RIEMANNIAN FOLIATIONS OF SPHERES

ALEXANDER LYTCHAK AND BURKHARD WILKING

Abstract. We show that a Riemannian foliation on a topological n-sphere has leaf dimension 1 or 3 unless $n = 15$ and the Riemannian foliation is given by the fibers of a Riemannian submersion to an 8-dimensional sphere. This allows us to classify Riemannian foliations on round spheres up to metric congruence.

1. Introduction

We are going to prove the final piece of the following theorem:

THEOREM 1.1. *Suppose $\mathcal{F}$ is a Riemannian foliation by k-dimensional leaves of a compact manifold (M, g) which is homeomorphic to $\mathbb{S}^n$. We assume $0 < k < n$. Then one of the following holds*

a) $k = 1$ *and the foliation is given by an isometric flow, with respect to some Riemannian metric.*
b) $k = 3$, $n \equiv 3 \bmod 4$ *and the generic leaves are diffeomorphic to* $\mathbb{RP}^3$ *or* $\mathbb{S}^3$.
c) $k = 7$, $n = 15$ *and $\mathcal{F}$ is given by the fibers of a Riemannian submersion* $(M, g) \to (B, \bar{g})$ *where* $(B, \bar{g})$ *is homeomorphic to* $\mathbb{S}^8$ *and the fiber is homeomorphic to* $\mathbb{S}^7$.

Furthermore all these cases can occur.

A big part of the Theorem follows by putting together various pieces in the literature: Ghys ([Ghy84]) showed that the generic leaves of a Riemannian foliation of a homotopy sphere are closed, unless the leaf dimension is 1 and the foliation is given by an isometric flow with respect to a possibly different Riemannian metric. Furthermore, the generic leaves are rational homotopy spheres. Haefliger ([Hae84]) observed that for any Riemannian foliation of a complete manifold M with closed leaves, one can find a space $\hat{M}$ homotopically equivalent to M, such that $\hat{M}$ is the total space of a fiber bundle, where the fibers are homeomorphic to the generic leaves of the foliation (see section 2

1991 *Mathematics Subject Classification.* 53C12, 57R30.

The first named author was partially supported by a Heisenberg grant of the DFG and both authors by the SFB "Groups, Geometry and Actions".

for further details). If M is a sphere then the fibers are contractible in $\hat{M}$. Spanier and Whitehead observed ([SW54]) that for any such fibration the fiber must be an H-space. Furthermore, closed manifolds which are H-spaces and rational homotopy spheres were classified by Browder ([Bro62]): they are homotopically equivalent to $\mathbb{S}^1$, $\mathbb{RP}^3$, $\mathbb{S}^3$, $\mathbb{RP}^7$ or $\mathbb{S}^7$. With Perelman's solution of the geometrization conjecture one can improve the statement further to diffeomorphic if $k = 3$.

We are left to consider 7-dimensional foliations of homotopy spheres. Our strategy will be to reduce the situation first to the case of $n = 15$ and then to show that the foliation is simple, i.e., given by the fibers of a Riemannian submersion. By a result of Browder ([Bro62]) this automatically rules out the possibility of an $\mathbb{RP}^7$-foliation.

To see that all examples can occur, we can again appeal to the literature for the only non-classical case: the existence of $\mathbb{RP}^3$ foliations on $\mathbb{S}^{4k+3}$. It was shown by Oliver ([Oli79]), that contrary to a previous conjecture, there are almost free smooth actions of $\mathsf{SO}(3) \cong \mathbb{RP}^3$ on $\mathbb{S}^{4k+3}$ for $k \geq 1$. The actions of Oliver extend to fixed point free smooth actions on the disc D^{4k+4}, different actions were later exhibited by Grove and Ziller ([GZ00]).

Our topological result allows us to classify Riemannian foliations of the round sphere up to metric congruence. We recall that Gromoll and Grove ([GG88]) classified Riemannian foliations of the sphere up to leaf dimension 3. Moreover, due to [Wil01], a Riemannian submersion of the round $\mathbb{S}^{15}$ with 7-dimensional fibers is metrically congruent to the Hopf fibration. Combining this work with Theorem 1.1 gives

Corollary 1.2. *Let $\mathcal{F}$ be a Riemannian foliation on a round sphere $\mathbb{S}^n$ with leaf dimension $0 < k < n$. Then, up to isometric congruence, either $\mathcal{F}$ is given by the orbits of an isometric action of $\mathbb{R}$ or S^3 with discrete isotropy groups or it is the Hopf fibration of $\mathbb{S}^{15} \to \mathbb{S}^8(1/2)$ with fiber $\mathbb{S}^7$.*

As has been pointed out by Gromoll and Grove, a real representation $\rho\colon \mathsf{S}^3 \to \mathsf{SO}(n)$ induces an almost free action of S^3 on the unit sphere if and only if all irreducible subrepresentations are even dimensional.

The paper is structured as follows. In Section 2 we recall the results stated after Theorem 1.1 and study the fibration $\hat{M} \to \hat{B}$ from a homotopy n-sphere $\hat{M}$ to the resolution $\hat{B}$ of the orbifold $B = M/\mathcal{F}$. The fiber of the fibration is $\mathcal{L}$, the principal leaf of $\mathcal{F}$, and we only need to consider the cases $\mathcal{L} = \mathbb{S}^7$ and $\mathcal{L} = \mathbb{RP}^7$. From this fibration we compute the cohomology of $\hat{B}$. The even-degree cohomology ring of $\hat{B}$ turns out to be a truncated polynomial ring $\mathbb{F}_p[a]$ at all odd primes p. Using Steenrod powers at $p = 3$ we deduce that n must be equal to 15. In the two subsequent sections, we exclude the possibility that the orbifold B is not a manifold. Here we use the local data of the orbfold to find non-trivial cohomology classes of $\hat{B}$ that cannot exist by the previous computations. We rely on the fact that all isotropy groups of B act freely on a 7-dimensional sphere or a projective space, a severe restriction on the possible group structure. In Section 3, we use the computation of the cohomology of $\hat{B}$ at the prime 2, to deduce that all isotropy groups are cyclic of odd order. Here we detect the forbidden classes by looking at single points of B, i.e., by finding non-zero restrictions of the cohomology classes to the classifying spaces of the isotropy groups. In Section 4, we exclude the possibility that the set B_p of points with non-trivial p-isotropy is non-empty, otherwise detecting forbidden cohomology classes by their non-trivial restriction to a component of B_p.

2. Topology

2.1. **Recollection.** Let $(M, \mathcal{F})$ be as in Theorem 1.1 and assume that the leaves have dimension $k \geq 2$. Due to [Ghy84], all leaves of $\mathcal{F}$ are closed. This in turn is equivalent to saying that $\mathcal{F}$ is a *generalized Seifert fibration* on M, i.e., the space of leaves $B = M/\mathcal{F}$ carries the natural structure of a smooth Riemannian orbifold, such that the induced Riemannian distance corresponds to the distance between leaves in M. Due to [Ghy84], the regular leaf $\mathcal{L}$ of $\mathcal{F}$ is a rational homology sphere. Following Haefliger, we consider the $\mathsf{SO}(n-k)$ bundle FrM over M given by all oriented horizontal frames in M. Then the Riemannian foliation $\mathcal{F}$ induces a fiber bundle structure on FrM with the fibers being diffeomorphic to $\mathcal{L}$ and with the base space being the oriented frame bundle FrB of the orbifold B. Furthermore, the natural fiber bundle map $FrM \to FrB$ is $\mathsf{SO}(n-k)$-equivariant.

