COMPOSITION ALGEBRA OF A WEIGHTED PROJECTIVE LINE

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Abstract. In this article, we deal with properties of the reduced Drinfeld double of the composition subalgebra of the Hall algebra of the category of coherent sheaves on a weighted projective line. This study is motivated by applications in the theory of quantized enveloping algebras of some Lie algebras. We obtain a new realization of quantized enveloping algebras of affine Lie algebras of simply-laced type and get new embeddings between such algebras. Moreover, our approach allows to derive new results on the structure of the quantized enveloping algebras of the toroidal algebras of types $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$. In particular, our method leads to a construction of a modular action and allows to define a PBW-type basis for that classes of algebras.

1. Introduction

By works of Ringel [33] and Green [15], the study of Hall algebras of the category of nilpotent representation of a finite quiver plays an important role in the theory of quantized Kac-Moody algebras. In this article, we deal with the Hall algebra of the category of coherent sheaves on a weighted projective line $\mathbb{X}$.

As it was established by Kapranov [22] and elaborated by Baumann and Kassel [2], the Hall algebra of the category of coherent sheaves on the classical projective line $\mathbb{P}^1$ is closely related with Drinfeld's new realization of the quantized enveloping algebra $U_v(\widehat{\mathfrak{sl}}_2)$. In our previous joint work [6] it was shown that the Drinfeld–Beck map $U_v\bigl(\widehat{\mathfrak{sl}}_2\bigr) \to U_v\bigl(\mathfrak{L}\mathfrak{sl}_2\bigr)$ (see [11, 3]) comes from the derived equivalence $D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr) \to D^b\bigl(\mathsf{Coh}(\mathbb{P}^1)\bigr)$, where $\overrightarrow{\Delta}$ is the Kronecker quiver. An attempt to generalize this result on other quantized enveloping algebras leads us to the study of Hall algebras of weighted projective lines.

Weighted projective lines and coherent sheaves on them were introduced by Geigle and Lenzing in [13]. This notion turned out to be quite useful from various points of view. In particular, the so-called *domestic* weighted projective lines provide a "geometric realization" of the derived category of representation of affine Dynkin quivers. Weighted projective lines of *tubular* type lead to a very interesting class of the canonical tubular algebras [32]. Geometrically, they correspond to the category of coherent sheaves on an elliptic curve with respect to the action of a finite automorphism group [13]. As it was shown in an earlier work of the second-named author [38], the Hall algebra of such weighted projective lines is closely related with the quantized enveloping algebras of the *toroidal algebras* of types $\widehat{\widehat{D}}_4$, $\widehat{\widehat{E}}_6$, $\widehat{\widehat{E}}_7$ and $\widehat{\widehat{E}}_8$.

The main results of this article are the following. Let $\mathbb{X}$ be a weighted projective line over a finite field $\mathbb{F}_q$, $H(\mathbb{X}) = H\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ be its Hall algebra. In a work of the second-named author [38], the composition subalgebra $U(\mathbb{X}) \subset H(\mathbb{X})$ was introduced. We give an alternative definition of $U(\mathbb{X})$ and show that is a *topological bialgebra*, i.e. it is a subalgebra of $H(\mathbb{X})$ closed under the comultiplication $H(\mathbb{X}) \to H(\mathbb{X})\widehat{\otimes} H(\mathbb{X})$.

Next, we prove that the subalgebra $\overline{U}(\mathbb{X})_{\mathrm{tor}}$ of the composition algebra $U(\mathbb{X})$, which generated by the classes of the skyscraper sheaves, is isomorphic to $\mathcal{Z} \otimes U_q^+\bigl(\widehat{\mathfrak{sl}}_{p_1}\bigr) \otimes \cdots \otimes U_q^+\bigl(\widehat{\mathfrak{sl}}_{p_n}\bigr)$, where $\mathcal{Z} = \mathbb{C}[z_1, z_2, \dots, z_l, \dots]$ is the Macdonald's ring of symmetric functions and $\underline{p} = (p_1, p_2, \dots, p_n)$ is the *weight type* of $\mathbb{X}$.

We study in details further properties of $U(\mathbb{X})$. In particular, we show that the functor $\mathsf{Coh}(\mathbb{P}^1) \to \mathsf{Coh}(\mathbb{X})$, which right adjoint to the functor of reduction of weights of Geigle and Lenzing [14], induces an injective morphism of the reduced Drinfeld doubles $U_q\bigl(\mathfrak{Lsl}_2\bigr) \cong DU(\mathbb{P}^1) \to DU(\mathbb{X})$. In particular, this implies that the reduced Drinfeld double of the composition algebra of an arbitrary weighted projective line $\mathbb{X}$ contains a subalgebra isomorphic to $U_q(\widehat{\mathfrak{sl}}_2)$. More generally, for any weighted projective line $\mathbb{Y}$ of type dominated by the type of $\mathbb{X}$, we construct an injective algebra homomorphism $DU(\mathbb{Y}) \to DU(\mathbb{X})$. These results are based on the notion of perpendicular categories introduced by Geigle and Lenzing in [14]. We also discuss some general properties of the reduced Drinfeld double of a perpendicular subcategory of a hereditary abelian category, which can be applied in the context of quivers as well. For example, this technique allows to construct a new embedding of the quantized enveloping algebras $U_v(\widehat{A}_3) \to U_v(\widehat{D}_4)$.

Using a recent result of Cramer [8], we show that for a domestic weighted projective line $\mathbb{X}$ of affine Dynkin type Δ, any derived equivalence $D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr) \to D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ induces an isomorphism of the reduced Drinfeld doubles of the composition algebras $DC(\overrightarrow{\Delta}) \to DU(\mathbb{X})$, commuting with the Coxeter transformation. This differs from the isomorphism given by Drinfeld [11] and Beck [3] and might be interesting from the point of view of quantum groups. The composition algebra of a weighted projective line of domestic type was also considered in a recent paper [9], where applications to the Drinfeld-Beck isomorphism were studied.

Next, for a tubular weighted projective line $\mathbb{X}$, we show that the group of the exact auto-equivalences of the derived category $\mathsf{Aut}\bigl(D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)\bigr)$ acts on the reduced Drinfeld double $DU(\mathbb{X})$ by algebra automorphisms. In particular, it leads to a very interesting modular action on $DU(\mathbb{X})$. Using this action, we construct a monomial basis of $DU(\mathbb{X})$.

Notation. Throughout the paper, $k = \mathbb{F}_q$ is a finite field with q elements and $\widetilde{\mathbb{Q}} = \mathbb{Q}[v, v^{-1}]/(v^{-2} - q) \cong \mathbb{Q}[\sqrt{q}]$. For a Kac-Moody Lie algebra $\mathfrak{g}$ we denote by $U_v(\mathfrak{g})$ the $\mathbb{Q}[v, v^{-1}]$ algebra, which is the integral form of the corresponding quantized enveloping algebra, whereas $U_q(\mathfrak{g}) = U_v(\mathfrak{g}) \otimes_{\mathbb{Q}[v,v^{-1}]} \widetilde{\mathbb{Q}}$ is the specialization of the integral form.

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2. Hereditary categories, their Hall algebras and Drinfeld doubles

In this section we recall some basic facts on Hall algebras of hereditary categories. Here we follow very closely the notations of our preceding article [6]. Let A be an essentially small hereditary abelian k-linear category such that for any pair $M, N \in \mathrm{Ob}(\mathsf{A})$ the k–vector spaces $\mathsf{Hom}_{\mathsf{A}}(M, N)$ and $\mathsf{Ext}^1_{\mathsf{A}}(M, N)$ are finite dimensional.

- Let $\mathsf{J} = \mathsf{J}_{\mathsf{A}} := \bigl(\mathrm{Ob}(\mathsf{A})/\cong\bigr)$ be the set of the isomorphy classes of objects in A.
- For an object $X \in \mathrm{Ob}(\mathsf{A})$, we denote by $[X]$ its image in J and set $a_X = \bigl|\mathsf{Aut}_{\mathsf{A}}(X)\bigr|$.
- For any triple of objects $X, Y, Z \in \mathrm{Ob}(\mathsf{A})$ we denote

$$P^Z_{X,Y} = \left|\Bigl\{(f,g) \in \mathsf{Hom}_{\mathsf{A}}(Y,Z) \times \mathsf{Hom}_{\mathsf{A}}(Z,X) \;\Big|\; 0 \to Y \xrightarrow{f} Z \xrightarrow{g} X \to 0 \;\text{ is exact}\Bigr\}\right|$$

  and set $F^Z_{X,Y} = \dfrac{P^Z_{X,Y}}{a_X \cdot a_Y}$. Note that $P^Z_{X,Y}$ and $F^Z_{X,Y}$ are integers.
- Let $K = K_0(\mathsf{A})$ be the K–group of A. For an object $X \in \mathrm{Ob}(\mathsf{A})$, we denote by $\overline{X}$ its image in K.
- Let $\widetilde{\mathbb{Q}}[K]$ be the group algebra of K. For a class $\alpha \in K$ we denote by K_α the corresponding element in $\widetilde{\mathbb{Q}}[K]$.
- Let $\langle -, - \rangle : K \times K \to \mathbb{Z}$ be the Euler form: for $X, Y \in \mathsf{Ob}(\mathsf{A})$ we have

$$\bigl\langle \overline{X}, \overline{Y} \bigr\rangle = \mathsf{dim}_k \,\mathsf{Hom}_{\mathsf{A}}(X,Y) - \mathsf{dim}_k \,\mathsf{Ext}^1_{\mathsf{A}}(X,Y).$$

  Next, let $(-, -) : K \times K \to \mathbb{Z}$ be the symmetrization of $\langle \overline{X}, \overline{Y} \rangle$, i.e. $(\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in K$.

Definition 2.1. Following a work of Ringel [33], the extended twisted *Hall algebra* of the abelian category A is an associative algebra over $\widetilde{\mathbb{Q}}$ defined as follows.

- As a vector space over $\widetilde{\mathbb{Q}}$, we have $\overline{H}(\mathsf{A}) := \bigoplus\limits_{Z \in \mathsf{J}} \widetilde{\mathbb{Q}}[Z]$ and $H(\mathsf{A}) := \overline{H}(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K]$.
- The product $\circ$ in $H(\mathsf{A})$ is defined by the following formulae:
  - For any $\alpha, \beta \in K$ we have: $K_\alpha \circ K_\beta = K_{\alpha+\beta}$.
  - For any $\alpha \in K$ and $[X] \in \mathsf{J}$ we have: $K_\alpha \circ [X] = v^{-(\alpha, \bar{X})}[X] \circ K_\alpha$.
  - For $[X], [Y] \in \mathsf{J}$ the product $\circ$ is defined to be

$$[X] \circ [Y] = v^{-\langle \bar{X}, \bar{Y} \rangle} \sum_{[Z] \in \mathsf{J}} F^Z_{X,Y} \, [Z].$$

- As it was shown in [33], the product $\circ$ is associative and the element $1 := [0] \otimes K_0$ is the unit element in $H(\mathsf{A})$. In what follows, we shall use the notation $[X]K_\alpha$ for the element $[X] \otimes K_\alpha \in H(\mathsf{A})$.

Definition 2.2. According to a work of Green [15], the Hall algebra $H(\mathsf{A})$ has a natural structure of a topological bialgebra endowed with a bialgebra pairing.

- There exists a comultiplication $H(\mathsf{A}) \xrightarrow{\Delta} H(\mathsf{A})\widehat{\otimes} H(\mathsf{A})$, given by the formula:

$$\Delta\bigl([Z]K_\alpha\bigr) = \sum_{[X],[Y] \in \mathsf{J}} v^{-\langle \bar{X}, \bar{Y} \rangle} \frac{P^Z_{X,Y}}{a_Z} [X] K_{\bar{Y}+\alpha} \otimes [Y] K_\alpha.$$

  Here we refer to [5, Section 2] for the definition of the completed tensor product $H(\mathsf{A})\widehat{\otimes} H(\mathsf{A})$. The map Δ is coassociative: $(\Delta \otimes \mathbb{1}) \circ \Delta = (\mathbb{1} \otimes \Delta) \circ \Delta$. Moreover, Δ is an algebra homomorphism.
- There exists a $\widetilde{\mathbb{Q}}$–linear algebra homomorphism $H(\mathsf{A}) \xrightarrow{\eta} \widetilde{\mathbb{Q}}$ given by the formula $\eta\bigl([Z]K_\alpha\bigr) = \delta_{Z,0}$. For any $a \in H(\mathsf{A})$ it satisfies the equality $(\eta \otimes \mathbb{1}) \circ \Delta(a) = (\mathbb{1} \otimes \eta) \circ \Delta(a) = a$.

- There exists a bilinear pairing $(\,-\,,\,-\,) : H(\mathsf{A}) \times H(\mathsf{A}) \to \widetilde{\mathbb{Q}}$ given by the formula
$$\Big([X]K_\alpha, [Y]K_\beta\Big) = v^{-(\alpha,\beta)}\frac{\delta_{X,Y}}{a_X}.$$
This pairing is non-degenerate on $\bar{H}(\mathsf{A})$ and symmetric. Moreover, for any elements $a, b, c \in H(\mathsf{A})$, the expression $\big(a \otimes b, \Delta(c)\big)$ takes a finite value and the equalities $(a \circ b,\, c) = \big(a \otimes b, \Delta(c)\big)$ and $(a, 1) = \eta(a)$ are fulfilled.
- If A is moreover a finite length hereditary category (for instance, the category of representations of a finite quiver) then $H(\mathsf{A})$ is a true bialgebra over $\widetilde{\mathbb{Q}}$ with the multiplication $\circ$, unit 1, comultiplication Δ and counit η. Moreover, by a work of Xiao [44], the Hall algebra $H(\mathsf{A})$ has a natural Hopf algebra structure.

Remark 2.3. The fact that the map Δ is a homomorphism of algebras, was proven by Green [15] in the case when A is the category of representations of a finite quiver. The case of general hereditary abelian categories can be treated in a similar way, see [35, 40].

Our next goal is to introduce the *reduced* Drinfeld double of the topological bialgebra $H(\mathsf{A})$. To define it, consider the pair of algebras $H^{\pm}(\mathsf{A})$, where we use the notation
$$H^+(\mathsf{A}) = \bigoplus_{Z \in \mathsf{J}} \widetilde{\mathbb{Q}}[Z]^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \text{ and } H^-(\mathsf{A}) = \bigoplus_{Z \in \mathsf{J}} \widetilde{\mathbb{Q}}[Z]^- \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K].$$
In these notations $H^{\pm}(\mathsf{A}) = H(\mathsf{A})$ viewed as $\widetilde{\mathbb{Q}}$-algebras. Let $a = [Z]K_\gamma$ and
$$\Delta(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)} = \sum_{[X],\,[Y] \in \mathsf{J}} v^{-\langle \bar{X}, \bar{Y}\rangle}\frac{P^Z_{X,Y}}{a_Z}[X]K_{\bar{Y}+\gamma} \otimes [Y]K_\gamma.$$
Then we denote
$$\Delta(a^{\pm}) = \sum_i a_i^{(1)\pm} \otimes a_i^{(2)\pm} = \sum_{[X],\,[Y] \in \mathsf{J}} v^{-\langle \bar{X}, \bar{Y}\rangle}\frac{P^Z_{X,Y}}{a_Z}[X]^{\pm}K_{\pm\bar{Y}+\gamma} \otimes [Y]^{\pm}K_\gamma.$$

Definition 2.4. The Drinfeld double of the topological bialgebra $H(\mathsf{A})$ with respect to the Green's pairing $(\,-\,,\,-\,)$ is the associative algebra $\widetilde{D}H(\mathsf{A})$, which is the free product of algebras $H^+(\mathsf{A})$ and $H^-(\mathsf{A})$ subject to the following relations $D(a, b)$ for all $a, b \in H(\mathsf{A})$:
$$\sum_{i,j} a_i^{(1)-} b_j^{(2)+}\big(a_i^{(2)}, b_j^{(1)}\big) = \sum_{i,j} b_j^{(1)+} a_i^{(2)-}\big(a_i^{(1)}, b_j^{(2)}\big).$$
The *reduced* Drinfeld double $DH(\mathsf{A})$ is the quotient of $\widetilde{D}H(\mathsf{A})$ by the two-sided ideal
$$I = \big\langle K_\alpha^+ \otimes K_{-\alpha}^- - \mathbb{1}^+ \otimes \mathbb{1}^- \mid \alpha \in K\big\rangle.$$
Note that if A is a finite length abelian category, then I is a Hopf ideal and the reduced Drinfeld double $DH(\mathsf{A})$ is again a Hopf algebra.

Proposition 2.5. *We have an isomorphism of $\widetilde{\mathbb{Q}}$–vector spaces*
$$\overline{H}^+(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \overline{H}^-(\mathsf{A}) \xrightarrow{\mathsf{mult}} DH(\mathsf{A})$$
also called the triangular decomposition of $DH(\mathsf{A})$.

Remark 2.6. The notion of the reduced Drinfeld double was introduced by Xiao in [44]. In the case of Hopf algebras, a proof of Proposition 2.5 can be found in a book of Joseph [21], whereas the case of topological bialgebras was treated in our previous joint paper [5].

The following important theorem was recently proven by Cramer [8].

Theorem 2.7. *Let* A *and* B *be two* k*-linear finitary hereditary categories such that there exists an equivalence of triangulated categories* $D^b(\mathsf{A}) \xrightarrow{\mathbb{F}} D^b(\mathsf{B})$*. Let* $D^b(\mathsf{A})/[2] \xrightarrow{\widehat{\mathbb{F}}} D^b(\mathsf{B})/[2]$ *be the induced equivalence of the root categories. Then there is an algebra isomorphism* $DH(\mathsf{A}) \xrightarrow{\mathbb{F}} DH(\mathsf{B})$ *uniquely determined by the following property. For any* $\alpha \in K$ *we have:* $\mathbb{F}(K_\alpha) = K_{\mathbb{F}(\alpha)}$*. Next, for any object* $X \in \mathrm{Ob}(\mathsf{A})$ *such that* $\mathbb{F}(X) \cong \widehat{X}[-n_{\mathbb{F}}(X)]$ *with* $\widehat{X} \in \mathrm{Ob}(\mathsf{B})$ *and* $n_{\mathbb{F}}(X) \in \mathbb{Z}$ *we have:*

$$\mathbb{F}\big([X]^{\pm}\big) = v^{n_{\mathbb{F}}(X)\langle \bar{X},\bar{X}\rangle}\,\big[\widehat{X}\big]^{\overline{n_{\mathbb{F}}(X)}} K^{n_{\mathbb{F}}(X)}_{\widehat{\mathbb{F}}(X^{\pm})}, \tag{1}$$

where $\overline{n_{\mathbb{F}}(X)} = +$ *if* $n_{\mathbb{F}}(X)$ *is even and* $-$ *if* $n_{\mathbb{F}}(X)$ *is odd.*

3. Perpendicular categories and Hall algebras

Definition 3.1. Let A be a hereditary abelian category and C be its full subcategory. Following Geigle and Lenzing [14], we define the *perpendicular category* $\mathsf{C}^\perp$ as follows:

$$\mathsf{C}^\perp := \big\{X \in \mathsf{Ob}(\mathsf{A}) \,\big|\, \mathsf{Hom}_{\mathsf{A}}(C,X) = 0 = \mathsf{Ext}^1_{\mathsf{A}}(C,X) \quad \text{for all} \quad C \in \mathsf{Ob}(\mathsf{C})\big\}\,.$$

Proposition 3.2. *In the above notations, the category* $\mathsf{C}^\perp$ *is abelian, hereditary and extension-closed in* A*.*

Proof. Let $X, Y \in \mathsf{Ob}(\mathsf{C}^\perp)$ be any pair of objects and $X \xrightarrow{f} Y$ be any morphism. We first show that $Z = \mathsf{im}(f)$ belongs to $\mathsf{C}^\perp$. Let $X \xrightarrow{p} Z$ and $Z \xrightarrow{\imath} Y$ be the canonical morphisms. Note that $f = \imath p$ and $T := \mathsf{ker}(p) \cong \mathsf{ker}(f)$. For any $C \in \mathsf{Ob}(\mathsf{C})$ we have exact sequences

$$0 \longrightarrow \mathsf{Hom}_{\mathsf{A}}(C,Z) \xrightarrow{\imath_*} \mathsf{Hom}_{\mathsf{A}}(C,Y) \quad \text{and} \quad \mathsf{Ext}^1_{\mathsf{A}}(C,X) \xrightarrow{p_*} \mathsf{Ext}^1_{\mathsf{A}}(C,Z) \longrightarrow \mathsf{Ext}^2_{\mathsf{A}}(C,T).$$

Since A is hereditary, we have: $\mathsf{Hom}_{\mathsf{A}}(C,Z) = 0 = \mathsf{Ext}^1_{\mathsf{A}}(C,Z)$, hence $Z \in \mathsf{Ob}(\mathsf{C}^\perp)$. Next, let $L = \mathsf{ker}(f) \cong T$ and $N = \mathsf{coker}(f)$. Then we have the following short exact sequences

$$0 \longrightarrow T \xrightarrow{\jmath} X \xrightarrow{p} Z \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Z \xrightarrow{\imath} Y \xrightarrow{q} N \longrightarrow 0.$$

Taking into account that $X, Y, Z \in \mathsf{Ob}(\mathsf{C}^\perp)$ and the assumption A to be hereditary, the induced long exact sequences of $\mathsf{Ext}^\bullet_{\mathsf{A}}$ imply that T, N belong to $\mathsf{C}^\perp$ as well. Hence, the perpendicular category $\mathsf{C}^\perp$ is an abelian extension-closed subcategory of A.

Using Baer's description of the bifunctor $\mathsf{Ext}^1_{\mathsf{C}^\perp}(-\,,\,-)$, it is easy to see that for any $X, Y \in \mathsf{Ob}(\mathsf{C}^\perp)$ the canonical morphism $\mathsf{Ext}^1_{\mathsf{C}^\perp}(X,Y) \to \mathsf{Ext}^1_{\mathsf{A}}(X,Y)$ is an isomorphism. It remains to show that the category $\mathsf{C}^\perp$ is hereditary. Let $\omega \in \mathsf{Ext}^2_{\mathsf{C}^\perp}(X,Y)$ be an extension class represented by an exact sequence

$$0 \longrightarrow Y \xrightarrow{a} E \xrightarrow{b} F \xrightarrow{c} X \longrightarrow 0,$$

where $E, F \in \mathsf{Ob}(\mathsf{C}^\perp)$. Let $I = \mathsf{im}(b) \cong \mathsf{ker}(c)$, then $I \in \mathsf{Ob}(\mathsf{C}^\perp)$. Since the extension ω is zero, viewed as an element of $\mathsf{Ext}^2_{\mathsf{A}}(X, Y)$, there exists an object J of A such that the following diagram is commutative

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r, "a"] \arrow[d, "="'] & E \arrow[r, "e"] \arrow[d] \arrow[dr, dotted, "b"] & I \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & Y \arrow[r] & J \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, "c"] & 0 \\
 & & X \arrow[r, "="] \arrow[d] & X \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}
$$

where all rows and columns are exact sequences in A. Since the subcategory $\mathsf{C}^\perp$ is extension-closed in A, it follows that $J \in \mathsf{Ob}(\mathsf{C}^\perp)$. Hence, $\mathsf{Ext}^2_{\mathsf{C}^\perp}(X, Y) = 0$ and the perpendicular category $\mathsf{C}^\perp$ is hereditary. $\square$

As a corollary, we obtain the following interesting result.

