COHEN–MACAULAY MODULES OVER THE ALGEBRA OF PLANAR QUASI–INVARIANTS AND CALOGERO–MOSER SYSTEMS

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ABSTRACT. In this paper, we study properties of the algebras of planar quasi–invariants. These algebras are Cohen–Macaulay and Gorenstein in codimension one. Using the technique of matrix problems, we classify all Cohen–Macaulay modules of rank one over them and determine their Picard groups. In terms of this classification, we describe the spectral module of a rational Calogero–Moser system of dihedral type. Finally, we elaborate the theory of the algebraic inverse scattering method, computing an explicit example of a deformed Calogero–Moser system.

1. INTRODUCTION

A prototype for our work is given by the following setting. For any $m \in \mathbb{N}$, consider the so–called rational Lamé operator $L_m := \dfrac{d}{dx^2} - \dfrac{m(m+1)}{x^2}$. It is well–known (see e.g. [10, 16]) that there exists a differential operator P_m of order $2m+1$ with meromorphic coefficients, such that $\bigl[P_m, L_m\bigr] = 0$ and $P_m^2 = L_m^{2m+1}$.

Let $A_m := \mathbb{C}[P_m, L_m] \cong \mathbb{C}[t^2, t^{2m+1}]$. Then the algebra A_m is Gorenstein and the following results are true (see for instance Theorem 7.1):

- There exists an isomorphism of algebraic groups:

$$\mathsf{Pic}(A_m) \cong K_m := \bigl(\mathbb{C}[\sigma]/(\sigma^m), \circ\bigr), \tag{1.1}$$

  where $\mathsf{Pic}(A_m)$ is the Picard group of A_m and $\gamma_1 \circ \gamma_2 := (\gamma_1 + \gamma_2)\cdot(1 + \sigma\gamma_1\gamma_2)^{-1}$ for any $\gamma_1, \gamma_2 \in K_m$.
- Let Q be a torsion free A_m–module of rank one. Then either Q is projective or there exists $m' < m$ and a projective module $A_{m'}$–module Q' of rank one such that Q is isomorphic to Q' viewed as a module over $A_m \subset A_{m'}$.

One goal of this work is to generalize the described picture on the two–dimensional case. As an input datum, we take any pair $(\Pi, \underline{\mu})$ (called *weighted line arrangement*), where

- $\Pi \subset \mathbb{C}$ is a *finite* subset satisfying the condition: $\alpha - \beta \notin \pi\mathbb{Z}$ for any $\alpha \neq \beta \in \Pi$.
- $\Pi \xrightarrow{\underline{\mu}} \mathbb{N}_0,\ \alpha \mapsto \mu_\alpha := \underline{\mu}(\alpha)$ is any *multiplicity function*.

For any $\alpha \in \Pi$, let us denote: $l_\alpha(z_1, z_2) := -\sin(\alpha)z_1 + \cos(\alpha)z_2 \in R := \mathbb{C}[z_1, z_2]$. The main object of our paper is the following $\mathbb{C}$–algebra of $(\Pi, \underline{\mu})$–quasi–invariant polynomials:

$$A = A\bigl(\Pi, \underline{\mu}\bigr) := \Bigl\{ f \in R \bigm| l_\alpha^{2\mu_\alpha+1} \text{ divides } \bigl(f - s_\alpha(f)\bigr) \text{ for all } \alpha \in \Pi \Bigr\}, \tag{1.2}$$

where $R \xrightarrow{s_\alpha} R$ is the involution associated with the reflection $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$, which keeps the line $l_\alpha = 0$ invariant. It is not difficult to show that A (called *algebra of planar quasi–invariants*) is a finitely generated $\mathbb{C}$–algebra of Krull dimension two.

A motivation to study ring– and module–theoretic properties of the algebra A comes from the theory of rational Calogero–Moser systems. Assume that $\bigl(\Pi, \underline{\mu}\bigr) = \bigl(\Lambda_n, m\bigr)$ is a

Coxeter weighted line arrangement, i.e.

$$\Pi = \Lambda_n := \bigl\{0, \frac{1}{n}\pi, \dots, \frac{n-1}{n}\pi\bigr\} \subset \mathbb{R} \;\text{ for some }\; n \in \mathbb{N},$$

whereas $\mu_\alpha = m$ for all $\alpha \in \Pi$ and some $m \in \mathbb{N}$. For any vector $\vec{\xi} \in \mathbb{C}^2$ such that $l_\alpha(\vec{\xi}) \neq 0$ for each $\alpha \in \Lambda_n$, consider the following *rational Calogero–Moser* operator

$$H = H\bigl((\Lambda_n, m); \vec{\xi}\bigr) := \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - \sum_{\alpha\in\Pi} \frac{m(m+1)}{l_\alpha^2(\vec{x}-\vec{\xi})} \tag{1.3}$$

According to a result of Chalykh and Veselov [11] (who proved a much more general statement for the real Coxeter groups of arbitrary rank), there exists an injective algebra homomorphism (defining a so–called algebraically integrable quantum system)

$$A\bigl(\Lambda_n, m\bigr) \xrightarrow{\Xi(\vec{\xi})} \mathfrak{D} := \mathbb{C}[\![x_1, x_2]\!][\partial_1, \partial_2]. \tag{1.4}$$

mapping the polynomial $z_1^2 + z_2^2 \in A$ to the operator H. In other words, H can be included into a large family of pairwise commuting differential operators (*quantum integrability*).

It was proven by Feigin and Veselov [18], that in the Coxeter case, the algebra A is Gorenstein (hence Cohen–Macaulay). This result was vastly generalized by Etingof and Ginzburg [15] on the case of arbitrary real Coxeter groups and multiplicity functions, invariant under the action of the Weyl group. In [19, 17], the authors proved that the algebra $A\bigl(\Pi, \underline{\mu}\bigr)$ is Gorenstein in several non–Coxeter cases.

Let $\mathfrak{B} := \mathrm{Im}\bigl(\Xi(\vec{\xi})\bigr) \subset \mathfrak{D}$. The $\mathfrak{B}$–module $F := \mathfrak{D}/(x_1, x_2)\mathfrak{D} \cong \mathbb{C}[\partial_1, \partial_2]$ (called *spectral module* of $\mathfrak{B}$) is the key object relating algebraic and analytic tools in the study of Calogero–Moser systems. Combining [10, Corollary 3.1] with [28, Theorem 3.1], one can conclude that F is a *Cohen–Macaulay* $\mathfrak{B}$–module *of rank one*. The analytic meaning of F can be illustrated by the following fact. For any algebra homomorphism $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$, consider the vector space

$$\mathsf{Sol}\bigl(\mathfrak{B}, \chi\bigr) := \Bigl\{f \in \mathbb{C}[\![x_1, x_2]\!] \big| P \diamond f = \chi(P) f \text{ for all } P \in \mathfrak{B}\Bigr\}, \tag{1.5}$$

where $\diamond$ denotes the usual action of $\mathfrak{D}$ on $\mathbb{C}[\![x_1, x_2]\!]$. Then there exists a canonical isomorphism of vector spaces (see Theorem 4.5)

$$F\big|_\chi := F \otimes_{\mathfrak{B}} \bigl(\mathfrak{B}/\mathrm{Ker}(\chi)\bigr) \cong \mathsf{Sol}\bigl(\mathfrak{B}, \chi\bigr)^*,$$

explaining why the A–module F is called spectral. The statement that F has rank one can be rephrased by saying that the vector space $\mathsf{Sol}\bigl(\mathfrak{B}, \chi\bigr)$ is one–dimensional for a "generic" character χ, i.e. the Calogero–Moser system (1.4) is *superintegrable*; see [11, 12, 10].

In their recent paper [17, Section 8], Feigin and Johnston raised a question about an *explicit* description of the spectral module F for the two–dimensional rational Calogero–Moser systems. This leads to the problem of classification of *all* Cohen–Macaulay modules of rank one over an arbitrary algebra of planar quasi–invariants $A = A\bigl(\Pi, \underline{\mu}\bigr)$. Another natural problem is to describe the Picard group $\mathsf{Pic}(A)$ of A. Obviously, the finitely generated projective A–modules form a proper subcategory of the category of Cohen–Macaualy A–modules. It turned out that it is in fact easier first to describe all Cohen–Macaulay A–modules of rank one and then specify those of them which are locally free.

Now, let us give a short overview of main results, which were obtained in our work.

1. Ring–theoretic properties of the algebra of planar quasi–invariants. For any weighted line arrangement $\bigl(\Pi, \underline{\mu}\bigr)$, the algebra $A = A\bigl(\Pi, \underline{\mu}\bigr)$ is a finitely generated, Cohen–Macaulay and of Krull dimension two. The main new feature about the algebra A is the result asserting that it is Gorenstein in codimension one; see Theorem 2.21.

2. The divisor class group of A. Let $\mathsf{CM}_1^{\mathrm{lf}}(A)$ be the abelian group of Cohen–Macaulay A–modules of rank one, which are *locally free in codimension one* [6, 7] (it is an analogue of the divisor class group of a normal domain [9, Section 7.3]). Then there exists an isomorphism of abelian groups

$$\mathsf{CM}_1^{\mathrm{lf}}(A) \longrightarrow K\bigl(\Pi, \underline{\mu}\bigr) := \prod_{\alpha \in \Pi} \bigl(\mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha}), \circ\bigr),$$

where the group law $\circ$ on $\mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha})$ is the same as in the case of the one–dimensional cuspidal curves (1.1); see Theorem 3.2.

3. Cohen–Macaulay A–modules of rank one. Let $M \in \mathsf{CM}_1(A)$ be a Cohen–Macaulay A–module of rank one, which is not locally free in the codimension one. Then there exists a multiplicity function $\Pi \xrightarrow{\underline{\mu}'} \mathbb{N}_0$ satisfying $\mu_\alpha \geq \mu'_\alpha$ for any $\alpha \in \Pi$ and $M' \in \mathsf{CM}_1^{\mathrm{lf}}(A')$ for $A' = A(\Pi, \underline{\mu}')$ such that M is isomorphic to M', where M' is viewed as a module over $A \subset A'$; see Corollary 3.8.

4. Description of a dualizing module of A. In the Coxeter case $\Pi = \bigl\{0, \frac{1}{n}\pi, \dots, \frac{n-1}{n}\pi\bigr\}$, we get an explicit description of a dualizing module of A for an arbitrary multiplicity function $\underline{\mu}$; see Theorem 3.10 and Lemma 3.12.

5. The Picard group of A. For any $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}$, we construct a certain homomorphism of abelian groups $\bigl(\mathbb{C}[\![z_1, z_2]\!], +\bigr) \xrightarrow{\Upsilon_{(\alpha,m)}} \bigl(\mathbb{C}[\![\rho]\!][\sigma]/(\sigma^m), \circ\bigr)$ (see Lemma 3.15), defining a homomorphism

$$\bigl(\mathbb{C}[\![z_1, z_2]\!], +\bigr) \xrightarrow{\Upsilon} \prod_{\alpha \in \Pi} \bigl(\mathbb{C}[\![\rho]\!][\sigma]/(\sigma^{\mu_\alpha}), \circ\bigr), \quad h \mapsto \bigl(\Upsilon_{(\alpha, 2\mu_\alpha)}(h)\bigr)_{\alpha \in \Pi}.$$

In these terms we have: $\mathsf{Pic}(A) \cong \mathrm{Im}(\Upsilon) \cap K^\diamond\bigl(\Pi, \underline{\mu}\bigr)$, where $K^\diamond\bigl(\Pi, \underline{\mu}\bigr) := \prod\limits_{\alpha \in \Pi} \mathbb{C}[\rho][\sigma]/(\sigma^{\mu_\alpha})$; see Theorem 3.17.

More explicitly, let $\Gamma\bigl(\Pi, \underline{\mu}\bigr) := \bigl\{h \in \mathbb{C}[\![z_1, z_2]\!] \big| \Upsilon(h) \in K^\diamond\bigl(\Pi, \underline{\mu}\bigr)\bigr\}$. Then for any such $h \in \Gamma\bigl(\Pi, \underline{\mu}\bigr)$, we have a projective A–module $P(h)$ of rank one, defined as

$$P(h) := \bigl\{f \in R \,\big|\, \exp(h) f \text{ is } (\Pi, \underline{\mu}) - \text{quasi–invariant}\bigr\}.$$

Conversely, for any $P \in \mathsf{Pic}(A)$, there exists $h \in \Gamma\bigl(\Pi, \underline{\mu}\bigr)$ such that $P \cong P(h)$. Moreover,

- $P(h_1) \cong P(h_2)$ if and only if $\Upsilon(h_1) = \Upsilon(h_2)$.
- The multiplication map $P(h_1) \otimes_A P(h_2) \longrightarrow P(h_1 + h_2)$, $f_1 \otimes f_2 \mapsto f_1 f_2$ is an isomorphism of A–modules.

6. The spectral module of a Calogero–Moser system of rank two. In the cases when there exists an embedding $A \xrightarrow{\Xi(\vec{\xi})} \mathfrak{D}$ (these are the so–called *Baker–Akhieser* weighted line arrangements $(\Pi, \underline{\mu})$, e.g. a Coxeter one; see Definition 4.1 and Example 4.2), we have an $(\mathfrak{B} - A)$–equivariant isomorphism $F \cong P(-u)$ (where $u(z_1, z_2) = \xi_1 z_1 + \xi_2 z_2$), i.e. an isomorphism of vector spaces $F \xrightarrow{\cong} P(-u)$ such that the diagram

$$\begin{tikzcd} F \arrow[r, "\cong"] \arrow[d, "- \circ \Xi(\vec{\xi})(f)"'] & P(-u) \arrow[d, "- \cdot f"] \\ F \arrow[r, "\cong"] & P(-u) \end{tikzcd}$$

is commutative for any $f \in A$, where $\circ$ (respectively $\cdot$) denotes the action of $\mathfrak{B}$ (respectively A) on F (respectively $P(-u)$). This answers a question of Feigin and Johnston [17] and proves that the spectral module F of the rational Calogero–Moser system $\mathfrak{B}$ is projective.

The key role in the proof of this result is plaid by the following formula for the Hilbert function $\mathbb{Z} \xrightarrow{H_{P(-u)}} \mathbb{N}_0$ of the filtered module $P(-u) \subset R$:

$$H_{P(-u)}(k) := \mathsf{dim}_{\mathbb{C}}\Big\{w \in P(-u) \;\Big|\; \deg(w) \le k\Big\} = \frac{(k-\mu+1)(k-\mu+2)}{2} \quad \text{for } k \in \mathbb{Z},$$

where $\mu := \sum\limits_{\alpha \in \Pi} \mu_\alpha$; see Theorem 4.14.

7. Elements of the higher–dimensional Sato theory. One motivation of our work was to find an appropriate generalization of Wilson's description [37] of *bispectral* commutative subalgebras of rank one of *ordinary* differential operators on the case of *partial* differential operators (the main point is that bispectrality can be characterized by the property of the spectral curve being rational with bijective normalization; see [37, Theorem 1]). Following the main idea of the works [38, 27, 28], it seems to be natural to replace the algebra $\mathfrak{D}$ by a bigger algebra. Namely, we introduce an algebra of formal power series of differential operators having the following special form:

$$\mathfrak{S} := \left\{ \sum_{k_1,k_2 \ge 0} a_{k_1,k_2}(x_1,x_2)\partial_1^{k_1}\partial_2^{k_2} \;\middle|\; \exists d \in \mathbb{Z}:\; k_1 + k_2 - \upsilon\big(a_{k_1,k_2}(x_1,x_2)\big) \le d \;\; \forall k_1,k_2 \ge 0 \right\},$$

where $\upsilon\big(a(x_1,x_2)\big)$ is the valuation of the power series $a(x_1,x_2) \in \mathbb{C}[\![x_1,x_2]\!]$; see Definition 5.1. It turns out that the algebra $\mathfrak{S}$ acts on $\mathbb{C}[\![x_1,x_2]\!]$ (this action extends the natural action of the algebra $\mathfrak{D}$ on $\mathbb{C}[\![x_1,x_2]\!]$) and contains some natural operators (e.g. operators of a change of variables, delta–functions and integration operators) which do not belong to $\mathfrak{D}$; see Example 5.4 and Example 5.5.

Inspired by the theory of Sato Grassmannian [34, 31, 37], we introduce the following sets:

$$\mathsf{Gr}_\mu(R) := \left\{ W \subseteq R \;\middle|\; \mathsf{dim}_{\mathbb{C}}\Big(w \in W \;\Big|\; \deg(w) \le k+\mu\Big) = \binom{k+2}{2} \text{ for any } k \in \mathbb{N}_0 \right\}$$

where $\mu \in \mathbb{N}_0$. It turns out that any *Schur pair* (W, A), where $W \in \mathsf{Gr}_\mu(R)$ and $A \subseteq R$ is a $\mathbb{C}$–subalgebra such that $W \cdot A = W$, determines an injective algebra homomorphism $A \longrightarrow \mathfrak{S}$, which is unique up to an appropriate inner automorphism of $\mathfrak{S}$; see Theorem 5.18. If $A = A\big(\Pi, \underline{\mu}\big)$ is the algebra of quasi–invariants of a Baker–Akhieser line arrangement and $W = P(-\xi_1 z_1 - \xi_2 z_2)$, then we recover in this way the algebra embedding (1.4).

8. *Deformations of Calogero–Moser systems arising from Cohen–Macaulay modules.* In Section 6, we illustrate the developed "algebraic inverse scattering method" by constructing an "isospectral deformation" of the simplest dihedral Calogero–Moser system associated with the operator

$$H = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - 2\left(\frac{1}{(x_1-\xi_1)^2} + \frac{1}{(x_2-\xi_2)^2}\right).$$

We tried to keep the exposition of this paper self–contained. Still, our work is based on the following two external ingredients. Firstly, we essentially use the theory of multivariate Baker–Akhieser functions of (generalized) Calogero–Moser systems [11, 12, 1, 14, 10]. The second ingredient is the "matrix–problem method" of [6, 7] to study Cohen–Macaulay modules over singular surfaces with non–isolated singularities.

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List of notations. For reader's convenience we make an account of the most important notations used in this paper.

1. A weighted line arrangement $(\Pi, \underline{\mu})$ is an input datum defining the corresponding algebra of quasi–invariant polynomials $A = A(\Pi, \underline{\mu}) \subset R = \mathbb{C}[z_1, z_2] = \mathbb{C}\bigl[\rho\cos(\varphi), \rho\sin(\varphi)\bigr]$. It consists of a finite set $\Pi \subset \mathbb{C}$ and a function $\Pi \xrightarrow{\underline{\mu}} \mathbb{N}_0$, $\alpha \mapsto \mu_\alpha = \underline{\mu}(\alpha)$. We put $\mu = \sum_{\alpha\in\Pi} \mu_\alpha$. For a Baker–Akhieser weighted line arrangement, the function $\underline{\mu}$ has to satisfy very strong constraints; see Section 4. In the Coxeter case, $\Pi = \Lambda_n = \bigl\{0, \frac{1}{n}\pi, \dots, \frac{n-1}{n}\pi\bigr\}$. Finally, X always denotes the affine surface, determined by the algebra A and $\mathcal{P}$ is the set of prime ideals in A of height one.

2. Next, $I = \mathsf{Ann}_A(R/A)$ is the conductor ideal of the algebra extension $A \subseteq R$. For any $m \in \mathbb{N}$, we denote $K_m = \mathbb{C}(\rho)[\sigma]/(\sigma^m)$ and $L_m = \mathbb{C}(\rho)[\varepsilon]/(\varepsilon^{2m})$. For any $\alpha \in \Pi$, we denote $K_\alpha = K_{\mu_\alpha}, L_\alpha = L_{\mu_\alpha}$ and $l_\alpha(z_1, z_2) = -\sin(\alpha)z_1 + \cos(\alpha)z_2 \in R$. For each $f \in \mathbb{C}\bigl[\rho\cos(\varphi), \rho\sin(\varphi)\bigr]$, $\alpha \in \Pi$ and $k \in \mathbb{N}_0$ we put $f_\alpha^{(k)} := \left.\dfrac{\partial^k f}{\partial\varphi^k}\right|_{\varphi=\alpha} \in \mathbb{C}[\rho]$.

3. Cohen–Macaulay modules in our work are always *maximal* Cohen–Macaulay and $\mathsf{CM}(A)$ denotes the category of Cohen–Macaulay A–modules, whereas $\mathsf{CM}^{\mathsf{lf}}(A)$ stands for its full subcategory consisting of those modules, which are locally free in codimension one. Next, $\mathsf{Pic}(A)$ is the Picard group of A, whereas $\mathsf{CM}_1(A)$ (respectively, $\mathsf{CM}_1^{\mathsf{lf}}(A)$) denotes the set of the isomorphism classes of rank one Cohen–Macaulay A–modules (respectively, the subset consisting of those modules, which are locally free in codimension one). We denote by $\mathsf{Tri}(A)$ the category of triples (see Definition 2.8).

Finally, $M(\vec{\nu}, \vec{\gamma})$ (respectively, $B(\vec{\gamma})$ and $P(h)$) denote elements of $\mathsf{CM}_1(A)$ (respectively, $\mathsf{CM}_1^{\mathsf{lf}}(A)$ and $\mathsf{Pic}(A)$), expressed through the corresponding classifying parameters; see formula (3.5) (respectively (3.3) and (3.21)).

4. Gothic letters stand for objects, related with partial differential operators. In particular, $\mathfrak{D} = \mathbb{C}[\![x_1, x_2]\!][\partial_1, \partial_2]$, whereas $\mathfrak{E} = \mathbb{C}[\![x_1, x_2]\!]((\partial_1^{-1}))((\partial_2^{-1}))$ is the algebra of pseudo–differential operators. Next, $\mathfrak{S}$ is another algebra of "infinite" partial differential operators

(see Section 5). We denote by $\diamond$ the natural left action of $\mathfrak{D}$ or $\mathfrak{S}$ on $\mathbb{C}[\![x_1, x_2]\!]$, whereas $\circ$ stands for the natural right action of $\mathfrak{D}$ or $\mathfrak{S}$ on $\mathbb{C}[\partial_1, \partial_2]$. Finally, $\mathfrak{B} \subset \mathfrak{D}$ is the commutative subalgebra of $\mathfrak{D}$ containing the Calogero–Moser operator H (1.3) and given by the formula (1.4), whereas F is the corresponding spectral module.

5. For any $\mu \in \mathbb{N}_0$, $\mathsf{Gr}_\mu(R)$ denotes the set of all subspaces $W \subset R$ with Hilbert polynomial $H_W(k+\mu) = \binom{k+2}{2}$; $S \in \mathfrak{S}$ is a Sato operator of such $W \in \mathsf{Gr}_\mu(R)$ (see Definition 5.14).

2. Ring–theoretic properties of the algebra of surface quasi–invariants

We refer to the monographs [9, 29, 35] for the definition and main properties of Cohen–Macaulay rings and modules.

2.1. **Some results on the Macaulayfication.** The exposition in this subsection closely follows the survey article [6], where a special attention to the study of Cohen–Macaulay modules on singular surfaces was paid. Let A be a finitely generated integral $\mathbb{C}$–algebra of Krull dimension two, $Q = Q(A)$ its field of fractions, X an affine surface, whose coordinate ring is isomorphic to A and $\mathcal{P} := \big\{\mathfrak{p} \in \mathsf{Spec}(A) \,\big|\, \mathsf{ht}(\mathfrak{p}) = 1\big\}$.

A proof of the following proposition can be for instance found in [27, Appendix 2].

Proposition 2.1. *Let* $A^\dagger := \bigcap\limits_{\mathfrak{p}\in\mathcal{P}} A_\mathfrak{p} \subset Q$. *Then the following statements are true.*

- $A^\dagger$ *is a finitely generated Cohen–Macaulay* $\mathbb{C}$*–algebra of Krull dimension two.*
- *We have:* $\mathsf{dim}_\mathbb{C}\big(A^\dagger/A\big) < \infty$.
- *Moreover,* $A = A^\dagger$ *if and only if* A *is Cohen–Macaulay.*

The $\mathbb{C}$*–algebra* $A^\dagger$ *is called* Macaulayfication *of* A.

Let M be a Noetherian torsion free A–module. Recall that the rank of M is defined to be $\mathrm{rk}(M) := \mathrm{rk}_Q\big(Q(M)\big)$, where $Q(M) := Q \otimes_A M$ is the rational envelope of M.

A proof of an analogous result for modules can be for instance found in [6, Section 3].

Proposition 2.2. *Let* $M^\dagger := \bigcap\limits_{\mathfrak{p}\in\mathcal{P}} M_\mathfrak{p} \subset Q(M)$. *Then the following statements are true.*

- $M^\dagger$ *is a Noetherian Cohen–Macaulay module (both over* $A^\dagger$ *and* A*).*
- *We have:* $\mathsf{dim}_\mathbb{C}\big(M^\dagger/M\big) < \infty$.
- *Moreover,* $M = M^\dagger$ *if and only if* M *is Cohen–Macaulay.*
- *Assume* A *is already Cohen–Macaulay. Let* Ω *be a* dualizing module *of* A. *Then we have:* $M^\dagger \cong M^{\vee\vee}$, *where* $M^\vee := \mathsf{Hom}_A(M, \Omega)$.

Similarly to Proposition 2.1, the $A^\dagger$*–module* $M^\dagger$ *is called* Macaulayfication *of* M.

From now on, we shall assume that the algebra A is Cohen–Macaulay. In what follows, $\mathsf{CM}(A)$ denotes the category of all Cohen–Macaulay A–modules.

Lemma 2.3. *Suppose that* A *is Gorenstein in codimension one, i.e.* $A_\mathfrak{p}$ *is Gorenstein for any* $\mathfrak{p} \in \mathcal{P}$. *Then for any torsion free Noetherian* A*–module* M *we have:* $M^\dagger \cong M^{**}$, *where* $M^* = \mathsf{Hom}_A(M, A)$.

Proof. We prove this fact here since the proof given in [6, Proposition 3.7] contained an error. First note that both modules $M^{\vee\vee}$ and M^{**} are Cohen–Macaulay; see for example [6, Lemma 3.1]. Let $M \xrightarrow{i} M^{\vee\vee}$ and $M \xrightarrow{j} M^{**}$ be the canonical morphisms. By the universal property of Macaulayfication [6, Proposition 3.2], there exists a unique morphism $M^{\vee\vee} \xrightarrow{f} M^{**}$ such that the diagram

$$\begin{tikzcd} & M \arrow[dl, "i"'] \arrow[dr, "j"] & \\ M^{\vee\vee} \arrow[rr, "f"] & & M^{**} \end{tikzcd}$$

is commutative. Since both morphisms i and j induce isomorphisms of the corresponding rational envelopes, we may conclude that f is a monomorphism. Next, $M_{\mathfrak{p}}$ is automatically Cohen–Macaulay over $A_{\mathfrak{p}}$ and $\Omega_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathcal{P}$. Therefore, the morphisms of $A_{\mathfrak{p}}$–modules $i_{\mathfrak{p}}$ and $j_{\mathfrak{p}}$ are isomorphisms, hence $f_{\mathfrak{p}}$ is an isomorphism, too. As a consequence, the cokernel of f is a finite dimensional A–module. Since both A–modules $M^{\vee\vee}$ and M^{**} have depth two, f is surjective by the Depth Lemma [9, Proposition 1.2.9]. □

Definition 2.4. A Cohen–Macaulay A–module M is called *locally free in codimension one* if $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$–module for any $\mathfrak{p} \in \mathcal{P}$. Equivalently, there exists a finite set $Z \subset X$ and a locally free sheaf N on the surface $X \setminus Z$ such that $M \cong \imath_*(N)$, where $X \setminus Z \xrightarrow{\imath} X$ is the canonical open inclusion (the proof of this statement is the same as in [6, Corollary 3.12]). In what follows, $\mathsf{CM}^{\mathsf{lf}}(A)$ denotes the category of all Cohen–Macaulay A–modules, which are locally free in codimension one.

Proposition 2.5. *Assume that the algebra A is Gorenstein in codimension one. Abusing the notation, let $\mathsf{CM}^{\mathsf{lf}}_1(A)$ be the set of the isomorphism classes of Cohen–Macaualy A–modules of rank one, which are locally free in codimension one. Then $\mathsf{CM}^{\mathsf{lf}}_1(A)$ is an abelian group (which is an analogue of the* divisor class group *of a normal domain* [9, Section 7.3]*) with respect to the operation*

$$M_1 \boxtimes_A M_2 := \big((M_1 \otimes_A M_2)/\,\mathsf{tor}\big)^{\dagger}, \tag{2.1}$$

where tor *denotes the torsion submodule of $M_1 \otimes_A M_2$. The neutral element of $\mathsf{CM}^{\mathsf{lf}}_1(A)$ is A, whereas the inverse element of M is $M^* = \mathsf{Hom}_A(M, A)$.*

Proof. For the associativity of $\boxtimes_A$, see [6, Proposition 3.15]. It is also clear that A is the neutral element of $\mathsf{CM}^{\mathsf{lf}}_1(A)$. Finally, let M be an element of $\mathsf{CM}^{\mathsf{lf}}_1(A)$. Then the evaluation morphism $M \otimes_A \mathsf{Hom}_A(M, A) \xrightarrow{\mathsf{ev}} A$ induces a morphism of Cohen–Macaulay modules $M \boxtimes_A \mathsf{Hom}_A(M, A) \xrightarrow{\mathsf{ev}'} A$ such that $\mathsf{ev}'_{\mathfrak{p}}$ is an isomorphism for any $\mathfrak{p} \in \mathcal{P}$. By [6, Lemma 3.6], ev' is an isomorphism. Hence, M^* is indeed inverse to M in $\mathsf{CM}^{\mathsf{lf}}_1(A)$. □

Remark 2.6. Let $Z_k \subset X$ be a finite subset, N_k be a locally free sheaf on $U_k := X \setminus Z_k$ and $M_k := \imath_{k*}(N_k) \in \mathsf{CM}^{\mathsf{lf}}_1(A)$ for $k = 1, 2$. Then we have: $M_1 \boxtimes_A M_2 \cong \imath_*\big(N_1\big|_U \otimes N_2\big|_U\big)$, where $U := X \setminus \{Z_1 \cup Z_2\}$ and $U \xrightarrow{\imath} X$ is the canonical embedding. The proof of this result follows from [6, Proposition 3.10].