Thus one also gets a fiber bundle with total space given by $\hat{M} = FrM \times_{\mathsf{SO}(n-k)} E\mathsf{SO}(n-k)$ with fiber $\mathcal{L}$ and with base space $\hat{B} := FrB \times_{\mathsf{SO}(n-k)} E\mathsf{SO}(n-k)$, $f\colon \hat{M} \to \hat{B}$. Clearly, $\hat{M}$ is homotopically equivalent to M and $\hat{B}$ is the so called resolution (or classifying space) of the orbifold B. Its cohomology is the so called orbifold cohomology of B. As has been oberserved by Haefliger the natural projection $\hat{B} \to B$ is a rational homotopy equivalence.

Since the fiber $\mathcal{L}$ is a k-dimensional manifold and $\hat{M} \sim_{heq} M \sim_{heq} \mathbb{S}^n$ is k-connected, we see that the fiber is contractible in $\hat{M}$. Therefore $\mathcal{L}$ is an H-space ([SW54]). Since $\mathcal{L}$ is a rational homology sphere, we may apply [Bro62] and deduce that $\mathcal{L}$ is homotopy equivalent to $\mathbb{RP}^3$, $\mathbb{S}^3$, $\mathbb{S}^7$ or $\mathbb{RP}^7$.

The geometrization conjecture shows that for $k = 3$ the generic leaf $\mathcal{F}$ is diffeomorphic to $\mathbb{RP}^3$ or $\mathbb{S}^3$. Moreover, the Gysin sequence with $\mathbb{Q}$-coefficients of the fibration $\hat{M} \to \hat{B}$ shows that the dimension n of M is divisibly by $k + 1 = 4$, cf. the argument in the next subsection. This finishes the proof of Theorem 1.1 in the case $k = 3$.

Thus we only need to consider the case $k = 7$. Hence, $\mathcal{L}$ is either homeomorphic to $\mathbb{S}^7$ or it is homotopy equivalent to $\mathbb{RP}^7$ and its double-cover is homeomorphic to $\mathbb{S}^7$. We call the first case the *spherical case* and the second case the *projective case*.

2.2. **Gysin sequence and dimension.** Let R be any ring with unit. In the projective case we assume in addition that 2 is invertible in R, e.g., $\mathbb{F}_3$ or $\mathbb{Q}$. Then $H^*(\mathcal{L}, R) = H^*(\mathbb{S}^7, R)$. Thus we find the Gysin sequence of the fibration f with coefficients in R. The Euler class must be a generator $a \in H^8(\hat{B}, R) \cong H^0(\hat{B}, R) \cong R$. Moreover, the cup product $\cup a : H^{2i}(\hat{B}) \to H^{2i+8}(\hat{B})$ is an isomorphism, if $2i \neq n - 7$.

Since $\hat{B}$ has finite rational cohomology, we use this isomorphism for $R = \mathbb{Q}$ to see that $n = 8l + 7$, for some positive integer l.

2.3. **Reduction to $n = 15$.** We want to show $l = 1$. Assume on the contrary $l \geq 2$. Then, due to the above isomorphism, we have $H^*(\hat{B}, \mathbb{F}_3) = \mathbb{F}_3[a]$ in degrees ≤ 16. To obtain a contradiction, we first show:

Lemma 2.1. *Under the assumptions above there exists a space X and an element $c \in H^8(X, \mathbb{F}_3)$ such that the cohomology ring $H^*(X, \mathbb{F}_3)$ equals the polynomial ring $\mathbb{F}_3[c]$ in degrees ≤ 24.*

Proof. For $l > 2$, one could just take $X = B$. In general, let E_f be the mapping cylinder of f, which is a fiber bundle over $\hat{B}$ with fiber being the cone over $\mathcal{L}$. Let X be the *Thom space* of the fibration f, which is obtained from E_f by identifying all points on the boundary of E_f. For the subbundle $E' = \hat{B}$ of the bundle $E_f \to \hat{B}$, we can apply [Hat02], Theorem 4.D.8. Using the fact that the bundle $\hat{M} \to \hat{B}$ is orientable, we deduce that there is an element $c \in H^8(E, E', \mathbb{F}_3) = H^8(X, \mathbb{F}_3)$ (the *Thom class* of the fibration), such that $b \to f^*(b) \cup c$ induces an isomorphism between $H^*(\hat{B})$ and the reduced cohomology $\tilde{H}^{*+8}(X, \mathbb{F}_3)$.

The claim follows from this isomorphism and the structure of $H^*(\hat{B})$. □

We now get a contradiction to the following application of Steenrod powers, cf. [Hat02], Theorem 4.L.9.

Lemma 2.2. *Let X be a topological space. Assume that $H^{12}(X, \mathbb{F}_3) = H^{20}(X, \mathbb{F}_3) = 0$. Then for all $c \in H^8(X, \mathbb{F}_3)$ we have $c^3 = 0$.*

Proof. Consider the Steenrod operations $P^i \colon H^n(X, \mathbb{F}_3) \to H^{n+4i}(X, \mathbb{F}_3)$. We have $c^3 = P^4(c)$. On the other hand, by the Adem relations, $P^4(c)$ is a linear combination of $P^1(P^3(c))$ and $P^3(P^1(c))$, which must both be zero, since $P^1(c)$ and $P^3(c)$ are zero by assumption. □

The contradiction shows $l = 1$, hence $n = 15$. Thus B has dimension 8 and $\hat{B}$ has the rational homology of $\mathbb{S}^8$.

2.4. **Cohomology of $\hat{B}$.** From the Gysin sequence of the fibration $f : \hat{M} \to \hat{B}$ we deduce:

Lemma 2.3. *Let p be a prime number, with $p \neq 2$ in the projective case. Then either $\hat{B}$ is an $\mathbb{F}_p$-homology sphere, or the $\mathbb{F}_p$-cohomology ring of $\hat{B}$ has the form*

$$H^*(\hat{B}, R) = R[a, b]/b^2,$$

where b has degree 15 and a has degree 8.

We will need:

Lemma 2.4. $H^4(\hat{B}, \mathbb{Z}) = 0$.

Proof. In the spherical case $\hat{B}$ is 7-connected. In the projective case, we know $\pi_2(\hat{B}) = \mathbb{Z}_2$ and $\pi_k(\hat{B}) = 0$ for $k = 1$ and $3 \leq k \leq 7$. Hence the canonical map from $\hat{B}$ to the Eilenberg-MacLane space $K(\mathbb{Z}_2, 2)$ induces isomorphisms on all cohomologies in all degrees ≤ 7. The result follows from the computations of the cohomology groups of $K(\mathbb{Z}_2, 2)$ (for instance, cf. [Cle02]). □

The last result about the cohomology of $\hat{B}$ which we extract from the fibre bundle $\hat{M} \to \hat{B}$ is the following application of the transgression theorem of Borel ([Bor53], Theorem 13.1). This theorem applies (cf. [Bro62], last paragraph on p. 370), since in the projective case, the fiber $\mathcal{L}$ has the cohomology of $\mathbb{RP}^7$.