Theorem 3.3. *Let A be a hereditary abelian category over k, C be some full subcategory of A and $\mathsf{C}^\perp$ be its perpendicular category. Then the embedding functor $\mathsf{C}^\perp \xrightarrow{\mathbb{I}} \mathsf{A}$ induces an injective algebra homomorphism of the reduced Drinfeld doubles $DH(\mathsf{C}^\perp) \to DH(\mathsf{A})$.*

Proof. By Proposition 3.2 we know that the category $\mathsf{C}^\perp$ is an extension-closed hereditary full abelian subcategory of A. Moreover, for any $X, Y \in \mathsf{Ob}(\mathsf{C}^\perp)$ and $i = 0, 1$ the canonical morphism $\mathsf{Ext}^i_{\mathsf{C}^\perp}(X, Y) \to \mathsf{Ext}^i_{\mathsf{A}}(X, Y)$ is an isomorphism. Hence, the functor $\mathbb{I}$ induces an isometry $K_0(\mathsf{C}^\perp) \to K_0(\mathsf{A})$ with respect to the Euler forms. Moreover, for any $X, Y, Z \in \mathsf{Ob}(\mathsf{C}^\perp)$ the Hall constants $F^Z_{X,Y}$, $P^Z_{X,Y}$ and a_X are preserved by $\mathbb{I}$. Hence, we get an injective algebra homomorphism $H(\mathsf{C}^\perp) \xrightarrow{\mathbb{I}} H(\mathsf{A})$.

Next, let X, Y be objects of $\mathsf{C}^\perp$ and Z be an object of A such that there exist an epimorphism $X \xrightarrow{p} Z$ and a monomorphism $Z \xrightarrow{\imath} Y$. As in the proof of Proposition 3.2 it can be shown that all three objects Z, $\mathsf{ker}(p)$ and $\mathsf{coker}(\imath)$ belong to $\mathsf{C}^\perp$. This shows that the relations $D(X^+, Y^-)$ of the Drinfeld double $DH(\mathsf{C}^\perp)$ are preserved by $\mathbb{I}$ and we have a well-defined injective algebra homomorphism $DH(\mathsf{C}^\perp) \xrightarrow{\mathbb{I}} DH(\mathsf{A})$. Note that $\mathbb{I}$ preserves the triangular decomposition:

$$\overline{H}^+(\mathsf{C}^\perp) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}\big[K_0(\mathsf{C}^\perp)\big] \otimes_{\widetilde{\mathbb{Q}}} \overline{H}^-(\mathsf{C}^\perp) \xrightarrow{\mathbb{I}} \overline{H}^+(\mathsf{A}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}\big[K_0(\mathsf{A})\big] \otimes_{\widetilde{\mathbb{Q}}} \overline{H}^-(\mathsf{A}).$$

$\square$

As the first application of Theorem 3.3, consider the following example.

Example 3.4. Let $\overrightarrow{\Delta}$ be a $\widehat{D}_4$–quiver with the orientation

$$\overrightarrow{\Delta} = \begin{tikzcd} 1 \arrow[dr] & & 4 \arrow[dl] \\ & 3 & \\ 2 \arrow[ur] & & 5 \arrow[ul] \end{tikzcd}$$

and $\mathsf{A} = \mathsf{Rep}(\overrightarrow{\Delta})$ be its category of representations. Consider the indecomposable object T with the dimension vector $\begin{smallmatrix} 0 & & 0 \\ & 1 & \\ 1 & & 1 \end{smallmatrix}$ and the perpendicular category $\mathsf{B} = T^{\perp}$. Let X_α, X_β, X_γ and X_δ be the indecomposable representations of $\overrightarrow{\Delta}$ with the dimension vectors

$$\alpha = \begin{smallmatrix} 0 & & 0 \\ & 1 & \\ 1 & & 0 \end{smallmatrix}, \quad \beta = \begin{smallmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 1 \end{smallmatrix}, \quad \gamma = \begin{smallmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{smallmatrix} \quad \text{and} \quad \delta = \begin{smallmatrix} 0 & & 1 \\ & 0 & \\ 0 & & 0 \end{smallmatrix}.$$

It is easy to see that X_α, X_β, X_γ and X_δ belong to $T^{\perp}$. Moreover, for any object X of the category B with the dimension vector $\begin{smallmatrix} m_1 & & m_4 \\ & m_3 & \\ m_2 & & m_5 \end{smallmatrix}$ there exists a short exact sequence

$$0 \longrightarrow Y \longrightarrow X \longrightarrow X_\gamma^{m_1} \oplus X_\delta^{m_4} \longrightarrow 0,$$

where Y has the dimension vector $\begin{smallmatrix} 0 & & 0 \\ & m_3 & \\ m_2 & & m_5 \end{smallmatrix}$. As in the proof of Proposition 3.2 one shows that Y belongs to B. Since Y is supported on a subquiver of type A_3, it is easy to deduce that Y splits into a direct sum of several copies of X_α and X_β. Hence, B is a finite length hereditary abelian category with four simple objects X_α, X_β, X_γ and X_δ.

Note that $\mathsf{Ext}^1_{\mathsf{A}}(X_\gamma, X_\alpha) = \mathsf{Ext}^1_{\mathsf{A}}(X_\gamma, X_\beta) = \mathsf{Ext}^1_{\mathsf{A}}(X_\delta, X_\alpha) = \mathsf{Ext}^1_{\mathsf{A}}(X_\delta, X_\beta) = k$, whereas all the remaining Ext^1–spaces between the simple objects of B vanish. In the notations of Keller's work [23] we have: $\mathsf{B} = \mathsf{Filt}(X_\alpha, X_\beta, X_\gamma, X_\delta)$. As in [23, Section 7] one can show that B is equivalent to $\mathsf{Rep}(\overrightarrow{\Delta}')$, where $\overrightarrow{\Delta}'$ is an $\widehat{A}_3$–quiver with the orientation

$$\overrightarrow{\Delta}' = \begin{tikzcd} & \alpha & \\ \gamma \arrow[ur] \arrow[dr] & & \delta \arrow[ul] \arrow[dl] \\ & \beta & \end{tikzcd}$$

By Theorem 3.3, we have an injective morphism of the reduced Drinfeld doubles

$$DH\big(\mathsf{Rep}(\overrightarrow{\Delta}')\big) \xrightarrow{\mathbb{I}} DH\big(\mathsf{Rep}(\overrightarrow{\Delta})\big). \tag{2}$$

Note that the following equalities are true in $H\big(\mathsf{Rep}(\overrightarrow{\Delta})\big)$:

$$[X_\alpha] = v^{-1}[S_1] \circ [S_3] - [S_3] \circ [S_1] \quad \text{and} \quad [X_\beta] = v^{-1}[S_5] \circ [S_3] - [S_3] \circ [S_5],$$

whereas $[X_\gamma] = [S_1]$ and $[X_\delta] = [S_4]$. Hence, the algebra homomorphism $\mathbb{I}$ restricts to an injective homomorphism $DC(\overrightarrow{\Delta}') \xrightarrow{\mathbb{I}} DC(\overrightarrow{\Delta})$ of the reduced Drinfeld doubles of the composition subalgebras.

Remark 3.5. Passing to the generic composition algebras, the map $\mathbb{I}$ constructed in Example 3.4, can be lifted to an injective algebra homomorphism of quantum affine algebras $U_v(\widehat{A}_3) \xrightarrow{\mathbb{I}} U_v(\widehat{D}_4)$. A similar game can be played with the category of representations of other affine Dynkin quivers. However, we prefer to postpone its further discussion to a future work.

4. Generalities on coherent sheaves on a weighted projective line

In this subsection, we recall some basic facts on the category of coherent sheaves on a weighted projective line, following a pioneering work of Geigle and Lenzing [13]. Let k be a base field. For a set of positive integers $\underline{p} = \{p_1, p_2, \dots, p_n\}$ and a set of pairwise distinct points $\underline{\lambda} = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ of $\mathbb{P}^1$ normalized in such a way that $\lambda_1 = \infty = (1:0)$, $\lambda_2 = 0 = (0:1)$, $\lambda_3 = 1 = (1:1)$ and $\lambda_i = (1:\lambda_i)$ for $i \ge 3$, consider the ideal $I = I(\underline{p}, \underline{\lambda}) := \langle x_2^{p_2} - \lambda_i x_1^{p_1} - x_i^{p_i} | i \ge 3 \rangle$ and the algebra $R = R(\underline{p}, \underline{\lambda}) = k[x_1, x_2, \dots, x_n]/I$. In what follows we shall assume that in the case of a finite field k its cardinality is bigger than the cardinality of $\underline{\lambda}$.

Let $\mathbb{L}(\underline{p})$ be the abelian group generated by the elements $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ subject to the relations $p_1\vec{x}_1 = p_2\vec{x}_2 = \cdots = p_n\vec{x}_n =: \vec{c}$. Then the polynomial algebra $k[x_1, \dots, x_n]$ is $\mathbb{L}(\underline{p})$–graded and I is an $\mathbb{L}(\underline{p})$–homogeneous ideal. Hence, the algebra R is $\mathbb{L}(\underline{p})$–graded, too. Note that $\mathbb{L}(\underline{p}) \cong \mathbb{Z} \oplus \big(\bigoplus_{i=1}^{t} \mathbb{Z}/p_i\mathbb{Z}\big)$ and any $\vec{x} \in \mathbb{L}(\underline{p})$ can be uniquely written as

$$\vec{x} = l\vec{c} + \sum_{i=1}^{n} a_i \vec{x}_i, \tag{3}$$

where $l \in \mathbb{Z}$ and $0 \le a_i < p_i$ for all $1 \le i \le n$. In what follows, the decomposition (3) will be called the canonical form of an element $\vec{x} \in \mathbb{L}(\underline{p})$. We say that $\vec{x} \in \mathbb{L}(\underline{p})_+$ if in its canonical decomposition (3) we have $l \ge 0$ and $a_i \ge 0$ for all $1 \le i \le n$.

Definition 4.1. The category of coherent sheaves $\mathsf{Coh}\big(\mathbb{X}(\underline{p}, \underline{\lambda})\big)$ on a weighted projective line $\mathbb{X}(\underline{p}, \underline{\lambda})$ is the Serre quotient $\mathsf{grmod}(R)/\mathsf{grmod}_0(R)$ of the category of graded Noetherian R–modules modulo the category of finite dimensional graded R–modules. For a graded R–module M we shall denote by $\widetilde{M}$ the corresponding object of $\mathsf{Coh}(\mathbb{X})$.

For an $\mathbb{L}(\underline{p})$–homogeneous prime ideal $\mathfrak{p}$ consider the ring

$$\mathcal{O}_{\mathbb{X},\mathfrak{p}} := R_{\mathfrak{p}} = \left\{ \frac{f}{g} \,\Big|\, g \in R \text{ is } \mathbb{L}(\underline{p})\text{–homogeneous}, \ g \notin \mathfrak{p} \right\}.$$

Note that $\mathcal{O}_{\mathbb{X},\mathfrak{p}}$ is again an $\mathbb{L}(\underline{p})$–graded discrete valuation ring and we have an exact functor $\mathsf{Coh}(\mathbb{X}) \to \mathsf{grmod}(R_{\mathfrak{p}})$ mapping a coherent sheaf $\mathcal{F}$ to the module $\mathcal{F}_{\mathfrak{p}}$.

The following observation is due to Geigle and Lenzing [13, Section 1.3].

Lemma 4.2. *Let $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$ be a weighted projective line over a field k. Then there are two types of homogeneous prime ideals of height one in $R(\underline{p}, \underline{\lambda})$:*

1. *Ideals of the form $\big(f(x_1^{p_1}, x_2^{p_2})\big)$, where $f(y_1, y_2) \in k[y_1, y_2]$ is an irreducible homogeneous polynomial in y_1, y_2, which is different from y_1 and y_2;*

(2) *Exceptional prime ideals* $(x_1), (x_2), \dots, (x_n)$.

Definition 4.3. A coherent sheaf $\mathcal{F}$ on a weighted projective line $\mathbb{X}$ is called

- locally free (or a vector bundle) if $\mathcal{F}_\mathfrak{p}$ is a projective object in $\mathsf{grmod}(R_\mathfrak{p})$ for all $\mathbb{L}(\underline{p})$–homogeneous prime ideals $\mathfrak{p}$;
- a skyscraper sheaf (or a torsion sheaf) if $\mathcal{F}_\mathfrak{p} = 0$ for all but finitely many $\mathbb{L}(\underline{p})$–homogeneous prime ideals $\mathfrak{p}$.

Let $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$ be a weighted projective line over a field k. It turns out that the category $\mathsf{Coh}(\mathbb{X})$ shares a lot of common properties with the category of coherent sheaves on a commutative smooth projective curve. We list some fundamental results on $\mathsf{Coh}(\mathbb{X})$, which are due to Geigle and Lenzing [13] and [14], see also a recent survey article [7].

1. The abelian category $\mathsf{Coh}(\mathbb{X})$ is a hereditary noetherian category, which is Ext–finite over the base field k.

2. The canonical functor $\mathsf{grmod}(R) \to \mathsf{Coh}(\mathbb{X})$ induces an equivalence $\mathsf{CM}(R) \to \mathsf{VB}(\mathbb{X})$ between the category of $\mathbb{L}(\underline{p})$–graded Cohen–Macaulay R–modules and the category of vector bundles on $\mathbb{X}$. For $\vec{x} \in \mathbb{L}(\underline{p})$ we denote $\mathcal{O}(\vec{x}) = \mathcal{O}_\mathbb{X}(\vec{x}) := \widetilde{R(\vec{x})}$. Then the map

$$R_{\vec{y}-\vec{x}} \cong \mathsf{Hom}_{\mathsf{grmod}(R)}\bigl(R(\vec{x}), R(\vec{y})\bigr) \xrightarrow{\mathsf{can}} \mathsf{Hom}_\mathbb{X}\bigl(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})\bigr)$$

is an isomorphism of vector spaces for all $\vec{x}, \vec{y} \in \mathbb{L}(\underline{p})$.

3. For any coherent sheaf $\mathcal{F}$ on a weighted projective line $\mathbb{X}$, the canonical exact sequence

$$0 \longrightarrow \mathsf{tor}(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\,\mathsf{tor}(\mathcal{F}) \longrightarrow 0$$

splits, i.e. any coherent sheaf on $\mathbb{X}$ is a direct sum of a vector bundle and a torsion sheaf.

4. The category of torsion coherent sheaves $\mathsf{Tor}(\mathbb{X})$ splits into a direct sum of blocks:

$$\mathsf{Tor}(\mathbb{X}) = \bigoplus_{\mathfrak{p} \in \mathsf{P}} \mathsf{Tor}_\mathfrak{p}(\mathbb{X}),$$

where P is the set of $\mathbb{L}(\underline{p})$–homogeneous prime ideals in R of height one and $\mathsf{Tor}_\mathfrak{p}(\mathbb{X})$ is the category of torsion coherent sheaves supported at the prime ideal $\mathfrak{p}$. Note that each block $\mathsf{Tor}_\mathfrak{p}(\mathbb{X})$ is equivalent to the category of finite length $\mathbb{L}(\underline{p})$–graded modules over $R_\mathfrak{p}$.

5. In the notations of Lemma 4.2, let $\mathfrak{p} = \bigl(f(x_1^{p_1}, x_2^{p_2})\bigr)$ be a homogeneous prime ideal of the first type. Then the category $\mathsf{grmod}_{\mathbb{L}(\underline{p})}(R_\mathfrak{p})$ is equivalent to the category of nilpotent representations over of the Jordan quiver over the field $k(\mathfrak{p}) := k[y]/f(y,1)$. Let d be the degree of the homogeneous form $f(y_1, y_2) \in k[y_1, y_2]$. Then the unique simple object $\mathcal{S}_\mathfrak{p}$ in $\mathsf{Tor}_\mathfrak{p}(\mathbb{X})$ has a locally free resolution

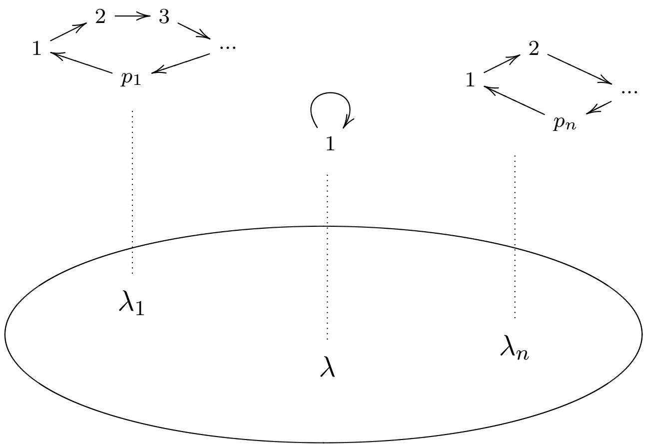
$$0 \longrightarrow \mathcal{O}(-d\vec{c}) \xrightarrow{f(x_1^{p_1}, x_2^{p_2})} \mathcal{O} \longrightarrow \mathcal{S}_\mathfrak{p} \longrightarrow 0.$$

6. If $\mathfrak{p} = (x_i)$ is an exceptional prime ideal for $1 \le i \le n$, then the category $\mathsf{Tor}_\mathfrak{p}(\mathbb{X})$ is equivalent to the category of nilpotent finite-dimensional representations of a cyclic quiver

$\overrightarrow{C}_{p_i}$ with p_i vertices over the base field k. In particular, the category $\mathsf{Tor}_{\mathfrak{p}}(\mathbb{X})$ has p_i simple objects $\mathcal{S}_i^{(j)}$, where $1 \le j \le p_i$. They have the following locally free resolutions:

$$0 \longrightarrow \mathcal{O}(-j\vec{x}_i) \xrightarrow{x_i} \mathcal{O}\bigl(-(j-1)\vec{x}_i\bigr) \longrightarrow \mathcal{S}_i^{(j)} \longrightarrow 0$$

for $1 \le j \le p_i$. Note that $\mathsf{Ext}^1\bigl(\mathcal{S}_i^{(j)}, \mathcal{S}_i^{(j+1)}\bigr) = k$ for all $1 \le j \le p_i$, where $\mathcal{S}_i^{(p_i+1)} := \mathcal{S}_i^{(1)}$. All other Ext^1–spaces between the simple objects $\mathcal{S}_i^{(j)}$ vanish. The structure of the category of torsion coherent sheaves $\mathsf{Tor}(\mathbb{X})$ can be visualized by the following picture



7. Consider the element $\vec{\omega} := (n-2)\vec{c} - \sum\limits_{i=1}^{n} \vec{x}_i \in \mathbb{L}(\underline{p})$. Then the functor $\mathcal{F} \mapsto \mathcal{F}(\vec{\omega}) =: \tau(\mathcal{F})$ is the Auslander–Reiten translation in the triangulated category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$. It means that for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ we have a bi-functorial isomorphism

$$\mathsf{Hom}(\mathcal{F}, \mathcal{G}) \cong \mathbb{D}\,\mathsf{Ext}^1\bigl(\mathcal{G}, \mathcal{F}(\vec{\omega})\bigr),$$

where $\mathbb{D} = \mathsf{Hom}_k(\,-\,, k)$ is the duality over the base field. Note that for any $\mathbb{L}(\underline{p})$–homogeneous prime ideal $\mathfrak{p}$ of the first type we have $\tau(\mathcal{S}_{\mathfrak{p}}) \cong \mathcal{S}_{\mathfrak{p}}$, whereas for the exceptional simple modules we have: $\tau\bigl(\mathcal{S}_i^{(j)}\bigr) \cong \mathcal{S}_i^{(j+1)}$ for all (i, j) such that $1 \le i \le n$ and $1 \le j \le p_i$. In these notations, $\mathcal{S}_i^{(p_i+1)} := \mathcal{S}_i^{(1)}$ for all $1 \le i \le n$.

8. The vector bundle $\mathcal{F} := \mathcal{O} \oplus \bigoplus\limits_{i=1}^{n}\Bigl(\bigoplus\limits_{l=1}^{p_i-1} \mathcal{O}(l\vec{x}_i)\Bigr) \oplus \mathcal{O}(\vec{c})$ is a *tilting object* in the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$. In particular, the derived category $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ is equivalent to $D^b\bigl(C(\underline{p}, \underline{\lambda}) - \mathsf{mod}\bigr)$, where $C(\underline{p}, \underline{\lambda}) = \mathsf{End}(\mathcal{F})^{\mathrm{op}}$ is the so-called *canonical algebra* of type $(\underline{p}, \underline{\lambda})$ introduced and studied by Ringel [33]. In particular, if $\mathbb{X}$ is a weighted projective line of domestic type Δ then $D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ is equivalent to $D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta}\bigr)$, where Δ is the corresponding affine Dynkin quiver.

9. The K–group $K_0(\mathbb{X})$ of the category $\mathsf{Coh}(\mathbb{X})$ is free of rank $\sum\limits_{i=1}^{n}(p_i - 1) + 2$ with a basis

$$\Bigl\{\overline{\mathcal{O}}, \overline{\mathcal{O}(\vec{x}_1)}, \ldots, \overline{\mathcal{O}\bigl((p_1-1)\vec{x}_1\bigr)}, \ldots, \overline{\mathcal{O}(\vec{x}_n)}, \ldots, \overline{\mathcal{O}\bigl((p_n-1)\vec{x}_n\bigr)}, \overline{\mathcal{O}(\vec{c})}\Bigr\}.$$

Let $x \in \mathbb{P}^1$ be a non-weighted point, $\mathcal{S}$ a simple torsion sheaf supported at x and $\delta = \overline{\mathcal{S}}$ its class in the K–group. Then for any $1 \le i \le n$ we have the following relations in $K_0(\mathbb{X})$: $\sum\limits_{j=1}^{p_i} \overline{\mathcal{S}_i^{(j)}} = \overline{\mathcal{O}} - \overline{\mathcal{O}(-\vec{c})} = \delta$. Moreover, the set $\Big\{\overline{\mathcal{S}_1^{(1)}}, \dots, \overline{\mathcal{S}_1^{(p_1-1)}}, \dots, \overline{\mathcal{S}_n^{(1)}}, \dots, \overline{\mathcal{S}_n^{(p_n-1)}}, \delta, \overline{\mathcal{O}}\Big\}$ is again a basis of $K_0(\mathbb{X})$. The rank function $K_0(\mathbb{X}) \xrightarrow{\mathrm{rk}} \mathbb{Z}$ is defined by the rule $\mathrm{rk}(\overline{\mathcal{O}}) = 1$ and $\mathrm{rk}\big(\overline{\mathcal{S}_i^{(j)}}\big) = 0$ for all $1 \le i \le n$ and $1 \le j \le p_i$. Moreover, for a coherent sheaf $\mathcal{F}$ the integer $\mathrm{rk}\big(\overline{\mathcal{F}}\big)$ coincides with the geometrical rank of $\mathcal{F}$ as defined in [13, Section 1.6].

10. Let $\mathsf{C} = \mathsf{add}\big(S_i^{(1)}\big)$ be the Serre subcategory of $\mathsf{Coh}(\mathbb{X})$ generated by an exceptional simple object supported at the point λ_i and $\mathsf{C}^\perp$ be the perpendicular subcategory. Let $\mathbb{L}(\underline{p}')$ be the abelian group generated by the elements $\vec{y}_1, \dots, \vec{y}_n$ subject to the relations $(p_1 - 1)\vec{y}_1 = p_2\vec{y}_2 = \dots = p_n\vec{y}_n$ and $\mathbb{Y} = \mathbb{X}(\underline{\lambda}, \underline{p}')$ be the corresponding weighted projective line. Then we have:

(1) The Serre quotient $\mathsf{Coh}(\mathbb{X})/\mathsf{C}$ is equivalent to $\mathsf{Coh}(\mathbb{Y})$. Moreover, the canonical functor $\mathsf{Coh}(\mathbb{X}) \xrightarrow{\mathbb{P}} \mathsf{Coh}(\mathbb{X})/\mathsf{C}$ has an *exact fully faithful* right adjoint functor $\mathbb{F}$.

(2) The functor $\mathsf{C}^\perp \to \mathsf{Coh}(\mathbb{X})/\mathsf{C}$ given by the composition of $\mathbb{P}$ and the embedding $\mathsf{C}^\perp \to \mathsf{Coh}(\mathbb{X})$, is an equivalence of categories. Moreover, the essential image of the functor $\mathsf{Coh}(\mathbb{X})/\mathsf{C} \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$ is the category $\mathsf{C}^\perp$.