2.2. **Category of triples.** Now we introduce a certain categorical construction [6, 7], playing the key role in our paper. Let A be a reduced *Cohen–Macaulay* $\mathbb{C}$–algebra of Krull dimension two, either finitely generated or complete. Let R be the normalization of A. Note that R is automatically Cohen–Macaulay [35, Theorem IV.D.11] and the algebra extension $A \subseteq R$ is finite.

We denote by $I = \mathsf{Ann}_A(R/A) = \big\{r \in A \,\big|\, rR \subseteq A\big\} \cong \mathsf{Hom}_A(R, A)$ the conductor ideal of the algebra extension $A \subseteq R$. Observe that I is an ideal both in A and R. Moreover, I is Cohen–Macaulay (both over A and R); see e.g. [6, Lemma 3.1]. Both $\mathbb{C}$–algebras

$\bar{A} = A/I$ and $\bar{R} = R/I$ are Cohen–Macaulay of Krull dimension one (but not necessarily reduced). Let $Q(\bar{A})$ and $Q(\bar{R})$ be the corresponding total rings of fractions. Then the algebra extension $\bar{A} \subseteq \bar{R}$ induces a canonical embedding $Q(\bar{A}) \subseteq Q(\bar{R})$.

Let $\mathsf{Ass}_R(I) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ be the set of the associated prime ideals of I in R. Since the algebra $Q(\bar{R})$ is Artinian, the Chinese Remainder theorem implies that

$$Q(\bar{R}) \cong \bar{R}_{\bar{\mathfrak{q}}_1} \times \dots \times \bar{R}_{\bar{\mathfrak{q}}_n},$$

where $\bar{\mathfrak{q}}_1, \dots, \bar{\mathfrak{q}}_n$ are the images of $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ in $\bar{S}$. Of course, a similar result holds for the algebra $Q(\bar{A})$, too.

The proof of the following result can be found in [7, Lemma 3.1].

Lemma 2.7. *For any $M \in \mathsf{CM}(A)$, the following statements are true:*

1. the canonical morphism of $Q(\bar{R})$–modules

$$\theta_M : Q(\bar{R}) \otimes_{Q(\bar{A})} \bigl(Q(\bar{A}) \otimes_A M\bigr) \longrightarrow Q(\bar{R}) \otimes_R \bigl(R \otimes_A M\bigr) \longrightarrow Q(\bar{R}) \otimes_R \bigl(R \boxtimes_A M\bigr)$$

is an epimorphism.

2. the adjoint morphism of $Q(\bar{A})$–modules

$$\tilde{\theta}_M : \; Q(\bar{A}) \otimes_A M \longrightarrow Q(\bar{R}) \otimes_{Q(\bar{A})} \bigl(Q(\bar{A}) \otimes_A M\bigr) \xrightarrow{\theta_M} Q(\bar{R}) \otimes_R \bigl(R \boxtimes_A M\bigr)$$

is a monomorphism.

Definition 2.8. Consider the following diagram of categories and functors:

$$\mathsf{CM}(R) \xrightarrow{Q(\bar{R})\otimes_R -} Q(\bar{R}) - \mathsf{mod} \xleftarrow{Q(\bar{R})\otimes_{Q(\bar{A})} -} Q(\bar{A}) - \mathsf{mod}$$

According to [30, Section II.6], the corresponding *comma category* $\mathsf{Comma}(A)$ is defined as follows. Its objects are *triples* (N, V, θ), where N is a Cohen—Macaulay R–module, V is a Noetherian $Q(\bar{A})$–module and $Q(\bar{R}) \otimes_{Q(\bar{A})} V \xrightarrow{\theta} Q(\bar{R}) \otimes_R N$ is a morphism of $Q(\bar{R})$–modules (called *gluing map*).

A morphism between two objects (N, V, θ) and (N', V', θ') of the category $\mathsf{Comma}(B)$ is given by a pair (f, g), where $N \xrightarrow{f} N'$ is a morphism of R–modules and $V \xrightarrow{g} V'$ is a morphism of $Q(\bar{A})$–modules such that the following diagram

$$\begin{tikzcd}
Q(\bar{R}) \otimes_{Q(\bar{A})} V \arrow[r, "\theta"] \arrow[d, "1\otimes g"'] & Q(\bar{R}) \otimes_R N \arrow[d, "1\otimes f"] \\
Q(\bar{R}) \otimes_{Q(\bar{A})} V' \arrow[r, "\theta'"] & Q(\bar{R}) \otimes_R N'
\end{tikzcd} \tag{2.2}$$

is commutative.

The *category of triples* $\mathsf{Tri}(A)$ is the full subcategory of $\mathsf{Comma}(A)$ consisting of those triples (N, V, θ), for which the gluing map θ satisfies the following two conditions:

- θ is an epimorphism;
- the adjoint morphism of $Q(\bar{A})$–modules $\tilde{\theta}$ defined as the composition

$$V \longrightarrow Q(\bar{R}) \otimes_{Q(\bar{A})} V \xrightarrow{\theta} Q(\bar{R}) \otimes_R N$$

is a monomorphism. $\square$

In the above terms, we have a commutative diagram of $\mathbb{C}$–algebras

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & Q(\bar{A}) \arrow[d, hook] \\
R \arrow[r] & Q(\bar{R}).
\end{tikzcd}
\tag{2.3}
$$

The main idea is to realize the category $\mathsf{CM}(A)$ as a "categorical gluing" of the categories $\mathsf{CM}(R)$ and $Q(\bar{A}) - \mathsf{mod}$ along the category $Q(\bar{R}) - \mathsf{mod}$. This is implemented by the following result [7, Theorem 3.5].

Theorem 2.9. *The functor* $\mathsf{CM}(A) \xrightarrow{\mathbb{F}} \mathsf{Tri}(A)$, *mapping a Cohen-Macaulay* A*–module* M *to the triple* $\big(R \boxtimes_A M, Q(\bar{A}) \otimes_A M, \theta_M\big)$, *is an equivalence of categories. The functor* $\mathbb{F}$ *establishes an equivalence of categories between* $\mathsf{CM}^{\mathsf{lf}}(A)$ *and the full subcategory* $\mathsf{Tri}^{\mathsf{lf}}(A)$ *of* $\mathsf{Tri}(A)$ *consisting of those triples* (N, V, θ) *for which the* $Q(\bar{A})$*–module* V *is free and the morphism* θ *is an isomorphism.*

Moreover, for any objects M_1 *and* M_2 *of* $\mathsf{CM}^{\mathsf{lf}}(A)$ *we have:*

$$\mathbb{F}(M_1 \boxtimes_A M_2) \cong \mathbb{F}(M_1) \otimes \mathbb{F}(M_2),$$

where $(N_1, V_1, \theta_1) \otimes (N_2, V_2, \theta_2) := \big(N_1 \boxtimes_R N_2, V_1 \otimes_{Q(\bar{A})} V_2, \theta_1 \otimes \theta_2\big)$ *for any two objects of the category* $\mathsf{Tri}^{\mathsf{lf}}(A)$.

Finally, for any maximal ideal $\mathfrak{m}$ *in the algebra* A, *we have a commutative diagram of categories and functors*

$$
\begin{tikzcd}
\mathsf{CM}(A) \arrow[r, "\mathbb{F}"] \arrow[d, "\widehat{(-)}_{\mathfrak{m}}"'] & \mathsf{Tri}(A) \arrow[d, "\mathbb{L}_{\mathfrak{m}}"] \\
\mathsf{CM}(\widehat{A}_{\mathfrak{m}}) \arrow[r, "\mathbb{F}_{\mathfrak{m}}"] & \mathsf{Tri}(\widehat{A}_{\mathfrak{m}}),
\end{tikzcd}
\tag{2.4}
$$

where $\mathbb{L}_{\mathfrak{m}}\big(N, V, \theta\big) = \big(\widehat{N}_{\mathfrak{m}}, \widehat{V}_{\mathfrak{m}}, \widehat{\theta}_{\mathfrak{m}}\big)$.

2.3. **Category of triples for the algebra of planar quasi–invariants.** From now on, we put $R := \mathbb{C}[z_1, z_2]$. As in Introduction, we fix the following datum $(\Pi, \underline{\mu})$, called *weighted line arrangement*:

- $\Pi \subset \mathbb{C}$ is a finite subset satisfying the condition: $\alpha - \beta \notin \pi\mathbb{Z}$ for any $\alpha \neq \beta \in \Pi$.
- $\Pi \xrightarrow{\underline{\mu}} \mathbb{N}_0,\ \alpha \mapsto \mu_\alpha = \underline{\mu}(\alpha)$ is a multiplicity function.

The above restriction on the elements of the set Π has the following meaning: the lines $V(l_\alpha)$ and $V(l_\beta)$ are distinct for any $\alpha \neq \beta \in \Pi$, where $l_\alpha = -\sin(\alpha) z_1 + \cos(\alpha) z_2 \in R$ and $V(f) \subset \mathbb{A}^2$ denotes the vanishing set of $f \in R$. Note that any complex line $V(l) \subset \mathbb{A}^2$ passing through the origin, with the only exception of two lines $V(z_1 \pm i z_2)$, can be written as $V(l_\alpha)$ for an appropriate $\alpha \in \mathbb{C}$.

For any $\alpha \in \Pi$, consider the reflection

$$\mathbb{C}^2 \xrightarrow{r_\alpha} \mathbb{C}^2,\ \vec{x} \mapsto \vec{z} - 2\big(\vec{z}, \vec{e}_\alpha\big)\vec{e}_\alpha,$$

where $\vec{e}_\alpha := \big(-\sin(\alpha), \cos(\alpha)\big)$ and $\big(\vec{z}, \vec{e}_\alpha\big) = -\sin(\alpha) z_1 + \cos(\alpha) z_2$. Let

$$R \xrightarrow{s_\alpha} R, \quad f(\vec{z}) \mapsto \big(s_\alpha(f)\big)(\vec{z}) := f\big(r_\alpha(\vec{z})\big)$$

be the involution associated with the reflection r_α.

Let $A = A\bigl(\Pi, \underline{\mu}\bigr)$ be the algebra of surface quasi–invariant polynomials (2.13) corresponding to the datum $\bigl(\Pi, \underline{\mu}\bigr)$. In what follows, we shall denote

$$\delta := \prod_{\alpha \in \Pi} l_\alpha^{\mu_\alpha} \quad \text{and} \quad \omega := z_1^2 + z_2^2.$$

Observe that both polynomials ω and δ^2 belong to the algebra A.

Although the following result is not original (compare with [2, Lemma 7.3] and [21, Theorem 3.3.2]), we provide its detailed proof for the sake of completeness and convenience of the reader.

Proposition 2.10. *The algebra A is a finitely generated homogeneous subalgebra of R of Krull dimension two. Next, $Q(A) = Q(R) = \mathbb{C}(z_1, z_2)$ and the algebra R is the normalization of A (i.e. the integral closure of A is $Q(A)$). Let X be an affine surface, whose coordinate ring is isomorphic to A. Then the corresponding normalization map $\mathbb{A}^2 \stackrel{\nu}{\longrightarrow} X$ is bijective.*

Proof. First note the following elementary fact: if $l \in R$ is a homogeneous polynomial dividing another (possibly, non–homogeneous) polynomial $f \in R$, then l divides the homogeneous component of f of the highest degree. It follows from the definition (2.13) that A is a homogeneous subalgebra of R.

To prove that the algebra A is finitely generated, put $A^\circ := \mathbb{C}[\omega, \delta^2] \subseteq A$. We claim that R is finite viewed as an A°–module. Indeed, let $J = \langle \omega, \delta^2 \rangle_R$. Since $V\bigl(z_1 \pm i z_2, l_\alpha\bigr) = \{0\}$ for any $\alpha \in \Pi$, we have: $V(J) = \{0\}$. Hence, the ideal J is (z_1, z_2)–primary implying that $\mathsf{dim}_{\mathbb{C}}(R/J) < \infty$. Let $h_1, \dots, h_m$ be a basis of R/J consisting of homogeneous polynomials. We claim that

$$R = \bigl\langle h_1, \dots, h_m \bigr\rangle_A. \tag{2.5}$$

Indeed, let $f \in R$ be an arbitrary homogeneous polynomial. Then there exist $\lambda_1 \dots, \lambda_m \in \mathbb{C}$ as well as homogeneous polynomials $g, h \in R$ satisfying $\deg(g) < \deg(f)$ and $\deg(h) < \deg(f)$, such that $f = \bigl(\sum\limits_{j=1}^{m} \lambda_j h_j\bigr) + g\omega + h\delta^2$. Proceeding inductively with g, h, we get the result (2.5). Since we have a tower of algebra extensions $A^\circ \subseteq A \subseteq R$ and R is finitely generated as A°–module, the algebra A is finitely generated of Krull dimension two.

Since for any $f \in R$, the polynomial $f\delta^2$ belongs to A, we have: $f \in Q(A)$. Hence, $Q(A) = Q(R)$. The normalization map $\mathbb{A}^2 \stackrel{\nu}{\longrightarrow} X$ is automatically surjective. We have to show that ν is injective. Our proof is a slightly modified version of the argument from [2, Lemma 7.3]. By Hilbert's Nullstellensatz, the injectivity of ν is equivalent to the statement that for any two maximal ideals $\mathfrak{m} \ne \mathfrak{n}$ of R we have: $A \cap \mathfrak{m} \ne A \cap \mathfrak{n}$. Equivalently, for any pair of points $p \ne q \in \mathbb{A}^2$, there exists $f \in A$ such that $f(p) = 0$ and $f(q) \ne 0$. Without loss of generality, we may assume that $q \ne (0,0)$. Then the following two cases can occur.

Case 1. $q \notin V(\delta) = \cup_{\alpha \in \Pi} V(l_\alpha)$. Again, take any $g \in R$ such that $g(p) = 0$ and $g(q) \ne 0$ and put $f := \delta g$. Then $f \in A$ and $f(p) = 0$, whereas $f(q) \ne 0$.

Case 2. $q \in V(\delta) = \cup_{\alpha \in \Pi} V(l_\alpha)$. Since $q \ne (0,0)$, there exists precisely one $\alpha \in \Pi$ such that $l_\alpha(q) = 0$. Again, take any $g \in R$ such that $g(p) = 0$ and $g(q) \ne 0$. Now we put

$$f = \Bigl(\prod_{\beta \ne \alpha} \bigl(s_\alpha(l_\beta) \cdot l_\beta\bigr)^{2\mu_\beta}\Bigr) \cdot \bigl(s_\alpha(g) \cdot g\bigr).$$

By construction, $l_\beta^{2\mu_\beta}\big|f$ for any $\beta \ne \alpha$, whereas $s_\alpha(f) = f$. Hence, $f \in A$. Obviously, $f(p) = 0$. On the other hand, $\big(s_\alpha(g)\big)(q) = g\big(s_\alpha(q)\big) = g(q) \ne 0$ and in a similar way, $\big(s_\alpha(l_\beta)\big)(q) = l_\beta\big(s_\alpha(q)\big) \ne 0$ for any $\beta \ne \alpha$. Therefore, $f(q) \ne 0$. $\square$

It is convenient to rewrite the definition of the algebra $A\big(\Pi, \underline{\mu}\big)$ in terms of polar coordinates. We put $z_1 = \rho\cos(\varphi), z_2 = \rho\sin(\varphi)$ and identify the algebra R with the subalgebra $\mathbb{C}\big[\rho\cos(\varphi), \rho\sin(\varphi)\big]$ of the algebra $\mathbb{C}\{\rho, \varphi\}$ of analytic functions on $\mathbb{C}\times\mathbb{C} = \mathbb{C}_\rho \times \mathbb{C}_\varphi$. For any $f \in \mathbb{C}\{\rho, \varphi\}$, $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$, consider the analytic function

$$f_\alpha^{(k)} := \left.\frac{\partial^k f}{\partial \varphi^k}\right|_{\varphi=\alpha} : \mathbb{C} \longrightarrow \mathbb{C}. \tag{2.6}$$

In these terms, we have an injective algebra homomorphism

$$\mathbb{C}\{\rho, \varphi\} \longrightarrow \mathbb{C}\{\rho\}[\![\varepsilon]\!], \quad f \mapsto T_\alpha(f) := \sum_{k=0}^{\infty} f_\alpha^{(k)} \frac{\varepsilon^k}{k!}.$$

In the polar coordinates, we have: $l_\alpha = \rho\sin(\varphi - \alpha)$ and

$$\big(s_\alpha(f)\big)(\rho, \varphi) = f(\rho, 2\alpha - \varphi) \quad \text{for any} \quad f \in R.$$

Lemma 2.11. *For $k \in \mathbb{N}$, let $P_k := \mathbb{C}[\rho, \varepsilon]/(\varepsilon^k)$ and*

$$T_{(\alpha,k)} : R \longrightarrow P_k, \quad f \mapsto \sum_{i=0}^{k-1} f_\alpha^{(i)} \frac{\varepsilon^i}{i!} \tag{2.7}$$

Then the following results are true.

- *The map $T_{(\alpha,k)}$ is an algebra homomorphism and $\mathsf{Ker}\big(T_{(\alpha,k)}\big) = \big(l_\alpha^k\big)$.*
- *The algebra inclusion $\widetilde{R}_{\alpha,k} := R/(l_\alpha^k) \xrightarrow{\tau} P_k$ induces an isomorphism $\tilde{\tau}$ of the corresponding total rings of fractions.*

Proof. The fact that $T_{(\alpha,k)}$ is an algebra homomorphism, is a basic property of Taylor series. If $f \in R$ is such that $f'_\alpha = 0$ then $l_\alpha \mid f$. The statement that $\mathsf{Ker}(T_{(\alpha,k)}) = \big(l_\alpha^k\big)$ can be easily proven by induction. Let

$$u := \rho\cos(\varphi - \alpha) \quad \text{and} \quad v := \rho\sin(\varphi - \alpha).$$

It is clear that $R = \mathbb{C}[u, v]$. We put:

$$\begin{cases} \bar{u} := \tau(u) = \rho\big(1 - \dfrac{\varepsilon^2}{2!} + \dfrac{\varepsilon^4}{4!} - \dots\big) \\ \bar{v} := \tau(v) = \rho\big(\varepsilon - \dfrac{\varepsilon^3}{3!} + \dfrac{\varepsilon^5}{5!} - \dots\big). \end{cases}$$

Next, we denote: $\bar{w}_0 = \bar{u}$ and $\bar{w}_i = \bar{u}^{1-i}\bar{v}^i \in Q(P_k)$ for $1 \le i \le k-1$. It is clear that

- $\bar{w}_i \in \mathrm{Im}\big(Q(\widetilde{R}_{\alpha,k}) \xrightarrow{\tilde{\tau}} Q(P_k)\big)$ and
- $\bar{w}_i = \rho \cdot s_i$, where $s_i = \varepsilon^i + \text{h.o.t} \in \mathbb{C}[\varepsilon]/(\varepsilon^k)$

for any $0 \le i \le k-1$. It is now easy to see that ρ and any element $s \in \mathbb{C}[\varepsilon]/(\varepsilon^k)$ belong to the image of $\tilde{\tau}$. Hence, $Q(\widetilde{R}_{\alpha,k}) = Q(P_k)$, as claimed. $\square$

Corollary 2.12. *We get the following description of the algebra of quasi–invariants:*

$$A = A\big(\Pi, \underline{\mu}\big) = \big\{f \in R \,\big|\, f'_\alpha = f'''_\alpha = \dots = f_\alpha^{(2\mu_\alpha - 1)} = 0 \;\text{ for all }\; \alpha \in \Pi\big\}. \tag{2.8}$$

Lemma 2.13. *Let $f \in R$ be such that $T_{(\alpha,2m)}(fg) \in \mathbb{C}[\rho, \varepsilon^2]/(\varepsilon^{2m})$ for any $g \in R$. Then we have: $T_{(\alpha,2m)}(f) = 0$.*

Proof. Taking $g = 1$, we see that $T_{(\alpha,2m)}(f) = \lambda_j \varepsilon^{2j} + \lambda_{j+1}\varepsilon^{2j+2} + \dots + \lambda_{m-1}\varepsilon^{2m-2}$ for some $\lambda_j, \dots, \lambda_{m-1} \in \mathbb{C}[\rho]$. Suppose that $j \le m-1$ and $\lambda_j \ne 0$. Take $g := v^{2(m-j)-1} \in R$. Then we have: $T_{(\alpha,2m)}(gf) = \rho^{2(m-j)-1}\lambda_j \cdot \varepsilon^{2m-1} \notin \mathbb{C}[\rho, \varepsilon^2]/(\varepsilon^{2m})$, giving a contradiction. $\square$

Corollary 2.14. *Let $I := \mathsf{Ann}_A(R/A) = \bigl\{f \in R \,|\, gf \in A \text{ for any } g \in R\bigr\} \cong \mathsf{Hom}_A(R, A)$ be the conductor ideal. Then we have: $I = (\delta^2)_R \cong R$. In particular, the ideal I is Cohen–Macaulay (both over A and over R, since $\mathsf{depth}_A(I) = \mathsf{depth}_R(I) = 2$) and we have:*

$$\mathsf{Ass}_R(I) = \bigl\{\mathfrak{q}_\alpha \,|\, \alpha \in \Pi\bigr\}, \quad \text{where} \quad \mathfrak{q}_\alpha := (l_\alpha)_R \subset R.$$

In other words, the affine variety $V_R(I) \subset \mathbb{A}^2$ is a union of n lines defined by the set Π.

Proof. It follows from the definition of the conductor ideal I that

$$I = \bigl\{f \in R \,\big|\, T_{(\alpha,2\mu_\alpha)}(gf) \in \mathbb{C}[\rho, \varepsilon^2]/(\varepsilon^{2\mu_\alpha}) \text{ for all } g \in R \text{ and } \alpha \in \Pi\bigr\}.$$

Hence, the statement immediately follows from Lemma 2.11 and Lemma 2.13. $\square$

Remark 2.15. The algebra $\bar{A} := A/I$ is *Cohen–Macaulay* of Krull dimension one, since we have a finite extension $\bar{A} \subseteq \bar{R} := R/I$ and the algebra $\bar{R}$ has these properties.

Lemma 2.16. *The map $\mathsf{Ass}_R(I) \longrightarrow \mathsf{Ass}_A(I)$, $\mathfrak{q} \mapsto \mathfrak{p} := A \cap \mathfrak{q}$ is a bijection. Moreover, for any $\mathfrak{q} \in \mathsf{Ass}_R(I)$ we have: $R_\mathfrak{p} = R_\mathfrak{q}$ and the algebra extension $A_\mathfrak{p} \subseteq R_\mathfrak{p}$ induces an isomorphism of the corresponding residue fields.*

Proof. The first statement is a consequence of the fact that the normalization morphism $\mathbb{A}^2 \xrightarrow{\nu} X$ is bijective; see Proposition 2.10. It follows that $\mathfrak{q}R_\mathfrak{p}$ is the unique maximal ideal of $R_\mathfrak{p}$, hence $R_\mathfrak{p} = R_\mathfrak{q}$.

Next, the morphism of affine curves $\mathbb{A}^1 \cong V(\mathfrak{q}) \xrightarrow{\nu} V(\mathfrak{p})$ is bijective, hence it is automatically birational. Therefore, the algebra extension $A/\mathfrak{p} \subseteq R/\mathfrak{q}$ induces an isomorphism of the corresponding fields of fractions and we have: $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong R_\mathfrak{q}/\mathfrak{q}R_\mathfrak{q}$. $\square$

Corollary 2.17. *Let $\bar{\mathfrak{q}} \in \mathsf{Ass}_{\bar{R}}(0)$ and $\bar{\mathfrak{p}} = \bar{\mathfrak{q}} \cap \bar{A}$. Then the algebra extension $\bar{A}_{\bar{\mathfrak{p}}} \subseteq \bar{R}_{\bar{\mathfrak{q}}}$ induces an isomorphism of the corresponding residue fields.*

Remark 2.18. If we view I as an ideal in A then the corresponding affine variety $V_A(I) \subset X$ is the locus of those points where the surface X is not normal. If we view I as an ideal in R then we have $\mathbb{A}^2 \supset V_R(I) = \nu^{-1}\bigl(V_A(I)\bigr)$. Since the map $\mathbb{A}^2 \setminus V_R(I) \xrightarrow{\nu} X \setminus V_A(I)$ is an isomorphism, $V_A(I)$ is precisely the singular locus of X. According to Corollary 2.14, the curve $V_R(I) \subset \mathbb{A}^2$ is a line arrangement consisting of n lines passing through the origin $(0,0)$, whose slopes are determined by the set Π. On the other hand, $V_A(I) = \nu\bigl(V_R(I)\bigr)$ is a union of n rational cuspidal curves (the order of each cusp is determined by the corresponding value of the multiplicity function $\underline{\mu}$) meeting at the common point $\nu(0,0)$.

Proposition 2.19. *For any $\alpha \in \Pi$, put $\mathfrak{q} = \mathfrak{q}_\alpha$, $\mathfrak{p} = \mathfrak{q} \cap A$ and consider the following diagram of $\mathbb{C}$–algebras:*

$$
\begin{tikzcd}
A \arrow[r, two heads] \arrow[d, hook] & \bar{A} \arrow[r] \arrow[d, hook] & \bar{A}_{\bar{\mathfrak{p}}} \arrow[d, hook] \arrow[ddd, bend left, "\bar{T}"] \\
R \arrow[r, two heads] \arrow[dr] \arrow[ddr, bend right, "T"'] & \bar{R} \arrow[r] & \bar{R}_{\bar{\mathfrak{q}}} \\
 & \widetilde{R} \arrow[u, two heads] \arrow[r, hook] \arrow[d, hook, "\tau"'] & \widetilde{R}_{\tilde{\mathfrak{q}}} \arrow[u, "\cong"'] \arrow[d, "\tilde{\tau}"] \\
 & P \arrow[r, hook] & L
\end{tikzcd}
\tag{2.9}
$$

where

- $\widetilde{R} = \widetilde{R}_{\alpha,2\mu_\alpha} := R/(l_\alpha^{2\mu_\alpha})$ *and $\tilde{\mathfrak{q}}$ is the image of $\mathfrak{q}$ in $\widetilde{R}$;*
- $P = P_{2\mu_\alpha} := \mathbb{C}[\rho, \varepsilon]/(\varepsilon^{2\mu_\alpha})$ *and* $T = T_{(\alpha,2\mu_\alpha)}$*;*
- $L = L_\alpha := \mathbb{C}(\rho)[\varepsilon]/(\varepsilon^{2\mu_\alpha})$.

Then we have: $\mathsf{Im}(\bar{T}) = K = K_\alpha := \mathbb{C}(\rho)[\varepsilon^2]/(\varepsilon^{2\mu_\alpha})$.

Proof. We first show that $\mathsf{Im}(\bar{T}) \subseteq K$. Indeed, for any $a \in A$ we have: $\bar{T}(\bar{a}) = T(a) \in K$. Next, observe that if $c \in L$ is invertible and $c \in K$, then $c^{-1} \in K$. Therefore, for any $\dfrac{\bar{a}}{\bar{b}} \in \bar{A}_{\bar{\mathfrak{p}}}$ we have: $\bar{T}\Big(\dfrac{\bar{a}}{\bar{b}}\Big) = T(a) \cdot T(b)^{-1} \in K$.

Let $K' := \mathsf{Im}(\bar{T})$, then we have to prove that $K' = K$. Note the following two facts.

(1) Consider the element $\delta_\alpha := l_\alpha^2 \cdot \prod\limits_{\beta \neq \alpha} \big(s_\alpha(l_\beta) \cdot l_\beta\big)^{2\mu_\beta} \in R$. Since $s_\alpha(\delta_\alpha) = \delta_\alpha$ and $l_\beta^{2\mu_\beta} \,\big|\, \delta_\alpha$ for all $\beta \neq \alpha$, we have: $\delta_\alpha \in A$. Moreover,

$$\bar{\delta}_\alpha := \bar{T}(\delta_\alpha) = \lambda_1 \varepsilon^2 + \cdots + \lambda_{m-1} \varepsilon^{2m-2} \in K'$$

for some $\lambda_1, \dots, \lambda_{m-1} \in \mathbb{C}(\rho)$ such that $\lambda_1 \neq 0$.

(2) By Corollary 2.17, the algebra extension $\bar{A}_{\bar{\mathfrak{p}}} \subseteq \bar{R}_{\bar{\mathfrak{q}}}$ induces an isomorphism of the residue fields. Next, according to Lemma 2.11, the morphism $\tilde{\tau}$ is an isomorphism. Therefore, we have an algebra extension

$$K' \subseteq K = \mathbb{C}(\rho)[\varepsilon^2]/(\varepsilon^{2\mu_\alpha}) \subset L = \mathbb{C}(\rho)[\varepsilon]/(\varepsilon^{2\mu_\alpha}),$$

which induces an isomorphism of the corresponding residue fields.

The last assumption implies that for any $g \in \mathbb{C}(\rho)$, there exists an element $\gamma_g \in K'$ of the form $\gamma_g = g + g_1 \varepsilon^2 + \cdots + g_{\mu_\alpha - 1} \varepsilon^{2\mu_\alpha - 2}$. If $\mu_\alpha = 1$ then we are done. Otherwise, if $\mu_\alpha \geq 2$, consider the element $\bar{\delta}_\alpha^{\mu_\alpha - 1} \cdot \gamma_g = \big(\lambda_1^{\mu_\alpha - 1} g\big) \cdot \varepsilon^{2\mu_\alpha - 2} \in K'$. It follows that for any $h \in \mathbb{C}(\rho)$ we have: $h\varepsilon^{2\mu_\alpha - 2} \in K'$. Proceeding inductively, we conclude that $K' = K$. $\square$

Corollary 2.20. *We have the following commutative diagram of $\mathbb{C}$–algebras:*

$$
\begin{tikzcd}
 & & K_{\alpha_1}\times\cdots\times K_{\alpha_n} \arrow[ddd, hook, "\mathsf{diag}"] \\
A \arrow[r] \arrow[urr] \arrow[d, hook] & Q(\bar{A}) \arrow[ur, "\cong"'] \arrow[d, hook] & \\
R \arrow[r] \arrow[drr, bend right, "T"'] & Q(\bar{R}) \arrow[dr, "\cong"] & \\
 & & L_{\alpha_1}\times\cdots\times L_{\alpha_n}
\end{tikzcd}
\tag{2.10}
$$

where $\{\alpha_1, \ldots \alpha_n\} = \Pi$ *and* $T(f) = \big(T_{(\alpha_1, 2\mu_{\alpha_1})}(f), \ldots, T_{(\alpha_n, 2\mu_{\alpha_n})}(f)\big)$ *for any* $f \in R$, *whereas* $K_\alpha = \mathbb{C}(\rho)[\varepsilon^2]/(\varepsilon^{2\mu_\alpha})$ *and* $L_\alpha = \mathbb{C}(\rho)[\varepsilon]/(\varepsilon^{2\mu_\alpha})$ *for each* $\alpha \in \Pi$.