Lemma 2.5. *Assume that $\mathcal{L}$ is homotopy equivalent to $\mathbb{RP}^7$. Then the cohomology ring $H^*(\hat{B}, \mathbb{F}_2)$ up to degree 14 is freely generated by elements u_2, u_3, u_5 of degree 2, 3 and 5 respectively. In particular, we have* $\dim H^{10}(\hat{B}, \mathbb{F}_2) = 4$ *and* $\dim H^{14}(\hat{B}, \mathbb{F}_2) = 6$.

3. Isotropy groups are cyclic groups of odd order

In this section we use characteristic classes to see that any 2-Sylow subgroup of any isotropy group is cyclic of order at most 4. Then we use that the isotropy groups act freely on the generic leaf $\mathcal{L}$ to show that all isotropy groups are cyclic groups of odd order.

Consider B as the quotient space $B = FrB/\mathsf{SO}(8)$, where FrB is the bundle of oriented frames of B with canonical action of $\mathsf{SO}(8)$. Recall that the space $\hat{B}$ is nothing else but the Borel construction $\hat{B} = FrB \times_{\mathsf{SO}(8)} E\mathsf{SO}(8)$. We will often consider the canonical 8-dimensional vector bundle (the *tangent bundle* of the orbifold)

$$T\hat{B} := FrB \times_{\mathsf{SO}(8)} E\mathsf{SO}(8) \times \mathbb{R}^8$$

over $\hat{B}$.

Lemma 3.1. *Let V be a vector bundle over $\hat{B}$. Then the Stiefel-Whitney classes $w_2(V)$ and $w_4(V)$ vanish.*

Proof. We first assume $w_2(V) = 0$ and prove that this implies $w_4(V) = 0$.

By stabilizing with a trivial bundle we may assume that the rank l of V is at least 5. Let $\mathrm{pr}\colon \hat{B} \to B\mathsf{SO}(l)$ be the classifying map of the bundle V. In particular, the Stiefel-Whitney classes of V are given by pull backs of Stiefel-Whitney classes of the universal bundle over $B\mathsf{SO}(l)$. Since $w_2(V) = 0$, pr can be lifted to a map $\tilde{pr}\colon \hat{B} \to B\mathsf{Spin}(l)$. Suppose now on the contrary that $w_4(V) \neq 0$. Then

$$\tilde{pr}^*\colon H^4(B\mathsf{Spin}(l), \mathbb{F}_2) \to H^4(\hat{B}, \mathbb{F}_2)$$

is not zero. Since $H^4(B\mathsf{Spin}(l), \mathbb{Z}) \cong \mathbb{Z}$ there is a natural map $B\mathsf{Spin}(l) \to K(\mathbb{Z}, 4)$ to the Eilenberg-MacLane space $K(\mathbb{Z}, 4)$ that induces an isomorphism on 4-th cohomology with integral coefficients. Since this map is 5-connected it also induces an isomorphism on 4-th cohomology with $\mathbb{F}_2$-coefficients. By composing this map with $\tilde{pr}$ we get a map $\hat{B} \to K(\mathbb{Z}, 4)$ which induces a nontrivial map on 4-th cohomology with $\mathbb{F}_2$-coeffcients. On the other hand, the homotopy classes of maps $\hat{B} \to K(\mathbb{Z}, 4)$ are classified by $H^4(\hat{B}, \mathbb{Z}) = 0$ (see Lemma 2.4) and thus any map $\hat{B} \to K(\mathbb{Z}, 4)$ is null homotopic – a contradiction.

Assume now $w_2(V) \neq 0$. Then $w_2(V)^2 \neq 0$ as well (cf. Lemma 2.5). Consider the bundle $W = V \oplus V$. Then the total Stiefel-Whitney classes satisfy $w_*(W) = w_*(V) \cdot w_*(V)$. Since $\hat{B}$ is simply connected, $w_1(V) = 0$. We deduce $w_2(W) = 0$ and $w_4(W) = w_2(V)^2$. Applying the previous observation to the bundle W, we deduce $w_4(W) = 0$. This contradicts $w_2(V)^2 \neq 0$.

□

Lemma 3.2. *Let $\Gamma_x \subset \mathsf{SO}(8)$ be an isotropy group. Then any element of order 2 is given by $-\mathrm{id} \in \mathsf{SO}(8)$. The 2-Sylow group of Γ_x is a cylic group of order at most 4.*

Proof. Let $\tilde{x} \in FrB$ be a point in the inverse image of $x \in B$ such that Γ_x is the isotropy group of the $\mathsf{SO}(8)$-action on FrB at $\tilde{x}$. Notice that the image of $\mathsf{SO}(8) \star \tilde{x} \times E\mathsf{SO}(8) \subset FrB \times E\mathsf{SO}(8)$ under the natural projection $FrB \times E\mathsf{SO}(8) \to \hat{B}$ can be naturally identified with the classifying space $B\Gamma_x \subset \hat{B}$ of the isotropy group Γ_x. If we restrict the canonical bundle $T\hat{B}$ over $\hat{B}$ to $B\Gamma_x$, we get an $\mathbb{R}^8$-bundle which is isomorphic to $E\Gamma_x \times_{\Gamma_x} \mathbb{R}^8$ where $\Gamma_x \subset \mathsf{SO}(8)$ is acting by the canonical representation on $\mathbb{R}^8$. Let $\Gamma_0 \subset \Gamma_x$ be a subgroup. If we pull back $T\hat{B}$ via the covering map $B\Gamma_0 \to B\Gamma_x \hookrightarrow B$, we thus get a bundle which is isomorphic to $V = E\Gamma_0 \times_{\Gamma_0} \mathbb{R}^8$ over $B\Gamma_0$. By Lemma 3.1, the second and the fourth Stiefel-Whitney classes of V vanish.

Suppose now that $\Gamma_0 \cong \mathbb{Z}_2$ and suppose the non-zero element $\iota \in \Gamma_0 \subset \mathsf{SO}(8)$ has $2k$ times the eigenvalue -1. Then $E\Gamma_0 \times_{\Gamma_0} \mathbb{R}^8$ is a bundle over $\mathbb{RP}^\infty \cong B\Gamma_0$ which decomposes as the sum of $2k$ canonical line bundles and $8-2k$ trivial line bundles. Thus the total Stiefel-Whitney class is given by $(1+w)^{2k} = (1+w^2)^k$, where 1 is the generator of $H^0(\mathbb{RP}^\infty, \mathbb{F}_2)$ and w is the generator of $H^1(\mathbb{RP}^\infty, \mathbb{F}_2) \cong \mathbb{F}_2$.

If k is odd, we get $w_2(V) \neq 0$ and if $k = 2$, we see that $w_4(V) \neq 0$. Since $w_2(V) = 0$ and $w_4(V) = 0$ we obtain a contradiction in both cases. This only leaves us with the possibility that ι has $2k = 8$ times the eigenvalue -1 and thus $\iota = -\mathrm{id}$.