(3) The functor $\mathsf{Coh}(\mathbb{Y}) \xrightarrow{\simeq} \mathsf{Coh}(\mathbb{X})/\mathsf{C} \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$ acts on objects as follows:
   (a) Let $\vec{y} = l\vec{c} + \sum_{i=1}^n b_i\vec{y}_i \in \mathbb{L}(\underline{p}')$ be written in its canonical form (3). Then $\mathbb{F}\big(\mathcal{O}_{\mathbb{Y}}(\vec{y})\big) \cong \mathcal{O}_{\mathbb{X}}\big(l\vec{c} + \sum_{i=1}^n b_i\vec{x}_i\big)$.
   (b) For any $2 \le i \le n$ and $1 \le j \le p_i$ we have: $\mathbb{F}\big(\mathcal{S}_i^{(j)}\big) \cong \mathcal{S}_i^{(j)}$.
   (c) Let X be a object of the category $\mathsf{Tor}_{\lambda_1}(\mathbb{Y})$. Assume it is given by a representation of the cyclic quiver $\overrightarrow{C}_{p_1-1}$

$$\begin{tikzcd} V_1 \arrow[r, "A_1"] & V_2 \arrow[r, "A_2"] & \cdots \arrow[r] & V_{p_1-1} \arrow[lll, bend left, "A_{p_1-1}"] \end{tikzcd}$$

   Then $\mathbb{F}(X)$ belongs to the category $\mathsf{Tor}_{\lambda_1}(\mathbb{X})$ and is given by the representation

$$\begin{tikzcd} V_1 \arrow[r, "I"] & V_1 \arrow[r, "A_1"] & V_2 \arrow[r, "A_2"] & \cdots \arrow[r] & V_{p_1-1} \arrow[llll, bend left, "A_{p_1-1}"] \end{tikzcd}$$

   of the cyclic quiver $\overrightarrow{C}_{p_1}$, where I is the identity operator.

11. More generally, let C be the Serre subcategory of $\mathsf{Coh}(\mathbb{X})$ generated by the simple objects $\mathcal{S}_1^{(1)}, \dots, \mathcal{S}_1^{(p_1-1)}, \dots, \mathcal{S}_n^{(1)}, \dots, \mathcal{S}_n^{(p_n-1)}$. Then the perpendicular category $\mathsf{C}^\perp$ is equivalent to the Serre quotient $\mathsf{Coh}(\mathbb{X})/\mathsf{C}$, which on its turn is equivalent to the category $\mathsf{Coh}(\mathbb{P}^1)$. The canonical functor $\mathsf{Coh}(\mathbb{X}) \xrightarrow{\mathbb{P}} \mathsf{Coh}(\mathbb{X})/\mathsf{C}$ has an exact fully faithful right adjoint functor $\mathbb{F}$, whose essential image is the perpendicular category $\mathsf{C}^\perp$. The functor $\mathsf{Coh}(\mathbb{P}^1) \xrightarrow{\simeq} \mathsf{Coh}(\mathbb{X})/\mathsf{C} \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$ acts on objects as follows:

(1) $\mathbb{F}\big(\mathcal{O}_{\mathbb{P}^1}(l)\big) \cong \mathcal{O}_{\mathbb{X}}(l\vec{c})$ for all $l \in \mathbb{Z}$.
(2) If $x \in \mathbb{P}^1$ is a regular point then for any $\mathcal{S} \in \mathsf{Tor}_x(\mathbb{P}^1)$ we have: $\mathbb{F}(\mathcal{S}) \cong \mathcal{S}$.
(3) If $x = \lambda_i$ is a special point, $1 \le i \le n$ then $\mathbb{F}\big(\mathcal{O}_{\mathbb{P}^1}/\mathfrak{m}_x^l\big)$ belongs to $\mathsf{Tor}_{\lambda_i}(\mathbb{X})$ and is given by the following representation of the cyclic quiver $\overrightarrow{C}_{p_i}$:

$$\begin{tikzcd} k^l \arrow[r, "I"] & k^l \arrow[r, "I"] & \cdots \arrow[r, "I"] & k^l \arrow[lll, bend left, "J"] \end{tikzcd}$$

where I is the identity matrix and J a nilpotent Jordan block of size $l \times l$.

Proposition 4.4. *Let $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$ be a weighted projective line, $\mathsf{C} = \mathsf{add}\big(S_1^{(1)}\big)$ be the Serre subcategory of $\mathsf{Coh}(\mathbb{X})$ generated by an exceptional simple object $S_1^{(1)}$ and $\mathbb{Y}$ be the weighted projective line such that $\mathsf{Coh}(\mathbb{Y})$ is equivalent to $\mathsf{C}^{\perp}$. Then the functor $\mathsf{Coh}(\mathbb{Y}) \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$ induces an injective homomorphism of the reduced Drinfeld doubles $DH(\mathbb{Y}) \xrightarrow{\mathbb{F}} DH(\mathbb{X})$.*

Proof. It is a consequence of Theorem 3.3 and results of Geigle and Lenzing quoted above. □

5. Composition subalgebra of a weighted projective line: Part I

In this section we recall the definition of the composition subalgebra $U(\mathbb{X})$ of the Hall algebra $H(\mathbb{X})$ of a weighted projective line $\mathbb{X}$ introduced by Schiffmann in [38]. The main result of this part is the fact that $U(\mathbb{X})$ is a *topological bialgebra*. We start with the case of the usual non-weighted projective line $\mathbb{P}^1$.

5.1. **Composition subalgebra of $\mathbb{P}^1$.** The composition subalgebra of the category of coherent sheaves on a projective line $\mathbb{P}^1$ was introduced by Kapranov [22]. Later, it was studied in details by Baumann and Kassel [2]. In this subsection, we recall its definition and main properties.

1. The map $K_0(\mathbb{P}^1) \xrightarrow{(\mathrm{rk}, \deg)} \mathbb{Z}^2$ is an isomorphism of abelian groups. Let $\delta = (0,1)$ be the class of the simple torsion sheaf supported at a k–point of $\mathbb{P}^1$. Then δ generates the radical of the Euler form $\langle -\,,\,- \rangle$.

2. For any integer $r \ge 1$ consider the element

$$\mathbb{1}_{r\delta} := \sum_{\mathcal{T} \in \mathsf{Tor}(\mathbb{P}^1):\, \overline{\mathcal{T}} = (0,r)} [\mathcal{T}] \in H(\mathbb{P}^1). \tag{4}$$

The elements $\big\{T_r\big\}_{r \ge 1}$ are determined by $\mathbb{1}_{r\delta}$ using the generating series

$$1 + \sum_{r=1}^{\infty} \mathbb{1}_{r\delta} t^r = \exp\Big(\sum_{r=1}^{\infty} \frac{T_r}{[r]_v} t^r\Big). \tag{5}$$

Finally, the elements $\big\{\Theta_r\big\}_{r \ge 1}$ are defined by the generating series

$$1 + \sum_{r=1}^{\infty} \Theta_r t^r = \exp\Big((v^{-1} - v) \sum_{r=1}^{\infty} T_r t^r\Big). \tag{6}$$

In what follows, we set $\mathbb{1}_{(0,r)} = T_0 = \Theta_0 = [0] = \mathbb{1}$.

3. In the above notations we have the following results, see for example [40, Chapter 4].

(a) These three sets $\{\mathbb{1}_{(0,r)}\}_{r\geq 1}$, $\{T_r\}_{r\geq 1}$ and $\{\Theta_r\}_{r\geq 1}$ generate the same subalgebra $U(\mathbb{P}^1)_{\mathrm{tor}}$ of the Hall algebra $H(\mathbb{P}^1)$;

(b) For any $r, s \geq 1$ we have the equalities:

$$\Delta(T_r) = T_r \otimes \mathbb{1} + K_{(0,r)} \otimes T_r, \quad \big(\Theta_r, T_r\big) = \frac{[2r]}{r} \quad \text{and} \quad \big(T_r, T_s\big) = \delta_{r,s}\frac{[2r]}{r(v^{-1}-v)}. \tag{7}$$

4. For any $n \in \mathbb{Z}$ we have the following formula:

$$\Delta\big(\big[\mathcal{O}_{\mathbb{P}^1}(n)\big]\big) = \big[\mathcal{O}_{\mathbb{P}^1}(n)\big] \otimes \mathbb{1} + \sum_{r=0}^{\infty} \Theta_r K_{(1,n-r)} \otimes \big[\mathcal{O}_{\mathbb{P}^1}(n-r)\big], \tag{8}$$

see also [22, Theorem 3.3] or [39, Section 12.2].

Definition 5.1. The composition algebra $U(\mathbb{P}^1)$ is the subalgebra of the Hall algebra $H(\mathbb{P}^1)$ generated by the elements $L_l := \big[\mathcal{O}_{\mathbb{P}^1}(l)\big]$, T_r and K_α, where $l \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$ and $\alpha \in K_0\big(\mathsf{Coh}(\mathbb{P}^1)\big) \cong \mathbb{Z}^2$. We also use the notations: $C = K_\delta$ and $O = K_{(1,0)}$. From the equalities (7) and (8) it follows that $U(\mathbb{P}^1)$ is a topological subbialgebra of $H(\mathbb{P}^1)$.

A complete list of relations between the generators of the composition algebra $U(\mathbb{P}^1)$ was obtained by Kapranov [22] and Baumann–Kassel [2], see also [40, Section 4.3].

Theorem 5.2. *The elements L_n, T_r, O and C satisfy the following relations:*

1. *C is central;*
2. *$[O, T_n] = 0 = [T_n, T_m]$ for all $m, n \in \mathbb{Z}_{>0}$;*
3. *$OL_n = v^{-2}L_nO$ for all $n \in \mathbb{Z}$;*
4. *$\big[T_r, L_n\big] = \frac{[2r]}{r}L_{n+r}$ for all $n \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$;*
5. *$L_mL_{n+1} + L_nL_{m+1} = v^2\big(L_{n+1}L_m + L_{m+1}L_n\big)$ for all $m, n \in \mathbb{Z}$.*

Let $U(\mathbb{P}^1)_{\mathrm{vec}}$ be the subalgebra of $U(\mathbb{P}^1)$ generated by the elements L_n ($n \in \mathbb{Z}$). Then the map $U(\mathbb{P}^1)_{\mathrm{vec}} \otimes_{\widetilde{\mathbb{Q}}} U(\mathbb{P}^1)_{\mathrm{tor}} \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \xrightarrow{\mathsf{mult}} U(\mathbb{P}^1)$ is an isomorphism. Next, the elements

$$B_{\underline{m},\,\underline{l},\,a,\,b} = \prod_{n\in\mathbb{Z}} L_n^{m_n} \circ \prod_{r\in\mathbb{Z}^+} T_r^{l_r} \circ K^a C^b,$$

where $a, b \in \mathbb{Z}$, $\underline{m} = (m_n)_{n\in\mathbb{Z}}$ and $\underline{l} = (l_r)_{r\in\mathbb{Z}_{>0}}$ are sequences of non-negative integers such that all but finitely entries are zero, form a basis of $U(\mathbb{P}^1)$ over the field $\widetilde{\mathbb{Q}}$.

5.2. **First results on the composition subalgebra of a weighted projective line.** Let $\mathbb{X}$ be a weighted projective line of type $(\underline{p}, \underline{\lambda})$ and C be the Serre subcategory of $\mathsf{Coh}(\mathbb{X})$ generated by the simple torsion sheaves $\mathcal{S}_1^{(1)}, \dots, \mathcal{S}_1^{(p_1-1)}, \dots, \mathcal{S}_n^{(1)}, \dots, \mathcal{S}_n^{(p_n-1)}$. Recall that the Serre quotient $\mathsf{Coh}(\mathbb{X})/\mathsf{C}$ is equivalent to the category $\mathsf{Coh}(\mathbb{P}^1)$ and the canonical functor $\mathsf{Coh}(\mathbb{X}) \xrightarrow{\mathbb{P}} \mathsf{Coh}(\mathbb{X})/\mathsf{C}$ has an exact fully faithful right adjoint functor $\mathsf{Coh}(\mathbb{P}^1) \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$. The second-named author suggested in [38] the following definition.

Definition 5.3. The composition algebra $U(\mathbb{X})$ is the subalgebra of the Hall algebra $H(\mathbb{X})$ generated by the elements $\big[\mathcal{S}_i^{(j)}\big]$ for $1 \le i \le n$ and $1 \le j \le p_i$, K_α for $\alpha \in K_0(\mathbb{X})$ and the image of the composition algebra $U(\mathbb{P}^1)$ under the homomorphism $H(\mathbb{P}^1) \xrightarrow{\mathbb{F}} H(\mathbb{X})$. For the sake of simplicity, we use the same notation T_r for a generator of $U(\mathbb{P}^1)$ and its image in $H(\mathbb{X})$ under the algebra homomorphism $\mathbb{F}$. Recall that for any $l \in \mathbb{Z}$ we have an isomorphism $\mathbb{F}\big(\mathcal{O}_{\mathbb{P}^1}(l)\big) \cong \mathcal{O}_{\mathbb{X}}(l\vec{c})$. Summing up, we set

$$U(\mathbb{X}) := \Big\langle \big[\mathcal{O}(l\vec{c})\big],\, T_r,\, \big[\mathcal{S}_i^{(j)}\big],\, K_\alpha \;\Big|\; l \in \mathbb{Z},\, r \in \mathbb{Z}_{>0},\, 1 \le i \le n,\, 1 \le j \le p_i,\, \alpha \in K_0(\mathbb{X}) \Big\rangle.$$

Remark 5.4. At the first glance, the definition of $U(\mathbb{X})$ depends on the choice of a Serre subcategory C and is not canonical. However, as we shall see later, this is *not* the case and $U(\mathbb{X})$ can be redefined in an *intrinsic* way as an invariant of the category $\mathsf{Coh}(\mathbb{X})$.

Lemma 5.5. *Let $\mathcal{L} \in \mathsf{Pic}(\mathbb{X})$ be a vector bundle. Then the element $[\mathcal{L}]$ belongs to the composition subalgebra $U(\mathbb{X})$.*

Proof. Any line bundle $\mathcal{L}$ on a weighted projective lines $\mathbb{X}$ can be written as $\mathcal{O}(\vec{x})$, where $\vec{x} = m\vec{c} + \sum_{l=1}^{n} a_l \vec{x}_l$ is an element of the group $\mathbb{L}(\underline{p})$ written in its canonical form (3). For any $0 \le i \le n$ consider the line bundle $\mathcal{L}_i = \mathcal{O}\big(m\vec{c} + \sum_{l=1}^{i} a_l \vec{x}_l\big)$. Note that $\mathcal{L}_0 = \mathcal{O}(m\vec{c})$ and $\mathcal{L}_n = \mathcal{L}$. Then for any $1 \le i \le n$ we have a short exact sequence

$$0 \longrightarrow \mathcal{L}_{i-1} \xrightarrow{x_i^{a_i}} \mathcal{L}_i \longrightarrow \mathcal{T}_i \longrightarrow 0,$$

where $\mathcal{T}_i$ is the torsion sheaf of length a_i supported at the weighted point λ_i and corresponding to the following representation of the cyclic quiver $\overrightarrow{C}_{p_i}$:

$$\begin{tikzcd} 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[r, "1"] & \cdots \arrow[r, "1"] & k \arrow[lllll, bend left, "0"] \end{tikzcd}$$

By a result of Ringel on the Hall algebra of a Dynkin quiver [33], the element $[\mathcal{T}_i]$ belongs to the subalgebra generated by $\big[\mathcal{S}_i^{(l)}\big]$ for $1 \le l \le p_i$. Note that we have the isomorphisms:

$$\mathsf{Ext}^1(\mathcal{T}_i, \mathcal{L}_{i-1}) = k \quad \text{and} \quad \mathsf{Ext}^1(\mathcal{L}_{i-1}, \mathcal{T}_i) = \mathsf{Hom}(\mathcal{T}_i, \mathcal{L}_{i-1}) = \mathsf{Hom}(\mathcal{L}_{i-1}, \mathcal{T}_i) = 0.$$

They yield the following identities in the Hall algebra $H(\mathbb{X})$:

$$\big[T_i\big] \circ \big[\mathcal{L}_{i-1}\big] = v\big(\big[\mathcal{L}_i\big] + \big[\mathcal{L}_{i-1} \oplus T_i\big]\big) \quad \text{and} \quad \big[\mathcal{L}_{i-1}\big] \circ \big[T_i\big] = \big[\mathcal{L}_{i-1} \oplus T_i\big].$$

Hence, for any $1 \le i \le n$ we obtain the equality $\big[\mathcal{L}_i\big] = v^{-1}\big[T_i\big] \circ \big[\mathcal{L}_{i-1}\big] - \big[\mathcal{L}_{i-1}\big] \circ \big[T_i\big]$ proving the statement of the lemma. □

5.3. **Composition subalgebra $U(\mathbb{X})_{\mathsf{tor}}$.**

Definition 5.6. Consider the subalgebra $U(\mathbb{X})_{\mathsf{tor}}$ of $U(\mathbb{X})$ defined as follows:

$$U(\mathbb{X})_{\mathsf{tor}} := \Big\langle T_r,\, \big[S_i^{(j)}\big],\, K_\alpha \;\Big|\; r \in \mathbb{Z}_{>0},\, 1 \le i \le n,\, 1 \le j \le p_i,\, \alpha \in K_0(\mathbb{X}) : \mathrm{rk}(\alpha) = 0 \Big\rangle.$$

In this subsection, we give another characterization of the subalgebra of $U(\mathbb{X})_{\mathsf{tor}}$ and show it is a Hopf algebra. For this purpose it is convenient to use the technique of stability conditions.

Let A be an abelian category of *finite length*, $\bigl(\Gamma, \{\,-\,,\,-\,\}\bigr)$ a lattice (i.e. a free abelian group equipped with a bilinear form) and $\bigl(K_0(\mathsf{A}), \langle\,-\,,\,-\,\rangle\bigr) \xrightarrow{\mathsf{ch}} \bigl(\Gamma, \{\,-\,,\,-\,\}\bigr)$ an isometry. Assume $\Gamma \xrightarrow{Z} \mathbb{R}^2_{\ge 0}$ is a group homomorphism such that $Z \circ \mathsf{ch}\bigl([X]\bigr) \ne 0$ for all non-zero objects X of A. For any $\alpha \in \Gamma$ we set

$$\mathbb{1}_\alpha := \sum_{[X]\in \mathsf{J}:\, \mathsf{ch}(\bar{X})=\alpha} [X] \quad \text{and} \quad \mathbb{1}^{\mathrm{ss}}_\alpha = \mathbb{1}^{\mathrm{ss}}_{\alpha,Z} := \sum_{[X]\in \mathsf{J}:\, X\in \mathsf{A}^{\mathrm{ss}}_\alpha} [X],$$

where $\mathsf{A}^{\mathrm{ss}}_\alpha$ is the category of semi-stable objects of A of class α with respect to the stability condition $Z\circ\mathsf{ch}$. The following result is well-known.

Proposition 5.7. *For any element $\alpha \in \Gamma$ we have the equality*

$$\mathbb{1}_\alpha = \mathbb{1}^{\mathrm{ss}}_\alpha + \sum_{t\ge 2} \sum_{\substack{\alpha_1+\dots+\alpha_t=\alpha \\ \mu(\alpha_1)\ge\dots\ge\mu(\alpha_t)}} v^{\sum_{i<j}\langle\alpha_i\alpha_j\rangle}\, \mathbb{1}^{\mathrm{ss}}_{\alpha_1}\circ\dots\circ\mathbb{1}^{\mathrm{ss}}_{\alpha_t}. \tag{9}$$

Note that the sum in the right-hand side of the equality is finite. Moreover, the elements $\bigl\{\mathbb{1}_\beta\bigr\}_{\beta\in\Gamma}$ and $\bigl\{\mathbb{1}^{\mathrm{ss}}_{\beta,Z}\bigr\}_{\beta\in\Gamma}$ generate the same subalgebra of the Hall algebra $H(\mathsf{A})$.

Proof. The equality (9) is an easy consequence of existence and uniqueness of the Harder-Narasimhan filtrations with respect to the stability condition $Z\circ\mathsf{ch}$, see [36]. In particular, for any $\alpha \in K_0(\mathsf{A})$ the element $\mathbb{1}_\alpha$ belongs to the algebra $\bigl\{\mathbb{1}^{\mathrm{ss}}_{\beta,Z}\bigr\}_{\beta\in\Gamma}$. From a "triangular" form of the equality (9) it follows that for any $\alpha \in K_0(\mathsf{A})$ the element $\mathbb{1}^{\mathrm{ss}}_{\alpha,Z}$ belongs to the algebra generated by $\bigl\{\mathbb{1}_\beta\bigr\}_{\beta\in\Gamma}$. $\square$

Remark 5.8. A result of Reineke [30, Theorem 5.1] provides an explicit formula expressing the elements $\mathbb{1}^{\mathrm{ss}}_\alpha$ via $\mathbb{1}_\beta$ for an arbitrary stability function Z:

$$\mathbb{1}^{\mathrm{ss}}_{\alpha,Z} = \mathbb{1}_\alpha + \sum_{t\ge 2}(-1)^{t-1} \sum_{\substack{\alpha_1+\dots+\alpha_t=\alpha:\, \forall\, 1\le s\le t-1 \\ \mu(\alpha_1)+\dots+\mu(\alpha_s)>\mu(\alpha)}} v^{\sum_{i<j}\langle\alpha_i\alpha_j\rangle}\, \mathbb{1}_{\alpha_1}\circ\dots\circ\mathbb{1}_{\alpha_t}.$$

Definition 5.9. Let $\mathbb{X}$ be a weighted projective line of type $(\underline{p}, \underline{\lambda})$ and $\mathsf{A} = \mathsf{Tor}(\mathbb{X})$ be the category of torsion coherent sheaves on $\mathbb{X}$. For the lattice $\bigl(K_0(\mathbb{X}), \langle\,-\,,\,-\,\rangle\bigr)$ let $K_0(\mathsf{A}) \xrightarrow{\mathsf{ch}} K_0(\mathbb{X})$ be the canonical morphism. We construct the linear map $K_0(\mathbb{X}) \xrightarrow{Z} \mathbb{R}_{\ge 0}$ as follows:

1. For any pair (i,j), where $1 \le i \le n$ and $1 \le j \le p_i$ we attach a positive weight $w_i^{(j)} \in \mathbb{R}_{>0}$.
2. Moreover, we assume that the following conditions are fulfilled:
   (a) For all $1 \le i \le n$ we have: $w_i^{(1)} > w_i^{(2)} > \dots > w_i^{(p_i)}$.
   (b) There exists $w \in \mathbb{R}_{>0}$ such that for all $1 \le i \le n$ we have $\sum_{j=1}^{p_i} w_i^{(j)} = w$.
   (c) For any $1 \le i \le n$ and $1 \le a \le b \le p_i$ we set:

$$\sigma_i^{(a,b)} := \frac{p_i}{b-a+1}\bigl(w_i^{(a)} + \dots + w_i^{(b)}\bigr).$$

Then $\sigma_i^{(a,b)} = \sigma_j^{(c,d)}$ if and only if $i = j$, $a = c$ and $b = d$. Note that this condition is automatically satisfied when we take arbitrary elements $w_1^{(1)}, \dots, w_1^{(p_1-1)}, \dots, w_n^{(1)}, \dots, w_n^{(p_n-1)}, w$ from $\mathbb{R}_{>0} \setminus \mathbb{Q}_{>0}$, which are linearly independent over $\mathbb{Q}$ and such that $w - \sum_{j=1}^{p_i-1} w_i^{(j)} > 0$ for all $1 \le i \le n$.

(3) We take the group homomorphism $K_0(\mathbb{X}) \xrightarrow{Z=(d,\delta)} \mathbb{R}^2$ given by the formulae:
   (a) $d\bigl(\overline{\mathcal{S}_i^{(j)}}\bigr) = w_i^{(j)}$ for all $1 \le i \le n$ and $1 \le j \le p_i$ and $d(\overline{\mathcal{O}}) = 1$.
   (b) $\delta\bigl(\overline{\mathcal{S}_i^{(j)}}\bigr) = \dfrac{1}{p_i}$ for all $1 \le i \le n$ and $1 \le j \le p_i$ and $\delta(\overline{\mathcal{O}}) = 0$.

