frac{1}{\pi} \cdot l^2$ in $\Lambda(\Gamma, B)$.

Moreover, any such map u_0 has a unique representative which extends continuously to $\bar{D}$.

Proof. Without loss of generality, we may assume $\kappa = 1$. By Reshetnyak's majorization theorem there exists a closed convex domain C in the open unit hemisphere and a 1-Lipschitz map $P : \bar{C} \to X$ which sends ∂C in a an arclength preserving bijective way onto Γ. Due to the isoperimetric inequality in the unit sphere we have

$$l^2 = \ell^2(\Gamma) > 2\pi \cdot \mathcal{H}^2(C) \; .$$

Any closed convex subset C of an open unit hemisphere contains a point p such that

$$r = \sup\{d(p,q) \; ; \; q \in C\} < \frac{\pi}{2} \; .$$

(Cf. [LRC11, 2.1.2].) Therefore, $P(\bar{C})$ and, in particular, Γ is contained in the ball $B = \bar{B}_r(P(p))$.

We find a conformal parametrization $f : \bar{D} \to C$. Then the energy $E^2(f)$ equals the area of C. Since the map P is 1-Lipschitz, the composition $u = P \circ f$ is an element of $\Lambda(\Gamma, B)$ and has energy

$$E^2(u) \leq E^2(f) = \mathcal{H}^2(C) < \frac{1}{2\pi} \cdot l^2 \; .$$

If B is compact, the existence of an energy minimizer in $\Lambda(\Gamma, B)$ is proved in [LW17]. The same classical argument, extended in [LW17] to proper metric spaces also works in the present non-compact case as follows.

As in the classical case, we can precompose with Moebius maps and restrict to the subspace $\Lambda_0(\Gamma)$ of all maps in $\Lambda(\Gamma)$ whose trace sends three fixed points in S^1 to three prescribed points in Γ, [LW17, Section 7]. Take an energy minimizing sequence u_n in $\Lambda_0(\Gamma)$. For any u_n, we find a unique harmonic map $v_n \in \Lambda_0(\Gamma)$ with the same trace as u_n, [Ser95], [Fug08, Theorem 4], since $r < \frac{\pi}{2}$. In particular, $E(v_n) \leq E(u_n)$ and v_n is an energy-minimizing sequence as well.

By the lemma of Courant–Lebesgue, the traces $tr(v_n)$ are uniformly continuous, [LW17, Section 7]. By [Ser95], [Fug08, Theorem 2], the maps v_n have unique representatives, which continuously extend to $\bar{D}$. Moreover, these representatives converge uniformly, once their traces converge uniformly. Thus, using the semi-continuity of energy, we obtain an energy minimizer in $\Lambda_0(\Gamma)$ by taking a uniform limit of a subsequence of the maps v_n. The so obtained map u_0 of minimal energy clearly satisfies

$$E^2(u_0) \leq E^2(u) < \frac{1}{2\pi} \cdot l^2 \ .$$

The statement about the continuous extendability of u_0 follows from [Fug08, Theorem 2]. $\square$

We call a continuous map $u_0 : \bar{D} \to B$ provided by the above result a *minimal filling* of Γ in B and are going to summarize its properties:

Theorem 4.11. *Let Γ be a Jordan curve of length l in a CAT(κ) space X, with $l < \frac{2\pi}{\sqrt{\kappa}}$ if $\kappa > 0$. As in Lemma 4.10, let $B = B_r(x)$ be a closed ball which contains Γ and let $u : \bar{D} \to B$ be a minimal filling of Γ.*

Then the following hold true:

1. *u is harmonic and $E^2(u) < \frac{1}{\pi} l^2$.*
2. *There exists a function $\varphi \in L^2(D)$, the* conformal factor *of u, such that the approximate metric differential satisfies $m_u(z, v) = \varphi(z) \cdot \|v\|$ for almost all $z \in D$ and all $v \in \mathbb{R}^2$.*
3. *The conformal factor φ can be chosen to be κ-log-subharmonic.*
4. *The completion Z of $\varphi \cdot D$ is a CAT(κ) space.*
5. *The space Z is homeomorphic to $\bar{D}$, the map $u : \varphi \cdot D \to X$ is 1-Lipschitz and extends to a majorization $v : Z \to X$ of Γ.*

Proof. (1) follows by definition of a minimal filling and Lemma 4.10.

(2) is verified in [LW17, Theorem 11.3]

(3) is verified in [Mes01].

In order to verify (4), we use (3) and Theorem 4.7 to see that $\varphi \cdot D$ is locally CAT(κ). The area of $\varphi \cdot D$ can be computed as $\mathcal{H}^2(\varphi \cdot D) = \int_D \varphi^2 = \frac{1}{2} E^2(u) < \frac{1}{2\pi} l^2$. In particular, if $\kappa > 0$, we have $\mathcal{H}^2(\varphi \cdot D) < \frac{2\pi}{\kappa}$. Thus, (4) is a consequence of Corollary 4.6.