Now we are prepared to prove the following statement.

Theorem 2.21. *The algebra of quasi–invariants* $A = A\big(\Pi, \underline{\mu}\big)$ *is Cohen–Macaulay and Gorenstein in codimension one. More precisely, let* $\mathfrak{m} \in \mathsf{Max}(A)$ *be the maximal ideal, corresponding to any point of the surface* $X \setminus \{p\}$, *where* $p = \nu(0,0)$ *for the normalization map* $\mathbb{A}^2 \xrightarrow{\nu} X$. *Then the local ring* $A_\mathfrak{m}$ *is Gorenstein.*

Proof. Let A' be the Macaulayfication of A (see Proposition 2.1) and $I' := \mathsf{Hom}_{A'}(R, A')$ be the corresponding conductor ideal. Since $Q(A) = Q(A')$, we have: $\mathsf{Hom}_{A'}(R, A') = \mathsf{Hom}_A(R, A')$. Moreover, the embedding $A \xrightarrow{j} A'$ induces a commutative diagram

$$
\begin{tikzcd}
I \arrow[r, "\cong"] \arrow[d, hook, "j^*_{\mid}"'] & \mathsf{Hom}_A(R, A) \arrow[d, hook, "j^*"] \\
I' \arrow[r, "\cong"] & \mathsf{Hom}_A(R, A').
\end{tikzcd}
$$

Since the $\mathbb{C}$–vector space A'/A is finite dimensional, $j^*_\mathfrak{p}$ is an isomorphism for any $\mathfrak{p} \in \mathcal{P}$. Therefore, the cokernel of $j^*_{\mid}$ is finite dimensional, too. On the other hand, both A–modules I and I' are Cohen–Macaulay; see Corollary 2.14 and [6, Lemma 3.1] respectively. From [6, Lemma 3.6] we deduce that $I = I'$.

Let $\bar{A}' := A'/I$. Then we have an algebra extension $\bar{A} \subseteq \bar{A}'$, where both algebras $\bar{A}$ and $\bar{A}'$ are Cohen–Macaualy and $\mathsf{dim}_\mathbb{C}\big(\bar{A}'/\bar{A}\big) < \infty$. This implies that $Q(\bar{A}) \cong Q(\bar{A}')$. Next, we have the following commutative diagram:

$$
\begin{tikzcd}
A' \arrow[r] \arrow[d, hook] & Q(\bar{A}') \arrow[r, "\cong"] \arrow[d, hook] & Q(\bar{A}) \arrow[r, "\cong"] \arrow[d, hook] & \bar{A}_{\bar{\mathfrak{p}}_1} \times \cdots \times \bar{A}_{\bar{\mathfrak{p}}_n} \arrow[r, "\cong"] \arrow[d, hook, "\mathsf{diag}"] & K_{\alpha_1} \times \cdots \times K_{\alpha_n} \arrow[d, hook, "\mathsf{diag}"] \\
R \arrow[r] & Q(\bar{R}) \arrow[r, "="] & Q(\bar{R}) \arrow[r, "\cong"] & \bar{R}_{\bar{\mathfrak{q}}_1} \times \cdots \times \bar{R}_{\bar{\mathfrak{q}}_n} \arrow[r, "\cong"] & L_{\alpha_1} \times \cdots \times L_{\alpha_n}
\end{tikzcd}
\tag{2.11}
$$

where $\mathfrak{p}_k := A \cap \mathfrak{q}_k$ for $1 \le k \le n$; compare with diagram (2.10).

Let $U_\circ := \mathbb{F}(A')$, where $\mathsf{CM}(A') \xrightarrow{\mathbb{F}} \mathsf{Tri}(A')$ is the equivalence of categories from Theorem 2.9. Clearly, $U_\circ = \big(R, Q(\bar{A}'), \theta\big)$, where $Q(\bar{A}') \xrightarrow{\theta} Q(\bar{R})$ is the canonical inclusion. In the terms of diagram (2.10) we have:

- $Q(\bar{A}') = \bigoplus_{\alpha\in\Pi} K_\alpha$,
- $\theta = \big((1), \dots, (1)\big)$.

Observe that $A' \cong \mathsf{Hom}_{A'}(A', A') \cong \mathsf{Hom}_{\mathsf{Tri}(A')}(U_\circ, U_\circ)$. Spelling out the definition (2.2) of morphisms in $\mathsf{Tri}(A')$, we obtain:

$$\mathsf{Hom}_{\mathsf{Tri}(A')}\big(U_\circ, U_\circ)\big) = \left\{ f \in R \;\middle|\; \forall \alpha \in \Pi \quad \begin{tikzcd} L_\alpha \arrow[d, "T_{(\alpha,2\mu_\alpha)}(f)"'] & L_\alpha \arrow[l, "1"'] \arrow[d, "g_\alpha"] \\ L_\alpha & L_\alpha \arrow[l, "1"] \end{tikzcd} \quad \text{for some } g_\alpha \in K_\alpha \right\}.$$

It follows that $R \supset A' = \big\{ f \in R \,\big|\, T_{(\alpha,2\mu_\alpha)}(f) \in \mathbb{C}[\rho][\varepsilon^2]/(\varepsilon^{2\mu_\alpha}) \text{ for all } \alpha \in \Pi \big\} = A\big(\Pi, \underline{\mu}\big)$. Hence, the algebra A is indeed Cohen–Macaulay.

It follows that the local ring $A_\mathfrak{m}$ is Cohen–Macaulay for any $\mathfrak{m} \in \mathsf{Max}(A)$. Moreover, $A_\mathfrak{m}$ is Gorenstein if and only if its completion $\widehat{A} := \widehat{A}_\mathfrak{m}$ is Gorenstein. Let $q \in X$ be the point corresponding to $\mathfrak{m}$. If q is smooth then $\widehat{A} \cong \mathbb{C}[\![u, v]\!]$ is regular.

Now, assume that the point q is singular and $\alpha \in \Pi$ is such that $q \in V(l_\alpha) \setminus \{p\}$. It follows from the formula $I = \mathsf{Hom}_A(R, A)$ that $\widehat{I} := \widehat{I}_\mathfrak{m}$ is the conductor ideal of the algebra extension $\widehat{A} \subseteq \widehat{R} = \mathbb{C}[\![u, v]\!]$. The diagram (2.11) for the algebra $\widehat{A}$ has the form

$$\begin{tikzcd} \widehat{A} \arrow[r] \arrow[d, hook] & \widehat{K}_\alpha = \mathbb{C}(\!(u)\!)[v^2]/(v^{2\mu_\alpha}) \arrow[d, hook] \\ \widehat{R} \arrow[r] & \widehat{L}_\alpha = \mathbb{C}(\!(u)\!)[v]/(v^{2\mu_\alpha}). \end{tikzcd} \tag{2.12}$$

Applying the same trick as in the proof of the Cohen–Macaulayness of A, we get

$$\widehat{A} \cong \mathsf{Hom}_{\widehat{A}}\big(\widehat{A}, \widehat{A}\big) = \mathsf{End}_{\mathsf{Tri}(\widehat{A})}(U_\bullet) \cong \mathbb{C}[\![u, v^2, v^{2\mu_\alpha+1}]\!],$$

where $U_\bullet := \big(\widehat{R}, \widehat{K}_\alpha, (1)\big) \in \mathsf{Tri}(\widehat{A})$ is the triple corresponding to the regular module $\widehat{A}$. Summing up, $\widehat{A} \cong \mathbb{C}[\![u, v^2, v^{2\mu_\alpha+1}]\!] \cong \mathbb{C}[\![u, z, t]\!]/(t^2 - z^{2\mu_\alpha+1})$ is a hypersurface singularity. Hence, $\widehat{A}$ is Gorenstein. Summing up, the algebra A is Gorenstein in codimension one. $\square$

Remark 2.22. As a consequence of Theorem 2.21, we get the following statement: the algebra of planar quasi–invariants $A(\Pi, \underline{\mu})$ is Gorenstein if and only if the its completion

$$\widehat{A}_p := \Big\{ f \in \mathbb{C}[\![z_1, z_2]\!] \mid l_\alpha^{2\mu_\alpha+1} \text{ divides } \big(f - s_\alpha(f)\big) \text{ for all } \alpha \in \Pi \Big\} \tag{2.13}$$

at its "most singular" point $p = \nu(0, 0)$ is Gorenstein. Another proof of the fact that $A(\Pi, \underline{\mu})$ is Cohen–Macaulay can be found in the thesis of Johnston [21, Theorem 3.3.2].

3. Rank one Cohen–Macaulay modules over the algebra of planar quasi–invariants

In this section, we classify all Cohen–Macaualy A–modules of rank one, specifying those of them, which are locally free in codimension one. Next, we give an explicit description of a dualizing module of A. Finally, we describe the Picard group $\mathsf{Pic}(A)$ viewed as a subgroup of the group $\mathsf{CM}_1^{\mathsf{lf}}(A)$, defined in Proposition 2.5.

3.1. **Description of the group $\mathsf{CM}_1^{\mathsf{lf}}(A)$.** For any $\alpha \in \Pi$ we denote $K_\alpha := \mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha})$ and $L_\alpha := \mathbb{C}(\rho)[\varepsilon]/(\varepsilon^{2\mu_\alpha})$. It what follows, we shall view K_α as a $\mathbb{C}(\rho)$–subalgebra of L_α via the identification $\sigma = \varepsilon^2$. Note that we have a direct sum decomposition $L_\alpha = K_\alpha \dot{+} \varepsilon K_\alpha$. The proof of the following lemma is a straightforward computation.

Lemma 3.1. *For any $\gamma', \gamma'' \in K_\alpha$ we put: $\gamma' \circ \gamma'' := (\gamma' + \gamma'') \cdot (1 + \sigma\gamma' \cdot \gamma'')$, where $+$ and $\cdot$ are the usual addition and multiplication operations in the $\mathbb{C}(\rho)$–algebra K_α. Then we have: $(K_\alpha, \circ)$ is an abelian group.*

For any $\alpha \in \Pi$ and $f \in R$, we define the following two elements $T^{\pm}_{(\alpha,2\mu_\alpha)}(f) \in K_\alpha$:

$$\left\{\begin{array}{lcl} T^{+}_{(\alpha,2\mu_\alpha)}(f) & = & \sum\limits_{j=0}^{\mu_\alpha-1} f_\alpha^{(2j)} \dfrac{\sigma^j}{(2j)!} \\ T^{-}_{(\alpha,2\mu_\alpha)}(f) & = & \sum\limits_{j=0}^{\mu_\alpha-1} f_\alpha^{(2j+1)} \dfrac{\sigma^j}{(2j+1)!}. \end{array}\right. \tag{3.1}$$

Note that $T_{(\alpha,2\mu_\alpha)}(f) = T^{+}_{(\alpha,2\mu_\alpha)}(f) + \varepsilon T^{-}_{(\alpha,2\mu_\alpha)}(f) \in L_\alpha$ for any $f \in R$. Moreover, $f \in A$ if and only if $T^{-}_{(\alpha,2\mu_\alpha)}(f) = 0$ for all $\alpha \in \Pi$.

Theorem 3.2. *There is an isomorphism of abelian groups*

$$\mathsf{CM}_1^{\mathsf{lf}}(A) \xrightarrow{\Theta} K\big(\Pi, \underline{\mu}\big) := \bigoplus_{\alpha\in\Pi} (K_\alpha, \circ) \tag{3.2}$$

such that for any element $\vec{\gamma} = (\gamma_\alpha)_{\alpha\in\Pi} \in K\big(\Pi, \underline{\mu}\big)$ we have:

$$B(\vec{\gamma}) := \Theta^{-1}(\vec{\gamma}) \cong \big\{ f \in R \,\big|\, T^{-}_{(\alpha,2\mu_\alpha)}(f) = \gamma_\alpha \cdot T^{+}_{(\alpha,2\mu_\alpha)}(f) \;\text{ for all }\; \alpha \in \Pi \big\}. \tag{3.3}$$

Proof. By Theorem 2.9, we have an equivalence of categories $\mathsf{CM}^{\mathsf{lf}}(A) \xrightarrow{\mathbb{F}} \mathsf{Tri}^{\mathsf{lf}}(A)$, preserving the monoidal structure on both sides. Let U be an object of $\mathsf{Tri}^{\mathsf{lf}}(A)$ corresponding to a Cohen–Macaulay A–module of rank one. Then we have: $U = \big(R, \oplus_{\alpha\in\Pi} K_\alpha, (\theta_\alpha)_{\alpha\in\Pi}\big)$, where $\theta_\alpha \in L_\alpha$ are some elements. Since the map $L_\alpha \xrightarrow{\theta_\alpha \cdot} L_\alpha$ is an isomorphism for any $\alpha \in \Pi$, we conclude that all elements θ_α are in fact invertible. Applying an appropriate automorphism of K_α, we can find a *uniquely determined* element $\gamma_\alpha \in K_\alpha$ such that

$$U \cong U(\vec{\gamma}) := \big(R, \oplus_{\alpha\in\Pi} K_\alpha, (1 + \varepsilon\gamma_\alpha)_{\alpha\in\Pi}\big).$$

Let $B(\vec{\gamma})$ be the unique (up to an isomorphism) element of the group $\mathsf{CM}_1^{\mathsf{lf}}(A)$ such that $\mathbb{F}\big(B(\vec{\gamma})\big) \cong U(\vec{\gamma})$. By Theorem 2.9 we have: $\mathbb{F}\big(B(\vec{\gamma}') \boxtimes_A B(\vec{\gamma}'')\big) \cong U(\vec{\gamma}') \otimes U(\vec{\gamma}'') \cong$

$$\big(R, \oplus_{\alpha\in\Pi} K_\alpha, \big((1 + \sigma\gamma'_\alpha\gamma''_\alpha) + \varepsilon(\gamma'_\alpha + \gamma''_\alpha)\big)_{\alpha\in\Pi}\big) \cong \big(R, \oplus_{\alpha\in\Pi} K_\alpha, (\gamma'_\alpha \circ \gamma''_\alpha)_{\alpha\in\Pi}\big) = U(\vec{\gamma}' \circ \vec{\gamma}'').$$

This implies that Θ is indeed an isomorphism of abelian groups. To get an explicit description of the module $B(\vec{\gamma})$, observe that

$$B(\vec{\gamma}) \cong \mathsf{Hom}_A\big(A, B(\vec{\gamma})\big) \cong \mathsf{Hom}_{\mathsf{Tri}(A)}\big(U_\circ, U(\vec{\gamma})\big),$$

where $U_\circ := \mathbb{F}(A) = \big(R, \oplus_{\alpha\in\Pi} K_\alpha, \big((1), \dots, (1)\big)\big)$. Writing down the definition of morphisms in the category $\mathsf{Tri}(A)$, we conclude that

$$\mathsf{Hom}_{\mathsf{Tri}(A)}\big(U_\circ, U(\vec{\gamma})\big) = \left\{ f \in R \;\middle|\; \forall \alpha \in \Pi \quad \begin{tikzcd} L_\alpha \arrow[d, "T_{(\alpha,2\mu_\alpha)}(f)"'] & L_\alpha \arrow[l, "1"'] \arrow[d, "g_\alpha"] \\ L_\alpha & L_\alpha \arrow[l, "1+\varepsilon\gamma_\alpha"] \end{tikzcd} \quad \text{for some } g_\alpha \in K_\alpha \right\}.$$

It is easy to see that the constraints on f are precisely the ones given by (3.3). $\square$

3.2. **Classification of all rank one Cohen–Macaulay A–modules.** We begin with the following preparatory results.

Let $\Bbbk$ be any field, $m \in \mathbb{N}$, $K = \Bbbk[\sigma]/(\sigma^m)$ and $L = \Bbbk[\varepsilon]/(\varepsilon^{2m})$. We view K as a $\Bbbk$–subalgebra of L, identifying σ with ε^2. For any $0 \le j \le m$ we put: $K_j := K/(\sigma^j)$ (in particular, $K_0 = 0$ and $K_m = K$).

Lemma 3.3. *Let V be a K–module and $V \xrightarrow{\widetilde{\theta}} L$ an injective map of K–modules such that the adjoint map of L–modules $L \otimes_K V \xrightarrow{\theta} L$ is surjective. Then there exists $0 \le j \le m$ such that $V \cong K \oplus K_j$.*

Proof. Let $\widetilde{V} := L \otimes_K V$. Then we have an isomorphism of K–modules $\widetilde{V} \cong V \oplus \varepsilon V$. By Nakayama's Lemma, the map $\widetilde{V} \xrightarrow{\theta} L$ is surjective if and only if the induced map $\widetilde{V}/\varepsilon\widetilde{V} \xrightarrow{\bar{\theta}} L/\varepsilon L$ is surjective. Note that $\widetilde{V}/\varepsilon\widetilde{V} \cong V/\varepsilon^2 V = V/\sigma V$.

Next, any K–linear map $K_j \xrightarrow{\widetilde{\psi}} L$ is fully determined by the element $a = \widetilde{\psi}(\bar{1}) \in L$, which has to satisfy the condition $\varepsilon^{2j}a = 0$, i.e. $a = \varepsilon^{2(m-j)}\tilde{a}$ for some $\tilde{a} \in L$. The induced map

$$\Bbbk \cong K_j/\sigma K_j \xrightarrow{\bar{\psi}} L/\varepsilon L \cong \Bbbk$$

sends 1 to $a(0)$, i.e. is zero for any $j < m$. Since any finitely generated K–module V splits into a finite direct sum of K_j-s, it follows that θ can be surjective only if V contains K as a direct summand.

In the next step, we prove that the K–linear map $\widetilde{\theta}$ can be injective only if V has at most two direct summands. Let $\mathbb{D} := \mathsf{Hom}_{\Bbbk}(\,-\,,\Bbbk) : K-\mathsf{mod} \longrightarrow K-\mathsf{mod}$ be the Nakayama functor. Obviously, $\mathbb{D}$ is an exact contravariant functor and $\mathbb{D}(K_j) \cong K_j$ for all $0 \le j \le m$. We have: $L \cong K^2$ and

$$0 \longrightarrow V \xrightarrow{\widetilde{\theta}} K \oplus K \quad \text{induces} \quad K \oplus K \xrightarrow{\widetilde{\theta}^*} V^* \longrightarrow 0.$$

It implies that V^* has at most two direct summands, hence V has at most two direct summands, too. Summing up, there exists $0 \le j \le m$ such that $V \cong K \oplus K_j$. $\square$

For $0 \le j \le m$ put $V := K \oplus K_j$. Let $V \xrightarrow{\widetilde{\theta}} L$ be a K–linear map and $L \otimes_K V \cong L \oplus L_j \xrightarrow{\theta} L$ be its adjoint map, where $L_j := L/(\varepsilon^{2j})$. Then both morphisms θ and $\widetilde{\theta}$ can be presented by a matrix $(a\,|\,b) \in \mathrm{Mat}_{(1\times 2)}(L)$, where $\varepsilon^{2j}b = 0$.

Definition 3.4. We call two such maps $\theta, \theta' \in \mathsf{Hom}_L(L \otimes_K V, L)$ *equivalent* if and only if there exists an automorphism $\varphi \in \mathsf{Aut}_K(V)$ such that $\theta' = \theta \circ \varphi$.

We also assume that $\theta = (a\,|\,b) \in \mathsf{Hom}_L(L \otimes_K V, L)$ is surjective and the corresponding map $\widetilde{\theta} \in \mathsf{Hom}_K(V, L)$ is injective:

$$\begin{tikzcd}
K \oplus K_j \arrow[r, hook, "(a\,|\,b)"] \arrow[d, hook] & L \arrow[d, "="] \\
L \oplus L_j \arrow[r, two heads, "(a\,|\,b)"] & L.
\end{tikzcd}$$

It is clear that both these properties of the matrix $(a\,|\,b)$ are preserved when we replace it by an equivalent matrix. Let us first treat the following two "boundary cases".

Lemma 3.5. *The following results are true.*

(1) *Assume $j = 0$, i.e. $V = K$. Then θ is equivalent to $(1 + \varepsilon\gamma)$ for some $\gamma \in K$. Moreover, $(1 + \varepsilon\gamma) \sim (1 + \varepsilon\gamma')$ if and only if $\gamma = \gamma'$.*
(2) *Assume $j = m$, i.e. $V = K \oplus K$. Then θ is equivalent to $(1 \mid \varepsilon)$.*

Proof. In the first case we have: $\theta = (a)$ for some $a \in L$. The surjectivity of θ is equivalent to the condition $a(0) \neq 0$, which also insures the injectivity of $\widetilde{\theta}$. Applying an appropriate automorphism of K, we get $(a) \sim (1 + \varepsilon\gamma)$, where $\gamma \in K$ is uniquely determined.

In the second case, first observe that $\mathrm{rk}_K(L) = 2$, hence the surjectivity of θ is equivalent to its bijectivity (which in its turn, implies the injectivity of $\widetilde{\theta}$). Since both elements $1, \varepsilon \in L$ belong to the image of the map θ, we can transform θ to $(1 \mid \varepsilon)$. □

Proposition 3.6. *As above, let $0 \le j \le m$ and $V = K \oplus K_j$. Let $V \xrightarrow{\widetilde{\theta}} L$ be an injective K–linear map such that its adjoint map $L \otimes_K V \cong L \oplus L_j \xrightarrow{\theta} L$ is surjective. Then there exists an element $\gamma = \alpha_0 + \alpha_1\varepsilon^2 + \dots + \alpha_{m-j-1}\varepsilon^{2(m-j-1)} \in L$ such that*

$$\theta \sim \vartheta_\gamma := (1 + \varepsilon\gamma \mid \varepsilon^{2(m-j)+1}). \tag{3.4}$$

Moreover, $\vartheta_\gamma \sim \vartheta_{\gamma'}$ if and only if $\gamma = \gamma'$.

Proof. Let $\theta = (a \mid b) \in \mathrm{Mat}_{(1\times 2)}(L)$, where $\varepsilon^{2j} b = 0$. By definition, the second component of θ is void if $j = 0$. Since the "boundary cases" $j = 0, m$ were already treated in Lemma 3.5, we can without loss of generality assume that $1 \le j \le m - 1$.

According to the proof of Lemma 3.3, the surjectivity of θ is equivalent to the non–vanishing $a(0) \neq 0$. Moreover, the action of the group $\mathsf{Aut}_K(V)$ leads to the following equivalence relations:

(1) $(a \mid b) \sim (\lambda a \mid b) \sim (a \mid \lambda b)$ for any $\lambda \in K^*$;
(2) $(a \mid b) \sim (a \mid \varepsilon^{2(m-j)}\nu a + b)$ for any $\nu \in K$;
(3) $(a \mid b) \sim (a + \mu b \mid b)$ for any $\mu \in K$.

Using transformations of the first type, we get: $(a \mid b) \sim (1 + \varepsilon c \mid \varepsilon^{2(m-j)} d)$ for some $c \in K$ and $d = \beta_0 + \varepsilon\beta_1 + \dots + \varepsilon^{2j-1}\beta_{2j-1} \in L$. Using an appropriate transformation of the second type, we can kill all coefficients $\beta_0, \beta_2, \dots, \beta_{2j-1}$ (i.e. entries at $1, \varepsilon^2, \dots, \varepsilon^{2j-2}$ of the element d). In other words, $\theta \sim \theta' = (1 + \varepsilon c \mid \varepsilon^{2(m-j)+1} e)$ for a certain $e = \xi_0 + \varepsilon^2\xi_1 + \dots + \varepsilon^{2(j-1)}\xi_{j-1} \in K \subset L$. Now observe that for $x = (0, \varepsilon^{2(j-1)}) \in K \oplus K_j$ we have: $\widetilde{\theta'}(x) = \varepsilon^{2m-1}\xi_0$. Since the map $\widetilde{\theta'}$ is injective, we conclude that $\xi_0 \neq 0$, i.e. $e \in K$ is a unit. Hence, we get: $\theta \sim \theta' \sim (1 + \varepsilon c \mid \varepsilon^{2(m-j)+1})$. Finally, using an appropriate transformation of the third type, we can kill all entries at $\varepsilon^{2(m-j)}, \dots, \varepsilon^{2(m-1)}$ of the element c and end up with a normal form $\theta \sim \vartheta_\gamma$ as in (3.4).

It is not difficult to see that the K–linear map $\widetilde{\theta}$ corresponding to the L–linear map $\theta = \vartheta_\gamma$ given by the formula (3.4), is injective for any $\gamma \in L$ as in the statement of Proposition. Next, consider the following K–modules

$$\left\{ \begin{array}{lcl} t(V) & = & \bigl\{v \in V \,|\, \sigma^j v = 0\bigr\} = \sigma^{m-j}K \oplus K_j \\ t(L) & = & \bigl\{u \in L \,|\, \varepsilon^{2j} u = 0\bigr\} = \varepsilon^{2(m-j)}L. \end{array} \right.$$

Since the submodule $t(V)$ is mapped to itself under arbitrary automorphisms of V, we obtain an induced map $V/t(V) \xrightarrow{\bar{\vartheta}_\gamma} L/t(L)$ such that

$$\begin{tikzcd} V/t(V) \arrow[rr, "\bar{\vartheta}_\gamma"] \arrow[d, "\cong"'] & & L/t(L) \arrow[d, "\cong"] \\ \Bbbk[\sigma]/(\sigma^{m-j}) \arrow[rr, "1+\varepsilon\gamma"] & & \Bbbk[\varepsilon]/(\varepsilon^{2(m-j)}). \end{tikzcd}$$

Hence, $\vartheta_\gamma \sim \vartheta_{\gamma'}$ if and only if $\gamma = \gamma'$. □

Now we are ready to prove the main result of this subsection.

Theorem 3.7. *For $\vec{\nu} = \bigl(\nu_\alpha\bigr)_{\alpha\in\Pi} \in \mathbb{N}_0^n$ such that $0 \le \nu_\alpha \le \mu_\alpha$ and $\vec{\gamma} = \bigl(\gamma_\alpha\bigr)_{\alpha\in\Pi}$ such that $\gamma_\alpha \in \mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha-\nu_\alpha})$ for any $\alpha \in \Pi$ we put:*

$$M(\vec{\nu},\vec{\gamma}) := \Bigl\{f \in R \Bigm| T^-_{(\alpha,2(\mu_\alpha-\nu_\alpha))}(f) = \gamma_\alpha \cdot T^+_{(\alpha,2(\mu_\alpha-\nu_\alpha))}(f) \quad \text{for all} \quad \alpha\in\Pi\Bigr\}. \tag{3.5}$$

Then the following results are true.

- *$M(\vec{\nu},\vec{\gamma})$ is a Cohen–Macaulay A–module of rank one.*
- *Conversely, any Cohen–Macaulay A–module of rank one is isomorphic to some $M(\vec{\nu},\vec{\gamma})$ for appropriate parameters $\vec{\nu},\vec{\gamma}$ as above.*
- *$M(\vec{\nu},\vec{\gamma}) \cong M(\vec{\nu}',\vec{\gamma}')$ if any only if $\vec{\nu} = \vec{\nu}'$ and $\vec{\gamma} = \vec{\gamma}'$.*

Proof. According to Theorem 2.9, the isomorphism classes of Cohen–Macaulay A–modules stand in bijection with the isomorphism classes of objects of the category of triples $\mathsf{Tri}(A)$. Let $U = (\widetilde{M}, V, \theta)$ be a rank one object of $\mathsf{Tri}(A)$ (i.e. an object corresponding to a Cohen–Macaulay A–module of rank one) then $\widetilde{M} \cong R$. Moreover, by Lemma 3.3, there exists a uniquely determined vector $\vec{\nu} \in \mathbb{N}_0^n$ as above such that

$$V \cong \bigoplus_{\alpha\in\Pi} V_\alpha = \bigoplus_{\alpha\in\Pi} K_\alpha \oplus \bigl(K_\alpha/(\sigma^{\mu_\alpha-\nu_\alpha})\bigr).$$

Next, Proposition 3.6 implies that there exists an automorphism of the triple U transforming every component of the gluing map θ into the canonical form

$$\theta_\alpha = (1+\varepsilon\gamma_\alpha \mid \varepsilon^{2(\mu_\alpha-\nu_\alpha)+1})$$

for an appropriate vector $\vec{\gamma} = \bigl(\gamma_\alpha\bigr)_{\alpha\in\Pi}$ as above. Since $\mathsf{Aut}_R(R) = \mathbb{C}^*$, in order to describe the isomorphism classes of rank one objects of $\mathsf{Tri}(A)$ it is sufficient to take into account only the action of the groups $\mathsf{Aut}_{K_\alpha}(V_\alpha)$ on the matrices θ_α. Proposition 3.6 then insures that the vector $\vec{\gamma}$ is in fact uniquely determined.

Summing up, consider the following object of the category $\mathsf{Tri}(A)$:

$$U(\vec{\nu},\vec{\gamma}) := \Bigl(R, \oplus_{\alpha\in\Pi}\bigl(K_\alpha \oplus K_\alpha/(\sigma^{\mu_\alpha-\nu_\alpha})\bigr), (1+\varepsilon\gamma_\alpha \mid \varepsilon^{2(\mu_\alpha-\nu_\alpha)+1})_{\alpha\in\Pi}\Bigr).$$

Then the following results are true.