Lemma 5.10. *In the above notations, the indecomposable semi-stable objects of the category $\mathsf{Tor}(\mathbb{X})$ with respect to the stability function $K_0\bigl(\mathsf{Tor}(\mathbb{X})\bigr) \xrightarrow{Z\circ\mathsf{ch}} \mathbb{R}^2_{\ge 0}$ are the following:*

(1) *Any indecomposable torsion sheaf supported at a regular point of $\mathbb{X}$. The slope of such a sheaf is w.*
(2) *For a special point $x = \lambda_i$ $(1 \le i \le n)$ the indecomposable semi-stable sheaves correspond to the following representations of the cyclic quiver $\overrightarrow{C}_{p_i}$:*
   (a) *Representations $T_i(l)$, $l \in \mathbb{Z}_{>0}$ of slope w:*

$$\begin{tikzcd} k^l \arrow[r, "I"] & k^l \arrow[r, "I"] & \cdots \arrow[r, "I"] & k^l \arrow[lll, bend left, "J"] \end{tikzcd}$$

   where I is the identity matrix and J is a nilpotent Jordan block.
   (b) *Representations $T_i^{(a,b)}$ of slope $\sigma_i^{(a,b)}$:*

$$\begin{tikzcd} 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & k \arrow[r, "1"] & \cdots \arrow[r, "1"] & k \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[llllllll, bend left, "0"] \end{tikzcd}$$

   where $1 \le a \le b \le p_i$ and $b - a < p_i - 1$. In these notations, the top of $T_i^{(a,b)}$ is $S_i^{(a)}$ and the socle of $T_i^{(a,b)}$ is $S_i^{(b)}$.

Proof. Let $x \in \mathbb{P}^1$ be a non-special point. Then the unique simple sheaf $\mathcal{S}_x$ supported at x is stable with respect to any stability condition. Any torsion sheaf supported at x has a filtration, whose factors are $\mathcal{S}_x$. Since any extension of semi-stable sheaves of a given slope is again semi-stable with the same slope, any torsion sheaf supported at x is semi-stable.

Let $x = \lambda_i$ be a special points and $1 \le a \le b \le p_i$ are such that $b - a < p_i - 1$. It is clear that any subrepresentation of $T_i^{(a,b)}$ is isomorphic to some $T_i^{(c,b)}$ with $a \le c \le b$. However, from the assumption $w_i^{(a)} > \dots > w_i^{(a)}$ it follows that the slope of $T_i^{(a,b)}$ is bigger than the slope of $T_i^{(c,b)}$. Hence, $T_i^{(a,b)}$ is stable with respect to the stability function $Z \circ \mathsf{ch}$. Since all representations $T_i^{(a,b)}$ are rigid, the category of semi-stable objects of $\mathsf{Tor}(\mathbb{X})$ of slope $\sigma_i^{(a,b)}$ is equivalent to the semi-simple abelian category $\mathsf{add}\bigl(T_i^{(a,b)}\bigr)$.

Next, consider the representation $T_i(1)$. Note that its slope is w. Moreover, any proper subrepresentation of $T_i(1)$ is isomorphic to $T_i^{(a,p_i)}$ for some $a \ge 2$. Since $w > \sigma_i^{(a,p_i)}$ for any $2 \le a \le p_i$, the representation $T_i(1)$ is stable. Moreover, for any $l \ge 1$ the representation

$T_i(l)$ can be written as a successive extension of $T_i(1)$. Hence, $T_i(l)$ is semi-stable of slope w for all $l \geq 2$.

It remains to check that all the remaining indecomposable representations of the cyclic quiver $\overrightarrow{C}_{p_i}$ are not semi-stable. Indeed, let X be an indecomposable representation, which is neither isomorphic to $T_i(l)$ nor to $T_i^{(a,b)}$. Then X either have a subrepresentation isomorphic to $T_i^{(1,b)}$ for some $1 \leq b < p_i$ or a quotient isomorphic to $T_i^{(a,p_i)}$ for some $2 \leq a \leq p_i$. In both cases, X is not semi-stable, what completes the proof. □

Proposition 5.11. *In the notations of this subsection we have:*

$$U(\mathbb{X})_{\mathsf{tor}} = \bigl\langle \mathbb{1}_\alpha, K_\alpha \bigm| \alpha \in K_0(\mathbb{X}) :\ \mathrm{rk}(\alpha) = 0 \bigr\rangle.$$

In particular, $U(\mathbb{X})_{\mathsf{tor}}$ is stable under the action of the Auslander-Reiten translation τ.

Proof. Let $\mathsf{Coh}(\mathbb{P}^1) \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$ be the exact fully faithful functor defined at the beginning of Subsection 5.2. For any $r \geq 1$ we set $\widetilde{\mathbb{1}}_{r\delta} := \mathbb{F}(\mathbb{1}_{r\delta}) \in H(\mathbb{X})$, where $\mathbb{1}_{r\delta} \in H(\mathbb{P}^1)$ was defined by the equality (4). Let $K_0(\mathbb{X}) \xrightarrow{Z} \mathbb{R}^2_{\geq 0}$ be the function introduced in Definition 5.9. Note that

1. For any $1 \leq i \leq n$ and $1 \leq a \leq b \leq p_i$ such that $b - a < p_i - 1$ the element $\bigl[\mathcal{T}_i^{(a,b)}\bigr]$ of $H(\mathbb{X})$ belongs to the subalgebra generated by the elements $\bigl[\mathcal{S}_i^{(a)}\bigr], \dots, \bigl[\mathcal{S}_i^{(b)}\bigr]$.

2. From the equality (5) it follows that the subalgebras $\bigl\langle \widetilde{\mathbb{1}}_{r\delta} \bigm| r \geq 1 \bigr\rangle$ and $\bigl\langle T_r \bigm| r \geq 1 \bigr\rangle$ of the Hall algebra $H(\mathbb{X})$ are equal.

3. By Proposition 5.7, we get the equality

$$\bigl\langle \mathbb{1}_\alpha \bigm| \alpha \in K_0(\mathbb{X}) :\ \mathrm{rk}(\alpha) = 0 \bigr\rangle = \bigl\langle \widetilde{\mathbb{1}}_{r\delta}, \bigl[\mathcal{S}_i^{(j)}\bigr] \bigm| r \in \mathbb{Z}_{>0},\ 1 \leq i \leq n,\ 1 \leq j \leq p_i \bigr\rangle.$$

This concludes the proof. □

Corollary 5.12. *The subalgebra $U(\mathbb{X})_{\mathsf{tor}}$ of the Hall algebra $H(\mathbb{X})$ is a Hopf algebra.*

Proof. This result follows from Proposition 5.11 and the formula

$$\Delta(\mathbb{1}_\gamma) = \sum_{\alpha+\beta=\gamma} v^{\langle \alpha, \beta \rangle} \mathbb{1}_\alpha K_\beta \otimes \mathbb{1}_\beta, \tag{10}$$

valid for any $\gamma \in K_0(\mathbb{X})$ such that $\mathrm{rk}(\gamma) = 0$, see for example [40, Lemma 1.7]. □

We conclude this subsection by the following important corollary.

Corollary 5.13. *The composition subalgebra $U(\mathbb{X})$ defined in* [38], *coincides with the following subalgebra $V(\mathbb{X})$ of the Hall algebra $H(\mathbb{X})$:*

$$V(\mathbb{X}) := \bigl\langle [\mathcal{O}(\vec{x})], \mathbb{1}_\alpha, K_\beta \bigm| \vec{x} \in \mathbb{L}(\underline{p}); \alpha \in K_0(\mathbb{X}) : \mathrm{rk}(\alpha) = 0; \beta \in K_0(\mathbb{X}) \bigr\rangle.$$

Proof. This result follows from Lemma 5.5 and Proposition 5.11. □

Remark 5.14. Corollary 5.13 gives an *intrinsic* description of the composition subalgebra $U(\mathbb{X})$. Moreover, it shows that $U(\mathbb{X})$ is invariant under the natural action of the groups $\mathsf{Pic}(\mathbb{X}) \cong \mathbb{L}(\underline{p})$ and $\mathsf{Aut}(\mathbb{X})$.

5.4. **Composition algebra $U(\mathbb{X})$ is a topological bialgebra.** In this subsection, we complete the proof of the fact that the composition subalgebra $U(\mathbb{X})$ is a topological bialgebra. Because of the equality $\Delta(K_\beta) = K_\beta \otimes K_\beta\bigl(\beta \in K_0(\mathbb{X})\bigr)$ and the formula (10) it is sufficient to compute the coproducts of the line bundles on $\mathbb{X}$. For any $\vec{x} \in \mathbb{L}(\underline{p})_+$ consider the element $\Theta_{\vec{x}} \in \overline{H}(\mathbb{X})_{\mathsf{tor}}$ defined as follows:

$$\Delta\bigl([\mathcal{O}]\bigr) = [\mathcal{O}] \otimes \mathbb{1} + \sum_{\vec{x} \in \mathbb{L}(\underline{p})_+} \Theta_{\vec{x}} K_{\overline{\mathcal{O}(-\vec{x})}} \otimes \bigl[\mathcal{O}(-\vec{x})\bigr]. \tag{11}$$

In this subsection we show that $\Theta_{\vec{x}} \in \overline{U}(\mathbb{X})_{\mathsf{tor}}$ for all $\vec{x} \in \mathbb{L}(\underline{p})_+$. By Corollary 5.13, the algebra $U(\mathbb{X})$ is closed under the action of the Picard group $\mathsf{Pic}(\mathbb{X})$. This will allows us to conclude that $U(\mathbb{X})$ is a topological bialgebra.

We start with the case when $\mathbb{X} = \mathbb{P}^1$ is the usual non-weighted projective line. Recall the following formula for the elements Θ_r defined in (6), see [40, Example 4.12]:

$$\Theta_r = v^{-r} \sum_{m=1}^{\infty} \sum_{\substack{x_1, \dots, x_m \in \mathbb{P}^1;\, x_i \neq x_j\, 1 \le i \neq j \le m \\ t_1, \dots, t_m \,:\, \sum_{i=1}^m t_i \deg(x_i) = r}} \prod_{i=1}^{m} \bigl(1 - v^{2\deg(x_i)}\bigr)\bigl[\mathcal{S}_{t_i, x_i}\bigr]. \tag{12}$$

Lemma 5.15. *For any $r \in \mathbb{Z}_{>0}$ consider the element $\vec{x} = r\vec{c} \in \mathbb{L}(\underline{p})_+$. Then we have: $\Theta_{r\vec{c}} = \mathbb{F}(\Theta_r)$, where $H(\mathbb{P}^1) \xrightarrow{\mathbb{F}} H(\mathbb{X})$ is the algebra homomorphism from Definition 5.3.*

Proof. Let $0 \to \mathcal{O}(-r\vec{c}) \to \mathcal{O} \to \mathcal{T} \to 0$ be a short exact sequence in $\mathsf{Coh}(\mathbb{X})$. Since the essential image of the functor $\mathsf{Coh}(\mathbb{P}^1) \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$ is closed under taking kernels and cokernels, the object $\mathcal{T}$ belongs to $\mathsf{Im}(\mathbb{F})$ as well. This means that all contributions to $\Theta_{r\vec{c}}$ come from $\mathbb{P}^1$ and Proposition 4.4 implies the claim. $\square$

Lemma 5.16. *Let $\overrightarrow{C} = \overrightarrow{C}_p$ be a cyclic quiver with vertices labeled by the natural numbers $\{1, 2, \dots, p\}$, $\mathsf{A} = \mathsf{Rep}(\overrightarrow{C})$ be the category of finite-dimensional nilpotent representations of $\overrightarrow{C}$ and $K = K_0(\mathsf{A})$ be its K–group. Let $\delta := \overline{S}_1 + \dots + \overline{S}_p \in K$ and $\alpha \in K$ be an element of the form $\overline{S}_1 + \dots + \overline{S}_t$ for some $1 \le t < p$. For any $l \in \mathbb{Z}_{>0}$ let $T_{l\delta+\alpha}$ be the unique indecomposable object of A of class $l\delta + \alpha$ with the simple top S_1. Then we have:*

$$\bigl[T_{l\delta+\alpha}\bigr] = \begin{cases} \bigl[T_\alpha\bigr] & \text{if } l = 0, \\ \bigl[T_{l\delta}\bigr] \circ \bigl[T_\alpha\bigr] - v^2\bigl[T_\alpha\bigr] \circ \bigl[T_{l\delta}\bigr] & \text{if } l > 0. \end{cases} \tag{13}$$

Proof. This formula is trivial for $l = 0$, so we consider the case $l \ge 1$. Since $\mathsf{Hom}_\mathsf{A}(S_i, S_i) = k = \mathsf{Ext}^1_\mathsf{A}(S_i, S_{i+1})$ for all $1 \le i \le p$ (as usual, we set $S_{p+1} := S_1$), whereas all other Hom and Ext^1 spaces between the simple objects are zero, we have: $\langle \delta, \beta \rangle = 0 = \langle \beta, \delta \rangle$ for all $\beta \in K$. It is easy to see that $\mathsf{Hom}_\mathsf{A}(T_{l\delta}, T_\alpha) = k$, hence $\mathsf{Ext}^1_\mathsf{A}(T_{l\delta}, T_\alpha) = k$ as well. Moreover, a generator of $\mathsf{Ext}^1_\mathsf{A}(T_{l\delta}, T_\alpha)$ corresponds to a non-split extension of the form $0 \to T_\alpha \to T_{l\delta+\alpha} \to T_{l\delta} \to 0$. Note that $\mathsf{Hom}_\mathsf{A}(T_\alpha, T_{l\delta}) = 0$, hence $\mathsf{Ext}^1_\mathsf{A}(T_\alpha, T_{l\delta}) = 0$. Summing up, we have the following equalities in the Hall algebra $H(\mathsf{A})$:

$$\bigl[T_{l\delta}\bigr] \circ \bigl[T_\alpha\bigr] = \bigl[T_{l\delta+\alpha}\bigr] + \bigl[T_{l\delta} \oplus T_\alpha\bigr] \quad \text{and} \quad \bigl[T_\alpha\bigr] \circ \bigl[T_{l\delta}\bigr] = v^{-2}\bigl[T_{l\delta} \oplus T_\alpha\bigr],$$

which conclude the proof. $\square$

Proposition 5.17. *Let $\underline{\mu}$ be a non-empty subset of the set $\underline{\lambda}$ of exceptional points of our weighted projective line $\mathbb{X}$. For a simplicity of notation, we assume that $\underline{\mu} = \{\lambda_1, \dots, \lambda_m\}$ for some $1 \le m \le n$. For any $1 \le i \le m$ we fix a number $1 \le a_i < p_i$ and for $r \in \mathbb{Z}_{>0}$ set $\vec{x} := r\vec{c} + \sum_{i=1}^{m} a_i \vec{x}_i$. Then we have:*

$$\Theta_{\vec{x}} = v^{-r-m} \sum_{\underline{\mu}=\underline{\mu}' \coprod \underline{\mu}''} \bigl((-1)^{m''} T_{\underline{\mu}'} \circ \Theta_{r\vec{c}} \circ T_{\underline{\mu}''} - v^{2m''}(-1)^{m'} T_{\underline{\mu}''} \circ \Theta_{r\vec{c}} \circ T_{\underline{\mu}'}\bigr), \tag{14}$$

where the sum is taken over all decompositions of $\underline{\mu}$ into a disjoint union of two (possibly empty) subsets, $m' = |\underline{\mu}'|$, $m'' = |\underline{\mu}''|$, $T_{\underline{\mu}'} := \prod_{i \in \underline{\mu}'} T_i^{(1,a_i)}$ and $T_{\underline{\mu}''} := \prod_{i \in \underline{\mu}''} T_i^{(1,a_i)}$. In particular, $\Theta_{\vec{y}}$ is an element of $\overline{U}(\mathbb{X})_{\mathsf{tor}}$ for all $\vec{y} \in \mathbb{L}(\underline{p})_+$.

Proof. Consider a short exact sequence in the category $\mathsf{Coh}(\mathbb{X})$:

$$0 \longrightarrow \mathcal{O}(-\vec{x}) \xrightarrow{a} \mathcal{O} \xrightarrow{b} \mathcal{T} \longrightarrow 0.$$

Note that we have the following equality for the Euler form: $\langle \overline{\mathcal{T}}, \overline{\mathcal{O}(-\vec{x})} \rangle = -\langle \overline{\mathcal{O}}, \overline{\mathcal{T}} \rangle = -r - m$. Our next goal is to compute the Hall constant $P^{\mathcal{O}}_{\mathcal{T}, \mathcal{O}(-\vec{x})}$. To do this, we assume

$$\mathcal{T} \cong \mathcal{S}_{l_1,x_1} \oplus \dots \oplus \mathcal{S}_{l_p,x_p} \oplus \mathcal{T}^{(1)}_{t_1\delta + a_1} \oplus \dots \oplus \mathcal{T}^{(m)}_{t_m\delta + a_m}, \tag{15}$$

where $x_1, \dots, x_p$ are distinct points of $\mathbb{P}^1 \setminus \underline{\mu}$, whereas $l_1, \dots, l_p \in \mathbb{Z}_{>0}$ and $t_1, \dots, t_m \in \mathbb{Z}_{\ge 0}$ are such that $\sum_{i=1}^{p} l_i \deg(x_i) + \sum_{i=1}^{m} t_i = r$. If $1 \le i \le p$ is such that the point x_i is non-special, then $\mathcal{S}_{l_i,x_i}$ is the unique torsion coherent sheaf of length l_i supported at x_i. If $x_i \in \underline{\lambda} \setminus \underline{\mu}$ then the sheaf $\mathcal{S}_{l_i,x_i}$ corresponds to the representation $T_i(l_i)$ of the cyclic quiver $\overrightarrow{C}_{p_i}$ (we follow the notations of Lemma 5.10). For any $1 \le i \le m$ the coherent sheaf $\mathcal{T}^{(i)}_{t_i\delta + a_i}$ is supported at the point $x_i \in \underline{\mu}$ and corresponds to the unique indecomposable representation of $\overrightarrow{C}_{p_i}$ of length $t_i p_i + a_i$ with the simple top $S_i^{(1)}$. Note that the decomposition type of the sheaf $\mathcal{T}$ given by (15) determines the map $\mathcal{O}(-\vec{x}) \xrightarrow{a} \mathcal{O}$ uniquely up to a non-zero constant. Hence, the Hall number $P^{\mathcal{O}}_{\mathcal{T}, \mathcal{O}(-\vec{x})}$ is equal to the number of epimorphisms from $\mathcal{O}$ to $\mathcal{T}$. Note that a morphism $\mathcal{O} \xrightarrow{a} \mathcal{T}$ is an epimorphism if and only if all its components $\mathcal{O} \to \mathcal{S}_{l_i,x_i}$ and $\mathcal{O} \to \mathcal{T}^{(j)}_{t_j\delta + a_j}$ are epimorphisms for all $1 \le i \le p$ and $1 \le j \le m$.

The number of epimorphisms from $\mathcal{O}$ to $\mathcal{S}_{l,x}$ is equal to $q^{\deg(x)l} - q^{\deg(x)(l-1)}$. Similarly, the number of epimorphisms from $\mathcal{O}$ to $\mathcal{T}^{(i)}_{t\delta + a}$ is equal to $q^{t+1} - q^t$. Hence, we obtain:

$$\Theta_{\vec{x}} = v^{-r-m} \sum_{\underline{x},\, \underline{l},\, \underline{t}} c_{\underline{x}} [\mathcal{S}_{l_1,x_1}] \circ \dots \circ [\mathcal{S}_{l_1,x_1}] \circ [\mathcal{T}^{(1)}_{t_1\delta + a_1}] \circ \dots \circ [\mathcal{T}^{(m)}_{t_m\delta + a_m}],$$

where $c_{\underline{x}} = \bigl(1 - v^{2\deg(x_1)}\bigr) \cdot \dots \cdot \bigl(1 - v^{2\deg(x_p)}\bigr)(1 - v^2)^m$ and the sum is taken over all decomposition types (15). Replacing each term $\bigl[\mathcal{T}^{(i)}_{t_i\delta + a_i}\bigr]$ by the expression given by (13), we end up with the formula (14). □

Theorem 5.18. *Let $\mathbb{X}$ be a weighted projective line. Then the composition algebra $U(\mathbb{X})$ is a topological subbialgebra of the Hall algebra $H(\mathbb{X})$. In particular, the canonical multiplication map $\overline{U}^+(\mathbb{X}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}^-(\mathbb{X}) \xrightarrow{\mathsf{mult}} DU(\mathbb{X})$ is an isomorphism of $\widetilde{\mathbb{Q}}$–vector* *spaces, where* $\overline{U}^{\pm}(\mathbb{X})$ *is the subalgebra of the reduced Drinfeld double* $DU(\mathbb{X})$ *generated by the elements* $\big[\mathcal{O}(l\vec{c})\big]^{\pm}$ $(l \in \mathbb{Z})$, $T_r^{\pm}$ $(r \in \mathbb{Z}_{>0})$ *and* $\big[\mathcal{S}_i^{(j)}\big]^{\pm}$ $(1 \le i \le n, 1 \le j \le p_i)$.

Proof. The composition algebra $U(\mathbb{X})$ is a topological bialgebra by Corollary 5.12 and formula (14). The triangular decomposition for $DU(\mathbb{X})$ is a general property of the reduced Drinfeld doubles of (topological) bialgebras, see Proposition 2.5 and references therein. □

5.5. **Further structure properties of the reduced Drinfeld double** $DU(\mathbb{X})$**.** First note that Theorem 5.18 implies the following interesting result.

Corollary 5.19. *The algebra* $DU(\mathbb{X})$ *is generated by one of the following sets:*

(1) *the elements* $\big[\mathcal{O}(l\vec{c})\big]^{\pm}$ $(l \in \mathbb{Z})$, $\big[\mathcal{S}_i^{(j)}\big]^{\pm}$ $(1 \le i \le n, 1 \le j \le p_i)$,
(2) *the classes of the line bundles* $\big[\mathcal{O}(\vec{x})\big]^{\pm}$ $\big(\vec{x} \in \mathbb{L}(\underline{p})\big)$,

together with the generators K_α $(\alpha \in K)$ *of the Cartan part* $\widetilde{\mathbb{Q}}[K]$.

Proof. Let $\mathsf{Coh}(\mathbb{P}^1) \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$ be the functor from Definition 5.3. By Proposition 4.4, we get the induced morphism of the reduced Drinfeld doubles $DH(\mathbb{P}^1) \xrightarrow{\mathbb{F}} DH(\mathbb{X})$, which restricts on the algebra homomorphism $DU(\mathbb{P}^1) \xrightarrow{\mathbb{F}} DU(\mathbb{X})$. In particular, for any $l \in \mathbb{Z}_{>0}$ we get the following relation in $DU(\mathbb{X})$: $\big[[\mathcal{O}]^+, [\mathcal{O}(-l\vec{c})]^-\big] = \frac{v}{v^{-1}-v}\,\Theta^+_{l\vec{c}} O C^{-l}$, where $O = K_{\overline{\mathcal{O}}}$ and $C = K_\delta$. Since the elements $\big\{\Theta_{l\vec{c}}\big\}_{l \ge 0}$ and $\big\{T_l\big\}_{l \ge 0}$ generate the same subalgebra of the composition algebra $U(\mathbb{X})$ (see Subsection 5.1), this shows that for any $r \in \mathbb{Z}_{>0}$ the generator T_r^+ can be expressed through the elements $\big[\mathcal{O}(l\vec{c})\big]^{\pm}$ $(l \in \mathbb{Z})$. The case of the elements T_r^- can be treated in the same way. Similarly, for any $1 \le i \le n$ and $1 \le j \le p_i$ we have the following equality in $DU(\mathbb{X})$: $\big[[\mathcal{O}((1-j)\vec{x}_i)]^+, [\mathcal{O}(-j\vec{x}_i)]^-\big] = \frac{v}{v^{-1}-v} K_{\overline{\mathcal{O}(-j\vec{x}_i)}}[\mathcal{S}_i^{(j)}]$. By Lemma 5.5, the element $[\mathcal{O}(\vec{x})]$ belongs to $U(\mathbb{X})$ for all $\vec{x} \in \mathbb{L}(\underline{p})$. □

Remark 5.20. Notice that the line bundles and the exceptional simple sheaves are rigid objects in the category $\mathsf{Coh}(\mathbb{X})$. Hence, Corollary 5.19 says that the reduced Drinfeld double $DU(\mathbb{X})$ is generated by classes of (certain) rigid objects. This observation will be elaborated one step further for weighted projective lines of domestic and tubular types.