Finally, (5) is verified in [LS17, Theorem 9] for $\kappa = 0$, hence also for $\kappa \leq 0$. The proof applies without changes to the case $\kappa > 0$. $\square$

5. Main result

5.1. **Local control of curvature under conformal changes.** Along the lines of [LS17], we are going to prove the following local version of Theorem 1.4.

Theorem 5.1. *For $c, C, \kappa, \lambda \in \mathbb{R}$ there exists some $\rho_0 = \rho_0(c, C, \kappa, \lambda) > 0$ with the following property.*

Let X be a CAT(κ) space and let $f : X \to [c, C]$ be a Lipschitz continuous λ-convex function. Further, let $Y = e^f \cdot X$ denote the conformally equivalent space. Then any closed ball of radius at most ρ_0 in Y is CAT($\bar{\kappa}$), where

- $\bar{\kappa} = e^{-2C} \cdot (\kappa - 4\lambda)$ *if* $\kappa - 4\lambda \leq 0$*;*
- $\bar{\kappa} = e^{-2c} \cdot (\kappa - 4\lambda)$ *if* $\kappa - 4\lambda \geq 0$*.*

Proof. Set $Y = e^f \cdot X$. Since f is bounded, the identity map from Y to X is a bilipschitz homeomorphism. Thus any complete subset of X is also complete in Y. Rescaling Y by the factor e^{-c}, thus subtracting the constant c from f we may assume that $c = 0$.

Choose a positive constant $\Lambda < e^{-C} \cdot \frac{2\pi}{\sqrt{\kappa}}$.

We claim that any Jordan curve Γ in Y of length $< \Lambda$ is majorized by a CAT($\bar{\kappa}$) space. Fix Γ and denote by $\hat{\Gamma}$ the curve Γ considered in X. Since the identity map $Y \to X$ is 1-Lipschitz, the length of $\hat{\Gamma}$ in X is at most $\Lambda < \frac{2\pi}{\sqrt{\kappa}}$.

By Lemma 4.10, we obtain a minimal filling u of $\hat{\Gamma}$ in X, whose properties are described in Theorem 4.11. Denote by φ the conformal factor of u. By Theorem 4.11, $u : \varphi \cdot D \to X$ extends to a majorization $v : Z \to X$, where the completion Z of $\varphi \cdot D$ is CAT(κ). Moreover, the area of Z is less than $\frac{1}{2\pi}\Lambda^2 < e^{-2C} \cdot \frac{2\pi}{\kappa}$.

Due to Theorem 1.2 and the conformality of u, the composition $f \circ u$ fulfills $\Delta(f \circ u) \geq 2\,\lambda \cdot \varphi^2$ weakly. Hence, Lemma 4.9 ensures that $e^{(f\circ v)} \cdot Z$ is CAT($\bar{\kappa}$). Moreover, the majorization $v : Z \to X$ of $\hat{\Gamma}$ defines a majorization $v : e^{(f\circ v)} \cdot Z \to e^f \cdot X = Y$ of Γ.

Thus, any Jordan curve Γ of length less than Λ in Y is majorized by a CAT($\bar{\kappa}$) space. We finish the proof by setting $\rho_0 = \frac{\Lambda}{4}$ and applying Lemma 4.4.

$\square$

5.2. **Global versions.** Now we can turn to the main theorems.

Proof of Theorem 1.4. Due to Theorem 5.1, the space Y is a complete length space, which is locally CAT($\bar{\kappa}$). It remains to globalize the statement.

Assume first that $\lambda > 0$ and consider the gradient flow Φ_t of the function f on the space X. Let Γ denote a rectifiable closed curve in Y. Considering Γ as a curve in X, we apply the gradient flow Φ_t to Γ and obtain closed curves Γ_t in X. The value of f (and hence of e^f) does not increase along flow lines of Φ. Since Φ_t contracts length in X, at least by a factor of $e^{-\lambda t}$, Lemma 3.1, we deduce the following two consequences. Firstly, the e^f-length of Γ_t (thus the length of Γ_t in Y) is non-increasing in t. Secondly, for any Γ as above, any $\epsilon > 0$ and any sufficiently large t, the length of Γ_t in Y is less than ϵ.

Taking ϵ to be smaller than ρ_0 in Theorem 5.1 and appyling the globalization Lemma 4.4, we deduce that Y is CAT($\bar{\kappa}$).