- Any rank one object of $\mathsf{Tri}(A)$ is isomorphic to some $U(\vec{\nu},\vec{\gamma})$ for appropriate parameters $\vec{\nu},\vec{\gamma}$ as in the statement of the theorem.
- $U(\vec{\nu},\vec{\gamma}) \cong U(\vec{\nu}',\vec{\gamma}')$ if any only if $\vec{\nu} = \vec{\nu}'$ and $\vec{\gamma} = \vec{\gamma}'$.

Let $M(\vec{\nu},\vec{\gamma}) := \mathbb{F}^{-1}\bigl(U(\vec{\nu},\vec{\gamma})\bigr)$, then we have: $M(\vec{\nu},\vec{\gamma}) \cong \mathsf{Hom}_{\mathsf{Tri}(A)}\bigl(U_\circ, U(\vec{\nu},\vec{\gamma})\bigr)$, where $U_\circ := \mathbb{F}(A) = \bigl(R, \oplus_{\alpha\in\Pi} K_\alpha, \bigl((1),\dots,(1)\bigr)\bigr)$. Following formula (2.2) we have: $f \in R$ belongs to $\mathsf{Hom}_{\mathsf{Tri}(A)}\bigl(U_\circ, U(\vec{\gamma})\bigr)$ if and only if for any $\alpha \in \Pi$ there exists a K_α–linear map $K_\alpha \xrightarrow{\left(\begin{smallmatrix} g_\alpha \\ h_\alpha\end{smallmatrix}\right)} K_\alpha \oplus \bigl(K_\alpha/(\sigma^{\mu_\alpha-\nu_\alpha})\bigr)$ such that

$$\begin{tikzcd}
L_\alpha \arrow[d, "T_{(\alpha,2\mu_\alpha)}(f)"'] & & L_\alpha \arrow[ll, "1"'] \arrow[d, "\left(\begin{smallmatrix} g_\alpha \\ h_\alpha\end{smallmatrix}\right)"] \\
L_\alpha & & L_\alpha \oplus \bigl(L_\alpha/(\varepsilon^{2(\mu_\alpha-\nu_\alpha)})\bigr). \arrow[ll, "(1+\varepsilon\gamma_\alpha \,|\, \varepsilon^{2(\mu_\alpha-\nu_\alpha)+1})"']
\end{tikzcd} \tag{3.6}$$

Writing down explicitly the constraints (3.6), we end up with the description (3.5). □

Corollary 3.8. *Let $(\vec{\nu},\vec{\gamma})$ be as in Theorem 3.7. Then the following results are true.*

- *The Cohen–Macaulay module $M(\vec{\nu},\vec{\gamma})$ is locally free in codimension one if and only if $\vec{\nu} = \vec{0}$. In notations of Theorem 3.2 we have: $M(\vec{0},\vec{\gamma}) = B(\vec{\gamma})$.*
- *Consider the weight function $\Pi \xrightarrow{\underline{\mu}'} \mathbb{N}_0$ given by the rule: $\underline{\mu}'(\alpha) = \mu_\alpha - \nu_\alpha$ for any $\alpha \in \Pi$. Let $A' := A\bigl(\Pi, \underline{\mu}'\bigr)$ be the corresponding algebra of quasi–invariant polynomials. Theorem 3.2 implies that $M(\vec{\nu},\vec{\gamma})$ is a Cohen–Macaulay A'–module of rank one, locally free in codimension one.*

Example 3.9. Consider the special case of a constant multiplicity function $\Pi \xrightarrow{\underline{\mu}} \mathbb{N}_0$ given by the rule: $\mu_\alpha = 1$ for all $\alpha \in \Pi$. Then the classification of Cohen–Macaulay A–modules of rank one takes the following form.

- Any object of $\mathsf{CM}_1^{\mathsf{lf}}(A)$ is isomorphic to some
$$B(\vec{\gamma}) = \bigl\{f \in R \,|\, f'_\alpha = \gamma_\alpha f_\alpha \quad \text{for all} \quad \alpha \in \Pi\bigr\}, \tag{3.7}$$
where $\vec{\gamma} = (\gamma_\alpha)_{\alpha\in\Pi} \in \mathbb{C}(\rho)^{\oplus n}$.
- Moreover, $B(\vec{\gamma}) \cong B(\vec{\gamma}')$ if and only if $\vec{\gamma} = \vec{\gamma}'$ and $B(\vec{\gamma}) \boxtimes_A B(\vec{\gamma}') \cong B(\vec{\gamma} + \vec{\gamma}')$. Note that $A = B(\vec{0})$.
- We get the full list of objects of $\mathsf{CM}_1(A)$ by the following rule: one takes any non–empty subset $\Pi^\circ \subseteq \Pi$ and just omits the conditions in (3.7) for $\alpha \in \Pi^\circ$. In particular, for $\Pi^\circ = \Pi$ (no conditions on f), we get the module R.

3.3. **Dualizing module of the algebra of dihedral quasi–invariants.** It was already pointed out by Etingof and Ginzburg in [15, Section 6] that the algebra $A = A\bigl(\Pi, \underline{\mu}\bigr)$ is not Gorenstein for a general weighted line arrangement $(\Pi, \underline{\mu})$. However, A is a finitely generated Cohen–Macaulay algebra (see Theorem 2.21), hence it has a dualizing module Ω (which is a Cohen–Macaualy A–module of rank one, uniquely determined up to a tensoring with an element of $\mathsf{Pic}(A)$). It is a natural question to describe Ω in terms of our classification.

We give an explicit description of Ω in the so–called *Coxeter* (or *dihedral* case), when

$$\Pi = \Lambda_n := \left\{0, \frac{1}{n}\pi, \dots, \frac{n-1}{n}\pi\right\} \tag{3.8}$$

for some $n \in \mathbb{N}$. For $m := \max\bigl\{\mu_\alpha \bigm| \alpha \in \Pi\bigr\}$ let $\Lambda_n \xrightarrow{\underline{\kappa}} \mathbb{N}_0$, $\alpha \mapsto m$ be the corresponding constant multiplicity function and $C := A(\Lambda_n, \underline{\kappa}\bigr)$ the corresponding ring of quasi–invariant polynomials. By [18, Corollary 5.6] (see also [15, Theorem 1.2]), the algebra C is a graded

Gorenstein domain. In particular, C viewed as a module over itself, is a dualizing module of C; see for instance [9, Section I.3.6]. Therefore,

$$\Omega := \mathsf{Hom}_C(A, C) \tag{3.9}$$

is a dualizing module of A, see for instance [4, Section X.9.3]. The following theorem is the main result of this subsection.

Theorem 3.10. *For any $\alpha \in \Lambda_n$ put: $\nu_\alpha := m - \mu_\alpha$. Then we have:*

$$\Omega \cong \bigl\{ f \in C \,\big|\, f_\alpha^{(0)} = f_\alpha^{(2)} = \cdots = f_\alpha^{(2(\nu_\alpha - 1))} = 0 \text{ for all } \alpha \in \Lambda_n \bigr\}. \tag{3.10}$$

Remark 3.11. It is a pleasant exercise in elementary calculus to verify directly that the right hand side of the expression (3.10) is an ideal in the algebra A given by (2.8).

Proof. For any $\alpha \in \Lambda_n$ we put: $\widetilde{K} := \mathbb{C}(\rho)[\sigma]/(\sigma^m)$ and $\widetilde{L} := \mathbb{C}(\rho)[\varepsilon]/(\varepsilon^{2m})$. Then we have the following

Claim. Let $U_\bullet := \mathbb{F}(A)$, where $\mathsf{CM}(C) \xrightarrow{\mathbb{F}} \mathsf{Tri}(C)$ is the equivalence of categories from Theorem 2.9. Then we have: $U_\bullet \cong (R, V, \theta)$, where $V = \oplus_{\alpha \in \Lambda_n} V_\alpha$ with

$$V_\alpha = \left\{ \begin{array}{cl} \widetilde{K} & \text{if} \quad \mu_\alpha = m \\ \widetilde{K} \oplus \bigl(\widetilde{K}/(\sigma^{\nu_\alpha})\bigr) & \text{if} \quad 0 \le \mu_\alpha \le m-1 \end{array} \right.$$

and

$$\theta_\alpha = \left\{ \begin{array}{cl} 1 & \text{if} \quad \mu_\alpha = m \\ (1 \,|\, \varepsilon^{2\mu_\alpha + 1}) & \text{if} \quad 0 \le \mu_\alpha \le m-1. \end{array} \right.$$

We prove this claim by computing the morphism space $\mathsf{Hom}_{\mathsf{Tri}(C)}(U_\circ, U_\bullet)$, where $U_\circ$ is the canonical triple corresponding to the regular module C. As in the proof of Theorem 3.7, we get: $f \in R$ belongs to $\mathsf{Hom}_{\mathsf{Tri}(C)}(U_\circ, U_\bullet)$ if and only if for any $\alpha \in \Lambda_n$ there exists a $\widetilde{K}$–linear map $\widetilde{K} \xrightarrow{\left(\begin{smallmatrix} g_\alpha \\ h_\alpha \end{smallmatrix}\right)} \widetilde{K} \oplus \bigl(\widetilde{K}/(\sigma^{\nu_\alpha})\bigr)$ making the diagram

$$\begin{tikzcd} \widetilde{L} \arrow[d, "T_{(\alpha,2m)}(f)"'] & \widetilde{L} \arrow[l, "1"'] \arrow[d, "\left(\begin{smallmatrix} g_\alpha \\ h_\alpha \end{smallmatrix}\right)"] \\ \widetilde{L} & \widetilde{L} \oplus \bigl(\widetilde{L}/(\varepsilon^{2\nu_\alpha})\bigr). \arrow[l, "(1 \,|\, \varepsilon^{2\mu_\alpha+1})"'] \end{tikzcd} \tag{3.11}$$

commutative, i.e. $T_{(\alpha,2m)}(f) = g_\alpha + \varepsilon^{2\mu_\alpha+1} h_\alpha$ for some $g_\alpha, h_\alpha \in \widetilde{L}$. This condition is equivalent to the vanishing $f_\alpha^{(2l-1)} = 0$ for any $\alpha \in \Lambda_n$ and $1 \le l \le \mu_\alpha$. Hence, (R, V, θ) is indeed isomorphic to $\mathbb{F}(A)$, as asserted (compare with Corollary 2.12).

Now, in virtue of (3.9), we have an isomorphism: $\Omega \cong \mathsf{Hom}_{\mathsf{Tri}(C)}(U_\bullet, U_\circ)$. A polynomial $f \in R$ belongs to the vector space $\mathsf{Hom}_{\mathsf{Tri}(C)}(U_\bullet, U_\circ)$ if and only if for every $\alpha \in \Lambda_n$ there exist elements $p_\alpha, q_\alpha \in \widetilde{K}$, making the diagram

$$\begin{tikzcd} \widetilde{L} \arrow[d, "T_{(\alpha,2m)}(f)"'] & \widetilde{L} \oplus \bigl(\widetilde{L}/(\varepsilon^{2\nu_\alpha})\bigr) \arrow[l, "(1 \,|\, \varepsilon^{2\mu_\alpha+1})"'] \arrow[d, "(\, p_\alpha | \varepsilon^{2\mu_\alpha} q_\alpha \,)"] \\ \widetilde{L} & \widetilde{L} \arrow[l, "1"'] \end{tikzcd} \tag{3.12}$$

commutative. In other words, for any $\alpha \in \Lambda_n$ there exist $p_\alpha, q_\alpha \in \widetilde{K}$ such that

$$\bigl(T_{(\alpha,2m)}(f) \,|\, \varepsilon^{2\mu_\alpha+1} T_{(\alpha,2m)}(f)\bigr) = \bigl(p_\alpha \,|\, \varepsilon^{2\mu_\alpha} q_\alpha\bigr).$$

The first condition $T_{(\alpha,2m)}(f) = p_\alpha$ just means that $T_{(\alpha,2m)}(f) \in \widetilde{K}$ for any $\alpha \in \Lambda_n$, i.e. f is an element of the algebra C. The second condition $\varepsilon^{2\mu_\alpha+1}T_{(\alpha,2m)}(f) = \varepsilon^{2\mu_\alpha} q_\alpha$ is equivalent to the vanishing $f_\alpha^{(0)} = f_\alpha^{(2)} = \dots = f_\alpha^{(2(\nu_\alpha-1))} = 0$ for all $\alpha \in \Lambda_n$. $\square$

As a further refinement of Theorem 3.10, we have the following result.

Lemma 3.12. *For any $k \in \mathbb{N}_0$, consider the multiplicity function $\Lambda_n \xrightarrow{\mu} \mathbb{N}_0$ given by the rule: $\mu_0 = k+1$ and $\mu_\alpha = 1$ for any $\alpha \in \Lambda_n^\circ := \Lambda_n \setminus \{0\}$. Let Ω be a dualizing module of the algebra $A = A\bigl(\Lambda_n, \underline{\mu}\bigr)$ given by (3.10). Then we have:*

$$\Omega \cong \left\{ g \in R \;\middle|\; \begin{array}{l} g_0' = g_0''' = \dots = g_0^{(2k+1)} = 0 \\ \left(\dfrac{g}{l_0^{2k}}\right)_\alpha' = 0 \quad \text{for all} \quad \alpha \in \Lambda_n^\circ. \end{array} \right\},$$

where $l_0 = x_1 = \rho\cos(\varphi)$.

Proof. In the notation of Theorem 3.10 we have: $m = k+1$, $\nu_\alpha = k$ for each $\alpha \in \Lambda_n^\circ$. Let $C = A\bigl(\Lambda_n, \underline{\kappa}\bigr)$, then

$$\Omega := \bigl\{ f \in C \,\big|\, f_\alpha^{(0)} = f_\alpha^{(2)} = \dots = f_\alpha^{(2k-2)} = 0 \text{ for all } \alpha \in \Lambda_n^\circ \bigr\}$$

is a dualizing module of A. More explicitly, for any $\alpha \in \Lambda_n^\circ$ and $f \in \Omega$ we have:

$$f_\alpha^{(l)} = 0 \quad \text{for any} \quad 0 \le l \le 2k-1 \quad \text{and} \quad f_\alpha^{(2k+1)} = 0.$$

The first condition is equivalent to the statement that $l_\alpha^{2k} \mid f$ for every $\alpha \in \Lambda_n^\circ$. Let

$$\delta_\circ := \prod_{\alpha \in \Lambda_n^\circ} l_\alpha^{2k} = \rho^{2k(n-1)} \Bigl(\prod_{l=1}^{n-1} \sin\Bigl(\varphi - \frac{l}{n}\pi\Bigr)\Bigr)^{2k}.$$

Then there exists a uniquely determined $g \in R$ such that $f = g \cdot \delta_\circ$. Note that $\delta_\circ(\rho, -\varphi) = \delta_\circ(\rho, \varphi)$, hence $(\delta_\circ)_0^{(2p+1)} = 0$ for all $p \in \mathbb{N}_0$. It is not difficult to show by induction that the condition $f_0^{(1)} = f_0^{(3)} = \dots = f_0^{(2k+1)} = 0$ (recall that $f \in C = A(\Lambda_n, \underline{\kappa})$) is equivalent to $g_0^{(1)} = g_0^{(3)} = \dots = g_0^{(2k+1)} = 0$.

Finally, it remains to interpret the constraint $f_\alpha^{(2k+1)} = 0$ for $\alpha \in \Lambda_n^\circ$ in terms of the polynomial g. Since $(\delta_\circ)_\alpha^{(l)} = 0$ for all $0 \le l \le 2k-1$, we have:

$$f_\alpha^{(2k+1)} = (2k+1)(\delta_\circ)_\alpha^{(2k)} g_\alpha' + (\delta_\circ)_\alpha^{(2k+1)} g_\alpha = 0. \tag{3.13}$$

As usual, we put $\delta^2 := \prod\limits_{\alpha \in \Lambda_n} l_\alpha^{2k} = l_0^{2k} \cdot \delta_\circ$. Then we have: $\delta^2(\rho, -\varphi) = \delta^2(\rho, \varphi)$, hence $\delta_0^{(2p+1)} = 0$ for all $p \in \mathbb{N}_0$. Moreover, $\delta^2(\rho, \varphi + \alpha) = \delta^2(\rho, \varphi)$ for any $\alpha \in \Lambda_n$ (here, we essentially use the fact that the image of the set Λ_n in $\mathbb{R}/\pi\mathbb{Z}$ is a subgroup). Therefore, $\delta_\alpha^{(2p+1)} = 0$ for any $\alpha \in \Lambda_n^\circ$ and $p \in \mathbb{N}_0$. In particular, we get:

$$\delta_\alpha^{(2k+1)} = (2k+1)(\delta_\circ)_\alpha^{(2k)} (l_0^{2k})_\alpha' + (\delta_\circ)_\alpha^{(2k+1)} (l_0^{2k})_\alpha = 0. \tag{3.14}$$

Comparing the equations (3.13) and (3.14), we see that the condition $f_\alpha^{(2k+1)} = 0$ is equivalent to $\left(\dfrac{g}{l_0^{2k}}\right)'_\alpha = 0$. Summing up, we obtain:

$$\Omega = \delta_\circ \cdot \left\{ g \in R \;\middle|\; \begin{array}{l} g_0' = g_0''' = \cdots = g_0^{(2k+1)} = 0 \\ \left(\dfrac{g}{l_0^{2k}}\right)'_\alpha = 0 \quad \text{for all} \quad \alpha \in \Lambda_n^\circ \end{array} \right\},$$

what implies the result. $\square$

Remark 3.13. According to a result of Feigin and Johnston [17, Theorem 7.14], the algebra of quasi–invariants from Lemma 3.12 is Gorenstein if and only if $k = 0$, what matches with our description of a dualizing module.

3.4. **Picard group of the algebra of quasi–invariants $A(\Pi, \underline{\mu})$.** In this subsection, we describe the Picard group $\mathsf{Pic}(A)$ of the algebra of planar quasi–invariants $A = A(\Pi, \underline{\mu})$.

We begin with the following elementary observation.

Lemma 3.14. *Let $\Bbbk$ be any ring, $m \in \mathbb{N}$, $K = \Bbbk[\varepsilon^2]/(\varepsilon^{2m})$ and $L = \Bbbk[\varepsilon]/(\varepsilon^{2m})$. Then for any $g \in L$, there exists a unique element $\gamma(g) \in K$ such that $g \cdot (1 + \varepsilon\gamma(g)) \in K$.*

Let $\widehat{R} := \mathbb{C}[\![z_1, z_2]\!] = \mathbb{C}[\![\rho\cos(\varphi), \rho\sin(\varphi)]\!]$. For any $h \in \widehat{R}$ and a pair $(\alpha, k) \in \mathbb{C} \times \mathbb{N}$, we define the power series $\widehat{T}_{(\alpha,k)}(h) \in \mathbb{C}[\![\rho]\!][\varepsilon]/(\varepsilon^{2\mu_\alpha})$ by the rule

$$T_{(\alpha,k)}\bigl(\exp(h)\bigr) = \widehat{T}_{(\alpha,k)}(h) \cdot \exp\bigl(h_\alpha^{(0)}\bigr). \tag{3.15}$$

It follows from the definition, that for any $h_1, h_2 \in \widehat{R}$ we have:

$$\widehat{T}_{(\alpha,k)}(h_1 + h_2) = \widehat{T}_{(\alpha,k)}(h_1) \cdot \widehat{T}_{(\alpha,k)}(h_2). \tag{3.16}$$

Lemma 3.15. *For $(\alpha, m) \in \mathbb{C} \times \mathbb{N}$, let $\widehat{R} \xrightarrow{\Upsilon_{(\alpha,m)}} \mathbb{C}[\![\rho]\!][\sigma]/(\sigma^m)$ be given by the composition*

$$\widehat{R} \xrightarrow{\widehat{T}_{(\alpha,2m)}} \mathbb{C}[\![\rho]\!][\varepsilon]/(\varepsilon^{2m}) \xrightarrow{\gamma} \mathbb{C}[\![\rho]\!][\sigma]/(\sigma^m), \tag{3.17}$$

where γ is the map from Lemma 3.14. Then we have: $(\widehat{R}, +) \xrightarrow{\Upsilon_{(\alpha,m)}} \bigl(\mathbb{C}[\![\rho]\!][\sigma]/(\sigma^m), \circ\bigr)$ is a group homomorphism.

Proof. Let $h_k \in \widehat{R}$ and $\gamma_k := \Upsilon_{(\alpha,m)}(h_k)$ for $k = 1, 2$. By definition, we have:

$$(1 + \varepsilon\gamma_k) \cdot \widehat{T}_{(\alpha,2m)}(h_k) \in \mathbb{C}[\![\rho]\!][\varepsilon^2]/(\varepsilon^{2m}) \quad \text{for} \quad k = 1, 2.$$

Note that we have the following identity in $\mathbb{C}[\![\rho]\!][\varepsilon^2]/(\varepsilon^{2m})$:

$$(1+\varepsilon\gamma_1)\cdot(1+\varepsilon\gamma_2)\cdot\widehat{T}_{(\alpha,2m)}(h_1)\cdot\widehat{T}_{(\alpha,2m)}(h_2) = \bigl((1+\varepsilon^2\gamma_1\gamma_2)+\varepsilon(\gamma_1+\gamma_2)\bigr)\cdot\widehat{T}_{(\alpha,2m)}(h_1+h_2).$$

Then we have: $\widehat{T}_{(\alpha,2m)}(h_1+h_2)\cdot\bigl(1+\varepsilon(\gamma_1+\gamma_2)\cdot(1+\varepsilon^2\gamma_1\gamma_2)^{-1}\bigr)$ belongs to $\mathbb{C}[\![\rho]\!][\varepsilon^2]/(\varepsilon^{2m})$. It follows from the definition of the operation $\circ$ that

$$\Upsilon_{(\alpha,m)}(h_1 + h_2) = \Upsilon_{(\alpha,m)}(h_1) \circ \Upsilon_{(\alpha,m)}(h_2),$$

proving the statement. $\square$

Lemma 3.16. *Let $B = \mathbb{C}[\![u, v^2, v^{2m+1}]\!]$ for some $m \in \mathbb{N}$. Then its normalization is $\widehat{R} = \mathbb{C}[\![u, v]\!]$ and the diagram (2.11) has the form*

$$\begin{array}{ccc} B & \longrightarrow & \widehat{K} = \mathbb{C}(\!(u)\!)[v^2]/(v^{2m}) \\ \big\downarrow & & \big\downarrow \\ \widehat{R} & \longrightarrow & \widehat{L} = \mathbb{C}(\!(u)\!)[v]/(v^{2m}) \end{array} \tag{3.18}$$

for some $m \in \mathbb{N}$. Let $U = \bigl(\widehat{R}, \widehat{K}, (1 + \bar{v}\gamma)\bigr)$ be an object of $\mathsf{Tri}(B)$ for some $\gamma \in \mathbb{C}(\!(u)\!)[v^2]/(v^{2m})$. Then we have: $\mathbb{F}(B) \cong U$ if and only if $\gamma \in \mathbb{C}[\![u]\!][v^2]/(v^{2m})$.

Proof. Recall that $\mathbb{F}(B) = U_\circ := (\widehat{R}, \widehat{K}, 1)$. Let $U = \bigl(\widehat{R}, \widehat{K}, (1 + \bar{v}\gamma)\bigr)$ be such that $\gamma \in \mathbb{C}[\![u]\!][v^2]/(v^{2m})$. Then we have an expansion $\gamma = \sum\limits_{j=0}^{m-1} \gamma_j(u)\bar{v}^{2j}$, where $\gamma_j \in \mathbb{C}[\![u]\!]$ for all $0 \le j \le m-1$. Let $g := 1 + v \cdot \bigl(\sum\limits_{j=0}^{m-1} \gamma_j v^{2j}\bigr) \in \widehat{R}$. Then g is a unit and $\bar{g}^{-1} \cdot (1 + \bar{v}\gamma) = 1$ in $\widehat{L}$. Hence $U_\circ \xrightarrow{(g,1)} U$ is an isomorphism in the category $\mathsf{Tri}(B)$.

On the other hand, if $\gamma_j \in \mathbb{C}(\!(u)\!)$ has a non–trivial Laurent part for some $0 \le j \le m-1$ then $U \ncong U_\circ$ (since we can not eliminate the Laurent part of γ by multiplying it with the image of a unit from $\widehat{R}$). $\square$

Theorem 3.17. *Let $\bigl(\Pi, \underline{\mu}\bigr)$ be any datum and $A = A\bigl(\Pi, \underline{\mu}\bigr)$ be the corresponding algebra of quasi–invariants. Consider the following homomorphism of abelian groups:*

$$\mathbb{C}[\![z_1, z_2]\!] \xrightarrow{\Upsilon} \prod_{\alpha \in \Pi} \bigl(\mathbb{C}[\![\rho]\!][\sigma]/(\sigma^{\mu_\alpha}), \circ\bigr), \quad h \mapsto \bigl(\Upsilon_{(\alpha, 2\mu_\alpha)}(h)\bigr)_{\alpha \in \Pi}. \tag{3.19}$$

Then we have:

$$\mathsf{Pic}(A) \cong \mathrm{Im}(\Upsilon) \cap K^\diamond\bigl(\Pi, \underline{\mu}\bigr), \tag{3.20}$$

where $K^\diamond\bigl(\Pi, \underline{\mu}\bigr) := \prod\limits_{\alpha \in \Pi} \bigl(\mathbb{C}[\rho][\sigma]/(\sigma^{\mu_\alpha}), \circ\bigr)$. More explicitly, let

$$\Gamma\bigl(\Pi, \underline{\mu}\bigr) := \bigl\{h \in \mathbb{C}[\![z_1, z_2]\!] \big| \Upsilon(h) \in K^\diamond\bigl(\Pi, \underline{\mu}\bigr)\bigr\}.$$

Then for any $h \in \Gamma\bigl(\Pi, \underline{\mu}\bigr)$, the corresponding projective A–module of rank one is given by

$$P(h) := \bigl\{f \in R \,\big|\, \exp(h)f \text{ is } (\Pi, \underline{\mu}) - \text{quasi–invariant}\bigr\} = B\bigl(\Upsilon(h)\bigr). \tag{3.21}$$

Conversely, for any $P \in \mathsf{Pic}(A)$, there exists $h \in \Gamma\bigl(\Pi, \underline{\mu}\bigr)$ such that $P \cong P(h)$. Moreover,

- *$P(h_1) \cong P(h_2)$ if and only if $\Upsilon(h_1) = \Upsilon(h_2)$.*
- *The multiplication map $P(h_1) \otimes_A P(h_2) \longrightarrow P(h_1 + h_2)$, $f_1 \otimes f_2 \mapsto f_1 f_2$ is an isomorphism of A–modules.*

Proof. Let $P \in \mathsf{CM}_1^{\mathsf{lf}}(A)$, then we have: $\mathbb{F}(P) := U \cong (R, V, \theta)$, where $V \cong \oplus_{\alpha \in \Pi} K_\alpha$ and $\theta = (\theta_\alpha)_{\alpha \in \Pi}$, where $\theta_\alpha = 1 + \varepsilon\gamma_\alpha \in L_\alpha$ for some $\gamma \in K_\alpha$.

Note that the module P is projective if and only if $\widehat{P}_\mathfrak{m} \cong \widehat{A}_\mathfrak{m}$ for any maximal ideal $\mathfrak{m} \in \mathsf{Max}(A)$. Let $p \in X$ be the point corresponding to $\mathfrak{m}$ and $q := \nu^{-1}(p) \in \mathbb{A}^2$. Assume that $q \notin \cup_{\alpha \in \Pi} V(l_\alpha)$. Then $\widehat{P}_\mathfrak{m} \cong \widehat{A}_\mathfrak{m}$ is automatically true.

Now, let $q \in \cup_{\alpha \in \Pi} V(l_\alpha)$. According to Theorem 2.9, $\widehat{P}_\mathfrak{m} \cong \widehat{A}_\mathfrak{m}$ if and only if the triples $\mathbb{L}_\mathfrak{m}(U)$ and $\mathbb{F}_\mathfrak{m}(\widehat{A}_\mathfrak{m})$ are isomorphic in the category $\mathsf{Tri}(\widehat{A}_\mathfrak{m})$.

<u>Case 1</u>. Assume that $q \ne (0,0)$. Then there exists uniquely determined $\alpha \in \Pi$ and $\rho_0 \in \mathbb{C}^*$ such that $q = \bigl(\rho_0 \cos(\alpha), \rho_0 \sin(\alpha)\bigr) \in V(l_\alpha)$. Denote $u = \rho\cos(\varphi-\alpha) - \rho_0$ and $v = \rho\sin(\varphi - \alpha)$. Obviously, we have: $R = \mathbb{C}[u,v]$, $\widehat{R} := \widehat{R}_{\mathfrak{m}} \cong \mathbb{C}[\![u,v]\!]$ and $\widehat{I} := \widehat{I}_{\mathfrak{m}} = (v^{2\mu_\alpha})$. Moreover, the following diagram is commutative:

$$\begin{tikzcd}
I \arrow[r, hook] \arrow[d, hook, "\mathsf{can}"'] & R \arrow[r, "T_{(\alpha,2\mu_\alpha)}"] \arrow[d, hook, "\mathsf{can}"] & \mathbb{C}(\rho)[\varepsilon]/(\varepsilon^{2\mu_\alpha}) \arrow[d] \arrow[dd, bend left=60, "\lambda_{\rho_0}"] \\
\widehat{I} \arrow[r, hook] \arrow[d, hook, "\cong"'] & \widehat{R} \arrow[r] \arrow[d, hook, "\cong"] & Q(\widehat{R}/\widehat{I}) \arrow[d, "\cong"] \\
(v^{2\mu_\alpha}) \arrow[r, hook] & \mathbb{C}[\![u,v]\!] \arrow[r] & \mathbb{C}(\!(u)\!)[v]/(v^{2\mu_\alpha})
\end{tikzcd}$$

where $\lambda_{\rho_0}(g) \in \mathbb{C}(\!(u)\!)[\sigma]/(\sigma^{\mu_\alpha})$ is the Laurent expansion of $g \in \mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha})$ at the point $\rho_0 \in \mathbb{A}^1 \cong V(l_\alpha)$. Note that we are in the setting of Lemma 3.16: $\widehat{A}_{\mathfrak{m}} \cong \mathbb{C}[\![u, v^2, v^{2\mu_\alpha+1}]\!]$. The key point is that we have the following formula for the localized and completed triple:

$$\mathbb{L}_{\mathfrak{m}}(U) \cong \bigl(\mathbb{C}[\![u,v]\!], \mathbb{C}(\!(u)\!)[v^2]/(v^{2\mu_\alpha}), 1 + \bar{v}\lambda_{\rho_0}(\gamma_\alpha)\bigr).$$

According to Lemma 3.16, the module P is locally free at the point $p \in X$ if and only if $\gamma_\alpha \in \mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha})$ has no pole at $\rho_0 \in \mathbb{A}^1$.