Theorem 5.21. *Let* $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$ *be a weighted projective line,* $\mathsf{C} = \mathsf{add}\big(\mathcal{S}_1^{(1)}\big)$ *and* $\mathsf{Coh}(\mathbb{Y}) = \mathsf{C}^\perp$ *be as in Proposition 4.4. Then the (injective) algebra homomorphism of the reduced Drinfeld doubles* $DH(\mathbb{Y}) \xrightarrow{\mathbb{F}} DH(\mathbb{X})$ *induced by the functor* $\mathsf{Coh}(\mathbb{Y}) \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$ *restricts to an injective algebra homomorphism* $DU(\mathbb{Y}) \xrightarrow{\mathbb{F}} DU(\mathbb{X})$.

Proof. The statement of theorem is a consequence of the following observations. Let D be the Serre subcategory of $\mathsf{Coh}(\mathbb{X})$ generated by the torsion sheaves $\mathcal{S}_1^{(1)}, \dots, \mathcal{S}_1^{(p_1-1)}, \dots, \mathcal{S}_n^{(1)}, \dots, \mathcal{S}_n^{(p_n-1)}$. Then the perpendicular category $\mathsf{D}^\perp$ is a full subcategory of $\mathsf{C}^\perp$ equivalent to $\mathsf{Coh}(\mathbb{P}^1)$. Consider the canonical inclusion functors $\mathsf{D}^\perp \xrightarrow{\mathbb{G}} \mathsf{C}^\perp \xrightarrow{\mathbb{F}} \mathsf{Coh}(\mathbb{X})$. By Proposition 4.4, we get an injective algebras homomorphisms of the reduced Drinfeld doubles $DH(\mathbb{P}^1) \xrightarrow{\mathbb{G}} DH(\mathbb{Y}) \xrightarrow{\mathbb{F}} DH(\mathbb{X})$. Note that

- For any $r \in \mathbb{Z}_{>0}$ we have: $\mathbb{F}\mathbb{G}(T_r) = T_r$.

- For any (i,j) such that $2 \le i \le n$ and $1 \le j \le p_i$ we have: $\mathbb{F}\bigl(\mathcal{S}_i^{(j)}\bigr) = \mathcal{S}_i^{(j)}$.
- Assume that $p_1 \ge 3$. Then for any $2 \le j \le p_1 - 1$ we have: $\mathbb{F}\bigl(\mathcal{S}_1^{(j)}\bigr) = \mathcal{S}_1^{(j+1)}$, whereas $\mathbb{F}\bigl(\mathcal{S}_1^{(1)}\bigr)$ corresponds to the representation
$$\begin{tikzcd} k \arrow[r, "1"] & k \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[llll, bend left] \end{tikzcd}$$
of the cyclic quiver $\overrightarrow{C}_{p_1}$. If $p_1 = 2$ then the point λ_1 is non-special for $\mathbb{Y}$.
- For any $l \in \mathbb{Z}$ we have: $\mathbb{F}\bigl(\mathcal{O}_{\mathbb{Y}}(l\vec{c}')\bigr) \cong \mathcal{O}_{\mathbb{X}}(l\vec{c})$.

By what was said above it follows that the image of the reduced Drinfeld double $DU(\mathbb{Y})$ of the composition algebra $U(\mathbb{Y})$ under the injective algebra homomorphism $DH(\mathbb{Y}) \xrightarrow{\mathbb{F}} DH(\mathbb{X})$ belongs to the algebra $DU(\mathbb{X})$. In the terms of Definition 5.3, the action of $\mathbb{F}$ on the generators of $DU(\mathbb{Y})$ is the following (for a simplicity we assume that $p_1 \ge 3$):

(1) For all $l \in \mathbb{Z}$ we have: $\mathbb{F}\bigl([\mathcal{O}_{\mathbb{Y}}(l\vec{c}')]^{\pm}\bigr) = \bigl[\mathcal{O}_{\mathbb{X}}(l\vec{c})\bigr]^{\pm}$.
(2) For all $r \in \mathbb{Z}_{>0}$ we have: $\mathbb{F}\bigl(T_r^{\pm}\bigr) = T_r^{\pm}$.
(3) For all $2 \le i \le n$ and $1 \le j \le p_i$ we have: $\mathbb{F}\bigl(\bigl[\mathcal{S}_i^{(j)}\bigr]^{\pm}\bigr) = \bigl[\mathcal{S}_i^{(j)}\bigr]^{\pm}$.
(4) We have: $\mathbb{F}\bigl(\bigl[\mathcal{S}_1^{(1)}\bigr]^{\pm}\bigr) = v^{-1}\bigl[\mathcal{S}_1^{(1)}\bigr]^{\pm} \circ \bigl[\mathcal{S}_1^{(2)}\bigr]^{\pm} - \bigl[\mathcal{S}_1^{(2)}\bigr]^{\pm} \circ \bigl[\mathcal{S}_1^{(1)}\bigr]^{\pm}$ and $\mathbb{F}\bigl(\bigl[\mathcal{S}_1^{(j)}\bigr]^{\pm}\bigr) = \bigl[\mathcal{S}_1^{(j+1)}\bigr]^{\pm}$ for all $2 \le j \le p_1 - 1$.
(5) Finally, $\mathbb{F}(O) = O$; $\mathbb{F}\bigl(K_i^{(j)}\bigr) = K_i^{(j)}$ for $2 \le i \le n$ and $1 \le j \le p_i$; $F\bigl(K_1^{(1)}\bigr) = K_1^{(1)}K_1^{(2)}$ and $\mathbb{F}\bigl(K_1^{(j)}\bigr) = K_1^{(j+1)}$ for all $2 \le j \le p_1 - 1$.

□

6. Composition subalgebra of a weighted projective line: Part II

The goal of this section is to clarify the structure of the subalgebra $U(\mathbb{X})_{\mathsf{tor}}$ of the composition algebra $U(\mathbb{X})$ and to derive some relations in the reduced Drinfeld double $DU(\mathbb{X})$.

6.1. **Remainder on the Hall algebra of a cyclic quiver.** Let
$$\overrightarrow{C} = \overrightarrow{C}_p = \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & p \arrow[lll, bend left] \end{tikzcd}$$
be a cyclic quiver with $p \ge 2$ vertices labeled by the natural numbers $1, 2, \dots, p$ and $\mathsf{A} = \mathsf{Rep}(\overrightarrow{C})$ be the category of its nilpotent representations. The following result is well-known, see for example [10].

Theorem 6.1. *Let $\overrightarrow{C}_\infty$ be an (infinite) quiver of type A_∞ with vertices labeled by the integers and linearly ordered arrows. Let $\mathsf{Rep}(\overrightarrow{C}_\infty)$ be the category of its finite-dimensional representations. Consider the exact functor $\mathsf{Rep}(\overrightarrow{C}_\infty) \xrightarrow{\mathbb{P}} \mathsf{Rep}(\overrightarrow{C}_p)$ sending a representation V into the representation $\mathbb{P}(V)$ such that $\mathbb{P}(V)_i = \bigoplus\limits_{j = i \bmod p} V_j$ for all $1 \le i \le p$. Then the indecomposable objects of $\mathsf{Rep}(\overrightarrow{C}_p)$ are precisely the images of the indecomposable objects of $\mathsf{Rep}(\overrightarrow{C}_\infty)$.*

Let $S_1, \dots, S_p$ be the simple objects of A. In what follows, we denote $K_i = K_{\overline{S}_i} \in H(\mathsf{A})$ and $\delta = \overline{S}_1 + \dots + \overline{S}_p \in K := K_0(\mathsf{A})$. The following result is due to Ringel [34].

Theorem 6.2. *We have:* $C(\overrightarrow{C}) := \big\langle [S_1], \dots, [S_n]; K_1^{\pm}, \dots, K_n^{\pm} \big\rangle \cong U_q\big(\mathfrak{b}(\widehat{\mathfrak{sl}}_p)\big)$, *where* $U_q\big(\mathfrak{b}(\widehat{\mathfrak{sl}}_p)\big) \cong U_q^+\big(\widehat{\mathfrak{sl}}_p\big) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K]$ *is the Borel part of the quantized enveloping algebra of* $\widehat{\mathfrak{sl}}_p$.

The structure of the complete Hall algebra $H(\overrightarrow{C})$ has been clarified by Schiffmann [37] and Hubery [19]. Let $\overline{H}(\overrightarrow{C})$ be the non-extended Hall algebra of A, i.e. $H(\overrightarrow{C}) = \overline{H}(\overrightarrow{C}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K]$. Then for any $1 \le i \le p$ we have a linear operator $e_i^* : \overline{H}(\overrightarrow{C}) \to \overline{H}(\overrightarrow{C})$ given by the rule $\big(e_i^*(x), y\big) = \big(x, [S_i] \otimes y\big)$ for any $x, y \in \overline{H}(\overrightarrow{C})$. Here we use the fact that the Green's form is non-degenerate on $\overline{H}(\overrightarrow{C})$. Let

$$Z(\overrightarrow{C}) = \bigcap_{i=1}^{p} \mathsf{ker}(e_i^*) = \big\{x \in \overline{H}(\overrightarrow{C}) \big| (\Delta(x), [S_i] \otimes -) = 0 \;\forall\, 1 \le i \le p\big\}.$$

The following results are due to Schiffmann [37] and Hubery [19].

Theorem 6.3. *In the above notations we have:*

(1) *The vector space* $Z(\overrightarrow{C})$ *is the center of the Hall algebra* $H(\overrightarrow{C})$. *In particular,* $Z(\overrightarrow{C})$ *is a commutative algebra.*
(2) *The canonical morphism* $Z(\overrightarrow{C}) \otimes_{\widetilde{\mathbb{Q}}} C(\overrightarrow{C}) \xrightarrow{\mathsf{mult}} H(\overrightarrow{C})$ *is an isomorphism of vector spaces over* $\widetilde{\mathbb{Q}}$.
(3) *The algebra* $Z(\overrightarrow{C})$ *is freely generated by the primitive elements of the Hall algebra* $H(\overrightarrow{C})$. *This means that* $Z(\overrightarrow{C}) = \widetilde{\mathbb{Q}}[z_1, z_2, \dots, z_r, \dots]$, *where* $z_r \in H(\overrightarrow{C})[r\delta]$ *are such that* $\Delta(z_r) = z_r \otimes \mathbb{1} + K_{r\delta} \otimes z_r$ *for any* $r \in \mathbb{Z}_{>0}$. *In particular,* $Z(\overrightarrow{C}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K_\delta^{\pm}]$ *is a commutative subbialgebra of the Hall algebra* $H(\overrightarrow{C})$.

In a work of Hubery [19] some further properties of the center $Z(\overrightarrow{C})$ were studied.

Theorem 6.4. *For any* $r \in \mathbb{Z}_{>0}$ *consider the element*

$$c_r := (-1)^r q^{-rp} \sum_{\substack{M:\, \underline{\mathsf{dim}}(M) = r\delta \\ \mathsf{top}(M) \text{ sq. free}}} (-1)^{\mathsf{dim}_k \mathsf{End}_{\mathsf{A}}(M)} \,|\mathsf{Aut}(M)|\, [M] \;\in\; H(\overrightarrow{C})[r\delta]. \tag{16}$$

Then the following statements are true:

(1) *The elements* c_r *are central and generate the algebra* $Z(\overrightarrow{C})$.
(2) *For any* $r \in \mathbb{Z}_{>0}$ *we have:* $\Delta(z_r) = \sum\limits_{t=0}^{r} c_t K_{(r-t)\delta} \otimes c_{r-t}$.
(3) *For any* $r, t \in \mathbb{Z}_{>0}$ *we have:* $(c_r, c_t) = \delta_{r,t}\, q^{-rp}\,(1 - q^{-p})$.

We set $z_1 = c_1$ *and for any* $r \ge 2$ *define the element* $z_r \in Z(\overrightarrow{C})[r\delta]$ *using the recursion*

$$z_r = r c_r - \sum_{l=1}^{r-1} z_l c_{r-l}. \tag{17}$$

Then the following formulae are true:

(1) $\Delta(z_r) = z_r \otimes \mathbb{1} + K_{r\delta} \otimes z_r$. *In other words,* z_r *is primitive for all* $r \in \mathbb{Z}_{>0}$.
(2) $(z_r, c_r) = \dfrac{q^{-rp}}{1 - q^{-rp}} = \dfrac{1}{q^{rp} - 1}$ *and* $(z_r, z_r) = r\dfrac{q^{-rp}}{1 - q^{-rp}} = \dfrac{r}{q^{rp} - 1}$.

(3) $(z_r, ab) = 0$ *for any pair* $a, b \in \overline{H}(\overrightarrow{C})$ *such that both* a *and* b *are non-scalar.*

Let $\overrightarrow{C}_1$ be the Jordan quiver. Then we have an exact functor $\mathsf{Rep}(\overrightarrow{C}_1) \xrightarrow{\mathbb{F}} \mathsf{Rep}(\overrightarrow{C}_p)$ mapping a representation (V, N) into the representation

$$\begin{tikzcd} \mathbb{F}(V,N) := V \arrow[r, "I"] & V \arrow[r, "I"] & \cdots \arrow[r, "I"] & V \arrow[lll, bend left, "N"] \end{tikzcd}$$

where I is the identity map. For any $r \in \mathbb{Z}_{>0}$ and any partition λ of r let I_λ be the corresponding representation of $\overrightarrow{C}_1$. Consider the following element of $H(\overrightarrow{C}_1)$:

$$p_r = \sum_{\lambda | r} n_q\big(l(\lambda)\big)\,[I_\lambda], \tag{18}$$

where $l(\lambda)$ is the length of λ and $n_q(i) = 1$ for $i = 1$ and $(1-q)\dots(1-q^{i-1})$ for $i \geq 2$. Recall the following standard facts on the classical Hall algebra $H(\overrightarrow{C}_1)$, see [28].

Theorem 6.5. *For the elements* $p_r \in H(\overrightarrow{C}_1)$ *the following properties are true:*

(1) $\Delta(p_r) = p_r \otimes \mathbb{1} + K_{r\delta} \otimes p_r$ *are primitive for all* $r \in \mathbb{Z}_{>0}$.

(2) $(p_r, p_t) = \delta_{r,t}\dfrac{r}{q^r - 1}$.

Proposition 6.6. *In the above notations, set* $t_r = \mathbb{F}(p_r) \in H(\overrightarrow{C}_p)$ *for all* $r \in \mathbb{Z}_{>0}$. *Then there exists an element* $u_r \in C(\overrightarrow{C}_p)$ *such that* $t_r = \dfrac{1}{1 - q^{-rp}} z_r + u_r$. *In particular, the difference* $\tau(t_r) - t_r$ *belongs to the composition subalgebra* $C(\overrightarrow{C}_p)$, *where* τ *is the Auslander-Reiten translation in* $\mathsf{Rep}(\overrightarrow{C}_p)$.

Proof. The existence of a constant $\gamma \in \widetilde{\mathbb{Q}}$ and an element $u_r \in C(\overrightarrow{C}_p)$ such that $t_r = \gamma z_r + u_r$ can be proven along the same lines as in [19, Theorem 14]. In order to determine the value of γ note that $(c_r, t_r) = q^{-rp}$, $(c_r, z_r) = q^{-rp}(1 - q^{-rp})$ and $(c_r, u_r) = 0$. Hence, $\gamma = \dfrac{1}{1 - q^{-rp}}$. It remains to observe that the translation τ acts as identity on the center $Z(\overrightarrow{C})$ of the Hall algebra $H(\overrightarrow{C})$ and maps the composition algebra $C(\overrightarrow{C})$ to itself. $\square$

Definition 6.7. Let $\mathbb{X}$ be a weighted projective line and $x \in \mathbb{X}$ be a closed point.

(1) Assume x is a non-special point of $\mathbb{X}$ and $d = \deg(x)$ is the degree of x. Then $\mathsf{Tor}_x(\mathbb{X})$ is equivalent to the category of representations of the cyclic quiver $\overrightarrow{C}_1$ over the field $\mathbb{F}_{q^d}$. We set $T_{r,x} = Z_{r,x} = d\frac{[r]}{r} p_{\frac{r}{d}}$, where $p_{\frac{r}{d}} \in H\big(\mathsf{Rep}_{\mathbb{F}_{q^d}}(\overrightarrow{C}_1)\big)$.

(2) Assume x is a special point of weight p. Then $\mathsf{Tor}_x(\mathbb{X})$ is equivalent to the category of representations of a cyclic quiver $\overrightarrow{C}_p$ and we set

$$T_{r,x} = \frac{[r]}{r} t_r \quad \text{and} \quad Z_{r,x} = \frac{[r]}{r}\frac{1}{1 - q^{-rp}} z_r.$$

The following lemma is a straightforward corollary of Theorem 6.4 and Theorem 6.5.

Lemma 6.8. *Let x be a closed point of $\mathbb{X}$. Then for any $r \in \mathbb{Z}_{>0}$ we have: $\bigl(T_{r,x}, T_{r,x}\bigr) = \frac{[r]^2}{r}\frac{d}{q^r-1}$, where d is the degree of x. Moreover, if x is a special point of $\mathbb{X}$ then $\bigl(Z_{r,x}, Z_{r,x}\bigr) = \frac{[r]^2}{r}\frac{1}{q^{rp}-1}$, where p is the weight of x.*

Next, for any $r \in \mathbb{Z}_{>0}$ we define:

$$T_r = \sum_{d|r} \sum_{x\in\mathbb{X}:\deg(x)=d} T_{r,x} \quad \text{and} \quad Z_r = \sum_{d|r} \sum_{x\in\mathbb{X}:\deg(x)=d} Z_{r,x}. \tag{19}$$

Note that this definition of the elements T_r coincides with the one given by Equation (5) and Definition 5.3, see [38, Section 5] for more details.

Proposition 6.9. *Let $\mathbb{X} = \mathbb{X}(\underline{p}, \underline{\lambda})$ be a weighted projective line. Then the following statements are true.*

1. *The algebra $V(\mathbb{X})_{\mathsf{tor}} := \bigl\langle Z_r, [\mathcal{S}_i^{(j)}], K_i^{(j)} \bigm| r \in \mathbb{Z}_{>0}, 1 \le i \le n, 1 \le j \le p_i \bigr\rangle$ is equal to the algebra $U(\mathbb{X})_{\mathsf{tor}}$. In particular, we have a decomposition:*

$$U(\mathbb{X})_{\mathsf{tor}} \cong \mathcal{Z} \otimes_{\widetilde{\mathbb{Q}}} U_q^+(\widehat{\mathfrak{sl}}_{p_1}) \otimes_{\widetilde{\mathbb{Q}}} \cdots \otimes_{\widetilde{\mathbb{Q}}} U_q^+(\widehat{\mathfrak{sl}}_{p_n}) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K], \tag{20}$$

where $\mathcal{Z} = \widetilde{\mathbb{Q}}[Z_1, Z_2, \dots, Z_r, \dots]$ is the ring of symmetric functions. The algebra $\mathcal{Z}$ is invariant under the action of the Picard group $\mathsf{Pic}(\mathbb{X}) \cong \mathbb{L}(\underline{p})$.

2. *For any $r \in \mathbb{Z}_{>0}$ we have the equalities:*

$$\alpha_r := (T_r, T_r) = \frac{1}{v^{-1} - v}\frac{[2r]}{r} \quad \text{and} \quad (T_r, \Theta_r) = \frac{[2r]}{r}.$$

Moreover, $\beta_r := (Z_r, T_r) = (Z_r, Z_r) = \alpha_r + \sum_{x\in\Pi} \mathrm{def}_x$, where $\mathrm{def}_x = \frac{[r]^2}{r}\Bigl(\frac{1}{q^{rp_x} - 1} - \frac{1}{q^r - 1}\Bigr)$ is the defect *of the special point x and $\gamma_r := (Z_r, \Theta_r) = (v^{-1} - v)\beta_r$.*

Proof. Since for any $r \in \mathbb{Z}_{>0}$ the difference $Z_r - T_r$ belongs to the algebra generated by the classes of the exceptional simple modules $\bigl[\mathcal{S}_i^{(j)}\bigr]$, $1 \le i \le n$, $1 \le j \le p_i$, we have the equality $V(\mathbb{X})_{\mathsf{tor}} = U(\mathbb{X})_{\mathsf{tor}}$. Next, $\mathcal{Z} = \widetilde{\mathbb{Q}}[Z_1, Z_2, \dots, Z_r, \dots]$ belongs to the center of $U(\mathbb{X})_{\mathsf{tor}}$. Hence, the decomposition (20) follows from Ringel's Theorem 6.2. Since for any closed point $x \in \mathbb{X}$, the element $Z_{r,x}$ is invariant under the action of the Auslander-Reiten translation in the category $\mathsf{Tor}_x(\mathbb{X})$, the element Z_r is invariant under the action of the Picard group $\mathsf{Pic}(\mathbb{X})$. The formulae for the value of the Green's form on the generators T_r, Θ_r and Z_r follow from the local formulae of Lemma 6.8 and the equality

$$\sum_{d|r} \sum_{x\in\mathbb{P}^1:\,\deg(x)=d} d = \mathbb{P}^1(\mathbb{F}_{q^r}) = (q^r + 1).$$

$\square$

Corollary 6.10. *Let $\mathbb{X}$ be a weighted projective line. The reduced Drinfeld double $DU(\mathbb{X})_{\mathsf{tor}}$ is a subalgebra of the reduced Drinfeld double $DU(\mathbb{X})$. Moreover, we have a decomposition:*

$$DU(\mathbb{X})_{\mathsf{tor}} \cong \mathcal{H} \otimes_{\mathcal{A}} U_q(\widehat{\mathfrak{sl}}_{p_1}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} U_q(\widehat{\mathfrak{sl}}_{p_t}), \tag{21}$$

where $\mathcal{A} = \widetilde{\mathbb{Q}}[C^{\pm}]$ *is the ring of Laurent polynomials in the variable* $C := K_\delta$ *and* $\mathcal{H}$ *is the* Heisenberg algebra*. More precisely,* $\mathcal{H} := \widetilde{\mathbb{Q}}\big\langle Z_r \,|\, r \in \mathbb{Z}\setminus\{0\}\big\rangle \otimes_{\widetilde{\mathbb{Q}}} \mathcal{A}$ *subject to the relations:*

$$[Z_r, Z_t] = \delta_{r+t,0}\, \beta_r\,(C^{-r} - C^r) \quad \text{and} \quad [Z_r, C^{\pm}] = 0 \quad r, t \in \mathbb{Z}\setminus\{0\}. \tag{22}$$

Proof. We have the equalities $\Delta(Z_r^{\pm}) = Z_r^{\pm} \otimes \mathbb{1} + C^{\pm r} \otimes Z_r^{\pm}$ for any $r \in \mathbb{Z}_{>0}$. Denoting $Z_r^{\pm} = Z_{\pm r}$ for $r \in \mathbb{Z}\setminus\{0\}$ it is easy to see that the first relation of (22) is just the structure relation $D(Z_r, Z_r)$ of the reduced Drinfeld double. The decomposition (21) is a corollary of Proposition 2.5. □

6.2. **Some further properties of** $U(\mathbb{X})$**.** For the reader's convenience, we give a proof of the following result from [38].