It remains to deal with the case $\lambda \leq 0$ and $\kappa - 4\lambda \leq 0$. But then $\kappa \leq 0$, hence X is simply connected. Since X is homeomorphic to Y, we deduce from the theorem of Cartan–Hadamard that Y is CAT($\bar{\kappa}$). $\square$

5.3. **Conclusions.** We can now easily prove:

Theorem 5.2. *Let x be a point in a CAT(0) space X. Define the function $f : X \to \mathbb{R}$ by $f(y) := \frac{1}{2}d^2(x,y)$. Then the space $Y = e^f \cdot X$ is CAT(0). Moreover, for any $R > 0$, the closed ball $\bar{B}_R(x)$ around x in Y is CAT(κ) for some $\kappa = \kappa(R) < 0$.*

Proof. The function f is 1-convex and Lipschitz continuous on bounded balls. Thus, Y is CAT(0) by [LS17].

By Lemma 4.1, the closed ball $\bar{B}_R(x)$ in Y has the form $e^f \cdot B$, where B is the closed ball $\bar{B}_r(x)$ in X and $r(R)$ is such that

$$\int_0^r e^{\frac{1}{2}t^2}\, dt = R\ .$$

From Theorem 1.4 we deduce that $e^f \cdot B$ is CAT(κ) with

$$\kappa = -4 \cdot e^{-r^2}\ .$$

$\square$

Finally we can provide

Proof of Theorem 1.1. Clearly, we may assume $\kappa > 0$ and, by rescaling, even $\kappa = 1$.

Thus let X be a CAT(1) space and let $O = B_r(x)$ be an open ball in X with $r < \frac{\pi}{2}$.

We can replace X by the closed ball $\bar{B}_r(x)$. In order to simplify the calculation we proceed as follows. First we improve the curvature bound on the closed ball to 0. In a second step, we change the metric on the open ball, to make it complete and simultaneously decrease the upper curvature bound to -1.

There exists $A > 0$ depending only on r, such that the function $g(y) = A \cdot d^2(x,y)$ is 1-convex on X. Due to Theorem 1.4, the space $Z = e^g \cdot X$ is a CAT(0) space. Moreover, by Lemma 4.1, the subset $e^g \cdot O \subset Z$ is an open ball in Z around the point x. Replacing the space X by Z we have reduced our task to the case $\kappa = 0$. In this case the function $g(y) = \frac{1}{2}d^2(x,y)$ is 1-convex on X.

Now consider the function $h : [0, \frac{r^2}{2}) \to \mathbb{R}$ given by

$$h(t) = -\log(\frac{r^2}{2} - t)\ .$$

Then the function h is convex and $\lim_{t \to \frac{1}{2}r^2} h(t) = \infty$. Moreover,

$$h'(t) = e^{h(t)}\ .$$

Consider the locally Lipschitz continuous function $f(y) := h(g(y))$ on $O = B_r(x)$ and the space $Y = e^f \cdot O$.

For an arbitrary point $y \in O$, we choose a small closed ball U around y, such that for all $z \in U$ holds

$$h'(g(z)) \geq \frac{1}{2}h'(g(y)) = \frac{1}{2}e^{h(g(y))} \quad \text{and} \quad h(g(z)) \leq 2h(g(y)).$$

Due to the convexity of h and the 1-convexity of g, the restriction of f to any geodesic γ in O is at least λ-convex, where λ denotes the minimum of h' on the image $g(\gamma) \subset [0, \frac{r^2}{2})$.

Hence for any such ball U, the space $e^f \cdot U$ is CAT(-1), by Theorem 1.4. This shows that the space Y is locally CAT(-1).

For any $s < r$ we deduce from Lemma 4.1, that the subset $e^f \cdot \bar{B}_s(x) \subset e^f \cdot O$ coincides with the closed ball in Y around x of radius

$$R(s) = \int_0^s h(\frac{1}{2}t^2)\, dt = -\int_0^s \log(\frac{r^2 - t^2}{2})\, dt$$

Moreover, this ball is CAT(0) by Theorem 1.4. Since $R(s)$ converges to infinity as s converges to r, we deduce that Y is CAT(0). In particular, it is complete, simply connected and geodesic. Since we have already seen that Y is locally CAT(-1), this finishes the proof. $\square$

References

[AB04] S. Alexander and R. Bishop. Curvature bounds for wrapped products in metrics spaces. *Geom. Funct. Anal.*, 14:1143–1181, 2004.

[AGS05] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

[AKP16] S. Alexander, V. Kapovitch, and A. Petrunin. Alexandrov geometry. *Preprint, http://anton-petrunin.github.io/book/all.pdf*, 2016.

[Bal04] W. Ballmann. On the geometry of metric spaces. *Preprint, lecture notes, http://people.mpim-bonn.mpg.de/hwbllmnn/archiv/sin40827.pdf*, 2004.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001.

[Ber83] V. Berestovskii. Borsuk's problem on metrization of a polyhedron. *Dokl. Akad. Nauk SSSR*, 268(2):273–277, 1983.