<u>Case 2</u>. For the point $q = (0,0)$ we have:

$$\mathbb{L}_{\mathfrak{m}}(U) \cong \bigl(\mathbb{C}[\![z_1,z_2]\!], \oplus_{\alpha\in\Pi}\mathbb{C}(\!(\rho)\!)[\sigma]/(\sigma^{\mu_\alpha}), (1+\varepsilon\gamma_\alpha)_{\alpha\in\Pi}\bigr),$$

where the element $\gamma_\alpha \in \mathbb{C}(\rho)[\sigma]/(\sigma^{\mu_\alpha})$ is viewed as an element of $\mathbb{C}(\!(\rho)\!)[\sigma]/(\sigma^{\mu_\alpha})$ for each $\alpha \in \Pi$. Similarly to the previous case we conclude that P can be projective only if $\gamma_\alpha \in K_\alpha$ is regular at 0 for each $\alpha \in \Pi$. Hence, $\vec{\gamma} = (\gamma_\alpha)_{\alpha\in\Pi} \in K^\diamond\bigl(\Pi, \underline{\mu}\bigr)$.

It follows from the definition of the category $\mathsf{Tri}(\widehat{A}_{\mathfrak{m}})$ that $\mathbb{L}_{\mathfrak{m}}(U) \cong \mathbb{F}_{\mathfrak{m}}(\widehat{A}_{\mathfrak{m}})$ if and only if there exists a *unit* $f \in \widehat{R}$ such that

$$T_{(\alpha,2\mu_\alpha)}(f)\cdot(1+\varepsilon\gamma_\alpha) \in \mathbb{C}[\![\rho]\!][\varepsilon^2]/(\varepsilon^{2\mu_\alpha}) \tag{3.22}$$

for every $\alpha \in \Pi$. Every unit in the algebra $\widehat{R}$ can be written as the exponential of some power series, hence $f = \exp(h)$ for some $h \in \widehat{R}$. In the notation of formula (3.15), the condition (3.22) can be rewritten as: $\widehat{T}_{(\alpha,2\mu_\alpha)}(h)\cdot(1+\varepsilon\gamma_\alpha) \in \mathbb{C}[\![\rho]\!][\varepsilon^2]/(\varepsilon^{2\mu_\alpha})$, i.e. $\Upsilon(h) = \vec{\gamma}$.

Note that the constraints on a polynomial $f \in R$ from the formula (3.3) defining the module $B(\vec{\gamma})$ and the ones from (3.22) are in fact the same. It implies that $P(h) = B(\vec{\gamma}) = B\bigl(\Upsilon(h)\bigr)$. Moreover, Theorem 3.2 implies that $P(h_1) \cong P(h_2)$ if and only if $\Upsilon(h_1) = \Upsilon(h_2)$. Finally, the diagram

$$\begin{tikzcd}
P(h_1)\otimes_A P(h_2) \arrow[r, "\mathsf{mult}"] \arrow[d, "="'] & P(h_1+h_2) \arrow[d, "="] \\
B\bigl(\Upsilon(h_1)\bigr) \boxtimes_A B\bigl(\Upsilon(h_2)\bigr) \arrow[r, "\cong"] \arrow[d, hook] & B\bigl(\Upsilon(h_1)\circ\Upsilon(h_2)\bigr) \arrow[d, hook] \\
R\otimes_R R \arrow[r, "\mathsf{mult}"] & R
\end{tikzcd} \tag{3.23}$$

is commutative. It implies that the multiplication map $P(h_1) \otimes_A P(h_2) \xrightarrow{\mathsf{mult}} P(h_1+h_2)$ is indeed an isomorphism of A–modules. Theorem is proven. $\square$

The following special cases of Theorem 3.17 are perhaps of independent interest.

Example 3.18. Let $\Pi = \left\{0, \dfrac{\pi}{2}\right\}$. Note that for any $f \in \mathbb{C}[\![x,y]\!]$, we have the following formulae for the directional derivatives (2.6):

$$f_0'(\rho) = \rho \frac{\partial f}{\partial y}(\rho, 0) \quad \text{and} \quad f'_{\frac{\pi}{2}}(\rho) = -\rho \frac{\partial f}{\partial x}(0, \rho).$$

(1) Let $\Pi \xrightarrow{\underline{\mu}} \mathbb{N}_0$ be given by the rule: $\underline{\mu}(0) = 1$ and $\underline{\mu}\left(\dfrac{\pi}{2}\right) = 0$. Then we have: $A = \mathbb{C}[x^2, x^3, y]$ and the Picard group $\mathsf{Pic}(A)$ has the following description:

$$\begin{tikzcd} \mathsf{CM}_1^{\mathsf{lf}}(A) \arrow[r, "\Theta"] & \big(\mathbb{C}(\rho), +\big) \\ \mathsf{Pic}(A) \arrow[u, hook] \arrow[r, "\cong"] & \big(\rho\mathbb{C}[\rho], +\big), \arrow[u, hook] \end{tikzcd}$$

where Θ is the isomorphism from Theorem 3.2. It is interesting to note that $\mathsf{Pic}\big(D[t]\big) \cong \mathsf{Pic}(D)$ for any Noetherian normal domain D; see for instance [36].

(2) Let $\Pi \xrightarrow{\underline{\mu}} \mathbb{N}_0$ be given by the rule: $\underline{\mu}(0) = \underline{\mu}\left(\dfrac{\pi}{2}\right) = 1$. Then we have:

$$A = \mathbb{C}[x^2, x^3, y^2, y^3] \cong \mathbb{C}[x^2, x^3] \otimes_{\mathbb{C}} \mathbb{C}[y^2, y^3].$$

Then the description of $\mathsf{Pic}(A)$ from Theorem 3.17 gets the following form:

$$\begin{tikzcd} \mathsf{CM}_1^{\mathsf{lf}}(A) \arrow[r, "\Theta"] & \big(\mathbb{C}(\rho), +\big) \oplus \big(\mathbb{C}(\rho), +\big) \\ \mathsf{Pic}(A) \arrow[u, hook] \arrow[r, "\cong"] & \Big\{(\rho f, \rho g) \in \rho\mathbb{C}[\rho] \oplus \rho\mathbb{C}[\rho] \,\Big|\, f'(0) + g'(0) = 0\Big\}. \arrow[u, hook] \end{tikzcd}$$

4. Spectral module of a rational Calogero–Moser system of dihedral type

In this section, we shall discuss a link between results on Cohen–Macaulay modules over an algebra planar quasi–invariants with the theory Calogero–Moser systems.

Definition 4.1. For any $\alpha \in \mathbb{C}$ we put: $e(\alpha) = \exp(2i\alpha)$. A weighted line arrangement $\big(\Pi, \underline{\mu}\big)$ is called *Baker–Akhieser* if for any $\alpha \in \Pi$ and $1 \le k \le \mu_\alpha$ we have:

$$\sum_{\substack{\beta \in \Pi \\ \beta \ne \alpha}} \frac{\mu_\beta \big(e(\beta) + e(\alpha)\big)^{2k-1}}{\big(e(\beta) - e(\alpha)\big)^{2k-1}} = 0 \quad \text{and} \quad \sum_{\substack{\beta \in \Pi \\ \beta \ne \alpha}} \frac{\mu_\beta(\mu_\beta + 1) e(\beta) \big(e(\beta) + e(\alpha)\big)^{2k-1}}{\big(e(\beta) - e(\alpha)\big)^{2k+1}} = 0; \tag{4.1}$$

see [12, 14] and [17, Lemma 2.1].

Example 4.2. A so–called *Coxeter* weighted line arrangement $\big(\Lambda_n, \underline{\mu}\big)$ defined below is Baker–Akhieser; see [12, 14, 17].

- $\Lambda_n := \big\{0, \frac{1}{n}\pi, \dots, \frac{n-1}{n}\pi\big\} \subset \mathbb{R}$ for some $n \in \mathbb{N}$.
- $\mu_\alpha = m$ for all $\alpha \in \Lambda_n$ and some $m \in \mathbb{N}$ (constant multiplicity function), or
- $n = 2\bar{n}$ and $\underline{\mu}\big(\frac{2k}{n}\pi\big) = m_1$ and $\underline{\mu}\big(\frac{2k+1}{n}\pi\big) = m_2$ for all $0 \le k \le \bar{n} - 1$ and some $m_1, m_2 \in \mathbb{N}$.

We put $\mu = \mu\bigl(\Pi, \underline{\mu}\bigr) := \sum_{\alpha\in\Pi} \mu_\alpha$ and $\delta(z_1, z_2) := \prod_{\alpha\in\Lambda_n} l_\alpha^{\mu_\alpha}(z_1, z_2)$. Note that $\delta(z_1, z_2)$ is a homogeneous polynomial of degree μ.

4.1. **Some results on two–dimensional rational Calogero–Moser system.** Let $\bigl(\Pi, \underline{\mu}\bigr)$ be a *Baker–Akhieser* weighted line arrangement. The *rational Calogero–Moser operator* $H = H(\Pi, \underline{\mu})$ is defined by the formula

$$H := \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - \sum_{\alpha\in\Pi} \frac{\mu_\alpha(\mu_\alpha + 1)}{l_\alpha^2(x_1, x_2)}. \tag{4.2}$$

According to a result of Chalykh and Veselov [11] (elaborated in their later joint work with Styrkas [12]) extending earlier results of Heckman and Opdam [25] and Heckman [24], the operator H can be included into a large family of pairwise commuting partial differential operators. In order to make this statement more precise, we recall the following formula of Berest [1, Theorem 2.8] (see also [14, Theorem 3.1]) for the so–called *multivariate Baker–Akhieser function*, corresponding to the datum $\bigl(\Pi, \underline{\mu}\bigr)$:

$$\Phi(x_1, x_2; z_1, z_2) := \frac{1}{2^\mu \cdot \mu!}\Bigl(H_{(x_1,x_2)} - z_1^2 - z_2^2\Bigr)^\mu \diamond \bigl(\delta(x_1, x_2) \cdot \exp(x_1 z_1 + x_2 z_2)\bigr). \tag{4.3}$$

Theorem 4.3. *The following results are true.*

(1) *The Baker–Akhieser function $\Phi(x_1, x_2; z_1, z_2)$ is an eigenfunction of the Calogero–Moser operator (4.2) in the following sense:*

$$H_{(x_1,x_2)} \diamond \Phi(x_1, x_2; z_1, z_2) = (z_1^2 + z_2^2) \cdot \Phi(x_1, x_2; z_1, z_2). \tag{4.4}$$

(2) *Moreover, there exists an injective algebra homomorphism*

$$A\bigl(\Pi, \underline{\mu}\bigr) \xrightarrow{\Xi} \mathbb{C}(x_1, x_2)[\partial_1, \partial_2]. \tag{4.5}$$

such that the highest symbol of $\Xi(f)$ is equal to the highest symbol of $f(\partial_1, \partial_2)$ and

$$\bigl(\Xi(f)\bigr)_{(x_1,x_2)} \diamond \Phi(x_1, x_2; z_1, z_2) = f(z_1, z_2) \cdot \Phi(x_1, x_2; z_1, z_2). \tag{4.6}$$

for any $f \in A\bigl(\Pi, \underline{\mu}\bigr)$. In particular, $H = \Xi(\omega)$ for $\omega = z_1^2 + z_2^2$.

(3) *The Baker–Akhieser function Φ has the following expansion:*

$$\Phi(x_1, x_2; z_1, z_2) = \bigl(\delta(z_1, z_2) + \sum_{i_1+i_2<\mu} c_{i_1,i_2}(x_1, x_2) z_1^{i_1} z_2^{i_2}\bigr) \cdot \exp(x_1 z_1 + x_2 z_2), \tag{4.7}$$

where $c_{i_1,i_2}(x_1, x_2) \in \mathbb{C}(x_1, x_2)$ for all (i_1, i_2). Moreover,

$$c_{0,0}(x_1, x_2) = \prod_{\alpha\in\Pi} \frac{1}{l_\alpha(x_1 - \xi_1, x_2 - \xi_2)^{2\mu_\alpha}}.$$

(4) *Let $z_1 = \rho\cos(\varphi)$ and $z_2 = \rho\sin\varphi$. Then we have:*

$$\Phi(x_1, x_2; \rho)_\alpha^{(2l-1)} := \left.\frac{\partial^{2l-1}\Phi}{\partial\varphi^{2l-1}}\right|_{\varphi=\alpha} = 0 \quad \text{for all } \alpha \in \Pi \text{ and } 1 \le l \le \mu_\alpha. \tag{4.8}$$

Comment to the proof. From the historical perspective, the development of results collected in Theorem 4.3 was slightly different. The notion of a multivariate Baker–Akhieser function $\Phi(x_1, x_2; z_1, z_2)$ corresponding to a Baker–Akhieser datum $\bigl(\Pi, \underline{\mu}\bigr)$ was axiomatized (in arbitrary dimension) by Chalykh, Styrkas and Veselov in [11, 12]. The properties (4.7) and (4.8) were stated as defining axioms, whereas the eigenfunction properties (4.5) and (4.6) were shown to be formal consequences of the proposed axiomatic.

In [1, Theorem 2.8], Berest discovered an explicit formula (4.3) for a multivariate Baker–Akhieser function; see also [14, Theorem 3.1]. There is a closed expression for the homomorphism Ξ; see [1, Section 2.3], [14, Theorem 1.3] or [16, Corollary 3.3].

In our exposition, we start with Berest's formula (4.3) for the Baker–Akhieser function $\Phi(x_1, x_2; z_1, z_2)$. The formula (4.7) can be deduced from (4.3) by induction on μ. We refer to a paper of Chalykh, Feigin and Veselov [14] for further details. $\square$

Our next goal is to study the rational Calogero–Moser operator (4.2) using methods of the higher–dimensional Krichever correspondence developed in [27, 38, 28]. To do this, we have to introduce the following minor modification of the operator (4.2). Let $\vec{\xi} = (\xi_1, \xi_2) \in \mathbb{C}^2$ be such that

- $l_\alpha(\vec{\xi}) = -\sin(\alpha)\xi_1 + \cos(\alpha)\xi_2 \neq 0$ for all $\alpha \in \Pi$.
- $\xi_1^2 + \xi_2^2 \neq 0$ (for example, one can simply take $\vec{\xi} \in \mathbb{R}^2 \setminus \{\vec{0}\}$).

The second condition on $\vec{\xi}$ implies that one can find $(\rho_0, \varphi_0) \in \mathbb{C}^* \times \mathbb{C}$ such that

$$\vec{\xi} = \big(\rho_0 \cos(\varphi_0), \rho_0 \cos(\varphi_0)\big).$$

For any such vector $\vec{\xi}$, we have an automorphism

$$\mathbb{C}(x_1, x_2) \xrightarrow{t(\vec{\xi})} \mathbb{C}(x_1, x_2), \quad \text{where } \big(t(\vec{\xi})(f)\big)(\vec{x}) = f(\vec{x} - \vec{\xi}),$$

which can obviously be extended to an automorphism $t(\vec{\xi})$ of the algebra of partial differential operators $\mathbb{C}(x_1, x_2)[\partial_1, \partial_2]$.

Summing up, we have an injective algebra homomorphism $\Xi(\vec{\xi})$ given as the composition

$$A\big(\Pi, \underline{\mu}\big) \xrightarrow{\Xi} \mathbb{C}(x_1, x_2)[\partial_1, \partial_2] \xrightarrow{t(\vec{\xi})} \mathbb{C}(x_1, x_2)[\partial_1, \partial_2]. \tag{4.9}$$

Then the perturbed Calogero–Moser operator $H = H\big((\Pi, \underline{\mu}), \vec{\xi}\big) := \big(\Xi(\vec{\xi})\big)(\omega)$ is given by the formula

$$H := \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - \sum_{\alpha \in \Pi} \frac{\mu_\alpha(\mu_\alpha + 1)}{l_\alpha^2(\vec{x} - \vec{\xi})}, \tag{4.10}$$

whereas the conventional Calogero–Moser operator (4.2) is $H\big((\Pi, \underline{\mu}), \vec{0}\big)$. Note that the potential of (4.10) is regular at the point $(0, 0)$. Moreover,

$$\mathfrak{B} := \mathsf{Im}\big(\Xi(\vec{\xi})\big) \subseteq \mathfrak{D} := \mathbb{C}[\![x_1, x_2]\!][\partial_1, \partial_2], \tag{4.11}$$

hence we get the embedding (1.4).

Definition 4.4. The $\mathfrak{B}$–module $F := \mathfrak{D}/(x_1, x_2)\mathfrak{D} \cong \mathbb{C}[\partial_1, \partial_2]$ is called *spectral module* of the algebra $\mathfrak{B}$.

Note that F is actually a *right* $\mathfrak{D}$–module. However, since the algebra $\mathfrak{B}$ is commutative, we shall view F as a *left* $\mathfrak{B}$–module, having the natural right action $\circ$ in mind.

Theorem 4.5. *The following results are true.*

(1) *F is a finitely generated Cohen–Macaulay $\mathfrak{B}$–module of rank one.*
(2) *For any character $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ (i.e. an algebra homomorphism), consider the vector space*

$$\mathsf{Sol}\big(\mathfrak{B}, \chi\big) := \Big\{ f \in \mathbb{C}[\![x_1, x_2]\!] \big| P \diamond f = \chi(P) f \text{ for all } P \in \mathfrak{B} \Big\}. \tag{4.12}$$

Then there exists a canonical isomorphism of vector spaces

$$F\big|_{\chi} := F \otimes_{\mathfrak{B}} \big(\mathfrak{B}/\mathrm{Ker}(\chi)\big) \cong \mathsf{Sol}\big(\mathfrak{B}, \chi\big)^{*}. \tag{4.13}$$

assigning to a class $\overline{\partial_1^{p_1}\partial_2^{p_2}} \in F\big|_{\chi}$ *the linear functional* $f \mapsto \dfrac{1}{p_1!p_2!}\dfrac{\partial^{p_1+p_2} f}{\partial x_1^{p_1}\partial x_2^{p_2}}\Big|_{(0,0)}$ *on the vector space* $\mathsf{Sol}\big(\mathfrak{B}, \chi\big)$*. In particular,* $\mathsf{dim}_{\mathbb{C}}\Big(\mathsf{Sol}\big(\mathfrak{B}, \chi\big)\Big) < \infty$ *for any* χ*.*

Proof. We get the first statement, combining [28, Theorem 3.1] with [10, Corollary 3.1]; see also [27, Theorem 2.1].

In the one–dimensional case, the isomorphism (4.13) is due to Mumford [32, Section 2]. For partial differential operators, we follow the exposition in [8, Theorem 1.14]. The key point is the following isomorphism of left $\mathfrak{D}$–modules:

$$\mathsf{Hom}_{\mathbb{C}}\big(F, \mathbb{C}\big) \xrightarrow{\mathfrak{t}} \mathbb{C}[\![x_1, x_2]\!], \quad l \mapsto \sum_{p_1,p_2=0}^{\infty} \frac{1}{p_1!p_2!} l\big(\partial_1^{p_1}\partial_2^{p_2}\big) x_1^{p_1} x_2^{p_2}, \tag{4.14}$$

where we take the right action $\circ$ on F and the usual right action $\diamond$ on $\mathbb{C}[\![x_1, x_2]\!]$ of the algebra $\mathfrak{D}$. Let $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$ be a character, then $\mathbb{C} = \mathbb{C}_{\chi} := \mathfrak{B}/\mathsf{Ker}(\chi)$ is a left $\mathfrak{B}$–module. Next, we have a $\mathfrak{B}$–linear map

$$\mathfrak{b} : \mathsf{Hom}_{\mathfrak{B}}(F, \mathbb{C}_{\chi}) \xrightarrow{\mathfrak{i}} \mathsf{Hom}_{\mathbb{C}}(F, \mathbb{C}) \xrightarrow{\mathfrak{t}} \mathbb{C}[\![x_1, x_2]\!], \tag{4.15}$$

where $\mathfrak{i}$ is the forgetful map. The image of $\mathfrak{i}$ consists of those $\mathbb{C}$–linear functionals, which are also $\mathfrak{B}$–linear, i.e.

$$\mathsf{Im}(\mathfrak{i}) = \big\{l \in \mathsf{Hom}_{\mathbb{C}}(F, \mathbb{C}) \;\big|\; l(P \diamond -\,) = \chi(P) \cdot l(-\,) \text{ for all } P \in \mathfrak{B}\big\}.$$

This implies that $\mathsf{Im}(\mathfrak{b}) = \mathsf{Sol}(\mathfrak{B}, \chi)$. By adjunction, we have a canonical isomorphism of $\mathfrak{B}$–modules: $\mathsf{Hom}_{\mathfrak{B}}(F, \mathbb{C}_{\chi}) \cong \mathsf{Hom}_{\mathbb{C}}\big(F \otimes_{\mathfrak{B}} (\mathfrak{B}/\mathsf{Ker}(\chi)), \mathbb{C}\big)$. Taking duals, we get an isomorphism of vector spaces $\mathsf{Sol}(\mathfrak{B}, \chi)^{*} \longrightarrow \big(F \otimes_{\mathfrak{B}} (\mathfrak{B}/\mathsf{Ker}(\chi))\big)^{**} \cong F\big|_{\chi}$. It remains to observe that $F\big|_{\chi} \xrightarrow{(\mathfrak{b}^{*})^{-1}} \mathsf{Sol}(\mathfrak{B}, \chi)^{*}$ is the map from the statement of the theorem. $\square$

Remark 4.6. By Hilbert's Nullstellensatz, the characters $\mathfrak{B} \longrightarrow \mathbb{C}$ stand in bijection with the points of the spectral surface X of the Calogero–Moser system $\mathfrak{B}$ (i.e. an affine surface, whose coordinate ring is isomorphic to $A\big(\Pi, \underline{\mu}\big) \cong \mathfrak{B}$). The finitely generated $\mathfrak{B}$–module F determines a coherent sheaf of X, so the $\mathbb{C}$–vector space $F\big|_{\chi}$ is the fiber of F over the point of X corresponding to the character $\mathfrak{B} \xrightarrow{\chi} \mathbb{C}$.

4.2. **Spectral module of a two–dimensional rational Calogero–Moser system.** In their recent paper [17, Section 8], Feigin and Johnston raised a question about an explicit description of the spectral module F of the algebra $\mathfrak{B}$ given by (4.11). In this subsection, we give a solution of this problem. To do this, we need a more concrete description of the algebra homomorphisms Ξ and $\Xi(\vec{\xi})$; see (4.5) and (4.9).

Lemma 4.7. *Consider the following variation of the function (4.3):*

$$\Psi(\vec{x}; \vec{z}; \vec{\xi}) := \frac{1}{2^{\mu} \cdot \mu!}\Big(H_{(x_1,x_2)} - z_1^2 - z_2^2\Big)^{\mu} \diamond \big(\delta(x_1 - \xi_1, x_2 - \xi_2) \cdot \exp(x_1 z_1 + x_2 z_2)\big). \tag{4.16}$$

Then the following results are true.

(1) *For any quasi–invariant polynomial $f \in A$ we have:*

$$\big(\Xi(\vec{\xi})(f)\big)_{(x_1,x_2)} \diamond \Psi(\vec{x};\vec{z};\vec{\xi}) = f(z_1,z_2)\cdot \Psi(\vec{x};\vec{z};\vec{\xi}). \tag{4.17}$$

(2) *For any $p_1, p_2 \in \mathbb{N}_0$ we have:*

$$w_{(p_1,p_2)}(z_1,z_2) := \left.\frac{\partial^{p_1+p_2}\Psi}{\partial x_1^{p_1}\partial x_2^{p_2}}\right|_{(x_1,x_2)=(0,0)} \in \mathbb{C}[z_1,z_2]. \tag{4.18}$$

Here, we view $w_{(p_1,p_2)}$ as a function of $\vec{z}$ depending on the parameter $\vec{\xi} \in \mathbb{C}^2$. Moreover, we have the following expansion:

$$w_{(p_1,p_2)}(z_1,z_2) = z_1^{p_1} z_2^{p_2}\cdot \delta(z_1,z_2) + \text{ lower order terms.} \tag{4.19}$$

(3) *For any $p_1, p_2 \in \mathbb{N}_0$, the function $\exp\big(-\rho\rho_0\cos(\varphi-\varphi_0)\big)\cdot w_{(p_1,p_2)}(\rho,\varphi)$ is quasi–invariant with respect to the datum $(\Pi, \underline{\mu})$, i.e.*

$$\Big(\exp\big(-\rho\rho_0\cos(\varphi-\varphi_0)\big)\cdot w_{(p_1,p_2)}(\rho,\varphi)\Big)_\alpha^{(2l-1)} = 0 \text{ for all } \alpha\in\Pi \text{ and } 1\le l\le \mu_\alpha, \tag{4.20}$$

where as usual, $(z_1,z_2) = \big(\rho\cos(\varphi), \rho\sin(\varphi)\big)$ *and* $(\xi_1,\xi_2) = \big(\rho_0\cos(\varphi_0), \rho_0\sin(\varphi_0)\big)$.

Proof. Observe that we have the following equality:

$$\Phi(\vec{x}-\vec{\xi},\vec{z}) = \Psi(\vec{x};\vec{z};\vec{\xi})\cdot\exp(-\xi_1 z_1 - \xi_2 z_2) = \Psi(\vec{x};\vec{z};\vec{\xi})\cdot \exp\big(-\rho\rho_0\cos(\varphi-\varphi_0)\big).$$

Hence, all statements of Lemma 4.7 are straightforward consequences of the corresponding results from Theorem 4.3. □

Let $\mathfrak{E} := \mathbb{C}[\![x_1,x_2]\!]((\partial_1^{-1}))((\partial_2^{-1}))$ be the algebra of partial *pseudo–differential operators*; see for instance [33] for a precise definition and main ring–theoretic properties. This algebra admits the following convenient characterization.

Lemma 4.8. *Let $\mathfrak{M} := \mathbb{C}[\![x_1,x_2]\!]((z_1^{-1}))((z_2^{-1}))\cdot\exp(x_1z_1+x_2z_2)$ be the so–called* Baker–Akhieser module. *Then we have an injective algebra homomorphism*

$$\mathfrak{E} \xrightarrow{\mathfrak{e}} \mathsf{End}_{\mathbb{C}((z_1^{-1}))((z_2^{-1}))}(\mathfrak{M}) \tag{4.21}$$

mapping $\partial_j^{\pm} \in \mathfrak{E}$ to $z_j^{\pm} \in \mathsf{End}_{\mathbb{C}((z_1^{-1}))((z_2^{-1}))}(\mathfrak{M})$ for $j = 1, 2$. Moreover, for any element $Q \in \mathfrak{M}$, there exists a uniquely determined element $S \in \mathfrak{E}$ such that

$$Q = S \diamond \exp(x_1z_1+x_2z_2) := \big(\mathfrak{e}(S)\big)\big(\exp(x_1z_1+x_2z_2)\big).$$

Remark 4.9. The recipe to construct the operator $S \in \mathfrak{E}$ corresponding to an element $Q \in \mathfrak{M}$ is as follows. Let $Q(x_1,x_2;z_1,z_2) = T(x_1,x_2;z_1,z_2)\cdot\exp(x_1z_1+x_2z_2)$, where

$$T(x_1,x_2;z_1,z_2) = \sum_{p_1,p_2} a_{p_1,p_2}(x_1,x_2) z^{p_1} z_2^{p_2} \in \mathbb{C}[\![x_1,x_2]\!]((z_1^{-1}))((z_2^{-1})). \tag{4.22}$$

Then we have:

$$S = \sum_{p_1,p_2} a_{p_1,p_2}(x_1,x_2)\partial_1^{p_1}\partial_2^{p_2} \in \mathbb{C}[\![x_1,x_2]\!]((\partial_1^{-1}))((\partial_2^{-1})). \tag{4.23}$$

Here, both sums (4.21) and (4.23) are taken in the appropriate sense.

Definition 4.10. Let $\Psi(x_1,x_2;z_1,z_2;\vec{\xi}) \in \mathfrak{M}$ be the Baker–Akhieser function of $\mathfrak{B}$ given by (4.16). Then the corresponding pseudo–differential operator $S \in \mathfrak{E}$, defined by the recipe (4.23), is called *Sato operator* of the algebra $\mathfrak{B}$.

Lemma 4.11. *For any quasi–invariant polynomial $f \in A$ we have:*

$$\big(\Xi(\vec{\xi})\big)(f) = S \cdot f(\partial_1, \partial_2) \cdot S^{-1}, \tag{4.24}$$

where both sides of (4.24) are viewed as elements of the algebra $\mathfrak{E}$.

Proof. Let $\Theta = \Theta(x_1, x_2; z_1, z_2; \vec{\xi}) := \Psi(x_1, x_2; z_1, z_2; \vec{\xi}) \cdot \exp(-x_1 z_1 - x_2 z_2)$. Then we have an expansion $\Theta = \delta(z_1, z_2) + \sum\limits_{i_1+i_2<\mu} b_{i_1,i_2}(x_1, x_2) z_1^{i_1} z_2^{i_2}$ for some coefficients $b_{i_1,i_2} \in \mathbb{C}[\![x_1, x_2]\!]$. In particular, the Sato operator of the algebra $\mathfrak{B}$ belongs to $\mathfrak{D}$:

$$S = \delta(\partial_1, \partial_2) + \sum_{i_1+i_2<\mu} b_{i_1,i_2}(x_1, x_2) \partial_1^{i_1} \partial_2^{i_2}. \tag{4.25}$$

Since the highest symbol $\delta(\partial_1, \partial_2)$ of S is a partial differential operator with constant coefficients, S is a unit in the algebra $\mathfrak{E}$; see for instance [33, Proposition 1].