Proposition 6.11. *Let* $\overline{U}(\mathbb{X})_{\mathsf{vec}}$ *be the subalgebra of* $U(\mathbb{X})$ *generated by the classes of the line bundles on* $\mathbb{X}$ *and* $\overline{U}(\mathbb{X})_{\mathsf{tor}}$ *be the subalgebra of* $U(\mathbb{X})$ *generated by the elements* T_r $(r \in \mathbb{Z}_{>0})$ *and* $\big[S_i^{(j)}\big]$ $(1 \le i \le n, 1 \le j \le p_i)$*. Then the canonical morphism*

$$\overline{U}(\mathbb{X})_{\mathsf{vec}} \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})_{\mathsf{tor}} \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \xrightarrow{\mathsf{mult}} U(\mathbb{X}) \tag{23}$$

is an isomorphism of vector spaces over $\widetilde{\mathbb{Q}}$*.*

Proof. Let $\overline{H}(\mathbb{X})_{\mathsf{vec}}$ be the subalgebra of the Hall algebra $H(\mathbb{X})$ generated by the classes of vector bundles and $\overline{H}(\mathbb{X})_{\mathsf{tor}}$ be the subalgebra of $H(\mathbb{X})$ generated by the classes of the torsion coherent sheaves. Then the map $\overline{H}(\mathbb{X})_{\mathsf{vec}} \otimes_{\widetilde{\mathbb{Q}}} \overline{H}(\mathbb{X})_{\mathsf{tor}} \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \xrightarrow{\mathsf{mult}} H(\mathbb{X})$ is an isomorphism of vector spaces. Hence, the corresponding map (23) for the composition subalgebra is at least injective. In order to show the surjectivity of mult first note that for any line bundle $\mathcal{L}$ and an arbitrary exceptional simple sheaf $\mathcal{S}_i^{(j)}$ there exist a line bundle $\mathcal{N}$ and constants $\alpha, \beta \in \widetilde{\mathbb{Q}}$ such that $\big[\mathcal{S}_i^{(j)}\big] \circ \big[\mathcal{L}\big] = \alpha\big[\mathcal{N}\big] + \beta\big[\mathcal{L}\big] \circ \big[\mathcal{S}_i^{(j)}\big]$. Next, we have the relation $T_r \circ \big[\mathcal{O}(l\vec{c})\big] = \big[\mathcal{O}(l\vec{c})\big] \circ T_r + \frac{[2r]}{r}\big[\mathcal{O}\big((l+r)\vec{c}\big)\big]$ for all $l \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$. Recall that for any $r \in \mathbb{Z}_{>0}$ the difference $Z_r - T_r$ belongs to the subalgebra of $U(\mathbb{X})_{\mathsf{tor}}$ generated by the exceptional simple sheaves $\big[\mathcal{S}_i^{(j)}\big]$. Moreover, the element Z_r is invariant under the action of $\mathsf{Pic}(\mathbb{X})$. Hence, the product $Z_r \circ [\mathcal{L}]$ belongs to the image of mult for any $\mathcal{L} \in \mathsf{Pic}(\mathbb{X})$ and $r \in \mathbb{Z}_{>0}$. It remains to note, that by Proposition 6.9, the algebra $\overline{U}(\mathbb{X})_{\mathsf{tor}}$ is generated by the elements Z_r $(r \in \mathbb{Z}_{>0})$ and $\big[\mathcal{S}_i^{(j)}\big]$ $(1 \le i \le n, 1 \le j \le p_i)$. This concludes the proof. □

Corollary 6.12. *Assume that* $\mathcal{F}_1, \dots, \mathcal{F}_t$ *are objects of* $\mathsf{VB}(\mathbb{X})$ *and* $\alpha_1, \dots, \alpha_t \in \widetilde{\mathbb{Q}}$ *are such that* $a := \sum_{i=1}^{t} \alpha_i[\mathcal{F}_i]$ *belongs to* $U(\mathbb{X})$*. Then* $a \in U(\mathbb{X})_{\mathsf{vec}}$*.*

Proof. Let $\overline{H}(\mathbb{X}) \xrightarrow{\mathsf{pr}} \widetilde{\mathbb{Q}}$ be the projection on the class of the zero object $[0]$. We consider pr as an endomorphism of $\overline{H}(\mathbb{X})$. Since $b \in \overline{U}(\mathbb{X})$, there exist elements $b_1, \dots, b_t \in \overline{U}(\mathbb{X})_{\mathsf{vec}}$ and $c_1, \dots, c_t \in \overline{U}(\mathbb{X})_{\mathsf{tor}}$ such that $a = b_1c_1 + \dots + b_tc_t$. Let $\mathsf{pr}(c_i) = \lambda_i[0]$ for some $\lambda_i \in \widetilde{\mathbb{Q}}$. Since there is a decomposition $\overline{H}(\mathbb{X}) = \overline{H}(\mathbb{X})_{\mathsf{vec}} \otimes_{\widetilde{\mathbb{Q}}} \overline{H}(\mathbb{X})_{\mathsf{tor}}$, we have: $b = \big(\mathbb{1} \otimes \mathsf{pr}\big)(b) = \sum_{i=1}^{t} \lambda_i b_i \otimes [0]$, where $\sum_{i=1}^{t} \lambda_i b_i \in \overline{U}(\mathbb{X})_{\mathsf{vec}}$. This implies the claim. □

Our next goal is to study the action of the central elements Z_r of the algebra $U(\mathbb{X})_{\mathsf{tor}}$ on the algebra $U(\mathbb{X})_{\mathsf{vec}}$. We have the following Hecke-type equality.

Theorem 6.13. *For any $\vec{x} \in \mathbb{L}(\underline{p})$ and any $r \in \mathbb{Z}_{>0}$ we have the following equality:*

$$\Big[Z_r, \big[\mathcal{O}(\vec{x})\big]\Big] = \gamma_r \big[\mathcal{O}(\vec{x} + r\vec{c})\big], \tag{24}$$

where $\beta_r = (Z_r, \Theta_r)$ is the constant introduced in Proposition 6.9.

Proof. Since the Picard group $\mathsf{Pic}(\mathbb{X})$ acts on $U(\mathbb{X})$ by algebra automorphisms and the element Z_r is stable under the this action, it is sufficient to prove the equality (24) in the special case $\vec{x} = 0$. By Proposition 6.11 there exist $a, b, c \in \widetilde{\mathbb{Q}}$, $T \in \big\langle [\mathcal{S}_i^{(j)}], Z_1, \dots, Z_{r-1}\big\rangle$, $c_{\vec{\alpha}} \in \widetilde{\mathbb{Q}}$ and $T_{\vec{\alpha}} \in \overline{U}(\mathbb{X})_{\mathsf{tor}}$ for $\vec{\alpha} \in \mathbb{L}(\underline{p}), 0 < \vec{\alpha} < r\vec{c}$ such that

$$Z_r \circ [\mathcal{O}] = a\,[\mathcal{O}(r\vec{c})] + \sum_{0 \le \vec{\alpha} \le r\vec{c}} c_{\vec{\alpha}}\,[\mathcal{O}(r\vec{c} - \vec{\alpha})] \circ T_{\vec{\alpha}} + b\,[\mathcal{O}] \circ Z_r + c\,[\mathcal{O}] \circ T.$$

We first show that $b = 1$ and $c = 0$. Let $Z_r = \sum\limits_{\mathcal{S}} d_{\mathcal{S}}[\mathcal{S}]$, where we sum over the isomorphism classes of torsion sheaves of class $r\delta$ in the K–group $K_0(\mathbb{X})$. Assume we have an extension

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \oplus \mathcal{T} \longrightarrow \mathcal{S} \longrightarrow 0,$$

where $\mathcal{T}$ is a torsion sheaf. Since $\mathsf{Hom}(\mathcal{O}, \mathcal{O}) = k$, it follows that $\mathcal{T} \cong \mathcal{S}$ and the above sequence splits. Note that $\mathsf{Ext}^1(\mathcal{O}, \mathcal{S}) = 0$, hence $\big\langle \overline{\mathcal{O}}, \overline{\mathcal{S}}\big\rangle = \mathsf{dim}_k\big(\mathsf{Hom}(\mathcal{O}, \mathcal{S})\big) = r$ and $F^{\mathcal{O}\oplus\mathcal{S}}_{\mathcal{S},\,\mathcal{O}} = q^r$. This implies that

$$\Big(\sum_{\mathcal{S}} d_{\mathcal{S}}[\mathcal{S}]\Big) \circ [\mathcal{O}] = v^{-r} \sum_{\mathcal{S}} d_{\mathcal{S}}\big[\mathcal{O} \oplus \mathcal{S}\big] + \;\; \text{terms involving} \;\; [\mathcal{O}(\vec{\gamma})],\ \vec{\gamma} > 0.$$

On the other hand, $[\mathcal{O}] \circ \big(\sum_{\mathcal{S}} d_{\mathcal{S}}[\mathcal{S}]\big) = v^{-r} \sum_{\mathcal{S}} d_{\mathcal{S}}\big[\mathcal{O} \oplus \mathcal{S}\big]$. Hence, $b = 1$ and $c = 0$ as stated. Our next goal is to show that $c_{\vec{\alpha}} = 0$ for all $0 < \alpha < r\vec{c}$. Recall that

$$\Delta\big([\mathcal{O}]\big) = [\mathcal{O}] \otimes \mathbb{1} + K_{\overline{\mathcal{O}}} \otimes [\mathcal{O}] + \sum_{\vec{x} > 0} \widetilde{\Theta}_{\vec{x}} \otimes \big[\mathcal{O}(-\vec{x})\big] \quad \text{and} \quad \Delta(Z_r) = Z_r \otimes \mathbb{1} + K_{r\,\delta} \otimes Z_r,$$

where in the notations of the equation (11) we denote $\widetilde{\Theta}_{\vec{x}} = \Theta_{\vec{x}} K_{\overline{\mathcal{O}(-\vec{x})}}$. Then we have:

$$\Delta\Big(\big[Z_r, [\mathcal{O}]\big]\Big) = \Big[\Delta(Z_r), \Delta([\mathcal{O}])\Big] = \Big[Z_r \otimes \mathbb{1} + K_{r\,\delta} \otimes Z_r, [\mathcal{O}] \otimes \mathbb{1} + \sum_{\vec{x} \ge 0} \widetilde{\Theta}_{\vec{x}} \otimes \big[\mathcal{O}(-\vec{x})\big]\Big] =$$

$$= \big[Z_r, [\mathcal{O}]\big] \otimes \mathbb{1} + K_{r\delta + \bar{\mathcal{O}}} \otimes \big[Z_r, [\mathcal{O}]\big] + \sum_{\vec{x} > 0} K_{r\delta} \widetilde{\Theta}_{\vec{x}} \otimes \big[Z_r, [\mathcal{O}(-\vec{x})]\big].$$

Here we benefit from the fact that $\big[Z_r, \widetilde{\Theta}_{\vec{x}}\big] = 0$ because the elements Z_r are central in $U(\mathbb{X})_{\mathsf{tor}}$. On the other hand, we have already shown that

$$\big[Z_r, [\mathcal{O}]\big] = a\,[\mathcal{O}(r\vec{c})] + \sum_{0 < \vec{\alpha} < r\vec{c}} c_{\vec{\alpha}}\,[\mathcal{O}(r\vec{c} - \vec{\alpha})] \circ T_{\vec{\alpha}}.$$

Apply the operator Δ to the right-hand side of this equality. Let $0 < \vec{\alpha} < r\vec{c}$ be a maximal element such that $c_\alpha \ne 0$. Then $\Delta\big(a\,[\mathcal{O}(r\vec{c})] + \sum\limits_{0 < \vec{\alpha} < r\vec{c}} c_{\vec{\alpha}}\,[\mathcal{O}(r\vec{c} - \vec{\alpha})] \circ T_{\vec{\alpha}}\big)$ contains a summand $c_{\vec{\alpha}}[\mathcal{O}(r\vec{c} - \vec{\alpha})]K_{\beta_2} \otimes T_{\vec{\alpha}}$ and the remaining part of the coproduct has no contributions to the graded piece $U(\mathbb{X})[\beta_1] \otimes U(\mathbb{X})[\beta_2]$, where $\beta_1 = \overline{\mathcal{O}(r\vec{c} - \vec{\alpha})} \in K_0(\mathbb{X})$ and $\beta_2 \in K_0(\mathbb{X})$ is the class of the summands of the element $T_{\vec{\alpha}}$. Contradiction. Hence, $c_{\vec{\alpha}} = 0$ for all $0 < \vec{\alpha} < r\vec{c}$ and we have the equality $\bigl[Z_r, [\mathcal{O}]\bigr] = a\,[\mathcal{O}(r\vec{c})]$ for a certain constant $a \in \widetilde{\mathbb{Q}}$. Our last goal is to determine a explicitly. Note that

$$a\bigl([\mathcal{O}(r\vec{c})], [\mathcal{O}(r\vec{c})]\bigr) = \bigl(Z_r \circ [\mathcal{O}] - [\mathcal{O}] \circ Z_r, [\mathcal{O}(r\vec{c})]\bigr) = \Bigl(Z_r \otimes [\mathcal{O}], \Delta\bigl([\mathcal{O}(r\vec{c})]\bigr)\Bigr)$$

$$= \bigl(Z_r, \Theta_r\bigr)\bigl([\mathcal{O}], [\mathcal{O}]\bigr) = \gamma_r\bigl([\mathcal{O}], [\mathcal{O}]\bigr).$$

Here we use the vanishing $\bigl([\mathcal{O}] \circ Z_r, [\mathcal{O}(r\vec{c})]\bigr) = 0$ following from fact that $\mathcal{O}$ is not a quotient of $\mathcal{O}(r\vec{c})$ for any $r \in \mathbb{Z}_{>0}$. Hence, $a = \gamma_r$ as stated. Theorem is proven. □

Lemma 6.14. *For any $\vec{x} \in \mathbb{L}(\underline{p})$ and $r \in \mathbb{Z}_{>0}$ we have the following equality in the reduced Drinfeld double $DU(\mathbb{X})$: $\bigl[Z_r^-, [\mathcal{O}(\vec{x})]^+\bigr] = \gamma_r[\mathcal{O}(\vec{x} - r\vec{c})]C^{-r}$.*

Proof. Recall that $\Delta\bigl([\mathcal{O}(\vec{x})]^+\bigr) = [\mathcal{O}(\vec{x})]^+ \otimes \mathbb{1} + K_{\overline{\mathcal{O}(\vec{x})}} \otimes [\mathcal{O}(\vec{x})]^+ + \sum_{\vec{y}>0} \widetilde{\Theta}^+_{\vec{y}} \otimes \bigl[\mathcal{O}(\vec{x} - \vec{y})\bigr]^+$ and $\Delta(Z_r^-) = Z_r^- \otimes \mathbb{1} + C^{-r} \otimes Z_r^-$. Note that $\Theta_{\vec{y}}$ contains contributions of sheaves of class $r\delta$ only for $\vec{y} = r\vec{c}$. Moreover, $\Theta_{r\vec{c}} = \Theta_r$ and $(\Theta_r, Z_r) = \gamma_r$. This implies the result. □

Remark 6.15. By Theorem 5.21 we know that for any $l, t \in \mathbb{Z}$ the following equalities are true in the reduced Drinfeld double $DU(\mathbb{X})$:

$$\bigl[[\mathcal{O}(l\vec{c})]^+, [\mathcal{O}(t\vec{c})]^-\bigr] = \begin{cases} \dfrac{v}{v - v^{-1}}\, \Theta^+_{l-t} O C^t & \text{if } \; l > t, \\ 0 & \text{if } \; l = t, \\ \dfrac{v}{v^{-1} - v}\, \Theta^-_{t-l} O^{-1} C^{-l} & \text{if } \; l < t. \end{cases}$$

Unfortunally, we have not succeeded to find an explicit formula expressing the elements Θ_r through the generators Z_t and $\bigl[\mathcal{S}_i^{(j)}\bigr]$ of the algebra $U(\mathbb{X})_{\mathsf{tor}}$, although by Proposition 5.11 it is known that such a formula does exist.

6.3. **Summary.** In this subsection we collect the main structure results on the composition algebra of a weighted projective line.

Let $\mathbb{X}$ be a weighted projective line of type $(\underline{\lambda}, \underline{p})$, where $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{P}^1(k)$ is a sequence of pairwise distinct points and $\underline{p} = (p_1, \dots, p_n) \in \mathbb{Z}_{>0}$ is a sequence of weights. Let $\mathbb{L}(\underline{p})$ be the abelian group generated by the elements $\vec{x}_1, \dots, \vec{x}_n$ subject to the relations $p_1\vec{x}_1 = \dots = \vec{x}_n = \vec{c}$. Next, let $K = K_0(\mathbb{X})$ be the K–group of the category $\mathsf{Coh}(\mathbb{X})$ and for any coherent sheaf $\mathcal{F}$ let $K_{\overline{\mathcal{F}}}$ be the corresponding element of the Cartan part $\widetilde{\mathbb{Q}}[K]$ of the Hall algebra $H(\mathbb{X})$.

1. For any $l \in \mathbb{Z}$ we denote $L_l = \bigl[\mathcal{O}(l\vec{c})\bigr] \in H(\mathbb{X})$. Then the composition algebra $U(\mathbb{X})$ is the subalgebra of the Hall algebra $H(\mathbb{X})$ defined as follows:

$$U(\mathbb{X}) := \Bigl\langle L_l,\, T_r,\, \bigl[\mathcal{S}_i^{(j)}\bigr],\, O, K_i^{(j)}, C \Bigm| l \in \mathbb{Z},\, r \in \mathbb{Z}_{>0},\, 1 \le i \le n,\, 1 \le j \le p_i \Bigr\rangle.$$

Here $K_i^{(j)} = K_{\overline{\mathcal{S}_i^{(j)}}}$, $O = K_{\overline{\mathcal{O}}}$ and $C = K_i^{(1)} \dots K_i^{(p_i)}$ for any $1 \le i \le n$. Note that the algebra $U(\mathbb{X})$ only depends on the weight sequence $\underline{p}$ and does *not* depend on the set $\underline{\lambda}$.

2. The subalgebra $V(\mathbb{X}) := \big\langle L_l,\, T_r,\, O, C \;\big|\; l \in \mathbb{Z},\, r \in \mathbb{Z}_{>0}\big\rangle$ of the composition algebra $U(\mathbb{X})$ is isomorphic to the composition algebra of a non-weighted projective line $\mathbb{P}^1$. The elements L_l, T_r, O and C satisfy the following relations:

(1) C is central;
(2) $[O, T_t] = 0 = [T_t, T_r]$ for all $r, t \in \mathbb{Z}_{>0}$;
(3) $OL_l = v^{-2} L_l O$ for all $l \in \mathbb{Z}$;
(4) $\big[T_r, L_l\big] = \frac{[2r]}{r} L_{l+r}$ for all $l \in \mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$;
(5) $L_m L_{l+1} + L_l L_{m+1} = v^2\big(L_{l+1}L_m + L_{m+1}L_l\big)$ for all $m, l \in \mathbb{Z}$.

Moreover, this is a complete list of relations of the subalgebra $V(\mathbb{X})$.

3. The algebra $U(\mathbb{X})$ is a topological bialgebra. Let $\overline{U}(\mathbb{X})$ be the subalgebra of $U(\mathbb{X})$ generated by L_l, T_r and $\big[\mathcal{S}_i^{(j)}\big]$ for $l \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$ and (i, j) such that $1 \le i \le n, 1 \le j \le p_i$. Then we have a triangular decomposition: $DU(\mathbb{X}) = \overline{U}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})^-$, where $DU(\mathbb{X})$ is the reduced Drinfeld double of $U(\mathbb{X})$. Moreover, the algebra $V(\mathbb{X})$ is a topological subbialgebra of $U(\mathbb{X})$. In particular, there is an injective algebra homomorphism

$$DV(\mathbb{X}) = \overline{V}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[\widetilde{K}] \otimes_{\widetilde{\mathbb{Q}}} \overline{V}(\mathbb{X})^- \longrightarrow \overline{U}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})^- = DU(\mathbb{X})$$

respecting the triangular decompositions of $DV(\mathbb{X})$ and $DU(\mathbb{X})$. Here $\widetilde{\mathbb{Q}}[\widetilde{K}] = \widetilde{\mathbb{Q}}[O^{\pm}, C^{\pm}]$ is the group algebra of the subgroup of $K_0(\mathbb{X})$ generated by $O = \overline{\mathcal{O}}$ and the class δ of a simple torsion sheaf supported at a non-special k–point of $\mathbb{X}$.

4. For any $\vec{x} \in \mathbb{L}(\underline{p})$ the element $\big[\mathcal{O}(\vec{x})\big]$ belongs to $U(\mathbb{X})$. Let $\overline{U}(\mathbb{X})_{\mathsf{vec}}$ be the subalgebra of $U(\mathbb{X})$ generated by the classes of line bundles and $\overline{U}(\mathbb{X})_{\mathsf{tor}}$ be the subalgebra generated by T_r, $\big[\mathcal{S}_i^{(j)}\big]$ for $r \in \mathbb{Z}_{>0}$ and $(i, j) : 1 \le i \le n, 1 \le j \le p_i$. Then the canonical morphism

$$\overline{U}(\mathbb{X})_{\mathsf{vec}} \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})_{\mathsf{tor}} \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \xrightarrow{\mathsf{mult}} U(\mathbb{X})$$

is an isomorphism of vector spaces over $\widetilde{\mathbb{Q}}$.

5. Let $U(\mathbb{X})_{\mathsf{tor}} = \overline{U}(\mathbb{X})_{\mathsf{tor}} \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K_{\mathsf{tor}}]$, where K_{tor} is the subgroup of $K_0(\mathbb{X})$ generated by the classes of torsion sheaves. Then $U(\mathbb{X})_{\mathsf{tor}}$ is a Hopf algebra. Moreover, it is a subbialgebra of the composition algebra $U(\mathbb{X})$. The algebra $\overline{U}(\mathbb{X})_{\mathsf{tor}}$ decays into a tensor product:

$$\overline{U}(\mathbb{X})_{\mathsf{tor}} = \mathcal{Z} \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})_{\mathsf{tor}}^{\mathsf{exc},\,1} \otimes_{\widetilde{\mathbb{Q}}} \cdots \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})_{\mathsf{tor}}^{\mathsf{exc},\,n}.$$

Here $\mathcal{Z} = \widetilde{\mathbb{Q}}[Z_1, Z_2, \dots, Z_r, \dots]$ is the ring of symmetric functions. It is generated by the elements $Z_r \in \overline{U}(\mathbb{X})_{\mathsf{tor}}[r\delta]$, which are central in $U(\mathbb{X})_{\mathsf{tor}}$ and primitive. The last condition means that $\Delta(Z_r) = Z_r \otimes \mathbb{1} + C^r \otimes Z_r$ for any $r \in \mathbb{Z}_{>0}$. Moreover, the Picard group $\mathsf{Pic}(\mathbb{X}) \cong \mathbb{L}(\underline{p})$ acts trivially on the algebra $\mathcal{Z}$.

For any $1 \le i \le n$, the algebra $\overline{U}(\mathbb{X})_{\mathsf{tor}}^{\mathsf{exc},\,i}$ is generated by the exceptional simple modules $\big[\mathcal{S}_i^{(1)}\big], \dots, \big[\mathcal{S}_i^{(p_i)}\big]$. It is isomorphic to the positive part of the quantized enveloping algebra $U_q^+(\widehat{\mathfrak{sl}}_{p_i})$. The algebra $U(\mathbb{X})_{\mathsf{tor}}^{\mathsf{exc},\,i}$, generated by $\overline{U}(\mathbb{X})_{\mathsf{tor}}^{\mathsf{exc},\,i}$ and the elements $K_i^{(j)}$ for $1 \le j \le p_i$, is a Hopf subalgebra of $U(\mathbb{X})_{\mathsf{tor}}$. All these Hopf algebras $U(\mathbb{X})_{\mathsf{tor}}^{\mathsf{exc},\,i}$ are embedded in the same Hopf algebra $U(\mathbb{X})_{\mathsf{tor}}$ and "share" the "same" central part $\mathcal{A} = \widetilde{\mathbb{Q}}[C^+, C^-]$. The reduced Drinfeld double $DU(\mathbb{X})_{\mathsf{tor}}$ decomposes into a tensor product of algebras:

$$DU(\mathbb{X})_{\mathsf{tor}} = \mathcal{H} \otimes_{\mathcal{A}} U_q(\widehat{\mathfrak{sl}}_{p_1}) \otimes_{\mathcal{A}} \cdots \otimes_{\mathcal{A}} U_q(\widehat{\mathfrak{sl}}_{p_n}),$$

where $\mathcal{H}$ is the Heiseberg algebra.