Thus there is at most one order 2 element in Γ_x. Hence a 2-Sylow subgroup $\mathsf{S}_2 \subset \Gamma_x$ does not contain any abelian non-cyclic subgroup. This implies that S_2 is either cyclic or generalized quaternionic ([Wol67], [Wal10]). In order to prove that S_2 is cyclic it suffices to rule out the possibility that we can realize the quaternion group with 8 elements Q_8 as a subgroup of an isotropy group $\Gamma_x \subset \mathsf{SO}(8)$. Suppose on the contrary we can. As before, the bundle $V_8 = E\mathsf{Q}_8 \times_{\mathsf{Q}_8} \mathbb{R}^8$ over $B\mathsf{Q}_8$ can be seen as a pull back bundle of the canonical bundle over $\hat{B}$. By Lemma 2.4, $H^4(\hat{B}, \mathbb{Z}) = 0$ and thus the first Pontryagin class of V_8 vanishes, $p_1(V_8) = 0$.

The embedding of $\mathsf{Q}_8 \subset \mathsf{SO}(8)$ is determined by the fact that the center of Q_8 is mapped to $\pm\mathrm{id}$. The representation of Q_8 decomposes into two equivalent 4-dimensional subrepresentations of Q_8. Thus V_8 is isomorphic to the sum of two copies of the 4-dimensional bundle

$V_4 = E\mathsf{Q}_8 \times_{\mathsf{Q}_8} \mathbb{R}^4$, where Q_8 acts by its unique 4-dimensional irreducible representation on $\mathbb{R}^4$. Since V_4 admits a complex structure, we have $c_1(V_4 \otimes_{\mathbb{R}} \mathbb{C}) = 0$ and thus the first Pointryagin class is additive: $2p_1(V_4) = p_1(V_8) = 0$. In other words, $p_1(V_4) \in \mathbb{Z}_2 \subset \mathbb{Z}_8 \cong H^4(B\mathsf{Q}_8, \mathbb{Z})$. If we pull back the bundle V_4 to $B\mathbb{Z}_4$ via the natural covering $B\mathbb{Z}_4 \to B\mathsf{Q}_8$ we get a bundle V_4^* which decomposes into two 2-dimensional subbundles, whose Euler classes are generators of $H^2(B\mathbb{Z}_4, \mathbb{Z}) \cong \mathbb{Z}_4$. This in turn implies that $p_1(V_4^*)$ is given by the order two element in $H^4(B\mathbb{Z}_4, \mathbb{Z}) \cong \mathbb{Z}_4$. On the other hand, $p_1(V_4^*)$ is given by the image of $p_1(V_4)$ under the natural homomorphism $H^4(B\mathsf{Q}_8, \mathbb{Z}) \cong \mathbb{Z}_8 \to \mathbb{Z}_4 \cong H^4(B\mathbb{Z}_4, \mathbb{Z})$ – a contradiction since any homomorphism $\mathbb{Z}_8 \to \mathbb{Z}_4$ has $p_1(V_4) \in \mathbb{Z}_2 \subset \mathbb{Z}_8$ in its kernel.

Thus the 2-Sylow group is cyclic. It remains to rule out that there are elements of order 8. Suppose on the contrary that $\Gamma_0 \subset \Gamma_x \subset \mathsf{SO}(8)$ is cyclic group of order 8 and fix a generator $\gamma \in \Gamma_0 \subset \mathsf{SO}(8)$. Let $\zeta \in \mathsf{S}^1 \subset \mathbb{C}$ be a primitive 8th root of unity, and choose numbers $m_1, \ldots, m_4 \in \mathbb{Z}$ such that $\zeta^{\pm m_i} \in \mathsf{S}^1 \subset \mathbb{C}$ $(i = 1, \ldots, 4)$ are the eigenvalues of $\gamma \in \mathsf{SO}(8)$ counted with multiplicity. Since we know $\gamma^4 = -\mathrm{id}$, all m_i are odd.

The bundle $W_8 = E\Gamma_0 \times_{\Gamma_0} \mathbb{R}^8$ over $B\Gamma_0$ decomposes into four orientable 2-dimensional subbundles whose Euler classes are given by $\pm m_i \eta$ $(i = 1, \ldots, 4)$ where η is a generator of $H^2(B\Gamma_0, \mathbb{Z}) \cong \mathbb{Z}_8$.

It follows that the first Pontryagin class of the bundle is given by $-(\sum_{i=1}^4 m_i^2)\eta^2$. As before $p_1(W_8) = 0$ and since η^2 is a generator of $H^4(B\Gamma_0, \mathbb{Z}) = \mathbb{Z}_8$, this implies $m_1^2 + m_2^2 + m_3^2 + m_4^2 \equiv 0 \mod 8$. But for any odd number we have $m_i^2 \equiv 1 \mod 8$ – a contradiction.

□

Lemma 3.3. *Any isotropy group is either cyclic or isomorphic to a semidirect product $\mathbb{Z}_q \rtimes \mathbb{Z}_4$, where $\mathbb{Z}_4$ acts on the cyclic group of odd order q by an automorphism of order 2. Moreover, if Γ has even order it has a nontrivial 4-periodic $\mathbb{F}_2$-cohomology.*

Proof. Let Γ be a (not necessarily proper) subgroup of an isotropy group. Since Γ acts freely on the generic leaf $\mathcal{L}$, either Γ or a $\mathbb{Z}_2$-extension of Γ acts freely on $\mathbb{S}^7$ and thus has 8-periodic cohomology (cf. [Wal10], [Wol67] for this fact and subsequent results about groups with periodic cohomology). Thus for all odd p, the p-Sylow groups are cyclic. By Lemma 3.2, the 2-Sylow group is cyclic as well.

A classical theorem of Burnside implies that such a group is metacyclic, that is, Γ is isomorphic to a semidirect product $\mathbb{Z}_q \rtimes \mathbb{Z}_r$ where q and r are relatively prime.

It remains to check that the homomorphism $\beta\colon \mathbb{Z}_r \to \mathrm{Aut}(\mathbb{Z}_q)$ does not contain any elements of odd prime order p. In fact then Lemma 3.2 implies that the image of β has order at most 2.

We argue by contradiction and assume that $\Gamma \cong \mathbb{Z}_q \rtimes \mathbb{Z}_r$ is a minimal counterexample. The minimality easily implies that q is a prime and that r is a prime power $r = p^k$, where $p \neq q$ are both odd.

We consider the normal covering $B\mathbb{Z}_q \to B\Gamma$ whose deck transformation group is generated by an element ι of order p^k. Since the order is prime to q, the induced map $H^*(B\Gamma, \mathbb{F}_q) \to H^*(B\mathbb{Z}_q, \mathbb{F}_q)$ is injective and its image is given by the fixed point set of ι^* where ι^* is the induced map on cohomology. Clearly ι^* acts on $H^2(B\mathbb{Z}_q, \mathbb{F}_q)$ by an element of order p. This in turn implies that $H^{2k}(B\mathbb{Z}_q, \mathbb{F}_q)$ is fixed by ι^* if and only k is divisible by p. Hence the minimal period of $H^*(\Gamma, \mathbb{Z})$ is divisible by $2p$ – a contradiction since we know that Γ has 8-periodic cohomology. Thus Γ is cyclic and has 2-periodic cohomology, or $\Gamma \cong \mathbb{Z}_q \rtimes \mathbb{Z}_4$, where $\mathbb{Z}_4$ acts by an automorphism ι of order two on $\mathbb{Z}_q$. To see that in the latter case Γ has 4-periodic cohomology we construct a free linear action of Γ on $\mathbb{S}^3$. Let $\mathbb{Z}_m \subset \mathbb{Z}_q$ be the fixed point set of ι. Since ι has order 2, the numbers m and q/m are relatively prime. In particular $\Gamma \cong \mathbb{Z}_m \times (\mathbb{Z}_{q/m} \rtimes \mathbb{Z}_4)$. We can now embed Γ into $\mathsf{U}(2)$ by mapping the factor $\mathbb{Z}_m$ injectively to a central subgroup of $\mathsf{U}(2)$ and by mapping $(\mathbb{Z}_{q/m} \rtimes \mathbb{Z}_4)$ injectively to a subgroup of $\mathsf{SU}(2)$. Clearly, the induced action on $\mathbb{S}^3$ is free and thus Γ has 4-periodic cohomology. The $\mathbb{F}_2$-cohomology of Γ can not be trivial as $H^1(B\Gamma, \mathbb{F}_2) \cong \mathbb{F}_2$. □