By definition, we have: $\Psi(x_1, x_2; z_1, z_2; \vec{\xi}) = S \diamond \exp(x_1 z_1 + x_2 z_2)$. Hence, the equality (4.17) can be rewritten in the form:

$$\Big(\big(\big(\Xi(\vec{\xi})\big)(f)\big) \cdot S\Big) \diamond \exp(x_1 z_1 + x_2 z_2) = f(z_1, z_2) \cdot \big(S \diamond \exp(x_1 z_1 + x_2 z_2)\big).$$

Since $\mathfrak{e}(S)$ is a $\mathbb{C}[z_1, z_2]$–linear endomorphism of the Baker–Akhieser module $\mathfrak{M}$, we have:

$$f(z_1, z_2) \cdot \big(S \diamond \exp(x_1 z_1 + x_2 z_2)\big) = S \diamond \big(f(z_1, z_2) \cdot \exp(x_1 z_1 + x_2 z_2)\big) =$$

$$S \diamond \big(f(\partial_1, \partial_2) \diamond \exp(x_1 z_1 + x_2 z_2)\big) = \big(S \cdot f(\partial_1, \partial_2)\big) \diamond \exp(x_1 z_1 + x_2 z_2).$$

Summing up, $\Big(\big(\big(\Xi(\vec{\xi})\big)(f)\big) \cdot S - S \cdot f(\partial_1, \partial_2)\Big) \diamond \exp(x_1 z_1 + x_2 z_2) = 0$, implying the result. $\square$

Remark 4.12. Using the identification $z_j = \partial_j$ for $j = 1, 2$, we can view the algebra of quasi–invariants $A \subset R = \mathbb{C}[z_1, z_2]$ as a subalgebra of the algebra of partial differential operators with constant coefficients $\mathbb{C}[\partial_1, \partial_2]$. If S the Sato operator of $\mathfrak{B}$ given by (4.23), then we have: $\mathfrak{B} = S \cdot A \cdot S^{-1}$.

Proposition 4.13. *Consider the vector space*

$$W := \big\langle w_{p_1,p_2} \big| (p_1, p_2) \in \mathbb{N}_0 \times \mathbb{N}_0 \big\rangle \subset R = \mathbb{C}[z_1, z_2], \tag{4.26}$$

where w_{p_1,p_2} are the elements given by (4.18). Then W is an A–module and the map

$$F := \mathbb{C}[\partial_1, \partial_2] \xrightarrow{-\circ S} W, \quad f(\partial_1, \partial_2) \mapsto f(\partial_1, \partial_2) \circ S \tag{4.27}$$

is a $(\mathfrak{B} - A)$–equivariant isomorphism, i.e. the following diagram

$$\begin{tikzcd} F \arrow[r, "-\circ S"] \arrow[d, "-\circ\Xi(\vec{\xi})(f)"'] & W \arrow[d, "-\cdot f"] \\ F \arrow[r, "-\circ S"] & W \end{tikzcd}$$

is commutative for any $f \in A$.

For any $j \in \mathbb{N}_0$ put $W_j := \big\{ w \in W \,\big|\, \deg(w) \le j \big\}$. Then we have the following formula for the Hilbert function H_W of the filtered A–module W:

$$H_W(\mu + k) := \mathsf{dim}_{\mathbb{C}}\big(W_{\mu+k}\big) = \frac{(k+1)(k+2)}{2} \quad \text{for } k \in \mathbb{N}_0. \tag{4.28}$$

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{D}/(x_1, x_2)\mathfrak{D} \arrow[r, "\cong"] \arrow[d, hook] & \mathbb{C}[\partial_1, \partial_2] \arrow[r, "="] \arrow[d, hook] & \mathbb{C}[z_1, z_2] \arrow[d, hook] \\
\mathfrak{E}/(x_1, x_2)\mathfrak{E} \arrow[r, "\cong"] & \mathbb{C}(\!(\partial_1^{-1})\!)(\!(\partial_2^{-1})\!) \arrow[r, "="] & \mathbb{C}(\!(z_1^{-1})\!)(\!(z_2^{-1})\!).
\end{tikzcd}$$

Let $W_0 = \mathbb{C}[z_1, z_2]$. Then $W := W_0 \circ S \subset \mathbb{C}(\!(z_1^{-1})\!)(\!(z_2^{-1})\!)$. Indeed, it follows from the definition of the action $\circ$ of the algebra $\mathfrak{E}$ on the vector space $\mathbb{C}(\!(\partial_1^{-1})\!)(\!(\partial_2^{-1})\!)$ that $\bigl(\partial_1^{p_1}\partial_2^{p_2}\bigr) \circ S = \left.\dfrac{\partial^{p_1+p_2}\Psi}{\partial x_1^{p_1}\partial x_2^{p_2}}\right|_{(x_1,x_2)=(0,0)}$ for any $(p_1, p_2) \in \mathbb{N}_0 \times \mathbb{N}_0$, implying the claim. Since the operator S is a unit in the algebra $\mathfrak{E}$, the $\mathbb{C}$–linear map (4.27) is an isomorphism. Next, by Lemma 4.11, for any $w \in F = W_0$ and $f \in A$ we have:

$$w \circ \Bigl(\bigl(\Xi(\vec{\xi})(f)\bigr) \cdot S\Bigr) = (w \circ S) \cdot f$$

Therefore, $W \cdot f \subseteq W$ and the map $F \xrightarrow{-\circ S} W$ is $(\mathfrak{B} - A)$–equivariant, as claimed.

Next, the highest degree homogeneous term of $w_{p_1,p_2}(z_1, z_2)$ is $z_1^{p_1} z_2^{p_2} \cdot \delta(z_1, z_2)$; see formula (4.19). This shows that

$$W_{\mu+k}/W_{\mu+k-1} \cong \mathbb{C}[z_1, z_2]_k := \bigl\{w \in \mathbb{C}[z_1, z_2] \,\big|\, w \text{ is homogeneous of degree } k\bigr\}.$$

Hence, $\mathsf{dim}_{\mathbb{C}}\bigl(W_{\mu+k}/W_{\mu+k-1}\bigr) = k + 1$, implying that

$$\mathsf{dim}_{\mathbb{C}}\bigl(W_{\mu+k}\bigr) = 1 + 2 + \cdots + (k + 1) = \frac{(k + 1)(k + 2)}{2}.$$

Proposition is proven. $\square$

Theorem 4.14. *Let* $\vec{\xi} = \bigl(\rho_0 \cos(\varphi_0), \rho_0 \sin(\varphi_0)\bigr) \in \mathbb{C}^2 \setminus \{\vec{0}\}$ *be such that* $\sin\bigl(2(\alpha - \varphi_0)\bigr) \neq 0$ *for all* $\alpha \in \Pi$. *Let* W *be the* A*–module defined by (4.26) (i.e.* W *is the spectral module of the algebra* $\mathfrak{B}$*). Then we have:*

(4.29) $W \cong P\bigl(-\xi_1 z_1 - x_2 z_2\bigr) := \bigl\{f \in R \,\big|\, \exp(-\xi_1 z_1 - x_2 z_2) f \text{ is } (\Pi, \underline{\mu})\text{–quasi–invariant}\bigr\}.$

In particular, the spectral module W *is projective; see Theorem 3.17.*

Proof. Let $u = \xi_1 z_1 + x_2 z_2 = \rho\rho_0 \cos(\varphi - \varphi_0)$. According to (4.20), we have an inclusion $W \subseteq P(-u)$. Our goal is to show that in fact $W = P(-u)$. By (4.28), $\mathsf{dim}_{\mathbb{C}}\bigl(W_{\mu+k}\bigr) = \dfrac{(k + 1)(k + 2)}{2}$ for any $k \in \mathbb{Z}$. Hence, it is sufficient to prove that

$$\mathsf{dim}_{\mathbb{C}}\bigl(P(-u)_{\mu+k}\bigr) \leq \frac{(k + 1)(k + 2)}{2} \tag{4.30}$$

for any $k \in \mathbb{Z}$. Let $g(\rho, \varphi) := \exp(-u) = \exp\bigl(-\rho\rho_0 \cos(\varphi - \varphi_0)\bigr)$ and $v = \rho\rho_0 \sin(\varphi - \varphi_0)$. Consider polynomials $t_n \in \mathbb{C}[u, v]$ defined by the rule: $\dfrac{\partial^n g}{\partial \varphi^n} = t_n \cdot g$. We have: $t_0 = 1$ and

$$t_{n+1}(u, v) := -v\frac{\partial t_n}{\partial u}(u, v) + u\frac{\partial t_n}{\partial v}(u, v) + v t_n(u, v) \tag{4.31}$$

for any $n \in \mathbb{N}$. Note that the highest order term of $t_n(u,v)$ is v^n. For any $\alpha \in \Pi$ and $j \in \mathbb{N}_0$ we put: $t_{(\alpha,j)} := t_n(u,v)\Big|_{\varphi=\alpha} \in \mathbb{C}[\rho]$. Then in notation (3.15) we have:

$$\widehat{T}_{(\alpha,2\mu_\alpha)}(-u) = \sum_{j=0}^{2\mu_\alpha-1} \frac{t_{(\alpha,j)}(\rho)}{j!}\varepsilon^j \in \mathbb{C}[\rho,\varepsilon]/(\varepsilon^{2\mu_\alpha}).$$

By definition, a polynomial $f \in R$ belongs to the subspace $P(-u)$ if and only if

$$\left(\sum_{k=0}^{2\mu_\alpha-1} \frac{f_\alpha^{(k)}}{k!}\varepsilon^k\right)\cdot\left(\sum_{j=0}^{2\mu_\alpha-1} \frac{t_{(\alpha,j)}}{j!}\varepsilon^j\right) \in \mathbb{C}[\rho,\varepsilon^2]/(\varepsilon^{2\mu_\alpha}). \tag{4.32}$$

The constraint (4.32) is equivalent to the following system of polynomial identities:

$$\begin{cases} f'_\alpha + t_{(\alpha,1)}f_\alpha & = & 0\\ \dfrac{f'''_\alpha}{3!} + \dfrac{f''_\alpha}{2!}\cdot\dfrac{t_{(\alpha,1)}}{1!} + \dfrac{f'_\alpha}{1!}\cdot\dfrac{t_{(\alpha,2)}}{2!} + f_\alpha\cdot\dfrac{t_{(\alpha,3)}}{3!} & = & 0\\ \vdots & & \\ \displaystyle\sum_{j=0}^{2\mu_\alpha-1} \frac{f_\alpha^{(2\mu_\alpha-1-j)}}{(2\mu_\alpha-1-j)!}\cdot\frac{t_{(\alpha,j)}}{j!} & = & 0. \end{cases} \tag{4.33}$$

Let $d = \deg(f)$ and $f = f_d + f_{d-1} + \dots + f_0$ be a decomposition of f into a sum of its homogeneous components. We prove the following

Claim. Suppose that $f \in R$ satisfies the system (4.33) for $m = \mu_\alpha$. Then l_α^m divides f_d.

Proof of the claim. It is instructive to begin with the special cases, when $m = 1$ or 2.

For $m = 1$, the system (4.33) consists only of one equation: $f'_\alpha + t_{(\alpha,1)}f_\alpha = 0$. We put $\varrho := \rho\rho_0\sin(\alpha-\varphi_0)$ (note, that the conditions on the vector $\vec{\xi}$ insure that $\varrho \ne 0$). Taking the homogeneous part of the top degree of the left–hand side, we obtain: $\varrho\cdot\big(f_d\big)_\alpha = 0$ (recall that the top degree homogeneous part of $t_{(\alpha,j)}(u,v)$ is v^j). Hence, we have the vanishing $\big(f_d\big)_\alpha = 0$, which implies that $l_\alpha \,\big|\, f_d$.

Let $m = 2$. In this case, the system (4.33) consists of two equations:

$$\begin{cases} f'_\alpha + t_{(\alpha,1)}f_\alpha & = & 0\\ \dfrac{f'''_\alpha}{3!} + \dfrac{f''_\alpha}{2!}\cdot\dfrac{t_{(\alpha,1)}}{1!} + \dfrac{f'_\alpha}{1!}\cdot\dfrac{t_{(\alpha,2)}}{2!} + f_\alpha\cdot\dfrac{t_{(\alpha,3)}}{3!} & = & 0. \end{cases} \tag{4.34}$$

We have already seen in the previous step, that the first equation of (4.34) implies that $\big(f_d\big)_\alpha = 0$. Taking the top degree (with respect to ρ) of both equations of (4.34), we get the following system:

$$\begin{cases} \big(f_d\big)'_\alpha + \varrho\big(f_{d-1}\big)_\alpha & = & 0\\ \dfrac{\varrho^2}{2!}\big(f_d\big)'_\alpha + \dfrac{\varrho^3}{3!}\big(f_{d-1}\big)_\alpha & = & 0. \end{cases}$$

Since $\det\begin{pmatrix} 1 & \varrho\\ \dfrac{\varrho^2}{2!} & \dfrac{\varrho^3}{3!}\end{pmatrix} = -\dfrac{1}{3}\varrho^3 \ne 0$, we conclude that $\big(f_d\big)'_\alpha = \big(f_{d-1}\big)_\alpha = 0$. The conditions $\big(f_d\big)_\alpha = \big(f_d\big)'_\alpha = 0$ imply that $l_\alpha^2 \,\big|\, f_d$; see Lemma 2.11.

Now we proceed to the general case. We prove by induction on m that

$$
\left\{
\begin{array}{ccccccccccc}
\bigl(f_d\bigr)_\alpha & = & \bigl(f_d\bigr)'_\alpha & = & \dots & = & \bigl(f_d\bigr)^{(m-2)}_\alpha & = & \bigl(f_d\bigr)^{(m-1)}_\alpha & = & 0 \\
\bigl(f_{d-1}\bigr)_\alpha & = & \bigl(f_{d-1}\bigr)'_\alpha & = & \dots & = & \bigl(f_{d-1}\bigr)^{(m-2)}_\alpha & = & 0 & & \\
\vdots & & & & & & & & & & \\
\bigl(f_{d-m+1}\bigr)_\alpha & = & 0. & & & & & & & &
\end{array}
\right.
\tag{4.35}
$$

Consider the following infinite matrix

$$
\left(
\begin{array}{ccccccccc}
\binom{1}{1} & \binom{1}{1}\lambda & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
\binom{3}{0} & \binom{3}{1}\lambda & \binom{3}{2}\lambda^2 & \binom{3}{3}\lambda^3 & 0 & 0 & 0 & 0 & \dots \\
\binom{5}{0} & \binom{5}{1}\lambda & \binom{5}{2}\lambda^2 & \binom{5}{3}\lambda^3 & \binom{5}{4}\lambda^4 & \binom{5}{5}\lambda^5 & 0 & 0 & \dots \\
\binom{7}{0} & \binom{7}{1}\lambda & \binom{7}{2}\lambda^2 & \binom{7}{3}\lambda^3 & \binom{7}{4}\lambda^4 & \binom{7}{5}\lambda^5 & \binom{7}{6}\lambda^5 & \binom{7}{7}\lambda^7 & \dots \\
\binom{9}{0} & \binom{9}{1}\lambda & \binom{9}{2}\lambda^2 & \binom{9}{3}\lambda^3 & \binom{9}{4}\lambda^4 & \binom{9}{5}\lambda^5 & \binom{9}{6}\lambda^5 & \binom{9}{7}\lambda^7 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\right).
\tag{4.36}
$$

To prove the induction step, it is sufficient to show that the first principal $(m \times m)$–minor of the matrix (4.36) in non–zero. Suppose it is not the case. Then the elements

$$
\overline{(1+\lambda)}, \overline{(1+\lambda)^3}, \dots, \overline{(1+\lambda)^{2m-1}} \in \mathbb{C}[\lambda]/(\lambda^m)
$$

are linearly dependent. Hence, there exist $c_0, c_1, \dots, c_{m-1} \in \mathbb{C}$ such that λ^m divides the polynomial $c_0 + c_1(1+\lambda)^2 + \dots + c_{m-1}(1+\lambda)^{2\cdot(m-1)}$. Let $1 \le \bar{m} \le m$ be such that $c_{\bar{m}-1} \ne 0$, whereas $c_{\bar{m}} = c_{\bar{m}+1} = \dots = 0$. Let $\zeta_1, \dots, \zeta_{\bar{m}-1} \in \mathbb{C}$ be such that

$$
\pi(t) := c_0 + c_1 t + \dots + c_{\bar{m}-1} t^{\bar{m}-1} = c_{\bar{m}-1}(t-\zeta_1)\cdot \dots \cdot (t - \zeta_{\bar{m}-1}) \in \mathbb{C}[t].
$$

Taking the substitution $t = (1+\lambda)^2$, we get:

$$
\lambda^m \;\Big|\; c_{\bar{m}-1} \prod_{j=1}^{\bar{m}-1} \bigl((1+\lambda)^2 - \zeta_j\bigr).
\tag{4.37}
$$

However, the order of vanishing at 0 of the polynomial in the right hand side of (4.37) is at most $\bar{m}-1$, contradiction. Hence, $c_0 = c_1 = \dots = c_{m-1} = 0$.

Therefore, for any $f \in P(-u)$ we have: $\bigl(f_d\bigr)_\alpha = \bigl(f_d\bigr)'_\alpha = \dots = \bigl(f_d\bigr)^{(m-1)}_\alpha = 0$. By Lemma 2.11, the polynomial l_α^m divides f_d, proving the claim.

Summing up, we have shown that the polynomial $\delta = \prod\limits_{\alpha \in \Pi} l_\alpha^{\mu_\alpha}$ divides the top homogeneous part of any element of the module $P(-u)$. Since $\deg(\delta) = \mu$, it implies that

$$
\mathsf{dim}_{\mathbb{C}}\bigl(P(-u)_\mu\bigr) \le 1 \quad \text{and} \quad \mathsf{dim}_{\mathbb{C}}\bigl(P(-u)_{\mu+k}/P(-u)_{\mu+k-1}\bigr) \le k+1
$$

for any $k \in \mathbb{N}$. Therefore,

$$
\mathsf{dim}_{\mathbb{C}}\bigl(P(-u)_{\mu+k}\bigr) \le 1 + 2 + \dots + (k+1) = \frac{(k+1)(k+2)}{2}.
$$

Theorem is proven. $\square$

Corollary 4.15. *Since the normalization map $\mathbb{A}^2 \xrightarrow{\nu} X$ is bijective, any character $A \xrightarrow{\chi} \mathbb{C}$ is given as the composition $A \hookrightarrow R \xrightarrow{\widetilde{\chi}} \mathbb{C}$, where $\widetilde{\chi}(P) = P(\zeta_1, \zeta_2)$ for any $P \in \mathbb{C}[z_1, z_2]$ and some uniquely determined $(\zeta_1, \zeta_2) \in \mathbb{A}^2$. For any $\vec{\xi} \in \mathbb{C}^2$ satisfying the conditions of Theorem 4.14, the power series $\Psi(x_1, x_2; \zeta_1, \zeta_2; \xi_1, \xi_2)$, given by formula*

(4.16), is non–zero and regular at $(x_1, x_2) = (0, 0)$*. Combining Theorem 4.5 with Theorem 3.7, we end up with the following result:*

$$\mathsf{Sol}(\mathfrak{B}, \chi) = \Big\{ f \in \mathbb{C}[\![x_1, x_2]\!] \Big| P \diamond f = P(\zeta_1, \zeta_2) f \text{ for all } P \in \mathfrak{B} \Big\} = \big\langle \Psi(x_1, x_2; \zeta_1, \zeta_2; \xi_1, \xi_2) \big\rangle_{\mathbb{C}}.$$

Remark 4.16. The A–module module $P(-u)$ from Theorem 4.14 appeared also in [2, Proposition 7.6], where another proof of its projectivity was given.

5. Elements of the higher–dimensional Sato theory

For any $n \in \mathbb{N}$, let $B = \mathbb{C}[x_1, \dots, x_n]$ and $\widehat{B} = \mathbb{C}[\![x_1, \dots, x_n]\!]$. For any $d \in \mathbb{Z}$, we denote by B_d be the vector space of homogeneous elements of B of degree d (in particular, $B_d = 0$ for $d \le 0$). Next, let $\mathfrak{m} = (x_1, \dots, x_n)$ be the unique maximal ideal of $\widehat{B}$ and $\widehat{B} \xrightarrow{\upsilon} \mathbb{N}_0 \cup \{\infty\}$ be the corresponding valuation. To simplify the notation, we denote $\Sigma := \mathbb{N}_0^n$ and for any $\underline{k} = (k_1, \dots, k_n) \in \Sigma$ we write:

- $\underline{x}^{\underline{k}} := x_1^{k_1} \dots x_n^{k_n}$ and $\underline{\partial}^{\underline{k}} := \partial_1^{k_1} \dots \partial_n^{k_n}$.
- $\underline{k}! = k_1! \dots k_n!$ and $|\underline{k}| = k_1 + \dots + k_n$.

We denote $\underline{0} := (0, \dots, 0) \in \Sigma$ and for $\underline{k}, \underline{l} \in \Sigma$ say that $\underline{k} \ge \underline{l}$ if any only if $k_i \ge l_i$ for all $1 \le i \le n$. Next, consider the following $\mathbb{C}$–vector space:

$$\mathfrak{W} := \mathbb{C}[\![x_1, \dots, x_n]\!][\![\partial_1, \dots, \partial_n]\!] = \left\{ \sum_{\underline{k} \ge \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}} \;\Big|\; a_{\underline{k}} \in \widehat{B} \text{ for all } \underline{k} \in \Sigma \right\}. \tag{5.1}$$

Note that $\mathfrak{W}$ has no natural $\mathbb{C}$–algebra structure since the natural product $\cdot$ is not defined on the whole vector space $\mathfrak{W}$.

Definition 5.1. For any element $P := \sum\limits_{\underline{k} \ge \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}} \in \mathfrak{W}$ we define its *order* to be

$$o(P) := \sup\big\{ |\underline{k}| - \upsilon(a_{\underline{k}}) \;\big|\; \underline{k} \in \Sigma \big\} \in \mathbb{Z} \cup \{\infty\}. \tag{5.2}$$

In particular, if $d = o(P) < \infty$ then we have:

$$\upsilon(a_{\underline{k}}) \ge |\underline{k}| - d = (k_1 + \dots + k_n) - d \quad \text{for any } \underline{k} \in \Sigma.$$

The key role in this section is plaid by the following subspace of the vector space $\mathfrak{W}$:

$$\mathfrak{S} := \big\{ Q \in \mathfrak{W} \,\big|\, o(Q) < \infty \big\}. \tag{5.3}$$

Note that for a partial differential operator

$$P = \sum_{|\underline{k}| = m} a_{\underline{k}} \underline{\partial}^{\underline{k}} + \sum_{|\underline{i}| < m} b_{\underline{i}} \underline{\partial}^{\underline{i}} \in \mathbb{C}[\![x_1, \dots, x_n]\!][\partial_1, \dots, \partial_n],$$

with constant highest symbol $0 \ne \sigma(P) = \sum_{|\underline{k}| = m} \alpha_{\underline{k}} \underline{\partial}^{\underline{k}} \in \mathbb{C}[\partial_1, \dots, \partial_n]$, the order of P taken in the sense (5.2) is equal to m and coincides with the usual definition of the order of a differential operator.

Let $P \in \mathfrak{S}$. Then for any $\underline{k}, \underline{i} \in \Sigma$, we have a uniquely determined $\alpha_{\underline{k}, \underline{i}} \in \mathbb{C}$ such that

$$P = \sum_{\underline{k}, \underline{i} \ge \underline{0}} \alpha_{\underline{k}, \underline{i}} \, \underline{x}^{\underline{i}} \underline{\partial}^{\underline{k}}. \tag{5.4}$$

For any $m \ge -d$ we put:

$$P_m := \sum_{\substack{\underline{k}, \underline{i} \in \Sigma \\ |\underline{i}| - |\underline{k}| = m}} \alpha_{\underline{k}, \underline{i}} \, \underline{x}^{\underline{i}} \underline{\partial}^{\underline{k}}$$

to be the m-th *homogeneous component* of P. Note that $o(P_m) = -m$ and we have a decomposition $P = \sum\limits_{m=-d}^{\infty} P_m$. Finally, $\sigma(P) := P_{-d}$ is the *symbol* of P (i.e. the sum of all components of P of maximal order). We say that $P \in \mathfrak{S}$ is *homogeneous* if $P = \sigma(P)$.

Example 5.2. Let $n = 1$. Then we have:

- The operator $\exp(x{*}\partial) := \sum\limits_{k=0}^{\infty} \dfrac{x^k}{k!}\partial^k$ belongs to $\mathfrak{S}$. Moreover, $\exp(x{*}\partial)$ is homogeneous of order zero.
- The element $\sum\limits_{k=0}^{\infty} \dfrac{x^k}{k!}\partial^{2k}$ of $\mathfrak{W}$ does not belong to $\mathfrak{S}$.

Theorem 5.3. *The following results are true.*

(1) *The vector space $\mathfrak{S}$ is a $\mathbb{C}$–algebra with respect to the natural operations $\cdot$ and $+$. In particular, $\mathfrak{S}$ contains the subalgebra $\mathfrak{D} := \mathbb{C}[\![x_1, \dots, x_n]\!][\partial_1, \dots, \partial_n]$ of partial differential operators.*

(2) *We have a natural isomorphism of $\mathbb{C}$–vector spaces $F := \mathfrak{S}/\mathfrak{m}\mathfrak{S} \longrightarrow \mathbb{C}[\partial_1, \dots, \partial_n]$.*

(3) *We have a natural injective algebra homomorphism $\mathfrak{S} \longrightarrow \mathsf{End}^{\mathsf{c}}_{\mathbb{C}}(\widehat{B})$, where $\mathsf{End}^{\mathsf{c}}_{\mathbb{C}}(\widehat{B})$ is the algebra of $\mathbb{C}$–linear endomorphisms of $\widehat{B}$, which are* continuous *in the $\mathfrak{m}$–adic topology. In particular, $\widehat{B}$ has a natural structure of a left $\mathfrak{S}$–module, which extends its natural structure of a left $\mathfrak{D}$–module.*

Proof. (1) The main point is to show that the natural product $\cdot$ is well–defined for any pair of elements $P, Q \in \mathfrak{S}$. Let $d = o(P)$ and $e = o(Q)$. Assume first that P and Q are homogeneous. Then we have presentations $P = \sum\limits_{\underline{k} \in \Sigma} a_{\underline{k}} \partial^{\underline{k}}$ and $Q = \sum\limits_{\underline{l} \in \Sigma} b_{\underline{l}} \partial^{\underline{l}}$, where $a_{\underline{k}} \in B_{|\underline{k}|-d}$ and $b_{\underline{l}} \in B_{|\underline{l}|-e}$ for any $\underline{k}, \underline{l} \in \Sigma$.
Having the Leibniz formula in mind, we *define*:

$$P \cdot Q := \sum_{\underline{k} \in \Sigma} \sum_{\underline{l} \in \Sigma} \sum_{\underline{0} \le \underline{i} \le \underline{k}} \binom{k_1}{i_1} \dots \binom{k_n}{i_n} a_{\underline{k}} \frac{\partial^{|\underline{i}|}\, b_{\underline{l}}}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} \partial^{\underline{k}+\underline{l}-\underline{i}}. \tag{5.5}$$

Since for any $\underline{j} \in \Sigma$, there exist only finitely many $\underline{k}, \underline{l}, \underline{i} \in \Sigma$ such that

$$\underline{j} = \underline{k} + \underline{l} - \underline{i} \quad \text{and} \quad \underline{k} \ge \underline{i},$$

the right–hand side of (5.5) is a well–defined *homogeneous* element of $\mathfrak{S}$. Moreover, $o(P \cdot Q) = o(P) + o(Q)$ provided $P \cdot Q \ne 0$.

Now, let $P, Q \in \mathfrak{S}$ be arbitrary elements and $P = \sum\limits_{m=-d}^{\infty} P_m$ respectively $Q = \sum\limits_{l=-e}^{\infty} Q_l$ be the corresponding homogeneous decompositions. Then we put:

$$P \cdot Q := \sum_{p=-(d+e)}^{\infty} \left(\sum_{\substack{m+l=p \\ m \ge -d \\ l \ge -e}} P_m \cdot Q_l \right). \tag{5.6}$$

It is a tedious but straightforward computation to verify that $\mathfrak{S}$ is indeed a $\mathbb{C}$–algebra with respect to the introduced operations $\cdot$ and $+$. Note that $\sigma(P \cdot Q) = \sigma(P) \cdot \sigma(Q)$, provided $\sigma(P) \cdot \sigma(Q) \ne 0$.

(2) Note that we have a well–defined injective $\mathbb{C}$–linear map

$$\mathfrak{S}/\mathfrak{m}\mathfrak{S} \longrightarrow \mathbb{C}[\![\partial_1, \dots, \partial_n]\!], \quad P = \sum_{\underline{k} \geq \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}} \mapsto P\Big|_{\underline{0}} := \sum_{\underline{k} \geq \underline{0}} a_{\underline{k}}(0) \underline{\partial}^{\underline{k}},$$

whose image contains the subspace $\mathbb{C}[\partial_1, \dots, \partial_n]$. Let $d = o(P)$, then by the definition we have: $\upsilon(a_{\underline{k}}) \geq |\underline{k}| - d$ for any $\underline{k} \in \Sigma$. In particular, $a_{\underline{k}}(0) = 0$ for any $\underline{k} \in \Sigma$ such that $|\underline{k}| \geq d+1$, hence $P\Big|_{\underline{0}} \in \mathbb{C}[\partial_1, \dots, \partial_n]$ as claimed.