6. More precisely, for any $r \in \mathbb{Z}_{>0}$ the difference $Z_r - T_r$ belongs to the subalgebra $\overline{U}(\mathbb{X})^{\mathsf{exc}}_{\mathsf{tor}} := \overline{U}(\mathbb{X})^{\mathsf{exc},\,1}_{\mathsf{tor}} \otimes_{\widetilde{\mathbb{Q}}} \cdots \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})^{\mathsf{exc},\,n}_{\mathsf{tor}}$. Next, Green's inner product takes the following values: $\beta_r := (Z_r, T_r) = (Z_r, Z_r) = \alpha_r + \sum\limits_{x \in \underline{\lambda}} \mathrm{def}_x$, where $\mathrm{def}_x = \dfrac{[r]^2}{r}\Bigl(\dfrac{1}{q^{rp_x} - 1} - \dfrac{1}{q^r - 1}\Bigr)$ is the defect of a special point x of weight p_x and $\gamma_r := (Z_r, \Theta_r) = (v^{-1} - v)\beta_r$. The Heisenberg algebra $\mathcal{H}$ is generated over $\widetilde{\mathbb{Q}}$ by the elements $\bigl\{Z_r\bigr\}_{r \in \mathbb{Z}\setminus\{0\}}$ and $C^{\pm 1}$ subject to the relations: $[Z_r, Z_t] = \delta_{r+t,0}\, \beta_r\,(C^{-r} - C^r)$ and $[Z_r, C^{\pm}] = 0$ for $r, t \in \mathbb{Z}\setminus\{0\}$.

7. For any $r \in \mathbb{Z}_{>0}$ and $\vec{x} \in \mathbb{L}(\underline{p})$ we have the following equalities in $DU(\mathbb{X})$:

$$\Bigl[Z_r^{\pm}, \bigl[\mathcal{O}(\vec{x})\bigr]^{\pm}\Bigr] = \gamma_r\,\bigl[\mathcal{O}(\vec{x} + r\vec{c})\bigr]^{\pm} \quad \text{and} \quad \Bigl[Z_r^{-}, \bigl[\mathcal{O}(\vec{x})\bigr]^{+}\Bigr] = \gamma_r\,\bigl[\mathcal{O}(\vec{x} - r\vec{c})\bigr]C^{-r}.$$

8. Let $\mathbb{Y}$ be a weighted projective line of type $(\underline{\mu}, \underline{q})$. Assume that $(\underline{\lambda}, \underline{p})$ dominates $(\underline{\mu}, \underline{q})$. This means that $\underline{\mu}$ is a subset of $\underline{\lambda}$ and for any $i \in \underline{\mu}$ we have: $q_i \le p_i$. Then the composition algebra $U(\mathbb{Y})$ is a topological subbialgebra of the composition algebra $U(\mathbb{X})$. Moreover, there exists an injective algebra homomorphism

$$\overline{U}(\mathbb{Y})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K_0(\mathbb{Y})] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{Y})^- \xrightarrow{\mathbb{F}} \overline{U}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K_0(\mathbb{X})] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})^-$$

preserving the triangular decomposition. Consider the following basic cases.
<u>Case 1</u>. Let $\underline{\mu} = \underline{\lambda} \setminus \{\lambda_1\} = \{\lambda_2, \dots, \lambda_n\}$ and $q_i = p_i$ for all $2 \le i \le n$. Then the homomorphism $U(\mathbb{Y}) \xrightarrow{\mathbb{F}} U(\mathbb{X})$ maps the generators O, C, L_l, T_r, $\bigl[\mathcal{S}_i^{(j)}\bigr]$ and $K_i^{(j)}$ of the algebra $U(\mathbb{Y})$ to the generators of $U(\mathbb{X})$ denoted by the same symbols for all $l \in \mathbb{Z}$, $r \in \mathbb{Z}_{>0}$ and all (i, j) such that $2 \le i \le n$ and $1 \le j \le p_i$.
<u>Case 2</u>. Assume $\underline{\mu} = \underline{\lambda}$ and $q_i = p_i$ for all $2 \le i \le n$ whereas $p_i = q_i + 1 \ge 3$. Then $\mathbb{F}$ maps the generators O, C, $L_l\,(l \in \mathbb{Z})$, $T_r\,(r \in \mathbb{Z}_{>0})$, $\bigl[\mathcal{S}_i^{(j)}\bigr]$ and $K_i^{(j)}$ $(2 \le i \le n, 1 \le j \le p_i)$ of $U(\mathbb{Y})$ to the generators of $U(\mathbb{X})$ denoted by the same symbols. Moreover, $\mathbb{F}\bigl(\bigl[\mathcal{S}_1^{(1)}\bigr]^{\pm}\bigr) = v^{-1}\bigl[\mathcal{S}_1^{(1)}\bigr]^{\pm} \circ \bigl[\mathcal{S}_1^{(2)}\bigr]^{\pm} - \bigl[\mathcal{S}_1^{(2)}\bigr]^{\pm} \circ \bigl[\mathcal{S}_1^{(1)}\bigr]^{\pm}$ and $\mathbb{F}\bigl(\bigl[\mathcal{S}_1^{(j)}\bigr]^{\pm}\bigr) = \bigl[\mathcal{S}_1^{(j+1)}\bigr]^{\pm}$ for all $2 \le j \le q_1$.

7. Composition Hall algebra of the domestic and tubular weighted projective lines

In the previous section, we derived some general properties of the composition algebra of a weighted projective line $\mathbb{X}$. In this section, we deal with two most important cases: domestic and tubular weighted projective lines. The composition algebra of a domestic weighted projective line was also studied in a recent article of Duoe, Jiang and Xiao [9].

We start with a remainder of some results on the reflection functors and the Hall algebras for quiver representations.

7.1. **Reflection functors and Hall algebras of quiver representations.** Let $\overrightarrow{\Delta} = (\Delta_0, \Delta_1, s, t)$ be a finite quiver without loops and oriented cycles. Here Δ_0 is the set of vertices of $\overrightarrow{\Delta}$, Δ_1 is its set of arrows and $s, t : \Delta_1 \to \Delta_0$ are the maps assigning to an arrow its source and target respectively. In this section we consider the Hall algebra of the category $\mathsf{A} = \mathsf{Rep}(k\overrightarrow{\Delta})$ of representations of $\overrightarrow{\Delta}$ over a finite field k. If A is the path algebra of $\overrightarrow{\Delta}$, then the categories $A - \mathsf{mod}$ and A are equivalent.

The following proposition is due to Happel, see [16, Section 4.6].

Proposition 7.1. *Let $\mathbb{D} := \mathsf{Hom}_k(\,-\,, k)$ be the duality over k. Then the derived functor*

$$\tau_D := \mathbb{L}\big(\mathbb{D}\,\mathsf{Hom}_A(\,-\,, A)\big)[-1] : D^b(\mathsf{A}) \longrightarrow D^b(\mathsf{A})$$

satisfies the following property: for any $X, Y \in \mathrm{Ob}\big(D^b(\mathsf{A})\big)$ we have an isomorphism

$$\mathsf{Hom}_{D^b(\mathsf{A})}\big(X, \tau_D(Y)\big) \longrightarrow \mathbb{D}\,\mathsf{Ext}^1_{D^b(\mathsf{A})}(Y, X),$$

functorial in both arguments. In other words, $\tau_D[1]$ is the Serre functor of $D^b(\mathsf{A})$.

Remark 7.2. Let $i \in \Delta_0$ be a vertex and $P_i = Ae_i$ be the indecomposable projective module, which is the projective cover of the simple module S_i. Then $\tau_D(P_i) = I_i[-1]$, where I_i is the injective envelope of S_i.

Definition 7.3. The left exact functor $\tau = \tau^+ := \mathbb{D}\,\mathsf{Ext}^1_A(\,-\,, A) : \mathsf{A} \to \mathsf{A}$ is called the Auslander–Reiten translation.

Proposition 7.4. *In the above notations we have:*

(1) *the functor τ is isomorphic to the composition $\mathsf{A} \xrightarrow{\mathsf{can}} D^b(\mathsf{A}) \xrightarrow{\tau_D} D^b(\mathsf{A}) \xrightarrow{H^0} \mathsf{A}$;*
(2) *Assume that there are no non-zero objects in A which are both projective and injective. Then we have an isomorphism of triangle functors $\tau_D \cong \mathbb{R}\tau$;*
(3) *The functor τ^+ has a left adjoint functor $\tau^- = \mathsf{Ext}^1_A\big(\mathbb{D}(A), -\big)$. Moreover, for any objects $X, Y \in \mathrm{Ob}(\mathsf{A})$ we have bi-functorial isomorphisms:*

$$\mathsf{Hom}_{\mathsf{A}}\big(X, \tau^+(Y)\big) \cong \mathbb{D}\,\mathsf{Ext}^1_{\mathsf{A}}(Y, X) \cong \mathsf{Hom}_{\mathsf{A}}\big(\tau^-(X), Y\big).$$

Proof. The first part of this proposition is trivial. To show the second statement, consider the right exact functor $\nu = \mathbb{D}\,\mathsf{Hom}_A(\,-\,, A)$. Note that $\tau \cong \mathbb{L}_1(\nu)$. Let I_i be an indecomposable injective module and P_j be an indecomposable projective module corresponding to the vertices $i, j \in \Delta_0$ respectively. Since I_i is non-projective, by [16, Section 4.7] we have: $\tau_D(I_i) = \tau(I_i) =: X \in \mathrm{Ob}(\mathsf{A})$. Hence, we have:

$$\mathsf{Hom}_A(I_i, P_j) \cong \mathsf{Hom}_{D^b(\mathsf{A})}\big(\tau_D(I_i), \tau_D(P_j)\big) \cong \mathsf{Hom}_{D^b(\mathsf{A})}\big(X, I_j[-1]\big) = 0.$$

Then by [17, Proposition I.7.4] we have: $\mathbb{L}\nu \cong \mathbb{R}\tau[1]$.

Let Y be an object of A. Then $\tau_D(Y)$ is a complex with at most two non-vanishing cohomologies. Moreover, $H^0\big(\tau_D(Y)\big) \cong \tau(Y)$ and $H^1\big(\tau_D(Y)\big) \cong \nu(Y)$. Using Proposition 7.1 we have:

$$\mathbb{D}\,\mathsf{Ext}^1_{\mathsf{A}}(X, Y) \cong \mathsf{Hom}_{D^b(\mathsf{A})}\big(X, \tau_D(Y)\big) \cong \mathsf{Hom}_{\mathsf{A}}\Big(X, H^0\big(\tau_D(Y)\big)\Big) \cong \mathsf{Hom}_{\mathsf{A}}\big(X, \tau^+(Y)\big),$$

where all the isomorphisms are bi-functorial. The proof of the third part is similar. $\square$

Remark 7.5. Let $\overrightarrow{\Delta} = (1 \longrightarrow 2)$ be a quiver of type A_2. Then the module $I := \bigl(k \xrightarrow{1} k\bigr)$ is both projective and injective. In particular, we have: $\mathbb{R}\tau(I) \cong \tau(X) = \mathbb{D}\,\mathsf{Ext}_A(I, A) = 0$. Since τ_D is an auto-equivalence of $D^b(\mathsf{A})$ and $\mathbb{R}\tau$ is not, we have: $\tau_D \ncong \mathbb{R}\tau$.

Definition 7.6. Let $\overrightarrow{\Delta}$ be a finite quiver without loops and oriented cycles and $* \in \Delta_0$ be a sink (i.e. there is no arrow $\alpha \in \Delta_1$ such that $s(\alpha) = *$). Let $\overleftarrow{\Delta}$ be the quiver obtained from $\overrightarrow{\Delta}$ by inverting all the arrows ending at $*$. Recall, that we have an adjoint pair of the so-called *reflection functors* of Bernstein, Gelfand and Ponomarev [4]

$$\mathbb{S}_*^+ : \mathsf{Rep}(\overrightarrow{\Delta}) \longrightarrow \mathsf{Rep}(\overleftarrow{\Delta}) \quad \text{and} \quad \mathbb{S}_*^- : \mathsf{Rep}(\overleftarrow{\Delta}) \longrightarrow \mathsf{Rep}(\overrightarrow{\Delta})$$

defined as follows. For any object $X = \bigl((V_i)_{i\in Q_0}, (A_\alpha)_{\alpha\in Q_1}\bigr) \in \mathrm{Ob}\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr)$ consider the exact sequence of vector spaces

$$0 \longrightarrow W_* \xrightarrow{\oplus_{\alpha\in Q_1 : t(\alpha)=*} B_\alpha} \bigoplus_{\alpha\in Q_1 : t(\alpha)=*} V_{s(\alpha)} \xrightarrow{\oplus_{\alpha\in Q_1 : t(\alpha)=*} A_\alpha} V_*.$$

Then the representation $Y = \mathbb{S}_*^+(X) = \bigl((U_i)_{i\in Q_0}, (C_\alpha)_{\alpha\in Q_1}\bigr) \in \mathrm{Ob}\bigl(\mathsf{Rep}(\overleftarrow{\Delta})\bigr)$ is defined by:

$$U_i = \begin{cases} V_i & \text{if} \quad i \ne * \\ W_* & \text{if} \quad i = * \end{cases} \quad \text{and} \quad C_\alpha = \begin{cases} A_\alpha & \text{if} \quad s(\alpha) \ne * \\ B_\alpha & \text{if} \quad s(\alpha) = *. \end{cases}$$

The definition of the adjoint reflection functor $\mathbb{S}_*^-$ is dual, see [4].

The following result is well-known, see for example [1, Section VII.5].

Theorem 7.7. *The derived functors* $D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr) \xrightarrow{\mathbb{R}\mathbb{S}_*^+} D^b\bigl(\mathsf{Rep}(\overleftarrow{\Delta})\bigr)$ *and* $D^b\bigl(\mathsf{Rep}(\overleftarrow{\Delta})\bigr) \xrightarrow{\mathbb{L}\mathbb{S}_*^-} D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr)$ *are mutually inverse equivalences of triangulated categories. Moreover, for an indecomposable object* $X \in \mathrm{Ob}\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr)$ *we have:*

$$\mathbb{R}^1\mathbb{S}_*^+(X) = \begin{cases} 0 & \textit{if} \quad X \ncong S_* \\ S_* & \textit{if} \quad X \cong S_* \end{cases}$$

and for an indecomposable object $Y \in \mathrm{Ob}\bigl(\mathsf{Rep}(\overleftarrow{\Delta})\bigr)$ *we have:*

$$\mathbb{L}^{-1}\mathbb{S}_*^-(Y) = \begin{cases} 0 & \textit{if} \quad Y \ncong S_* \\ S_* & \textit{if} \quad Y \cong S_*. \end{cases}$$

In particular, the reflection functors $\mathbb{S}_*^+$ *and* $\mathbb{S}_*^-$ *yield mutually inverse equivalences between the categories* $\mathsf{Rep}\bigl(\overrightarrow{\Delta}\bigr)^\circ$ *and* $\mathsf{Rep}\bigl(\overleftarrow{\Delta}\bigr)^\circ$*, which are the full subcategories of* $\mathsf{Rep}\bigl(\overrightarrow{\Delta}\bigr)$ *and* $\mathsf{Rep}\bigl(\overleftarrow{\Delta}\bigr)$ *consisting of objects without direct summands isomorphic to* S_*.

The following fundamental result gives a link between reflection functors and Auslander–Reiten translations.

Theorem 7.8. *Let* $\overrightarrow{\Delta}$ *be a finite quiver without loops and oriented cycles with a prescribed labeling* $\Delta_0 = \{1, 2, \dots, n\}$*. Assume that for any* $1 \le i < j \le n$ *there is no oriented path starting at* j *and ending at* i *(such a labeling is called admissible). Then there is an isomorphism of functors* $\tau \cong \mathbb{T} \circ \mathbb{S}_1^+ \circ \dots \circ \mathbb{S}_n^+$*, where* $\mathbb{T} : \mathsf{Rep}(\overrightarrow{\Delta}) \to \mathsf{Rep}(\overrightarrow{\Delta})$ *is defined by the following rule. For* $X = \bigl((V_i)_{i\in\Delta_0}, (A_\alpha)_{\alpha\in\Delta_1}\bigr) \in \mathrm{Ob}\bigl(\mathsf{Rep}\,\overrightarrow{\Delta}\bigr)$ *we set:* $\mathbb{T}(X) =$

$\big((U_i)_{i\in Q_0},(B_\alpha)_{\alpha\in Q_1}\big)$, *where* $U_i = V_i$ *and* $B_\alpha = -A_\alpha$. *In a similar way, we have an isomorphism of functors:* $\tau^- = \mathbb{T}\circ \mathbb{S}_n^- \circ \dots \circ \mathbb{S}_1^-$. *In particular, the Coxeter functors* $\mathbb{A}^+ = \mathbb{S}_1^+\circ\dots\circ\mathbb{S}_n^+$ *and* $\mathbb{A}^- = \mathbb{S}_n^-\circ\dots\circ\mathbb{S}_1^-$ *do not depend on the choice of an admissible labeling of vertices.*

For a proof of this result, we refer to [12, Section 5.3] and [43, Proposition II.3.2].

The following important statement seems to be well-known. Nevertheless, we have not succeeded to find its proof in the literature and therefore give it here.

Theorem 7.9. *Let* $\overrightarrow{\Delta}$ *be a finite quiver without loops and oriented cycles and* $\Delta_0 = \{1,2,\dots,n\}$ *be an admissible labeling of its vertices. Assume* $\overrightarrow{\Delta}$ *is not a Dynkin quiver of type* A_n *with linear ordering. Then we have an isomorphism of triangle functors*

$$\tau_D \cong T\circ \mathbb{R}\mathbb{S}_1^+\circ\dots\circ\mathbb{R}\mathbb{S}_n^+.$$

Proof. By Proposition 7.4 and Theorem 7.8 we know that $\tau_D \cong \mathbb{R}\tau \cong T\circ\mathbb{R}\big(\mathbb{S}_1^+\circ\dots\circ\mathbb{S}_n^+\big)$. Using the universal property of the right derived functor of a left exact functor, we obtain a sequence of natural transformation of triangle functors

$$\mathbb{R}\big(\mathbb{S}_1^+\circ\dots\circ\mathbb{S}_n^+\big) \xrightarrow{\xi} \mathbb{R}\mathbb{S}_1^+\circ\mathbb{R}\big(\mathbb{S}_2^+\circ\dots\circ\mathbb{S}_n^+\big) \xrightarrow{\zeta} \cdots \xrightarrow{\kappa} \mathbb{R}\mathbb{S}_1^+\circ\mathbb{R}\mathbb{S}_2^+\circ\dots\circ\mathbb{R}\mathbb{S}_n^+.$$

Let us show that the first natural transformation ξ is an isomorphism (the proof for the remaining ones is the same). By Theorem 7.7, for a non-zero indecomposable representation X we have: if $\mathbb{S}_*^+(X)$ is non-zero, then it is indecomposable and $\mathbb{R}^1\mathbb{S}_*^+(X) = 0$. Next, by Theorem 7.8 we have: $\mathbb{T}\circ\mathbb{S}_1^+\circ\dots\circ\mathbb{S}_n^+(X)\cong\tau(X)$. Hence, if X is a non-projective object of $\mathsf{Rep}(\overrightarrow{\Delta})$ then $\mathbb{S}_1^+\circ\dots\circ\mathbb{S}_n^+(X)\ne 0$. In particular, for any injective module $I\in\mathsf{Rep}(\overrightarrow{\Delta})$ (which is automatically not projective) we get: $\mathbb{R}^1\mathbb{S}_1^+\big(\mathbb{S}_2^+\circ\dots\circ\mathbb{S}_n^+(I)\big) = 0$. Hence, the functor $\mathbb{S}_2^+\circ\dots\circ\mathbb{S}_n^+$ maps injective modules into $\mathbb{S}_1^+$–acyclic modules. This shows that the natural transformation ξ is an isomorphism of functors. □

Definition 7.10. Let $C\in\mathsf{Mat}_{n\times n}(\mathbb{Z})$ be a symmetric matrix such that $c_{ii} = 2$ for all $1\le i\le n$ and $c_{ij} < 0$ for all $1\le i\ne j\le n$. Consider the $\widetilde{\mathbb{Q}}$–algebra $U_q(C)$ generated by the elements $E_1,\dots,E_n$; $F_1,\dots,F_n$ and $K_1^\pm,\dots,K_n^\pm$ subject to the following relations:

- $K_i^\pm K_i^\mp = 1 = K_i^\mp K_i^\pm$, $1\le i\le n$;
- $K_iK_j = K_jK_i$, $1\le i,j\le n$;
- $K_iE_j = v^{-c_{ij}}E_jK_i$ and $K_iF_j = v^{c_{ij}}F_jK_i$, $1\le i,j\le n$;
- $\big[E_i,F_j\big] = \delta_{ij}v\dfrac{K_i-K_i^{-1}}{v-v^{-1}}$, $1\le i,j\le n$;
- $\sum_{k=0}^{1-c_{ij}}(-1)^kE_i^{(k)}E_jE_i^{(1-c_{ij}-k)} = 0$ for $1\le i\ne j\le n$;
- $\sum_{k=0}^{1-c_{ij}}(-1)^kF_i^{(k)}F_jF_i^{(1-c_{ij}-k)} = 0$ for $1\le i\ne j\le n$.

The following result is due to Ringel [33] and Green [15].

Theorem 7.11. *Let* $\overrightarrow{\Delta}$ *be a quiver without loops and oriented cycles,* $|\Delta_0| = n$ *and and* $C = C(\Delta)\in\mathsf{Mat}_{n\times n}(\mathbb{Z})$ *be the Cartan matrix of* $\mathsf{Rep}(\overrightarrow{\Delta})$, *i.e.* $c_{ij} = (\bar S_i,\bar S_j) = \langle\bar S_i,\bar S_j\rangle + \langle\bar S_j,\bar S_i\rangle$, $1\le i,j\le n$. *Then there exists an isomorphism of algebras* $U_q(C)\xrightarrow{\pi_{\overrightarrow{\Delta}}} DC(\overrightarrow{\Delta})$ *mapping* E_i *to* $[S_i]^+$, F_i *to* $[S_i]^-$ *and* $K_i^\pm$ *to* $K_{\pm\overline{S}_i}$ *for* $1\le i\le n$.

By Theorem 2.7, the reflection functors $\mathbb{RS}_*^+$ and $\mathbb{LS}_*^-$ induce mutually inverse algebra isomorphisms of $DH(\overrightarrow{\Delta})$ and $DH(\overleftarrow{\Delta})$. The following result was first time proven by Sevenhant and van den Bergh [42].