Lemma 3.4. *The isotropy groups are cyclic groups of odd order.*

Proof. By Lemma 3.3 it suffices to show the isotropy groups have odd order. By Lemma 3.2 the subset $B_2 \subset B$ of points whose isotropy groups have even order is finite $B_2 = \{p_1, \ldots, p_h\}$. Let $\Gamma_1, \ldots, \Gamma_h$ denote the corresponding isotropy groups. Suppose on the contrary that B_2 is not empty.

Let FrB_2 denote the inverse image of B_2 in the frame bundle $FrB \to B$ and $\hat{B}_2 = FrB_2 \times_{\mathsf{SO}(8)} E\mathsf{SO}(8)$ the corresponding subset in the Borel construction $\hat{B} = FrB \times_{\mathsf{SO}(8)} E\mathsf{SO}(8)$. By assumption there is a tubular neighbourhood U of $\hat{B}_2$ in $\hat{B}$ which is homeomorphic to the normal bundle of $\hat{B}_2$ in $\hat{B}$. By excision and Thom isomorphism the relative cohomology group $H^*(\hat{B}, \hat{B} \setminus \hat{B}_2, \mathbb{F}_2)$ is given by $\oplus_{j=1}^h H^{*-8}(B\Gamma_j, \mathbb{F}_2)$. Furthermore, the $\mathbb{F}_2$-cohomology of $\hat{B} \setminus \hat{B}_2$ coincides with the $\mathbb{F}_2$-cohomology of $B \setminus B_2$ and thus is zero in degrees above 8. Since Γ_i has nontrivial 4-periodic $\mathbb{F}_2$-cohomology we can combine all this with the exact sequence of the relative cohomology of the pair $(\hat{B}, \hat{B} \setminus \hat{B}_2)$ to see that $\hat{B}$ has nontrival 4-periodic $\mathbb{F}_2$-cohomology in all degrees ≥ 9.

In the spherical case we get a contradiction to Lemma 2.3. In the projective case this contradicts Lemma 2.5. □

Remark 3.1. Once one has established that any order two element in an isotropy group is given by $-\mathrm{id}$, one can also proceed differently to rule out isotropy groups of even order altogether: As above, there are only finitely many points $x_i \in B$ whose isotropy groups $\Gamma_i, i = 1, \dots, h$ have even order. Moreover, the 2-Sylow group of Γ_i is either cyclic or generalized quaternionic. By a theorem of Swan [Swa60] this implies that the $\mathbb{F}_2$-cohomology of Γ_i is nontrivial and 4-periodic. One can then directly pass to the proof of Lemma 3.4.

4. All isotropy groups are trivial

We have seen in the last section that all isotropy groups are cyclic groups of odd order, Lemma 3.4. We fix an odd prime p. In this section we plan to prove that the order of any isotropy group is not divisible by p. We argue by contradiction and assume the set B_p of points in B whose isotropy group has p-torsion is not empty.

In any isotropy group Γ_x with $x \in B_p$ there is a unique normal subgroup of Γ_x which is isomorphic to $\mathbb{Z}_p$. This implies that B_p is a smooth suborbifold of B. Let X denote a connected component of B_p.

Let FrX denote the inverse image of X in the frame bundle $FrB \to B$ and $\hat{X} = FrX \times_{\mathsf{SO}(8)} E\mathsf{SO}(8)$ the corresponding subset in the Borel construction $\hat{B} = FrB \times_{\mathsf{SO}(8)} E\mathsf{SO}(8)$. By assumption there is a tubular neighbourhood U of $\hat{X}$ in $\hat{B}$ which is homeomorphic to the normal bundle of $\hat{X}$ in $\hat{B}$.

Lemma 4.1. *The image $H^*(\hat{B}, \mathbb{F}_p) \to H^*(\hat{X}, \mathbb{F}_p)$ contains the kernel of $H^*(\hat{X}, \mathbb{F}_p) \to H^*(\nu^1 \hat{X}, \mathbb{F}_p)$, where $\nu^1 \hat{X}$ denotes the unit normal bundle of $\hat{X}$ in $\hat{B}$. If the normal bundle is orientable and $e \in H^*(\hat{X}, \mathbb{F}_p)$ denotes its Euler class then the kernel of the latter map is given by the image of $H^*(\hat{X}, \mathbb{F}_p) \to H^*(\nu^1 \hat{X}, \mathbb{F}_p)$, $x \mapsto x \cup e$.*

Proof. Consider the Mayer Vietoris sequence of $\hat{B} = U \cup (\hat{B} \setminus \hat{X})$

$$H^*(\hat{B}) \xrightarrow{j} H^*(U) \oplus H^*(\hat{B} \setminus \hat{X}) \to H^*(U \setminus \hat{X}).$$

Since U is homotopy equivalent to $\hat{X}$ and $H^*(U \setminus \hat{X})$ is homotopy equivalent to $\nu^1(\hat{X})$ the first statement follows. The second statement is an immediate consequence of the exactness of the Gysin sequence. □

We will use that the cohomology $H^l(B\mathbb{Z}_p, \mathbb{Z})$ is given by 0 for all odd l and by $\mathbb{Z}_p$ for all even positive l. It is generated by elements in degree 0 and 2. Furthermore $H^*(B\mathbb{Z}_p, \mathbb{F}_p) \cong \mathbb{F}_p[x, y]/x^2\mathbb{F}_p[x, y]$ where x has degree 1 and y has degree 2.

We distinguish among three cases.

4.1. **Case 1. The normal bundle of $\hat{X}$ is orientable.** Let $x \in X$ be a point and let $B\Gamma_x \subset \hat{X}$ be the fiber of x with respect to the natural projection $\hat{B} \to B$.