(3) In order to define the natural left action of the $\mathbb{C}$–algebra $\mathfrak{S}$ on $\widehat{B}$, take first $P \in \mathfrak{S}$ homogeneous of order $d \in \mathbb{Z}$ and $f \in B_e$ for some $e \in \mathbb{N}_0$. Then we have an expansion $P = \sum\limits_{\underline{k} \geq \underline{0}} a_{\underline{k}} \underline{\partial}^{\underline{k}}$ with $a_{\underline{k}} \in B_{|\underline{k}|-d}$ for any $\underline{k} \in \Sigma$. Since $\underline{\partial}^{\underline{k}} \circ f = 0$ for any $\underline{k} \in \Sigma$ such that $|\underline{k}| \geq e+1$, we have a well–defined element $P \circ f \in B_{e-d}$.

Now, let $P \in \mathfrak{S}$ and $f \in \widehat{B}$ be arbitrary elements and $d = o(P)$. Since we have homogeneous decompositions $P = \sum\limits_{m=-d}^{\infty} P_m$ and $f = \sum\limits_{e=0}^{\infty} f_e$, we can define:

$$P \circ f := \sum_{k=0}^{\infty} \Bigl(\sum_{\substack{m \geq -d,\, e \geq 0 \\ m+e=k}} P_m \circ f_e \Bigr).$$

It follows from the definition that $P \circ f \in \mathfrak{m}^k$ provided $f \in \mathfrak{m}^{k+d}$. This shows that the action of $\mathfrak{S}$ on $\widehat{B}$ is indeed continuous in the $\mathfrak{m}$–adic topology.

It remains to prove that the algebra homomorphism $\mathfrak{S} \longrightarrow \mathsf{End}^{\mathrm{c}}_{\mathbb{C}}(\widehat{B})$ is injective. For this, it is sufficient to show that for any homogeneous operator

$$P = \sum_{\substack{\underline{k}, \underline{i} \geq \underline{0} \\ |\underline{k}| - |\underline{i}| = d}} \alpha_{\underline{k}, \underline{i}}\, \underline{x}^{\underline{i}} \underline{\partial}^{\underline{k}} \in \mathfrak{S}$$

of order d, there exists $f \in \widehat{R}$ such that $P \circ f \neq 0$. Let $\underline{l}$ be an element of the set

$$\bigl\{ \underline{k} \in \Sigma \bigm| \text{there exists } \underline{i} \in \Sigma \text{ such that } \alpha_{\underline{k}, \underline{i}} \neq 0 \bigr\}$$

with $|\underline{l}|$ smallest possible. Then $P \circ \underline{x}^{\underline{l}} = \underline{l}!\, \alpha_{\underline{l}, \underline{i}}\, \underline{x}^{\underline{i}} \neq 0$, implying the statement. □

One evidence that the algebra $\mathfrak{S}$ deserves a further study is due to the fact that it contains several operators, which do not belong to the subalgebra $\mathfrak{D}$ but act "naturally" on $\widehat{B}$.

Example 5.4. Let $n = 1$. For any $u \in x\mathbb{C}[\![x]\!]$, consider the following operator:

$$\exp(u * \partial) := \sum_{k=0}^{\infty} \frac{u^k}{k!} \partial^k. \tag{5.7}$$

Obviously, $\exp(u*\partial)$ is an operator of the algebra $\mathfrak{S}$ of non–positive order. Moreover, for any $f \in \widehat{B}$ we have:

$$\exp(u*\partial) \circ f(x) = f\bigl(u + x\bigr), \tag{5.8}$$

i.e. the operator $\exp(u*\partial)$ can realize an arbitrary $\mathbb{C}$–linear endomorphism of $\mathbb{C}[\![x]\!]$. Indeed,

$$\exp(u*\partial) \circ x^m = \left(\sum_{k=0}^{m} \frac{u^k}{k!} \partial^k \right) \circ x^m = \sum_{k=0}^{m} \binom{m}{k} u^k x^{m-k} = \bigl(u + x\bigr)^m,$$

implying the statement.

In particular, let $E := \exp((-x)*\partial)$. Then $E \circ f(x) = f(0)$, i.e. the operator E is Dirac's delta–function.

Now, let $n \in \mathbb{N}$ be arbitrary. For any $1 \le i \le n$ and $u \in \mathfrak{m}$, consider the operator

$$\exp(u*\partial_i) := \sum_{k=0}^{\infty} \frac{u^k}{k!}\partial_i^k. \tag{5.9}$$

Then for any $f \in \widehat{B}$, we have the formula:

$$\exp(u*\partial_i) \circ f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = f\bigl(x_1, \dots, x_{i-1}, x_i + u, x_{i+1}, \dots, x_n\bigr).$$

As a special case, for $E_i := \exp((-x_i)*\partial_i)$ we have the following formula:

$$E_i \circ f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = f\bigl(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n\bigr). \tag{5.10}$$

Finally, note that the formula (5.10) implies that $\partial_i \cdot E_i = 0$ in $\mathfrak{S}$. As a consequence, $\bigl(E_i \cdot \partial_i\bigr)^2 = 0$ for any $1 \le i \le n$.

Example 5.5. Again, let us first assume that $n = 1$. Consider the operator

$$G := \bigl(1 - \exp((-x)*\partial)\bigr) \cdot \partial^{-1} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!}(-\partial)^k. \tag{5.11}$$

Then for any $m \in \mathbb{N}_0$ we have:

$$G \circ x^m = \sum_{k=0}^{m} (-1)^k \frac{x^{k+1}}{(k+1)!}\bigl(\partial^k \circ x^m\bigr) = \frac{x^{m+1}}{m+1} \sum_{k=0}^{m} (-1)^{k+1} \binom{m+1}{k+1} = \frac{x^{m+1}}{m+1}.$$

Hence,

$$G \circ \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} \frac{a_m}{m+1} x^{m+1}, \tag{5.12}$$

i.e. G acts on $\widehat{B}$ as the integration operator. In particular, we have: $\partial \cdot G = 1$ in $\mathfrak{S}$.

Similarly, for $n \in \mathbb{N}$ and any $1 \le i \le n$, the operator $G_i := \bigl(1 - \exp((-x_i)*\partial_i)\bigr) \cdot \partial_i^{-1}$ is the operator of indefinite integration in the i-th variable.

In what follows, we shall study more precisely the right action of the algebra $\mathfrak{S}$ on the $\mathbb{C}$–vector space $F = \mathfrak{S}/\mathfrak{m}\mathfrak{S} = \mathbb{C}[\partial_1, \dots, \partial_n]$.

Definition 5.6. Let $P \in \mathfrak{S}$ be an operator of order d given by the expansion (5.4). Then we have another form of the formal power series expansion of P called *slice decomposition*:

$$P = \sum_{\underline{i} \ge 0} \frac{x^{\underline{i}}}{\underline{i}!} P_{(\underline{i})}, \quad \text{where} \quad P_{(\underline{i})} = \underline{i}! \sum_{\substack{\underline{k} \ge 0 \\ |\underline{k}| - |\underline{i}| \le d}} \alpha_{\underline{k}, \underline{i}}\, \partial^{\underline{k}}. \tag{5.13}$$

For any $\underline{i} \in \Sigma$, the partial differential operator with constant coefficients $P_{\underline{i}} \in \mathbb{C}[\partial_1, \dots, \partial_n]$ is called $\underline{i}$-th slice of P.

Remark 5.7. Note that for any $\underline{i} \in \Sigma$ we have the following identity $\partial^{\underline{i}} \circ P = P_{\underline{i}}$, where $P_{\underline{i}}$ is viewed as an element of the module F. In particular, for any $P, Q \in \mathfrak{S}$, the following statement is true:

$$P = Q \quad \text{if and only if} \quad \partial^{\underline{i}} \circ P = \partial^{\underline{i}} \circ Q \quad \text{for any} \quad \underline{i} \in \Sigma.$$

In other words, the algebra homomorphism $\mathfrak{S} \longrightarrow \mathsf{End}_{\mathbb{C}}(F)$ is injective.

Definition 5.8. An element $P \in \mathfrak{S}$ is called *regular* if the $\mathbb{C}$–linear map $F \xrightarrow{-\circ\sigma(P)} F$ is injective. In particular, P is regular if and only if its symbol $\sigma(P)$ is regular.

Lemma 5.9. *Let $P \in \mathfrak{S}$. Then the following results are true.*

(1) *The operator P is regular if and only if for any $m \in \mathbb{N}_0$, the elements of the set $\bigl\{\underline{\partial}^{\underline{k}} \circ \sigma(P) \,\big|\, \underline{k} \in \Sigma : |\underline{k}| = m\bigr\} \subset F$ are linearly independent.*
(2) *Assume that P is regular. Then P is not a right zero divisor in $\mathfrak{S}$, i.e. the equation $Q \cdot P = 0$ in $\mathfrak{S}$ implies that $Q = 0$.*

Proof. (1) Let $d := o(P) = o\bigl(\sigma(P)\bigr)$. Then for any $\underline{k} \in \Sigma$ we have: $o\bigl(\underline{\partial}^{\underline{k}} \circ \sigma(P)\bigr) = |\underline{k}| + d$ provided $\underline{\partial}^{\underline{k}} \circ \sigma(P) \neq 0$. Therefore, the linear map $F \xrightarrow{-\circ\sigma(P)} F$ splits into a direct sum of its graded components $F_m \xrightarrow{-\circ\sigma(P)} F_{m+d}$, implying the first statement.
(2) Let $Q \neq 0$ be such that $Q \cdot P = 0$. Then $\sigma(Q) \neq 0$ as well, whereas $\sigma(Q) \cdot \sigma(P) = 0$. Next, there exists $\underline{k} \in \Sigma$ such that $\overline{\underline{\partial}^{\underline{k}} \cdot \sigma(Q)} \neq 0$ in F. On the other hand,

$$\overline{\underline{\partial}^{\underline{k}} \cdot \sigma(Q)} \circ \sigma(P) = \overline{\underline{\partial}^{\underline{k}} \cdot \sigma(Q) \cdot \sigma(P)} = 0,$$

hence P is not regular, contradiction. □

Definition 5.10. Let $\mathfrak{S}_- := \bigl\{P \in \mathfrak{S} \,\big|\, o(P) \leq 0\bigr\}$ and $\mathfrak{S}_-^*$ be the group of units of $\mathfrak{S}_-$.

Lemma 5.11. *The following results are true.*

(1) *Let $P \in \mathfrak{S}_-$. Then we have: $P \in \mathfrak{S}_-^*$ if and only if $\sigma(P) \in \mathfrak{S}_-^*$.*
(2) *Let $Q \in \mathfrak{S}_-^*$. Then $o(Q) = 0$ and $\overline{\sigma(Q)} = Q_{(\underline{0})} \in \mathbb{C}^*$.*

Remark 5.12. It is not true that any unit in the algebra $\mathfrak{S}$ belongs to its subalgebra $\mathfrak{S}_-$. Indeed, let $P = \exp\bigl((-x)*\partial\bigr) \cdot \partial \in \mathfrak{S}$. Then $P^2 = 0$, hence $1 + P$ is a unit in $\mathfrak{S}$, which is not an element of $\mathfrak{S}_-$.

Definition 5.13. Let $W \subseteq F = \mathbb{C}[\partial_1, \dots, \partial_n]$ be a $\mathbb{C}$–linear subspace.

(1) For any $k \in \mathbb{Z}$, we put: $W_k := \bigl\{w \in W \,\big|\, o(w) \leq k\bigr\}$.
(2) $H_W(k) := \mathsf{dim}_{\mathbb{C}}\bigl(W_k\bigr)$ is the *Hilbert function* of W.

Definition 5.14. Let $\mu \in \mathbb{N}_0$.

(1) We put: $\mathsf{Gr}_\mu(F) := \left\{W \subseteq F \,\middle|\, H_W(\mu + k) = \binom{n+k}{n} \quad \text{for any} \quad k \in \mathbb{N}_0\right\}$ (recall that $H_F(k) = \binom{n+k}{n}$).
(2) Let $W \in \mathsf{Gr}_\mu(F)$. Then $S \in \mathfrak{S}$ is a *Sato operator* of W if the following conditions are fulfilled:
   - S is regular and $o(S) = \mu$.
   - We have: $W = F \circ S$.

Proposition 5.15. *Let $\mu \in \mathbb{N}_0$, $W \in \mathsf{Gr}_\mu(F)$ and T be a Sato operator of W. Then the following results are true.*

(1) *$U \cdot T$ is a Sato operator of W for any $U \in \mathfrak{S}_-^*$.*
(2) *For any $m \in \mathbb{N}_0$, the elements $\bigl\{T_{(\underline{k})} \,\big|\, \underline{k} \in \Sigma \text{ such that } |\underline{k}| \leq m\bigr\}$ form a basis of the vector space $W_{\mu+m}$, where $T_{(\underline{k})} \in F$ is the $\underline{k}$-th slice of the operator T.*
(3) *Moreover, for any $m \in \mathbb{N}_0$, the elements $\bigl\{\overline{T}_{(\underline{k})} \,\big|\, \underline{k} \in \Sigma \text{ such that } |\underline{k}| = m\bigr\}$ form a basis of the vector space $W_{\mu+m}/W_{\mu+m-1}$, where $\overline{T}_{(\underline{k})}$ denotes the class of $T_{(\underline{k})}$.*
(4) *The linear map $F \xrightarrow{-\circ T} W$ is a bijection.*

Proof. (1) If $U \in \mathfrak{S}_-^*$ then $U \cdot T \neq 0$ and $o(U \cdot T) = o(U) + o(T) = \mu$. Next, $F \circ (U \cdot T) = (F \circ U) \circ T = F \circ T = W$. Finally, $\sigma(U \cdot T) = \sigma(U) \cdot \sigma(T)$ and $\sigma(U) \in \mathbb{C}^*$. Hence, the linear map $F \xrightarrow{-\circ\sigma(U\cdot T)} F$ is injective, implying that the operator $U \cdot T$ is regular. Hence, $U \cdot T$ is indeed a Sato operator of W.

(2) Recall that we have a slice expansion: $T = \sum\limits_{\underline{k} \geq \underline{0}} \dfrac{\underline{x}^{\underline{k}}}{\underline{k}!} T_{(\underline{k})}$. Since $o(T) = \mu$, we have: $o\bigl(T_{(\underline{k})}\bigr) \leq \mu + |\underline{k}|$, i.e. $T_{(\underline{k})} \in W_{|\underline{k}|+\mu}$ for any $\underline{k} \in \Sigma$. Moreover, there exists $\underline{l} \in \Sigma$ such that $o\bigl(T_{(\underline{l})}\bigr) = \mu + |\underline{l}|$.

Next, for any $\underline{k} \in \Sigma$ we put: $\bar{\sigma}\bigl(T_{(\underline{k})}\bigr) = \begin{cases} \sigma\bigl(T_{(\underline{k})}\bigr) & \text{if} \quad o\bigl(T_{(\underline{k})}\bigr) = \mu + |\underline{k}| \\ 0 & \text{if} \quad o\bigl(T_{(\underline{k})}\bigr) < \mu + |\underline{k}|. \end{cases}$ Then we have the formula: $\sigma(T) = \sum\limits_{\underline{k} \geq \underline{0}} \dfrac{\underline{x}^{\underline{k}}}{\underline{k}!} \bar{\sigma}\bigl(T_{(\underline{k})}\bigr)$, which implies that $\underline{\partial}^{\underline{k}} \circ \sigma(T) = \bar{\sigma}\bigl(T_{(\underline{k})}\bigr)$ for all $\underline{k} \in \Sigma$. Since T is regular, $\bar{\sigma}\bigl(T_{(\underline{k})}\bigr) \neq 0$ and $o\bigl(T_{(\underline{k})}\bigr) = \mu + |\underline{k}|$ for all $\underline{k} \in \Sigma$, hence

$$\sigma(T) = \sum_{\underline{k} \geq \underline{0}} \frac{\underline{x}^{\underline{k}}}{\underline{k}!} \sigma\bigl(T_{(\underline{k})}\bigr). \tag{5.14}$$

By assumption, $\mathsf{dim}_{\mathbb{C}}(W_{m+\mu}) = \binom{n+m}{n}$. Therefore, to prove the second statement, it is sufficient to show that for any $m \in \mathbb{N}_0$, the elements $\bigl\{T_{(\underline{k})} \bigm| \underline{k} \in \Sigma \text{ such that } |\underline{k}| \leq m\bigr\}$ are linearly independent. We prove it by induction on m. The case $m = 0$ is clear: since $\mathsf{dim}_{\mathbb{C}}(W_\mu) = 1$ and $o\bigl(T_{(\underline{k})}\bigr) = |\underline{k}| + \mu$, we have $W_\mu = \bigl\langle T_{(\underline{0})}\bigr\rangle_{\mathbb{C}}$. Next, let $\bigl\{\beta_{\underline{k}} \in \mathbb{C} \bigm| \underline{k} \in \Sigma : |\underline{k}| \leq m\bigr\}$ be such that $\sum\limits_{|\underline{k}| \leq m} \beta_{\underline{k}} T_{(\underline{k})} = 0$. Then we have: $\sum\limits_{|\underline{k}| = m} \beta_{\underline{k}}\, \sigma\bigl(T_{(\underline{k})}\bigr) = 0$ in the vector space $F_{m+\mu}$. From formula (5.14) and Lemma 5.9 we deduce that $\beta_{\underline{k}} = 0$ for any $\underline{k} \in \Sigma$ such that $|\underline{k}| = m$. Proceeding by induction, we get the second claim.

(3) Analogously, assume that $\sum\limits_{|\underline{k}| = m} \gamma_{\underline{k}}\, \overline{T}_{(\underline{k})} = 0$ in the quotient vector space $W_{m+\mu}/W_{m+\mu-1}$. Then we get: $\sum\limits_{|\underline{k}| = m} \gamma_{\underline{k}}\, \sigma\bigl(T_{(\underline{k})}\bigr) = 0$ in $F_{m+\mu}$, hence $\beta_{\underline{k}} = 0$ for any $\underline{k} \in \Sigma$ such that $|\underline{k}| = m$. The third claim follows from the fact that $\mathsf{dim}_{\mathbb{C}}\bigl(W_{m+\mu}/W_{m+\mu-1}\bigr) = \binom{n+m-1}{m}$.

(4) For any $m \in \mathbb{N}_0$, the linear map $F_m \xrightarrow{-\circ T} W_{m+\mu}$, $\underline{\partial}^{\underline{k}} \mapsto T_{(\underline{k})}$ is an isomorphism by the dimension reasons. This implies the fourth statement. □

Theorem 5.16. *Let $\mu \in \mathbb{N}_0$ and $W \in \mathsf{Gr}_\mu(F)$. Then the following statements are true.*

1. *The vector space W possesses a Sato operator S.*
2. *If T is another Sato operator for W then there exists a uniquely determined $U \in \mathfrak{S}_-^*$ such that $S = U \cdot T$.*

In other words, a Sato operator of W exists and is unique up to a unit of the algebra $\mathfrak{S}_-$.

Proof. (1) Our construction of a Sato operator S is algorithmic and depends on the following choice. Namely, for any $\underline{k} \in \Sigma$, we choose $w_{\underline{k}} \in W_{\mu+|\underline{k}|}$ such that for any $m \in \mathbb{N}_0$, the set $\bigl\{\bar{w}_{\underline{k}} \bigm| \underline{k} \in \Sigma : |\underline{k}| = m\bigr\}$ forms a basis of the vector space $W_{m+\mu}/W_{m+\mu-1}$ (at this place, we essentially use the assumption on the Hilbert function of W). Then the following statements are true:

- $o\bigl(w_{\underline{k}}\bigr) = \mu + |\underline{k}|$ for any $\underline{k} \in \Sigma$;
- the set $\bigl\{w_{\underline{k}} \bigm| \underline{k} \in \Sigma : |\underline{k}| \leq m\bigr\}$ is a basis of the vector space $W_{m+\mu}$.

Consider the operator $S := \sum\limits_{\underline{k}\geq 0} \dfrac{x^{\underline{k}}}{\underline{k}!} S_{(\underline{k})} \in \mathfrak{S}$ such that $S_{(\underline{k})} = w_{\underline{k}}$ for all $\underline{k} \in \Sigma$. By construction of S we have:

- $\sigma(S) = \sum\limits_{\underline{k}\geq 0} \dfrac{x^{\underline{k}}}{\underline{k}!} \sigma\big(S_{(\underline{k})}\big)$, hence $o(S) = \mu$.
- $\underline{\partial}^{\underline{k}} \circ S = w_{\underline{k}}$ for all $\underline{k} \in \Sigma$, hence $W = F \circ S$.
- $\underline{\partial}^{\underline{k}} \circ \sigma(S) = \sigma\big(w_{\underline{k}}\big)$ for all $\underline{k} \in \Sigma$. Since $\big\{\bar{w}_{\underline{k}} \,\big|\, \underline{k} \in \Sigma : |\underline{k}| = m\big\}$ form a basis of the vector space $W_{m+\mu}/W_{m+\mu-1}$, the vectors from the set $\big\{\sigma\big(S_{\underline{k}}\big) \,\big|\, \underline{k} \in \Sigma : |\underline{k}| = m\big\}$ are linearly independent. According to Lemma 5.9, the operator S is regular.

Summing up, S is a Sato operator of the vector space W.

(2) Let S and T be two Sato operators of W and

$$S = \sum_{\underline{k}\geq 0} \frac{x^{\underline{k}}}{\underline{k}!} S_{(\underline{k})} \quad \text{respectively} \quad T = \sum_{\underline{k}\geq 0} \frac{x^{\underline{k}}}{\underline{k}!} T_{(\underline{k})}$$

be the corresponding slice decompositions. According to Proposition 5.15, the following statements are true:

- $o\big(S_{(\underline{k})}\big) = o\big(T_{(\underline{k})}\big) = \mu + |\underline{k}|$ for any $\underline{k} \in \Sigma$.
- For any $m \in \mathbb{N}_0$, the elements of the set $\big\{T_{(\underline{k})} \,\big|\, \underline{k} \in \Sigma \text{ such that } |\underline{k}| \leq m\big\}$ form a basis of the vector space $W_{\mu+m}$.

Therefore, for any $\underline{i} \in \Sigma$ we can find (uniquely determined) scalars $\big\{\gamma_{\underline{i},\underline{k}} \in \mathbb{C} \,\big|\, \underline{k} \in \Sigma : |\underline{k}| \leq |\underline{i}|\big\}$ such that $S_{(\underline{i})} = \sum\limits_{|\underline{k}|\leq|\underline{i}|} \gamma_{\underline{i},\underline{k}} T_{(\underline{k})}$. Moreover, there exists at least one $\underline{l} \in \Sigma$ such that $|\underline{l}| = |\underline{i}|$ and $\gamma_{\underline{i},\underline{l}} \neq 0$. Now we put: $U := \sum\limits_{\underline{i}\geq 0} \dfrac{x^{\underline{i}}}{\underline{i}!} \cdot \sum\limits_{|\underline{k}|\leq|\underline{i}|} \gamma_{\underline{i},\underline{k}} \underline{\partial}^{\underline{k}} \in \mathfrak{S}$. It is clear that $o(U) = 0$, hence $U \in \mathfrak{S}_-$. We claim that $S = U \cdot T$. According to Remark 5.7, it is equivalent to the statement that $\underline{\partial}^{\underline{i}} \circ S = \underline{\partial}^{\underline{i}} \circ (U \cdot T) = \big(\underline{\partial}^{\underline{i}} \circ U\big) \circ T$ for all $\underline{i} \in \Sigma$.
By the construction of U, we have:

$$\big(\underline{\partial}^{\underline{i}} \circ U\big) \circ T = \sum_{|\underline{k}|\leq|\underline{i}|} \gamma_{\underline{i},\underline{k}}\, \underline{\partial}^{\underline{k}} \circ T = \sum_{|\underline{k}|\leq|\underline{i}|} \gamma_{\underline{i},\underline{k}} T_{(\underline{k})} = S_{(\underline{i})},$$

so $S = U \cdot T$ as asserted. In a similar way, we can find $V \in \mathfrak{S}_-$ such that $T = V \cdot S$. Therefore, we have: $(1 - U \cdot V) \cdot S = 0$. Since S is regular, Lemma 5.9 implies that $U \cdot V = 1$. In a similar way, $V \cdot U = 1$, hence $U, V \in \mathfrak{S}_-^*$ as claimed.

The uniqueness of the unit U also follows from Lemma 5.9. Theorem is proven. □

Definition 5.17. Let $\mu \in \mathbb{N}_0$ and $F = \mathbb{C}[\partial_1, \dots, \partial_n]$ (viewed as the polynomial algebra). We say that (W, A) is a *Schur pair* of index μ if $W \in \mathsf{Gr}_\mu(F)$ and $A \subseteq F$ is a subalgebra such that $W \cdot A = W$.

Theorem 5.18. *Let (W, A) be a Schur pair of index $\mu \in \mathbb{N}_0$ and S be a Sato operator of W. Then the following statements are true.*

(1) *For any polynomial $f \in A$, there exists a uniquely determined operator $\mathtt{L}_S(f) \in \mathfrak{S}$ such that $S \cdot f = \mathtt{L}_S(f) \cdot S$. Moreover, $o\big(\mathtt{L}_S(f)\big) = \deg(f)$ for any $f \in A$.*
(2) *Next, for any $f_1, f_2 \in A$ and $\lambda_1, \lambda_2 \in C$ we have:*

$$\mathtt{L}_S(f_1 \cdot f_2) = \mathtt{L}_S(f_1) \cdot \mathtt{L}_S(f_2) \quad \text{and} \quad \mathtt{L}_S(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \mathtt{L}_S(f_1) + \lambda_2 \mathtt{L}_S(f_2)$$

In other words, the map $A \longrightarrow \mathfrak{S}, f \mapsto \mathtt{L}_S(f)$ *is a homomorphism of* $\mathbb{C}$*–algebras, which is moreover injective.*

(3) *Finally, for any polynomial* $f \in A$*, the following diagram of* $\mathbb{C}$*–linear maps*

$$\begin{tikzcd} F \arrow[r, "- \circ S"] \arrow[d, "- \circ \mathtt{L}_S(f)"'] & W \arrow[d, "\cdot f"] \\ F \arrow[r, "- \circ S"] & W \end{tikzcd} \tag{5.15}$$

is commutative.

Proof. (1) Let $S = \sum\limits_{\underline{k} \geq \underline{0}} \dfrac{\underline{x}^{\underline{k}}}{\underline{k}!} S_{(\underline{k})}$ be the slice decomposition of S and f be an element of A with $d = \deg(f)$. Then for any $\underline{i} \in \Sigma$ we have:

$$\underline{\partial}^{\underline{i}} \circ S = S_{(\underline{i})} \in W_{|\underline{i}|+\mu} \quad \text{and} \quad \underline{\partial}^{\underline{i}} \circ (S \cdot f) = S_{(\underline{i})} \cdot f \in W_{|\underline{i}|+d+\mu}.$$

According to Proposition 5.15, there exists uniquely determined scalars $\bigl\{\gamma_{\underline{i},\underline{k}} \in \mathbb{C} \,\big|\, \underline{k} \in \Sigma : |\underline{k}| \leq |\underline{i}| + d\bigr\}$ such that $S_{(\underline{i})} \cdot f = \sum\limits_{|\underline{k}| \leq |\underline{i}|+d} \gamma_{\underline{i},\underline{k}} S_{(\underline{k})} \in W_{|\underline{i}|+d+\mu}$. In these terms, we put:

$$\mathtt{L}_S(f) := \sum_{\underline{i} \geq \underline{0}} \frac{\underline{x}^{\underline{i}}}{\underline{i}!} \cdot \sum_{|\underline{k}| \leq |\underline{i}|+d} \gamma_{\underline{i},\underline{k}}\, \underline{\partial}^{\underline{k}} \in \mathfrak{S}. \tag{5.16}$$

Since for any $\underline{i} \in \Sigma$, there exists at least one $\underline{l} \in \Sigma$ such that $|\underline{l}| = |\underline{i}| + d$ and $\gamma_{\underline{i},\underline{l}} \neq 0$, we have: $o\bigl(\mathtt{L}_S(f)\bigr) = d$. The identity $S \cdot f = \mathtt{L}_S(f) \cdot S$ in the algebra $\mathfrak{S}$ follows from the fact that $\underline{\partial}^{\underline{i}} \circ (S \cdot f) = \underline{\partial}^{\underline{i}} \circ \bigl(\mathtt{L}_S(f) \cdot S\bigr)$ for any $\underline{i} \in \Sigma$ (which is true by the construction of $\mathtt{L}_S(f)$). The uniqueness of $\mathtt{L}_S(f)$ follows from the regularity of S and Lemma 5.9.

(2) Let $f_1, f_2 \in A$. By construction, we have:

$$S \cdot (f_1 \cdot f_2) = \bigl(\mathtt{L}_S(f_1) \cdot S\bigr) \cdot f_2 = \bigl(\mathtt{L}_S(f_1) \cdot \mathtt{L}_S(f_2)\bigr) \cdot S.$$

Since the operator $L_S(f)$ is uniquely determined by $f \in A$, we get: $\mathtt{L}_S(f_1) \cdot \mathtt{L}_S(f_2) = \mathtt{L}_S(f_1 \cdot f_2)$. The proof of the second statement is analogous, hence $A \longrightarrow \mathfrak{S}$, $f \mapsto L_S(f)$ is indeed a homomorphism of $\mathbb{C}$–algebras. For $f \neq 0$ we have: $\sigma(S \cdot f) = \sigma(S) \cdot \sigma(f) \neq 0$, hence $\mathtt{L}_S(f) \neq 0$, too.

(3) The commutativity of diagram (5.15) is a reformulation of the first assertion of this theorem. Note that by Proposition 5.15, the linear map $- \circ S$ is an isomorphism. □

Remark 5.19. Let (W, A) be a Schur pair of index $\mu \in \mathbb{N}_0$. According to Theorem 5.18, any choice of Sato operator $S \in \mathfrak{S}$ of the vector space W specifies an injective algebra homomorphism $A \xrightarrow{\mathtt{L}_S} \mathfrak{S}$. However, a Sato operator is determined only up to a unit of the algebra $\mathfrak{S}_-$; see Theorem 5.16. If $V \in \mathfrak{S}_-^*$ and $T = V \cdot S$ is any other Sato operator of W, then we have: $\mathtt{L}_T = V \cdot \mathtt{L}_S \cdot V^{-1}$. In other words, any Schur pair defines an injective algebra homomorphism $A \xrightarrow{\mathtt{L}} \mathfrak{S}$, which is unique up to an appropriate inner automorphism of the algebra $\mathfrak{S}$.