[BFH$^+$16] C. Breiner, A. Fraser, L. Huang, C. Mese, P. Sargent, and Y. Zhang. Regularity of harmonic maps from polyhedra to *cat*(1) spaces. *arXiv: 1610.07829*, 2016.

[BH99] M. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1999.

[Che95] J. Chen. On energy minimizing mappings between and into singular spaces. *Duke Math. J.*, 79(1):77–99, 1995.

[DM10] G. Daskalopoulos and C. Mese. Harmonic maps between singular spaces I. *Comm. Anal. Geom.*, 18(2):257–337, 2010.

[Fug05] B. Fuglede. The Dirichlet problem for harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature. *Trans. Amer. Math. Soc.*, 357(2):757–792, 2005.

[Fug08] B. Fuglede. Harmonic maps from Riemannian polyhedra to geodesic spaces with curvature bounded from above. *Calc. Var. Partial Differential Equations*, 31(1):99–136, 2008.

[GW17] C. Guo and S. Wenger. Area minimizing discs in locally non-compact metric spaces. *Comm. Anal. Geom., to appear, preprint*, 2017.

[HKST15] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. Tyson. *Sobolev spaces on metric measure spaces*, volume 27 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2015.

[Ish79] Tôru Ishihara. A mapping of Riemannian manifolds which preserves harmonic functions. *J. Math. Kyoto Univ.*, 19(2):215–229, 1979.

[Ken91] W. Kendall. Convexity and the hemisphere. *J. London Math. Soc. (2)*, 43(3):567–576, 1991.

[KL97] B. Kleiner and B. Leeb. Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings. *Inst. Hautes Études Sci. Publ. Math.*, 86:115–197, 1997.

[Kle99] B. Kleiner. The local structure of spaces with curvature bounded above. *Math. Z.*, 231:409–456, 1999.

[Kra11] L. Kramer. On the local structure and homology of $CAT(\kappa)$ spaces and buidlings. *Advances in Geometry*, 11:347–369, 2011.

[KS93] N. Korevaar and R. Schoen. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.*, 1(3-4):561–659, 1993.

[LN18] A. Lytchak and K. Nagano. Topological regularity of spaces with an upper curvature bound. *arXiv:1809.06183*, 2018.

[LN19] A. Lytchak and K. Nagano. Geodesically complete spaces with an upper curvature bound. *Geom. Funct. Anal.*, 29:295–342, 2019.

[LRC11] Bernhard Leeb and Carlos Ramos-Cuevas. The center conjecture for spherical buildings of types F_4 and E_6. *Geom. Funct. Anal.*, 21(3):525–559, 2011.

[LS17] A. Lytchak and S. Stadler. Conformal deformation of $CAT(0)$ spaces. *Math. Ann.*, Online first, 2017.

[LW17] A. Lytchak and S. Wenger. Area minimizing discs in metric spaces. *Arch. Ration. Mech. Anal.*, 223(3):1123–1182, 2017.

[LW18] Alexander Lytchak and Stefan Wenger. Isoperimetric characterization of upper curvature bounds. *Acta Math.*, 221(1):159–202, 2018.

[Lyt05] A. Lytchak. Open map theorem for metric spaces. *Algebra i Analiz*, 17(3):139–159, 2005.

[May98] U. Mayer. Gradient flows on nonpositively curved metric spaces and harmonic maps. *Comm. Anal. Geom.*, 6(2):199–253, 1998.

[Mes01] C. Mese. The curvature of minimal surfaces in singular spaces. *Comm. Anal. Geom.*, 9(1):3–34, 2001.

[MT02] R. Mazzeo and M. Taylor. Curvature and uniformization. *Israel J. Math.*, 130:323–346, 2002.

[Oht09] S. Ohta. Gradient flows on Wasserstein spaces over compact Alexandrov spaces. *Amer. J. Math.*, 131(2):475–516, 2009.

[OP17] S. Ohta and M. Pálfia. Gradient flows and a Trotter-Kato formula of semi-convex functions on CAT(1)-spaces. *Amer. J. Math.*, 139(4):937–965, 2017.

[Pet07] A. Petrunin. Semiconcave functions in Alexandrov's geometry. In *Surveys in differential geometry. Vol. XI*, pages 137–201. Int. Press, Somerville, MA, 2007.

[Res93] Yu. G. Reshetnyak. Two-dimensional manifolds of bounded curvature. In *Geometry, IV*, volume 70 of *Encyclopaedia Math. Sci.*, pages 3–163. Springer, Berlin, 1993.

[Ser95] T. Serbinowski. Harmonic maps into metric spaces of curvature bounded from above. *Thesis, University of Utah*, 1995.

[Yok16] T. Yokota. Convex functions and barycenter on CAT(1)-spaces of small radii. *J. Math. Soc. Japan*, 68(3):1297–1323, 2016.