Then there is a unique normal subgroup $\mathbb{Z}_p \subset \Gamma_x$ and there are natural maps $B\mathbb{Z}_p \to B\Gamma_x \to \hat{X}$. Consider the induced map $\alpha^*\colon H^*(\hat{X}, \mathbb{F}_p) \to H^*(B\mathbb{Z}_p, \mathbb{F}_p)$. The Euler class $e \in H^t(\hat{X}, \mathbb{F}_p)$ of the normal bundle of $\hat{X} \subset \hat{B}$ is mapped to the Euler class $\alpha^* e$ of the bundle $E\mathbb{Z}_p \times_\rho \nu_x(\hat{B})$, where $\rho\colon \mathbb{Z}_p \to O(\nu_x(\hat{B}))$ denotes the natural representation. The representation ρ decomposes into 2-dimensional irreducible subrepresentation and, by construction, each of these 2-dimensional subrepresentations is effective. This in turn implies that the Euler class $\alpha^* e$ of the bundle is a generator of $H^t(B\mathbb{Z}_p, \mathbb{F}_p)$, where t is the codimension of X. Hence $(\alpha^* e)^k$ is not zero, for all $k \geq 0$. By Lemma 4.1, this non-zero element lies in the image of $H^*(\hat{B}, \mathbb{Z}) \to H^*(B\mathbb{Z}_p, \mathbb{F}_p)$. We deduce that $H^{kt}(\hat{B}, \mathbb{F}_p)$ does not vanish for all $k \in \mathbb{N}$. Combining with Lemma 2.3 this gives $t = 8$. Thus X is a single point and $\hat{X} = B\Gamma_x$.

Since Γ_x is cyclic we have $H^l(B\Gamma_x, \mathbb{F}_p) \cong \mathbb{F}_p$ for all $l \geq 0$. Finally, since cupping with the Euler class induces an isomorphism, we can use Lemma 4.1 once more to see that $H^l(\hat{B}, \mathbb{F}_p) \neq 0$ for all $l \geq 8$ – this contradicts Lemma 2.3.

4.2. **Case 2.** $\dim(X) \neq 4$ **and the normal bundle of $\hat{X}$ is not orientable.** Since B is an orientable orbifold, this assumption implies that X is a nonorientable orbifold and, in particular, X is not a point. Thus $t = (8 - \dim(X)) \in \{2, 6\}$.

We consider the twofold cover $\tilde{X} \to \hat{X}$ such that the pull back of the normal bundle is orientable. The map $H^*(\hat{X}, \mathbb{F}_p) \to H^*(\tilde{X}, \mathbb{F}_p)$ is injective and its image is given by the fixed point set of ι^*, where ι^* is the map induced by the nontrivial deck transformation ι of $\tilde{X}$.

By the non-orientability assumption, the Euler class e of the pull back bundle satisfies $\iota^* e = -e$. As before we deduce that the image of e in $H^*(B\mathbb{Z}_p, \mathbb{F}_p)$ does not vanish. Therefore, $e^l \in H^{lt}(\tilde{X}, \mathbb{F}_p)$ is a nontrivial element in the kernel of $H^{lt}(\tilde{X}, \mathbb{F}_p) \to H^{lt}(\nu^1(\tilde{X}), \mathbb{F}_p)$ for $l \geq 1$. If l is even e^l is the pull back of an element $f^{l/2} \in H^{lt}(\hat{X}, \mathbb{F}_p)$, with $f \in H^{2t}(\tilde{X}, \mathbb{F}_p)$. Clearly, $f^{l/2}$ is in kernel of $H^{lt}(\hat{X}, \mathbb{F}_p) \to H^{lt}(\nu^1(\hat{X}), \mathbb{F}_p)$

and, by Lemma 4.1, $H^{lt}(\hat{B}, \mathbb{F}_p) \neq 0$ for all even l. Since $t \in \{2, 6\}$, this is a contradiction to Lemma 2.3.

4.3. **Case 3.** $\dim(X) = 4$ **and the normal bundle of $\hat{X}$ is not orientable.** This case is technically more involved and we subdivide its discussion into several steps.

Step 1. Each normal space $\nu_y(\hat{X})$ of a point $y \in \hat{X}$ decomposes into two inequivalent 2-dimensional subrepresentations of $\mathbb{Z}_p \subset \Gamma_y$.

It is clear that $\nu_y(\hat{X})$ decomposes into two subrepresentations of $\mathbb{Z}_p \subset \Gamma_y$. If the two representations would be equivalent then each element $g \in \mathbb{Z}_p$ would naturally induce a complex structure J on the normal space, and up to the sign the complex structure would not depend on the choice of g. Since $\pm J$ induce the same orientation on 4-dimensional spaces, this would imply that $\nu(\hat{X})$ is orientable – a contradiction.

Again, instead of working directly with $\hat{X}$ we go to a suitable cover $\tilde{X}$. This time we consider a fourfold cover in which the pull-back of the bundle ν is orientable and decomposes into the sum of two orientable 2-dimensional subbundles determined by the first step above. We summarize the properties of this cover $\tilde{X}$, which are intuitively rather clear, but whose exact derivation requires some tedious considerations:

Step 2. There is a normal cover $\tilde{X}$ of $\hat{X}$ whose group of deck transformation is generated by one element ι of order 4, such that the following holds true:

(1) The pull-back bundle $\nu(\tilde{X})$ of ν to $\tilde{X}$ is orientable and sum of two orientable 2-dimensional subbundles. The map ι exchanges the subbundles and the map ι^2 changes the orientation of each of them.
(2) The unit bundle $\nu^1(\tilde{X})$ has vanishing cohomology in degrees ≥ 8 with coefficients in $\mathbb{F}_p$.
(3) $\tilde{X}$ is the total space of a fiber bundle $\tilde{X} \to \tilde{Y}$ with fiber $B\mathbb{Z}_p$ and connected structure group.
(4) The restrictions of both 2-dimensional subbundles of $\nu(\tilde{X})$ to a fiber $B\mathbb{Z}_p$ have Euler classes which generate $H^2(B\mathbb{Z}_p, \mathbb{Z})$.

Moreover, $p \equiv 1 \mod 4$.

Proof. As before, $FrX \subset FrB$ denotes the inverse image of X in the frame bundle of B. Let $x \in X$ be a point, with isotropy group $\Gamma_x \subset \mathsf{SO}(8)$. Let Γ be the unique normal subgroup of Γ_x isomorphic to $\mathbb{Z}_p$.

We have seen above that Γ acts on $\mathbb{R}^8$ as the sum of two inequivalent representations and a trivial four-dimensional representation. Therefore, the normalizer N of Γ which is contained in $\mathsf{O}(4)\times\mathsf{O}(4)\cap\mathsf{SO}(8)$ has connected component $\mathsf{N}^0 = \mathsf{SO}(4) \times \mathsf{T}^2$. Moreover, N^0 coincides with the centralizer of Γ. We see that N has either two or four connected components.

Let $L \subset FrX$ be a fixed point component of Γ, whose projection to X is surjective. Then L is N^0-invariant. If L is not N-invariant, or if N has only two connected components, then we could make a continuous choice of pairs $\{g, g^{-1}\} \in \Gamma$ along L. We can then argue, similarly to the first paragraph of Case 3, that the normal bundle of $\hat{X}$ is orientable, in contradiction to the assumption.

We deduce that N/N^0 has 4 elements. Thus N is isomorphic to $\mathsf{SO}(4) \rtimes (\mathsf{T}^2 \rtimes \mathbb{Z}_4)$ where $\mathbb{Z}_4$ acts effectively on T^2 and $\mathsf{T}^2 \rtimes \mathbb{Z}_4$ acts on $\mathsf{SO}(4)$ as $\mathbb{Z}_2$. Moreover, N acts on Γ as the group $\mathbb{Z}_4$. In particular, $p \equiv 1 \bmod 4$ because otherwise $\mathrm{Aut}(\mathbb{Z}_p)$ does not contain elements of order 4.