Remark 5.20. The modern algebro–geometric study of commuting differential operators was initiated by Krichever [22, 23] and investigated by many authors in what follows. Our work was especially influenced by the approaches of Mumford [32] and Mulase [31]. In particular, the higher–dimensional Sato theory developed in this section was inspired by [31]. The idea to enlarge the algebra of differential operators in the context of a generalized Krichever correspondence was suggested by the second–named author in [38], see also

[27, 28]. The algebra $\mathfrak{S}$ introduced in Definition 5.1 deviates from its cousins studied in [38] (on the one–hand side it is larger, on the other hand it is much more symmetric). The construction of commutative subalgebras of $\mathfrak{S}$ based on Schur pairs (W, A) can be thought as an attempt to generalize Wilson's theory of bispectral commutative subalgebras of ordinary differential operators of rank one [37].

6. On the algebraic inverse scattering method in dimension two

In this section, we are going to discuss some examples of the theory developed in the previous section in the special case $n = 2$. Let $R = \mathbb{C}[z_1, z_2]$ and $\mathfrak{D} = \mathbb{C}[\![x_1, x_2]\!][\partial_1, \partial_2]$, whereas

$$\mathfrak{S} := \left\{ \sum_{k_1,k_2 \geq 0} a_{k_1,k_2}(x_1, x_2)\partial_1^{k_1}\partial_2^{k_2} \,\Big|\, \exists d \in \mathbb{Z} : \, k_1 + k_2 - \upsilon\big(a_{k_1,k_2}(x_1, x_2)\big) \leq d \;\; \forall k_1, k_2 \geq 0 \right\}$$

is the algebra from Definition 5.1 (here, $\upsilon\big(a(x_1, x_2)\big)$ is the valuation of the power series $a(x_1, x_2) \in \mathbb{C}[\![x_1, x_2]\!]$).

Lemma 6.1. *Let $\big(\Pi, \underline{\mu}\big)$ be a Baker–Akhieser weighted line arrangement, $A = A\big(\Pi, \underline{\mu}\big)$ the corresponding algebra of quasi–invariants and $W = P(-\xi_1 z_1 - \xi_2 z_2) \subset R$ the projective A–module from Theorem 4.14 for an appropriate $(\xi_1, \xi_2) \in \mathbb{C}^2$. Then the embedding $A \longrightarrow \mathfrak{S}$ determined by the Schur pair (W, A) (see Theorem 5.18) coincides (up to an appropriate inner automorphism of $\mathfrak{S}$) with the embedding $\Xi(\vec{\xi})$ of Chalykh and Veselov defined by the formula (4.9).*

Proof. The fact that $W \in \mathsf{Gr}_\mu(R)$ (where $\mu := \sum_{\alpha\in\Pi} \mu_\alpha$) was established in Proposition 4.13. The same Proposition implies that the differential operator S of order μ from Lemma 4.11 is a Sato operator of the vector space W in the sense of Definition 5.14. Recall that for any $f \in A$, we have the following equality $S \cdot f(\partial_1, \partial_2) = \big(\Xi(\vec{\xi})\big)(f) \cdot S$ in $\mathfrak{D} \subset \mathfrak{S}$; see formula (4.24). This implies the result. □

Remark 6.2. Let $A = A\big(\Pi, \underline{\mu}\big)$ be an algebra of planar quasi–invariants and $W \in \mathsf{Gr}_\mu(R)$ be a finitely generated torsion free A–module of rank one (as usual, $\mu := \sum_{\alpha\in\Pi} \mu_\alpha$).

1. It is not true that W is automatically a Cohen–Macaulay A–module. Indeed, let $A := \mathbb{C}[x^2, x^3, y^2, y^3]$ and $W := \big\langle x^2, x^3, y^3, y^4 \big\rangle_A$. Obviously, W is a finitely generated torsion free A–module of rank one and $W \in \mathsf{Gr}_2(R)$. Moreover, $A/W = \langle 1, \bar{y}^2\rangle_{\mathbb{C}}$ is non–zero and finite dimensional. Hence, the module W is not Cohen–Macaulay (in fact, the regular module A is the Macaulayfication of W).

2. Nevertheless, there are good reasons to focus on those Schur pairs (W, A) of index μ, for which W is a Cohen–Macaulay A–module of rank one. Assume that $\Pi, \underline{\mu})$ is a Baker–Akhieser weighted line arrangement (see Definition 4.1) and $A = A(\Pi, \underline{\mu})$ is the corresponding algebra of quasi–invariants. Consider the Rees algebra (respectively, the Rees module)

$$\widetilde{A} = \bigoplus_{k=0}^{\infty} A_k t^k \subset A[t] \quad \text{respectively} \quad \widetilde{W} = \bigoplus_{k=0}^{\infty} W_{k+\mu} t^k \subset W[t].$$

Let $\widetilde{X} := \mathsf{Proj}(\widetilde{A})$ (projective spectral surface), $C := V(t) \subset \widetilde{X}$ and $\mathcal{F} := \mathsf{Proj}(\widetilde{W})$ (projective spectral sheaf). Then the following statements are true; see [38, Lemma 3.3 and Lemma 3.8] as well as [27, Theorem 2.1].

(1) There exists an isomorphism of algebraic varieties $X \cong \widetilde{X} \setminus C$. In particular, we have an isomorphism of $\mathbb{C}$–algebras $A \cong \Gamma\bigl(\widetilde{X} \setminus C, \mathcal{O}_{\widetilde{X}}\bigr)$.
(2) Moreover, there exists a natural isomorphism of A–modules $W \cong \Gamma\bigl(\widetilde{X} \setminus C, \mathcal{F}\bigr)$.
(3) The variety C is an integral projective curve. Moreover, there exists $d \in \mathbb{N}$ such that $C' = dC$ is a Cartier divisor, and $\mathcal{L} = \mathcal{O}_{\widetilde{X}}(C')$ is an ample line bundle.
(4) $\mathcal{F}$ is a torsion free coherent sheaf of rank one on X. Moreover, we have:

$$\chi\bigl(\widetilde{X}, \mathcal{F} \otimes \mathcal{L}^{\otimes k}\bigr) = \binom{kd+1}{2} \text{ for any } k \in \mathbb{N}_0.$$

Let M_χ be the moduli space of stable torsion free coherent sheaves on X with Hilbert polynomial $\chi = \binom{kd+1}{2}$ with respect to the ample line bundle $\mathcal{L}$. Then M_χ is a projective variety (see e.g. [26, Theorem 4.3.4]) and $\mathcal{F} \in M_\chi$. More precisely, let $\mathcal{U} \in \mathsf{Coh}(X \times M_\chi)$ be a universal family of the moduli functor $\underline{M}_\chi$. Then any Schur pair (W, A) as above defines a point $p = p(W) \in M_\chi$ such that $\mathcal{U}_p := \mathcal{U}\big|_{X \times \{p\}} \cong \mathcal{F} := \mathsf{Proj}(\widetilde{W})$.

Now, let $\mathfrak{B} := \mathsf{Im}\bigl(\Xi(\vec{\xi})\bigr) \subset \mathfrak{D}$ be the algebra defined by (4.9). Then the corresponding projective spectral sheaf $\mathcal{F}$ is *Cohen–Macaulay*; see [28, Theorem 3.1]. Let $p \in M_\chi$ be the corresponding point. Then by [20, Théorème 12.2.1], there exists an open neighbourhood $p \in U \subset M_\chi$ such that for any $q \in U$, the coherent sheaf $\mathcal{U}_q$ is Cohen–Macaulay of rank one. As a consequence, the A–module $W_q := \Gamma(X, \mathcal{U}_q)$ is Cohen–Macaulay of rank one, too. Therefore, in order to construct algebra embeddings $A \longrightarrow \mathfrak{S}$ arising from Schur pairs (W, A), which are "deformations" of the standard Calogero–Moser system $A \xrightarrow{\Xi(\vec{\xi})} \mathfrak{D}$ given by (4.9), it is natural to take those rank one torsion free A–modules $W \in \mathsf{Gr}_\mu(R)$, which are *Cohen–Macaulay*. □

In the remaining part of this section, we illustrate the "algebraic inverse scattering method" of Theorem 5.18 by constructing an "isospectral deformation" of the simplest dihedral Calogero–Moser system associated with the operator

$$H = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) - 2\left(\frac{1}{(x_1 - \xi_1)^2} + \frac{1}{(x_2 - \xi_2)^2}\right), \tag{6.1}$$

where $(\xi_1, \xi_2) \in \mathbb{C}^2$ is such that $\xi_1 \xi_2 \neq 0$. In this case we have:

- $A = A(\Pi, \underline{\mu}) = \mathbb{C}[z_1^2, z_1^3, z_2^2, z_2^3]$, where $\Pi = \Lambda_2 = \left\{0, \dfrac{\pi}{2}\right\}$ and $\underline{\mu}(0) = \underline{\mu}\left(\dfrac{\pi}{2}\right) = 1$.
- It follows from Theorem 4.14 that the spectral module F of the corresponding Calogero–Moser system has the following description:

$$F = \left\{ f \in R \;\middle|\; \begin{array}{l} \dfrac{\partial f}{\partial z_1}(0, \rho) = \xi_1 \rho f(0, \rho) \\ \dfrac{\partial f}{\partial z_1}(\rho, 0) = \xi_2 \rho f(\rho, 0) \end{array} \right\}.$$

- Let $K = \mathbb{C}(\rho)$ and $L = K[\varepsilon]/(\varepsilon^2)$. Then the diagram (2.3) for the algebra A has the following form:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & K \times K \arrow[d, hook] \\ R \arrow[r, "T"] & L \times L, \end{tikzcd} \tag{6.2}$$

where $T(f) = \left(f(\rho,0) + \varepsilon\rho\dfrac{\partial f}{\partial z_2}(\rho,0), f(0,\rho) - \varepsilon\rho\dfrac{\partial f}{\partial z_1}(0,\rho)\right)$.

- In terms of Theorem 3.2 we have: $F \cong \mathcal{B}(\vec{\gamma})$, where $\vec{\gamma} = (\xi_2\rho, -\xi_1\rho)$.

Lemma 6.3. *For $(\xi_1,\xi_2) \in \mathbb{C}^2$ such that $\xi_1\xi_2 \neq 0$ and $\beta \neq \xi_1\xi_2 \in \mathbb{C}$, consider the Cohen–Macaulay A–module $W_\beta := B\big(\vec{\gamma}_\beta\big) \in \mathsf{CM}_1^{\mathsf{lf}}(A)$, where*

$$\vec{\gamma}_\beta = \left(\frac{\xi_2^2\rho}{\xi_2+\beta\rho}, -\frac{\xi_1^2\rho}{\xi_1+\beta\rho}\right) \in \mathbb{C}(\rho)\oplus\mathbb{C}(\rho).$$

Then $W_\beta \in \mathsf{Gr}_2(R)$. Moreover, W_β is not projective over A for $\beta \neq 0$.

Proof. By definition, we have:

$$W_\beta = \left\{ f \in R \;\middle|\; \begin{array}{l} \dfrac{\partial f}{\partial z_1}(0,\rho) = \dfrac{\xi_1^2\rho}{\xi_1+\beta\rho} f(0,\rho) \\ \dfrac{\partial f}{\partial z_2}(\rho,0) = \dfrac{\xi_2^2\rho}{\xi_2+\beta\rho} f(\rho,0) \end{array} \right\}. \tag{6.3}$$

A lengthy but straightforward computation shows that

$$W_\beta = \mathbb{C}\cdot w + (\xi_2 + \xi_2^2 z_2 + \beta z_1)z_1^2\mathbb{C}[z_1] + (\xi_1 + \xi_1^2 z_1 + \beta z_2)z_2^2\mathbb{C}[z_2] + z_1^2 z_2^2\mathbb{C}[z_1,z_2], \tag{6.4}$$

where $w = 1+\xi_1 z_1+\xi_2 z_2+(\xi_1\xi_2-\beta)\cdot\left(z_1z_2 + \left(\dfrac{z_1^2}{\xi_2^2} + \dfrac{z_2^2}{\xi_1^2}\right)\right)$. The description (6.4) implies that $W_\beta \in \mathsf{Gr}_2(R)$. Since the rational functions $\gamma_1(\rho)$ and $\gamma_2(\rho)$ have a pole provided $\beta \neq 0$, Lemma 3.16 (see also the proof of Theorem 3.17) implies that the A–module W_β is not projective. $\square$

Remark 6.4. It is interesting to note that the A–module W_β is actually *locally free* at the point $p := \nu(0,0) \in X$ (the "most singular" point of X), where $\mathbb{A}^2 \xrightarrow{\nu} X$ is the normalization of the spectral surface X of the algebra A. Indeed, the Cohen–Macaulay A–module W_β corresponds to the following object

$$\left(R, K\oplus K, \left(1+\varepsilon\frac{\xi_2^2\rho}{\xi_2+\beta\rho}, 1-\varepsilon\frac{\xi_1^2\rho}{\xi_1+\beta\rho}\right)\right).$$

of the category of triples $\mathsf{Tri}(A)$; see Theorem 3.2. As in the course of the proof of Theorem 3.17 one can show, that W_β is locally free at the point p if and only if there exist $h \in \mathbb{C}[\![z_1,z_2]\!]$ and $f,g \in \mathbb{C}(\!(\rho)\!)$ making the following two diagrams

$$\begin{tikzcd} L \arrow[r, "1"] \arrow[d, "f"'] & L \arrow[d, "(1+\varepsilon\rho\frac{\partial h}{\partial z_2}(\rho,0))\cdot\exp(h(\rho,0))"] \\ L \arrow[r, "1+\varepsilon\frac{\xi_2^2\rho}{\xi_2+\beta\rho}"'] & L \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "1"] \arrow[d, "g"'] & L \arrow[d, "(1-\varepsilon\rho\frac{\partial h}{\partial z_1}(0,\rho))\cdot\exp(h(0,\rho))"] \\ L \arrow[r, "1-\varepsilon\frac{\xi_1^2\rho}{\xi_1+\beta\rho}"'] & L \end{tikzcd}$$

commutative in the category of L–modules. To achieve this, we have to put $f := \exp(h(\rho,0))$ and $g := \exp(h(0,\rho))$, whereas the power series h has to fulfil the constraint

$$\left(\frac{\partial h}{\partial z_1}(0,\rho), \frac{\partial h}{\partial z_2}(\rho,0)\right) = \left(\frac{\xi_1^2}{\xi_1+\beta\rho}, \frac{\xi_2^2}{\xi_2+\beta\rho}\right),$$

which (as one can easily see) is consistent. $\square$

Theorem 6.5. *The operator* $S_\beta := S_0 + \beta T \in \mathfrak{S}$, *where*

$$S_0 := \partial_1\partial_2 + \frac{1}{\xi_2 - x_2}\partial_1 + \frac{1}{\xi_1 - x_1}\partial_2 + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)}$$

and

$$\begin{aligned} T =& \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)}\left(\frac{1}{\xi_2}\left(E_2\partial_1 + (\xi_1 - x_1)E_2\partial_1^2\right) + \frac{1}{\xi_1}\left(E_1\partial_2 + (\xi_2 - x_2)E_1\partial_2^2\right)\right) + \\ & \frac{1}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)} E_1E_2\left(1 + \beta\left(\frac{\partial_1}{\xi_2} + \frac{\partial_2}{\xi_1}\right)\right), \end{aligned}$$

with $E_1, E_2 \in \mathfrak{S}$ *defined in Example 5.4, is a Sato operator of the space* W_β, *given by formula (6.4).*

Proof. It is easy to see that $o(S_\beta) = 2$ and $\sigma(S_\beta) = \partial_1\partial_2 + \frac{\beta}{\xi_2^2} E_2\partial_1^2 + \frac{\beta}{\xi_1^2} E_1\partial_2^2$. Let $i, j \in \mathbb{N}_0$ be such that $i + j = m$. Then we have:

$$\mathbb{C}[\partial_1, \partial_2] \ni \partial_1^i\partial_2^j \xrightarrow{-\circ\,\sigma(S_\beta)} \begin{cases} \partial_1^{i+1}\partial_2^{j+1} & \text{if} \quad i \cdot j \neq 0 \\ (-1)^m \dfrac{\beta}{\xi_2^2}\partial_1^{m+2} + \partial_1^{m+1}\partial_2 & \text{if} \quad (i, j) = (m, 0) \\ (-1)^m \dfrac{\beta}{\xi_1^2}\partial_2^{m+2} + \partial_2^{m+1}\partial_1 & \text{if} \quad (i, j) = (0, m). \end{cases}$$

By Lemma 5.9, the operator S_β is regular.

Let $\Theta_0(x_1, x_2; z_1, z_2; \xi_1, \xi_2) = z_1z_2 + \frac{z_1}{\xi_2 - x_2} + \frac{z_2}{\xi_1 - x_1} + \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)}$. It follows from Berest's formula [1] that

$$\Psi_0(x_1, x_2; z_1, z_2; \xi_1, \xi_2) := \Theta_0(x_1, x_2; z_1, z_2; \xi_1, \xi_2) \cdot \exp(x_1z_1 + x_2z_2)$$

is a Baker–Akhieser function of the Calogero–Moser operator H given by (6.1). This implies that S_0 is a Sato operator of the unperturbed space W_0. For general β, consider the formal power series $\Psi_\beta = \Psi_0 + \beta\overline{\Psi}$, where $\overline{\Psi}(x_1, x_2; z_1, z_2; \xi_1, \xi_2) =$

$$\begin{aligned} \frac{1 + \beta\left(\dfrac{z_1}{\xi_2} + \dfrac{z_2}{\xi_1}\right)}{(\xi_1\xi_2 - \beta)(\xi_1 - x_1)(\xi_2 - x_2)} &+ \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_1z_1)z_1 + (\xi_1 - x_1)\exp(x_1z_1)z_1^2\right) + \\ & \frac{1}{(\xi_1 - x_1)(\xi_2 - x_2)\xi_2}\left(\exp(x_2z_2)z_2 + (\xi_2 - x_2)\exp(x_2z_2)z_2^2\right). \end{aligned}$$

A straightforward computation shows that Ψ_β satisfies the same equations (6.3)

$$\begin{cases} \left.\dfrac{\partial\Psi_\beta}{\partial z_1}\right|_{(z_1,z_2)=(0,\rho)} = \dfrac{\xi_1^2\rho}{\xi_1 + \beta\rho}\Psi_\beta\Big|_{(z_1,z_2)=(0,\rho)} \\ \left.\dfrac{\partial\Psi_\beta}{\partial z_2}\right|_{(z_1,z_2)=(\rho,0)} = \dfrac{\xi_2^2\rho}{\xi_2 + \beta\rho}\Psi_\beta\Big|_{(z_1,z_2)=(\rho,0)} \end{cases}$$

which define the vector space W_β. Thus,

$$W_\beta \supseteq W'_\beta := \mathbb{C}[\partial_1, \partial_2] \circ S_\beta = \left\langle \left.\frac{\partial^{p_1+p_2}\Psi_\beta}{\partial x_1^{p_1}\partial x_2^{p_2}}\right|_{(x_1,x_2)=(0,0)} \;\middle|\; (p_1, p_2) \in \mathbb{N}_0 \times \mathbb{N}_0 \right\rangle_{\mathbb{C}}.$$

Since both vector spaces W_β and W'_β belong to $\mathsf{Gr}_2(R)$, we conclude that $W'_\beta = W_\beta$. $\square$

Example 6.6. Let $A \xrightarrow{\mathsf{L}} \mathfrak{S}$ be the algebra homomorphism corresponding to the Sato operator S_β from Theorem 6.5 for the special value $(\xi_1, \xi_2) = (1,1)$ and $H_\beta := \mathsf{L}(\omega)$, where $\omega = z_1^2 + z_2^2 \in A$. Proceeding along the lines of the proof of Theorem 5.18, we get the formula $H_\beta = H_0 + \beta(Z + Z')$, where $H_0 = H$ is the unperturbed operator (6.1),

$$Z = \left(\frac{1}{1-x_1}E_1 + \frac{1}{1-x_2}E_2\right)\partial_1\partial_2 + \frac{\beta}{(1-\beta)(1-x_1)(1-x_2)}E_1E_2(\partial_1+\partial_2)$$
$$-\frac{1}{1-x_1}E_1\partial_2 - \frac{1}{1-x_2}E_2\partial_1 + \frac{2\beta}{\beta-1}E_1E_2 - (G_2E_1\partial_1 + G_1E_2\partial_2)$$

and Z' is some operator of negative order.

For any $f \in A$ of the form $f = z_1^2 z_2^2 \cdot g$ for some $g \in R$ it can be shown that

$$\mathsf{L}(f) = S_\beta \cdot g(\partial_1, \partial_2) \cdot \left(\partial_1 + \frac{1}{x_1 - 1}\right) \cdot \left(\partial_2 + \frac{1}{x_2 - 1}\right).$$

Finally, the formal power series $\Psi_\beta(x_1, x_2; z_1, z_2) = \Psi_\beta(x_1, x_2; z_1, z_2; 1, 1)$ introduced in the course of the proof of Theorem 6.5, is a Baker–Akhieser function of the *deformed* Calogero–Moser system $\mathsf{L}(A) \subset \mathfrak{S}$: for any $f \in A$ we have:

$$\mathsf{L}(f)_{(x_1,x_2)} \diamond \Psi_\beta(x_1, x_2; z_1, z_2) = f(z_1, z_2) \cdot \Psi_\beta(x_1, x_2; z_1, z_2).$$

Remark 6.7. Let $A = A\big(\Pi, \underline{\mu}\big)$ be the algebra of planar quasi–invariants of Baker–Akhieser type. A description of those $\vec{\gamma} \in K\big(\Pi, \underline{\mu}\big)$, for which the corresponding Cohen–Macaualy module $B(\vec{\gamma}) \in \mathsf{CM}_1^{\mathsf{lf}}(A)$ belong to $\mathsf{Gr}_\mu(R)$ (see Theorem 3.2 for the used notations), is a non–trivial problem, which will be studied in a future work. Theorem 5.16 on existence and uniqueness of the corresponding Sato operator can be understood as an analogue of the axiomatic description (in terms of the works [13, 12]) of a Baker–Akhieser function for the corresponding deformed Calogero–Moser system $A \xrightarrow{\mathsf{L}} \mathfrak{S}$.

7. Appendix: the compactified Picard variety of an affine cuspidal curve

For any $m \in \mathbb{N}$, let $A_m := \mathbb{C}[t^2, t^{2m+1}]$. Although the (compactified) Picard variety of an algebraic curve is a well–studied object, we were not able to find a precise reference in the literature for an explicit description of the Picard group $\mathsf{Pic}(A_m)$, mentioned in Introduction. For the reader's convenience we give its proof below.

Theorem 7.1. *There is an isomorphism of algebraic groups:*

$$\mathsf{Pic}(A_m) \cong K_m := \big(\mathbb{C}[\sigma]/(\sigma^m), \circ\big), \tag{7.1}$$

where $\gamma_1 \circ \gamma_2 := (\gamma_1 + \gamma_2) \cdot (1 + \sigma\gamma_1\gamma_2)^{-1}$ for any $\gamma_1, \gamma_2 \in K_m$. Next, let Q be a torsion free A_m–module of rank one. Then either Q is projective or there exists $m' < m$ and a projective module of rank one Q' over $A_{m'}$ such that Q is isomorphic to the restriction of Q' on $A_m \subset A_{m'}$.

Proof. To simplify the notation, we denote $A = A_m$. Then the normalization of the algebra A is $R = \mathbb{C}[t]$ and we have:

$$A = \big\{f \in R \,\big|\, f'(0) = f'''(0) = \dots = f^{(2m-1)}(0) = 0\big\}.$$

Let $I := \mathsf{Ann}_A(R/A)$ be the conductor ideal, then we have: $I = (t^{2m})_R$ and $K := \mathbb{C}[\sigma]/(\sigma^m) \cong A/I$, whereas $L := \mathbb{C}[\varepsilon]/(\varepsilon^{2m}) \cong R/I$. The canonical inclusion $A/I \subset R/I$ realizes K as a subalgebra of L via the identification $\sigma = \varepsilon^2$. Note that $L = K \dotplus \varepsilon K$.

Let $\mathsf{TF}(A)$ be the category of finitely generated torsion free A–modules. Similarly to Theorem 2.9, we have an equivalence of categories

$$\mathsf{TF}(A) \xrightarrow{\mathbb{F}} \mathsf{Tri}(A), \quad M \mapsto \Big(\big(R \otimes_A M\big)/\mathsf{tor}, K \otimes_A M, \theta_M\Big),$$

where tor denotes the torsion part of the R–module $R \otimes_A M$. Here, the category of triples $\mathsf{Tri}(A)$ is the one–dimensional prototype of its two–dimensional descendent from Definition 2.8. Namely, it is the full subcategory of the comma category associated with the pair of functors

$$\mathsf{TF}(R) \xrightarrow{L\otimes_R -} L - \mathsf{mod} \xleftarrow{L\otimes_K -} K - \mathsf{mod}.$$

consisting of those triples $(\widetilde{M}, V, \theta)$, for which the morphism of L–modules (gluing map)

$$L \otimes_K V \xrightarrow{\theta} L \otimes_R \widetilde{M}$$

is surjective and its adjoint morphism of K–modules $V \longrightarrow L \otimes_K V \xrightarrow{\theta} L \otimes_R \widetilde{M}$ is injective. The functor $\mathbb{F}$ restricts to an equivalence of categories $\mathsf{Pro}(A) \xrightarrow{\mathbb{F}} \mathsf{Tri}^{\mathsf{lf}}(A)$, where $\mathsf{Pro}(A)$ is the category of finitely generated projective modules and $\mathsf{Tri}^{\mathsf{lf}}(A)$ consists of those triples $(\widetilde{M}, V, \theta)$, for which the gluing map θ is an isomorphism (this condition actually implies that V is a free K–module); see [5, Theorem 1.3 and Theorem 3.2], [3, Theorem 16], [7, Theorem 2.5] as well as the beginning of [7, Chapter 2] for a survey of similar constructions.

It follows from the definition of the functor $\mathbb{F}$ that $\mathbb{F}(P_1 \otimes_A P_2) \cong \mathbb{F}(P_1) \otimes \mathbb{F}(P_2)$ for any finitely generated projective A–modules P_1 and P_2, where the definition of the monoidal structure on the category $\mathsf{Tri}^{\mathsf{lf}}(A)$ is straightforward.

Let $P \in \mathsf{Pic}(A)$. Then $\mathbb{F}(P) \cong \big(R, K, \theta\big)$, where the gluing map $L \xrightarrow{\theta} L$ can be written in the normal form: $\theta \sim \vartheta_\gamma = 1 + \varepsilon \cdot \gamma$ for a uniquely determined $\gamma \in K$. Conversely, we put $P_\gamma := \mathbb{F}^{-1}\big(R, K, \vartheta_\gamma\big)$ for any $\gamma \in K$. Then we have:

$$\mathbb{F}\big(P_{\gamma_1} \otimes_A P_{\gamma_2}\big) \cong \Big(R, K, \big((1+\varepsilon^2\gamma_1\cdot\gamma_2)+\varepsilon\cdot(\gamma_1+\gamma_2)\big)\Big) \cong \big(R, K, 1+\varepsilon(\gamma_1+\gamma_2)\cdot(1+\varepsilon^2\gamma_1\gamma_2)^{-1}\big).$$

This implies that $P_{\gamma_1} \otimes_A P_{\gamma_2} \cong P_{\gamma_1 \circ \gamma_2}$ for any $\gamma_1, \gamma_2 \in K$, proving the isomorphism (7.1). It is clear that the regular module $A = P_0$ corresponds to the triple $(A, K, 1)$. Hence, for any $\gamma \in K$, we have the following realization of the module P_γ:

$$P_\gamma \cong \mathsf{Hom}_A(A, P_\gamma) \cong \mathsf{Hom}_{\mathsf{Tri}(A)}\big((A, K, 1), (A, K, 1+\varepsilon\gamma)\big) = \big\{f \in R \,\big|\, T^-_{2m}(f) = \gamma\cdot T^+_{2m}(f)\big\},$$

where the elements $T^\pm_{2m}(f) \in K$ are defined by the rules:

$$T^+_{2m}(f) = \sum_{j=0}^{m-1} \frac{f^{(2j)}(0)}{(2j)!}\sigma^j \quad \text{and} \quad T^-_{2m}(f) = \sum_{j=0}^{m-1} \frac{f^{(2j+1)}(0)}{(2j+1)!}\sigma^j.$$

Let $\overline{\mathsf{Pic}}(A)$ be the set of the isomorphism classes of finitely generated torsion free A–modules of rank one (compactified Picard variety) and $Q \in \overline{\mathsf{Pic}}(A)$. Then $\mathbb{F}(Q) \cong \big(R, V, \theta\big)$, where $V \cong K \oplus K/(\sigma^i)$ for some uniquely determined $1 \le i \le m$; see Lemma 3.3. By Proposition 3.6, we can transform the gluing map θ into a uniquely determined normal form $\theta \sim \vartheta_\gamma := (1 + \varepsilon\gamma \mid \varepsilon^{2(m-j)+1})$, where $\gamma = \alpha_0 + \alpha_1\varepsilon^2 + \cdots + \alpha_{m-j-1}\varepsilon^{2(m-j-1)} \in L$. This implies that

$$Q \cong \mathsf{Hom}_{\mathsf{Tri}(A)}\big((A, K, 1), (A, K \oplus K/(\sigma^j), \vartheta_\gamma)\big) = \big\{f \in R \,\big|\, T^-_{2(m-j)}(f) = \gamma \cdot T^+_{2(m-j)}(f)\big\}.$$

Hence, the vector space $Q \subset R$ is stable under the multiplication with the elements of the algebra $A_{m-j} = \mathbb{C}[t^2, t^{2(m-j)+1}]$. Moreover, from the above description of the Picard group $\mathsf{Pic}(A)$ it follows that Q is a projective module over the algebra $A_{m-j} \supset A_m = A$. $\square$

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