The generator ι of this group $\mathbb{Z}_4$ exchanges the 2-dimensional Γ-invariant subspaces of $\mathbb{R}^4 \subset \mathbb{R}^8$. The square ι^2 preserves the subspaces and changes the orientation on each of them.

Since all isotropy groups of points in L with respect to the $\mathsf{SO}(8)$-action on FrX are contained in N and the $\mathsf{SO}(8)$-orbit through any point of FrX intersects L, we see that FrX is $\mathsf{SO}(8)$-equivariantly diffeomorphic to $L \times_{\mathsf{N}} \mathsf{SO}(8)$. This in turn shows that $\hat{X} = FrX \times_{\mathsf{SO}(8)} E\mathsf{SO}(8)$ is homotopy equivalent to $L \times_{\mathsf{N}} E\mathsf{N}$.

We now consider the 4-fold cyclic cover $\tilde{X} = L \times_{\mathsf{N}^0} E\mathsf{N}$ of $\hat{X}$ with the group of deck transformations $\mathsf{N}/\mathsf{N}^0 = \mathbb{Z}_4$. Note that the normal bundle $\nu(L)$ decomposes as a sum of N^0-invariant orientable 2-dimensional subbundles. Hence, the bundle $\nu(L) \times_{\mathsf{N}^0} E\mathsf{N}$ decomposes as a sum of orientable 2-dimensional subbundles. But this bundle is just the pull-back to $\tilde{X}$ of the normal bundle of $\hat{X}$.

The description the action of ι on $\mathbb{R}^4$ above finishes the proof of (1).

The unit bundle $\nu^1(\tilde{X})$ is a covering of the unit bundle $\nu^1(\hat{X})$. The latter space is homotopy equivalent to the resolution of a 7-dimensional orbifold without p-isotropy. This implies (2).

In order to see (3), observe that $\Gamma = \mathbb{Z}_p$ lies in the kernel of the action of N on L. Thus we have a canonical action of N/Γ (which is isomorphic to N) on L. Consider now the canonical action of N on $E\mathsf{N}$ and via N/Γ on $E(\mathsf{N}/\Gamma)$. Then, for the diagonal action of N on $L \times E\mathsf{N} \times E(\mathsf{N}/\Gamma)$, we see that $\tilde{X}$ is homotop to $L \times_{\mathsf{N}^0} (E\mathsf{N} \times E(\mathsf{N}/\Gamma))$. The canonical projection of this space to $\tilde{Y} := L \times_{\mathsf{N}^0} E(\mathsf{N}/\Gamma)$ is a fiber bundle with fiber $B\Gamma$. Moreover, the structure group of this bundle is the connected group N^0.

The restriction of each of the 2-dimensional subbundles to the fiber $B\mathbb{Z}_p$ is given by $E\mathbb{Z}_p \times_{(\mathbb{Z}_p,\rho_i)} \mathbb{R}^2$ where ρ_1 and ρ_2 are the two inequivalent faithful representations mentioned at the beginning. This proves (4). □

The last statement, namely $p \equiv 1 \mod 4$, implies that any endomorphism of order 4 on any finite-dimensional $\mathbb{F}_p$-vector space is diagonalizable with eigenvalues $\lambda \in \mathbb{F}_p$ satisfying $\lambda^4 = 1 \in \mathbb{F}_p$. In particular, it applies to the endomorphism ι^* of $H^*(\tilde{X}, \mathbb{F}_p)$.

If e denotes the Euler class (with any coefficients) of the bundle $\nu(\tilde{X})$ and e_i denote the Euler classes of the two 2-dimensional subbbundles, then the first statement of the above lemma reads as follows: $e_1 \cup e_2 = e$; ι^* preserves the set of four elements $\{\pm e_1, \pm e_2\}$; and $(\iota^*)^2(e_i) = -e_i$, for $i = 1, 2$.

Step 3. Let $\mathcal{I}^*$ denote the graded subalgebra of $H^*(\tilde{X}, \mathbb{F}_p)$ that consists of ι^*-invariant elements divisible by the Euler class e of $\nu(\tilde{X})$. Then $\dim(\mathcal{I}^8) = 1$ and $\mathcal{I}^k = 0$ for $0 < k < 15, k \neq 8$.

The natural map $H^*(\hat{X}, \mathbb{F}_p) \to H^*(\tilde{X}, \mathbb{F}_p)$ is injective and as in Case 2 its image is given by the ι^*-invariant elements. The subalgebra $\mathcal{I}^*$ is thus isomorphic to the kernel of $H^*(\hat{X}, \mathbb{F}_p) \to H^*(\nu^1(\hat{X}), \mathbb{F}_p)$. Combining Lemma 4.1 and Lemma 2.3, Step 3 follows.

Step 4. $H^1(\tilde{X}, \mathbb{F}_p) = 0$.

Otherwise, choose a non-zero eigenvector $w \in H^1(\tilde{X}, \mathbb{F}_p)$ of ι^*. In the subspace $H^2(\tilde{X}, \mathbb{F}_p)$ spanned by e_1 and e_2 we can find an eigenvector f of ι^* which is not in kernel of the restriction to $H^2(B\mathbb{Z}_p, \mathbb{F}_p)$. Of course, the Euler class e satisfies $\iota^* e = -e$. Since f^2 restricts to a generator of $H^4(B\mathbb{Z}_p, \mathbb{F}_p)$, we see that $\iota^* f = hf$ with $h^2 \equiv -1 \mod p$.

We claim that $w \cup f^l \cup e \neq 0$, for all $l \geq 0$. We choose a circle $\mathbb{S}^1 \subset \tilde{Y}$ in the base of the fiber bundle $\tilde{X} \to \tilde{Y}$ (see Step 2 (3)), such that w restricts to a nonzero element in the first $\mathbb{F}_p$ cohomology group of the inverse image $\tilde{S}$ of $\mathbb{S}^1$ in $\tilde{X}$. We get a fiber bundle $B\mathbb{Z}_p \to \tilde{S} \to \mathbb{S}^1$ and since the structure group is connected, this bundle must be trivial. Since f and e restrict to nonzero elements of the $\mathbb{F}_p$-cohomology of the fiber $B\mathbb{Z}_p$ the claim follows.

Depending on the eigenvalue of w, we can choose some $l \in \{0, 1, 2, 3\}$ such that $w \cup f^l \cup e$ is fixed by ι^*. The existence of this non-zero element of $\mathcal{I}^k$ with $k \in 5, 7, 9, 11$ contradicts Step 3.

Step 5. For all $j > 0$, we have $\dim(H^{2j}(\tilde{X}, \mathbb{F}_p)) \geq 2$.

Proof. By the previous step $H^1(\tilde{X}, \mathbb{F}_p) = 0$. The group $H_1(\tilde{X}, \mathbb{Z})$ is finite without p-torsion, thus $H^2(\tilde{X}, \mathbb{Z})$ does not have p-torsion either.

Let R be the ring obtained by localizing $\mathbb{Z}$ at p, i.e.

$$R = \mathbb{Z}\big[\{1/q \mid q \text{ is a prime with } q \neq p\}\big] \subset \mathbb{Q}$$

From the universal coefficient theorem $H^1(\tilde{X}, R) = 0$ and $H^2(\tilde{X}, R) = R^r$, for some r. Let $\hat{e}_1, \hat{e}_2 \in H^2(\tilde{X}, R)$ denote the Euler classes with R coefficients of the two 2-dimensional subbundles of $\nu(\tilde{X})$. Due to Step 2, they restrict to generators of $H^2(B\mathbb{Z}_p, R) \cong \mathbb{Z}_p$. In particular $\hat{e}_i \neq 0$. Moreover $(\iota^*)^2 \hat{e}_i = -\hat{e}_i$. Thus ι^* acts as an endomorphism of order four on $H^2(\tilde{X}, R) = R^r$. Therefore $r \geq 2$.

We consider the fibration $B\mathbb{Z}_p \to \tilde{X} \to \tilde{Y}$. Clearly $H^2(\tilde{Y}, R)$ has rank at least 2 as well. We look at the cohomology Serre spectral sequence with coefficients in R corresponding to this fibration. Since the action of the fundamental group on the cohomology of the fiber is trivial, the E_2 page is given by $E_2^{i,j} = H^i(\tilde{Y}, H^j(B\mathbb{Z}_p, R))$. The 0-th column $E_2^{0,j}$ survives throughout the sequence since $H^*(\tilde{X}, R) \to H^*(B\mathbb{Z}_p, R)$ is surjective. Therefore also the 0-th entry $E_2^{2,0}$ of the second column survives throughout. In the second column of the E_2-page, all odd entries are zero while the even positive entries are all isomorphic to $H^2(\tilde{Y}, \mathbb{Z}_p)$. For each of these even dimensional entries the natural image of $H^2(\tilde{Y}, R) \to H^2(\tilde{Y}, \mathbb{Z}_p)$ coincides with the image of $E_2^{0,2j} \otimes E_2^{2,0}$ in $E_2^{2,2j}$ with respect to the multiplicative structure since the multiplicative structure is induced by the cup product. Clearly these subgroups survive till the E_∞ term. Notice that the image of $H^2(\tilde{Y}, R)$ in $H^2(\tilde{Y}, \mathbb{Z}_p)$ is given by $(\mathbb{Z}_p)^r$ for some $r \geq 2$. Therefore $H^{2k}(\tilde{X}, R)$ is the domain of a surjective homomorphism to $(\mathbb{Z}_p)^2$ for all positive k. ☐

A contradiction in Case 3 now arises as follows. Since $\nu^1(\tilde{X})$ can be seen as a resolution of a 7-dimensional orbifold whose isotropy groups do not have p-torsion, it follows that $H^i(\nu^1(\tilde{X}), \mathbb{F}_p) = 0$ for all $i \geq 8$. We see from the Gysin sequence for $\nu^1(\tilde{X})$ that cupping with e induces an isomorphism of the cohomology groups in degrees ≥ 5. Since $e = e_1 \cup e_2$ the same holds for cupping with e_1. Moreover, cupping with e is surjective onto $H^8(\tilde{X}, \mathbb{F}_p)$.

By Step 5 we can choose an ι^*-eigenvector $w \in H^8(\tilde{X}, \mathbb{F}_p)$, which is linear independent to the fixed point e^2.

If $\iota^* w = w$, then $\dim(\mathcal{I}^8) \geq 2$. If $\iota^* w = -w$, then $w \cup e \in H^{12}(\tilde{X}, \mathbb{F}_p)$ would be a nonzero element of $\mathcal{I}^{12}$. In both cases we get a contradiction to Step 3.

Otherwise we have that $(\iota^*)^2 w = -w$. Then $w \cup e_1$ is a nonzero fixed point of $(\iota^*)^2$. This in turn implies that $H^{10}(\tilde{X}, \mathbb{F}_p)$ contains an eigenvector of ι^* to the eigenvalue of 1 or -1. In the latter case cupping with e gives a nontrivial element of $\mathcal{I}^{14}$. In the former case we have a nonzero element in $\mathcal{I}^{10}$, providing a contradiction to Step 3 in both cases.

5. Final Remarks.

In summary we have ruled out all orbifold singularties in B. Thus B is a Riemannian manifold, and $\mathcal{F}$ is given by the fibers of a Riemannian submersion $M \to B$. By [Bro62] (or Lemma 2.5), it follows that we are in the spherical case. From the homotopy sequence of the fiber bundle, we see that the base B of the submersion is a homotopy sphere, hence B is a topological sphere. This finishes the proof of Theorem 1.1.

Remark 5.1. It is well known ([Shi57]) that there are many exotic 15-spheres that fiber over $\mathbb{S}^8$. Of course, the base manifold B in part c) of the main theorem can also be an exotic sphere, in fact one can just pull back the Hopf fibration to the exotic 8-sphere by a smooth degree 1 map from the exotic 8-sphere to $\mathbb{S}^8$. What is not known however is whether the fibers of such a fibration can be exotic 7-spheres. This seems to be closely related to the question on how closely the diffeomorphism group of an exotic 7-sphere is linked to the diffeomorphism group of $\mathbb{S}^7$.

References

[Bor53] A. Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de lie compacts. *Ann. of Math.*, 57:115–207, 1953.

[Bro62] Browder. Higher torsion in h-spaces. *Ann. Math.*, 116:70–97, 1962.

[Cle02] A. Clement. *Integral cohomology of finite Postnikov towers.* PhD thesis, Lausanne, 2002.

[GG88] D. Gromoll and K. Grove. The low-dimensional metric foliations of euclidean spheres. *J. Differential Geom.*, 28:143–156, 1988.

[Ghy84] E. Ghys. Feuilletages riemanniens sur les variétés simplement connexes. *Annales de l'institut Fourier*, 34:203–223, 1984.

[GZ00] K. Grove and W. Ziller. Curvature and symmetry of milnor spheres. *Ann. of Math.*, 152:331–367, 2000.

[Hae84] A. Haefliger. Groupoids d'holonomie et classifiants. *Asterisque*, 116:70–97, 1984.

[Hat02] A. Hatcher. *Algebraic topology.* Cambridge university press, 2002.

[Oli79] R. Oliver. Weight systems for $so(3)$ actions. *Ann. of Math.*, 110:227–241, 1979.

[Shi57] N. Shimada. Differentiable structure on the 15 sphere and pontrjagin classes of certain manifolds. *Nagoya Math. J.*, 12:59–69, 1957.

[SW54] E.H. Spanier and Whitehead. On fiber spaces in which the fiber is contractible. *Ann.*, 116:70–97, 1954.

[Swa60] R. Swan. A p-period of a finite group. *Ill. J. Math.*, 4:341–346, 1960.

[Wal10] C. Wall. On the structure of groups with periodic cohomology. Preprint, 2010.

[Wil01] B. Wilking. Index parity of closed geodesics and rigidity of hopf fibrations. *Invent. Math.*, 148:281–295, 2001.

[Wol67] J. Wolf. *Spaces of constant curvature*. McGraw-Hill, 1967.

Mathematisches Institut, Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany,

E-mail address: `alytchak@math.uni-koeln.de`

Mathematisches Institut, Universität Münster, Einsteinstr. 62, 48147 Münster, Germany,

E-mail address: `wilking@uni-muenster.de`