Theorem 7.12. *Let $\overrightarrow{\Delta}$ be a finite quiver without loops and oriented cycles, $* \in \Delta_0$ be a sink and $\overleftarrow{\Delta}$ be the quiver obtained from $\overrightarrow{\Delta}$ by inverting all the arrows ending at $*$. Then the derived reflection functor $\mathbb{RS}_*^+$ induces an algebra isomorphism of the Drinfeld doubles of the composition algebras $DC(\overrightarrow{\Delta}) \xrightarrow{\mathbb{S}_*^+} DC(\overleftarrow{\Delta})$, whose inverse is the isomorphism induced by the reflection functor $\mathbb{LS}_*^-$. Moreover, they induce a pair of mutually inverse automorphisms $\mathbb{S}_*^\pm$ of the quantized enveloping algebra $U_q(C)$ determined by the quiver $\overrightarrow{\Delta}$:*

$$\begin{tikzcd} U_q(C) \arrow[r, "\mathbb{S}_*^+"] \arrow[d, "\pi_{\overrightarrow{\Delta}}"'] & U_q(C) \arrow[d, "\pi_{\overleftarrow{\Delta}}"] \\ DC(\overrightarrow{\Delta}) \arrow[r, "\mathbb{S}_*^+"] & DC(\overleftarrow{\Delta}) \end{tikzcd} \qquad \begin{tikzcd} U_q(C) \arrow[r, "\mathbb{S}_*^-"] \arrow[d, "\pi_{\overleftarrow{\Delta}}"'] & U_q(C) \arrow[d, "\pi_{\overrightarrow{\Delta}}"] \\ DC(\overleftarrow{\Delta}) \arrow[r, "\mathbb{S}_*^-"] & DC(\overrightarrow{\Delta}) \end{tikzcd}$$

which are given by the following formulae:

$E_i \xrightarrow{\mathbb{S}_*^+} E_i, \quad F_i \xrightarrow{\mathbb{S}_*^+} F_i$	if $c_{i*} = 0$	$E_i \xrightarrow{\mathbb{S}_*^-} E_i, \quad F_i \xrightarrow{\mathbb{S}_*^-} F_i$
$E_* \xrightarrow{\mathbb{S}_*^+} v^{-1}K_*^{-1}F_*, \quad F_* \xrightarrow{\mathbb{S}_*^+} vE_*K_*$	if $c_{i*} = 2$	$E_* \xrightarrow{\mathbb{S}_*^-} v^{-1}F_*K_*, \quad F_* \xrightarrow{\mathbb{S}_*^-} vK_*^{-1}E_*$
$E_i \xrightarrow{\mathbb{S}_*^+} \sum\limits_{a+b=-c_{i*}} (-1)^a v^{-b} E_*^{(a)} E_i E_*^{(b)}$	if $c_{i*} < 0$	$E_i \xrightarrow{\mathbb{S}_*^-} \sum\limits_{a+b=-c_{i*}} (-1)^a v^{-b} E_i^{(a)} E_* E_i^{(b)}$
$F_i \xrightarrow{\mathbb{S}_*^+} \sum\limits_{a+b=-c_{i*}} (-1)^a v^{-b} F_*^{(a)} F_i F_*^{(b)}$	if $c_{i*} < 0$	$F_i \xrightarrow{\mathbb{S}_*^-} \sum\limits_{a+b=-c_{i*}} (-1)^a v^{-b} F_i^{(a)} F_* F_i^{(b)}$
$K_i^\pm \xrightarrow{\mathbb{S}_*^+} K_i^\pm$	if $c_{i*} = 0$	$K_i^\pm \xrightarrow{\mathbb{S}_*^-} K_i^\pm$
$K_i^\pm \xrightarrow{\mathbb{S}_*^+} K_i^{\mp c_{i*}} K_*^\pm$	if $c_{i*} \neq 0$	$K_i^\pm \xrightarrow{\mathbb{S}_*^-} K_*^{\mp c_{i*}} K_i^\pm$.

As it was explained in [42, Section 13], *up to a certain twist these automorphisms coincide with the symmetries discovered by Lusztig* [27].

In the next subsection, the following notion will be important.

Definition 7.13. Let $\overrightarrow{\Delta}$ be a quiver without loops and oriented cycles and $\Delta_0 = \{1, 2, \dots, n\}$ be an admissible labeling of vertices. Then $\mathbb{A} := \mathbb{S}_1^+ \circ \cdots \circ \mathbb{S}_n^+ : DC(\overrightarrow{\Delta}) \to DC(\overrightarrow{\Delta})$ is the *Coxeter automorphism* of $DC(\overrightarrow{\Delta})$. Using Corollary 7.12, we also obtain the corresponding automorphism of the algebra $U_q(C)$, given by the commutative diagram

$$\begin{tikzcd} U_q(C) \arrow[rrrr, "\mathbb{A}^+"] \arrow[d, "\pi_{\overrightarrow{\Delta}}"'] & & & & U_q(C) \arrow[d, "\pi_{\overrightarrow{\Delta}}"] \\ DC(\overrightarrow{\Delta}) \arrow[r, "\mathbb{S}_n^+"] & DC(\overrightarrow{\Delta}') \arrow[r] & \cdots \arrow[r] & DC(\overrightarrow{\Delta}'') \arrow[r, "\mathbb{S}_1^+"] & DC(\overrightarrow{\Delta}). \end{tikzcd}$$

The inverse automorphism $U_q(C) \xrightarrow{\mathbb{A}^-} U_q(C)$ is defined in a similar way.

7.2. **Composition algebra of a domestic weighted projective line.** Let $\mathbb{X} = \mathbb{X}(\underline{p})$ be a weighted projective line of domestic type $\underline{p} = (p_1, \dots, p_n)$. The weight sequence $\underline{p}$ determines the affine Dynkin diagram $\widehat{\Delta} = \widehat{\Delta}(\underline{p})$ by the following table.

weight sequence $\underline{p}$	affine Dynkin diagram $\widehat{\Delta}$
$(p, q): \ \min(p, q) \ge 2$	$\widehat{A}_{p+q-1}$
$(2, 2, n): \ (n \ge 2)$	$\widehat{D}_{n+2}$
$(2, 3, 3)$	$\widehat{E}_6$
$(2, 3, 4)$	$\widehat{E}_7$
$(2, 3, 5)$	$\widehat{E}_8$

The following theorem follows from a result of Geigle and Lenzing [13, Proposition 4.1].

Theorem 7.14. *Let $\mathbb{X} = \mathbb{X}(\underline{p})$ be a weighted projective line of domestic type and Δ be the corresponding affine Dynkin diagram. For a sake of simplicity assume that $\overrightarrow{\Delta}$ has the "star-shaped" orientation in the case of $\widehat{D}_{n+2}$ $(n \ge 2)$, $\widehat{E}_6$, $\widehat{E}_7$ and $\widehat{E}_8$ and it has p subsequent arrows going clockwise and q subsequent arrows going anti-clockwise in the case $\underline{p} = (p, q)$. Then there exists a derived equivalence $D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr) \xrightarrow{\mathbb{G}} D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr)$.*

Applying Cramer's Theorem 2.7, we immediately obtain the following corollary.

Corollary 7.15. *The equivalence $D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr) \xrightarrow{\mathbb{G}} D^b\bigl(\mathsf{Coh}(\mathbb{X})\bigr)$ induces an algebra isomorphism of the reduced Drinfeld doubles of the Hall algebras $DH(\overrightarrow{\Delta}) \xrightarrow{\mathbb{G}} DH(\mathbb{X})$.*

The next result is a refinement of this statement.

Theorem 7.16. *The equivalence $D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr) \xrightarrow{\mathbb{G}} D^b\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr)$ induces an algebra isomorphism of the reduced Drinfeld doubles of the composition Hall algebras $DC(\overrightarrow{\Delta}) \xrightarrow{\mathbb{G}} DU(\mathbb{X})$. Moreover, the following diagram of algebra homomorphisms is commutative:*

$$\begin{tikzcd} DC(\overrightarrow{\Delta}) \arrow[r, "\mathbb{G}"] \arrow[d, "\mathbb{A}_{\overrightarrow{\Delta}}"'] & DU(\mathbb{X}) \arrow[d, "\mathbb{A}_{\mathbb{X}}"] \\ DC(\overrightarrow{\Delta}) \arrow[r, "\mathbb{G}"] & DU(\mathbb{X}). \end{tikzcd} \tag{25}$$

Here $\mathbb{A}_{\overrightarrow{\Delta}}$ is the Coxeter transformation introduced in Definition 7.13 and $\mathbb{A}_{\mathbb{X}}$ is the automorphism of $DU(\mathbb{X})$ induced by the Auslander-Reiten translation $\mathcal{F} \mapsto \mathcal{F}(\vec{\omega})$.

Proof. First recall that the equivalence $\mathbb{G}$ induces an isomorphism of the K–groups $K_0(\overrightarrow{\Delta}) \xrightarrow{\mathbb{G}} K_0(\mathbb{X})$. By [38, Proposition 7.4], the composition algebra $U(\mathbb{X})$ contains classes of all indecomposable locally free sheaves on $\mathbb{X}$. Next, by [19, Theorem 3] it is known that for any indecomposable preprojective or preinjective object $X \in \mathsf{Ob}\bigl(\mathsf{Rep}(\overrightarrow{\Delta})\bigr)$, the element $[X]$ belongs to the composition algebra $C(\overrightarrow{\Delta})$. Moreover, $\mathbb{G}(X) \cong \mathcal{F}[i]$, where $\mathcal{F}$ is some indecomposable vector bundle and i is some integer. Hence, the algebra homomorphism $DH(\overrightarrow{\Delta}) \xrightarrow{\mathbb{G}} DH(\mathbb{X})$ restricts to an injective homomorphism $DC(\overrightarrow{\Delta}) \xrightarrow{\mathbb{G}} DU(\mathbb{X})$. By Corollary 5.19, the reduced Drinfeld double $DU(\mathbb{X})$ is generated by the elements $\bigl[\mathcal{O}(\vec{x})\bigr]^{\pm}$

for $\vec{x} \in \mathbb{L}(\underline{p})$ and the Cartan part $\widetilde{\mathbb{Q}}\big[K_0(\mathbb{X})\big]$. Hence, the map $DC(\overrightarrow{\Delta}) \xrightarrow{\mathbb{G}} DU(\mathbb{X})$ is surjective, hence an isomorphism. The commutativity of the diagram (25) follows from the general fact that an equivalence of categories $D^b\big(\mathsf{Rep}(\overrightarrow{\Delta})\big) \xrightarrow{\mathbb{G}} D^b\big(\mathsf{Rep}(\overrightarrow{\Delta})\big)$ commutes with Serre functors: $\mathbb{A}_{\mathbb{X}} \circ \mathbb{G} \cong \mathbb{G} \circ \mathbb{A}_{\overrightarrow{\Delta}}$, see for example [31]. $\square$

Remark 7.17. The automorphism $\mathbb{A} = \mathbb{A}_{\mathbb{X}}$ preserves the triangular decomposition $DU(\mathbb{X}) = \overline{U}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}\big[K_0(\mathbb{X})\big] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})^+$. Moreover, it maps the algebra $DU(\mathbb{X})_{\mathsf{tor}}$ to itself. This action is given by the following formulae:

$$\mathbb{A}(Z_r^{\pm}) = Z_r^{\pm}, \quad \mathbb{A}(C) = C, \quad \mathbb{A}\big(\big[\mathcal{S}_i^{(j)}\big]^{\pm}\big) = \big[\mathcal{S}_i^{(j+1)}\big]^{\pm} \quad \text{and} \quad \mathbb{A}\big(K_i^{(j)}\big) = K_i^{(j+1)}$$

for all $r \in \mathbb{Z}_{>0}$ and (i,j) such that $1 \le i \le n, 1 \le j \le p_i$ (as usual, we set $(i, p_i{+}1) = (i,1)$). The explicit action of the Coxeter automorphism $\mathbb{A}_{\overrightarrow{\Delta}}$ is given by Theorem 7.9 and Theorem 7.12. The commutative diagram (25) yields a practical rule to compute the images of elements of the algebra $DU(\mathbb{X})$ in the algebra $DC(\overrightarrow{\Delta})$ under the map $\mathbb{G}^{-1}$.

Recall that for any object $\mathcal{F}$ of the category $\mathsf{Coh}(\mathbb{X})$ such that $\mathsf{Hom}(\mathcal{F},\mathcal{F}) = k$ and $\mathsf{Ext}^1(\mathcal{F},\mathcal{F}) = 0$ we have the following equality in the Hall algebra $H(\mathbb{X})$:

$$\big[\mathcal{F}^{\oplus n}\big] = v^{n(n-1)}\frac{[\mathcal{F}]^n}{[n]!} = v^{n(n-1)}[\mathcal{F}]^{(n)}.$$

Definition 7.18. Let $\mathcal{P} := \big\{\alpha \in K_0(\mathbb{X}) \big| \langle \alpha, \alpha\rangle = 1 \text{ and } \mathrm{rk}(\alpha) > 0\big\}$.

It is well-known that an indecomposable vector bundle $\mathcal{F}$ on a domestic projective line $\mathbb{X}$ is determined by its class in the K–group $\alpha = \overline{\mathcal{F}} \in \mathcal{P} \subset K_0(\mathbb{X})$. On the other hand, any real root $\alpha \in \mathcal{P}$ corresponds to some indecomposable vector bundle $\mathcal{F} = \mathcal{F}_\alpha$. It is also known that for any pair of indecomposable vector bundles $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathbb{X}$ we have: either $\mathsf{Ext}^1(\mathcal{F}_1,\mathcal{F}_2) = 0$ or $\mathsf{Ext}^1(\mathcal{F}_2,\mathcal{F}_1) = 0$. Define an ordering on the set $\mathcal{P}$ of real roots of positive rank: for $\alpha \ne \beta$ we say that $\alpha > \beta$ if $\langle \alpha, \beta\rangle > 0$. Note that this condition implies that $\langle \beta, \alpha\rangle < 0$. If $\alpha \ne \beta$ and $\langle \alpha, \beta\rangle = 0$ than we can define the ordering between α and β in an arbitrary way.

Let $\mathcal{F}$ be an arbitrary vector bundle on $\mathbb{X}$. Then it splits into a direct sum of indecomposable ones: $\mathcal{F} \cong \mathcal{F}_{\alpha_1}^{m_1} \oplus \dots \oplus \mathcal{F}_{\alpha_t}^{m_t}$ for some uniquely determined $\alpha_1, \dots, \alpha_t \in \mathcal{P}$ such that $\alpha_1 > \dots > \alpha_t$ and $m_1, \dots, m_t \in \mathbb{Z}_{>0}$. Then we have the following equality in the Hall algebra $H(\mathbb{X})$: $\big[\mathcal{F}\big] = v^m\big[\mathcal{F}_{\alpha_1}\big]^{(m_1)} \circ \dots \circ \big[\mathcal{F}_{\alpha_t}\big]^{(m_t)}$, where $m = m_1(m_1-1) + \dots + m_t(m_t-1) + \sum_{i<j} m_i m_j \langle \alpha_i, \alpha_j\rangle$. This implies the following corollary on the structure of the composition algebra $U(\mathbb{X})$.

Corollary 7.19. *Let $\mathbb{X} = \mathbb{X}(\underline{p})$ be a weighted projective line of a domestic type $\underline{p} = (p_1, \dots, p_n)$. Then the composition algebra $U(\mathbb{X})$ decomposes into a tensor product of vector spaces $U(\mathbb{X}) = \overline{U}(\mathbb{X})_{\mathsf{vec}} \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})_{\mathsf{tor}} \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K]$. Moreover, $\overline{U}(\mathbb{X})_{\mathsf{tor}} = \mathcal{Z} \otimes_{\widetilde{\mathbb{Q}}} U_q^+\big(\widehat{\mathfrak{sl}}_{p_1}\big) \otimes_{\widetilde{\mathbb{Q}}} \dots \otimes_{\widetilde{\mathbb{Q}}} U_q^+\big(\widehat{\mathfrak{sl}}_{p_n}\big)$, where $\mathcal{Z} = \widetilde{\mathbb{Q}}[Z_1, \dots, Z_r, \dots]$ is the ring of symmetric functions. The subalgebra $\overline{U}(\mathbb{X})_{\mathsf{vec}}$ has a monomial basis $\big[\mathcal{F}_{\alpha_1}\big]^{(n_1)} \circ \dots \circ \big[\mathcal{F}_{\alpha_t}\big]^{(n_t)}$ parameterized by the sequences $\big((\alpha_1, m_1), \dots, (\alpha_t, m_t)\big)$, where $\alpha_1 > \dots > \alpha_t$ are from $\mathcal{P}$ and $m_1, \dots, m_t \in \mathbb{Z}_{>0}$.*

7.3. **Composition algebra of a tubular weighted projective line.** The most beautiful applications of our approach concern the case of a weighted projective line $\mathbb{X}$ of a tubular type $(\underline{\lambda}, \underline{p})$. There are actually only four cases of such curves:

(1) $\underline{\lambda} = (0, \infty, 1, \lambda)$ and $\underline{p} = (2,2,2,2)$ (type $\widehat{\widehat{D}}_4$).

(2) $\underline{\lambda} = (0, \infty, 1)$ and $\underline{p} = (3,3,3)$ (type $\widehat{\widehat{E}}_6$), $(2,4,4)$ (type $\widehat{\widehat{E}}_7$) or $(2,3,6)$ (type $\widehat{\widehat{E}}_8$).

As it was already observed by Geigle and Lenzing in [13, Example 5.8], in the case of the base field $k = \mathbb{C}$, the category $\mathsf{Coh}(\mathbb{X})$ is equivalent to the category of equivariant coherent sheaves on an elliptic curve $\mathbb{E}$ with respect to an appropriate finite group action. For example, if $\mathbb{E}$ is the elliptic curve defined by the cubic equation $zy^2 = x(x-z)(x-\lambda z)$ in $\mathbb{P}^2$ for some $\lambda \in k \backslash \{0,1\}$ and $\mathbb{Z}_2$ is a cyclic group acting by the rule $(x:y:z) \mapsto (x:-y:z)$ then $\mathsf{Coh}\big(\mathbb{X}((2,2,2,2),\lambda)\big)$ is equivalent to $\mathsf{Coh}^{\mathbb{Z}_2}(\mathbb{E})$.

Below we give a list of the most important properties of the category of coherent sheaves on a tubular weighted projective line $\mathbb{X}$, which are due to Geigle and Lenzing [13], Lenzing and Meltzer [24, 25], and Meltzer [29].

1. Let $\underline{p} = (p_1, \dots, p_n)$ be the weight type of $\mathbb{X}$ and p be the least common multiple of $p_1, \dots, p_n$. Then there exists an isomorphism of functors $\tau^p \xrightarrow{\simeq} \mathbb{1}$, where $\mathsf{Coh}(\mathbb{X}) \xrightarrow{\tau} \mathsf{Coh}(\mathbb{X})$ is the Auslander–Reiten translation in $\mathsf{Coh}(\mathbb{X})$. In particular, all Auslander–Reiten components of $\mathsf{Coh}(\mathbb{X})$ are the so-called *tubes*, hence the name “tubular”.

2. Let $Z = (\mathrm{rk}, -\deg)$ be the standard stability condition on $\mathsf{Coh}(\mathbb{X})$. Then any indecomposable coherent sheaf is automatically semi-stable with respect to Z. For any $\mu \in \mathbb{Q} \cup \{\infty\}$ let $\mathsf{SS}^\mu = \mathsf{SS}^\mu(\mathbb{X})$ be the abelian category of semi-stable coherent sheaves of slope μ (in these notations, $\mathsf{SS}^\infty = \mathsf{Tor}(\mathbb{X})$). Then the Auslander-Reiten functor τ maps SS^μ to itself.

3. The derived category $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$ has a rich group of exact auto-equivalences. Let $\mathcal{F}$ be an arbitrary coherent sheaf such that $\mathsf{End}(\mathcal{F}) = k$ and $p(\mathcal{F})$ be the smallest positive integer such that $\tau^{p(\mathcal{F})}(\mathcal{F}) \cong \mathcal{F}$. Then $\mathcal{F}$ induces the following auto-equivalence $T_\mathcal{F}$ of the derived category $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$: for any object $\mathcal{H}$ of $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$ we have

$$T_\mathcal{F}(\mathcal{H}) \cong \mathsf{cone}\Big(\bigoplus_{i\in\mathbb{Z}} \bigoplus_{l=1}^{p(\mathcal{F})} \mathsf{Hom}_{D^b(\mathbb{X})}\big(\tau^l(\mathcal{F}[-i]), \mathcal{H}\big) \otimes_k \tau^l(\mathcal{F}[-i]) \xrightarrow{\mathsf{ev}} \mathcal{H}\Big).$$

Moreover, $T_\mathcal{F}$ induces an isometry of the K–group $K_0(\mathbb{X})$ given by the formula

$$K_0(\mathbb{X}) \xrightarrow{T_\mathcal{F}} K_0(\mathbb{X}), \quad a \mapsto a - \sum_{l=1}^{p(\mathcal{F})} \big\langle \tau^l(\mathcal{F}), a \big\rangle \big[\tau^l(\mathcal{F})\big].$$

This auto-equivalence $T_\mathcal{F}$ is called *tubular mutation* or *twist functor*.

4. Let $R := \big\{a \in K_0(\mathbb{X}) \,|\, \langle a, -\rangle = 0\big\}$ be the left radical of $K_0(\mathbb{X})$. Then $R \cong \langle \delta, \omega \rangle \cong \mathbb{Z}^2$, where δ is the class of a simple torsion sheaf supported at a non-special point of $\mathbb{X}(k)$ and $\omega = \overline{\mathcal{O}} + \tau(\overline{\mathcal{O}}) + \dots + \tau^{p(\mathcal{O})-1}(\overline{\mathcal{O}})$. Moreover, the canonical group homomorphism $\mathsf{Aut}\big(D^b\big(\mathsf{Coh}(\mathbb{X})\big)\big) \xrightarrow{\pi} \mathsf{Isom}\big(K_0(\mathbb{X})\big)$ restricts to a surjective group homomorphism $\mathsf{Aut}\big(D^b\big(\mathsf{Coh}(\mathbb{X})\big)\big) \to \mathsf{SL}(2,\mathbb{Z}) \subseteq \mathsf{Aut}(R)$. Here, $\mathsf{Isom}\big(K_0(\mathbb{X})\big)$ is the group of isometries of $K_0(\mathbb{X})$, i.e. the group of linear automorphisms of $K_0(\mathbb{X})$ respecting the Euler form $\langle -, - \rangle$.

5. Let $\mathcal{S}$ be the simple torsion sheaf supported at a non-special point of $\mathbb{X}(k)$. Then the tubular mutations $\mathbb{F} := T_{\mathcal{O}}$ and $\mathbb{G} := T_{\mathcal{S}}$ satisfy the braid group relation: $\mathbb{FGF} \cong \mathbb{GFG}$, see also [41] for a discussion of twist functors and the induced braid group actions in more general situations. If the field k is algebraically closed then there exists an exact sequence

$$1 \longrightarrow \mathsf{Pic}^0 \rtimes \mathsf{Aut}(\mathbb{X}) \longrightarrow \mathsf{Aut}\big(D^b\big(\mathsf{Coh}(\mathbb{X})\big)\big) \longrightarrow B_3 \longrightarrow 1$$

where B_3 is the braid group on three strands and $\mathsf{Pic}^0(\mathbb{X})$ is the group of line bundles on $\mathbb{X}$ of degree zero. Moreover, $\mathsf{Pic}^0 \rtimes \mathsf{Aut}(\mathbb{X})$ consists of all elements of $\mathsf{Aut}\big(D^b\big(\mathsf{Coh}(\mathbb{X})\big)\big)$ having finite order.

6. For any slope $\mu \in \mathbb{Q}$, the category of semi-stable sheaves $\mathsf{SS}^\mu(\mathbb{X})$ is equivalent to the category of torsion sheaves $\mathsf{Tor}(\mathbb{X}) = \mathsf{SS}^\infty(\mathbb{X})$. Any element $\mathbb{F} \in \mathsf{Aut}\big(D^b(\mathsf{Coh}(\mathbb{X})\big)$ establish an equivalence between the category SS^μ and $\mathsf{SS}^\nu[i]$ for some $\nu \in \mathbb{Q} \cup \{\infty\}$ and $i \in \mathbb{Z}$.

Lemma 7.20. *Let $\mathbb{X}$ be a weighted projective line of tubular type. Then there is a canonical isomorphism of vector spaces over $\widetilde{\mathbb{Q}}$:*

$$\overrightarrow{\bigotimes}_{\mu\in\mathbb{Q}\cup\{\infty\}} \overline{H}(\mathsf{SS}^\mu) \xrightarrow{\mathsf{mult}} \overline{H}(\mathbb{X}), \tag{26}$$

where $\overline{H}(\mathsf{SS}^\mu)$ is the non-extended Hall algebra of the abelian category SS^μ of semi-stable sheaves of slope μ and $\overrightarrow{\otimes}$ is the restricted directed tensor product. This means that a simple tensor in the left-hand side of (26) has all but finitely many entries equal to $\mathbb{1}$.

Proof. Since any indecomposable sheaf on $\mathbb{X}$ is automatically semi-stable, for any $\mathcal{F} \in \mathsf{Ob}\big(\mathsf{Coh}(\mathbb{X})\big)$ there exist a unique decomposition $\mathcal{F} \cong \mathcal{F}_1 \oplus \dots \oplus \mathcal{F}_t$, where all sheaves $\mathcal{F}_1, \dots, \mathcal{F}_t$ are semi-stable and $\mu(\mathcal{F}_1) < \dots < \mu(\mathcal{F}_t)$. Then we have: $\mathsf{Ext}^1(\mathcal{F}_i, \mathcal{F}_l) = 0 = \mathsf{Hom}(\mathcal{F}_l, \mathcal{F}_i)$ for any $1 \le i < l \le t$. Hence, there is the following equality in the Hall algebra $H(\mathbb{X})$: $\big[\mathcal{F}\big] = v^m \big[\mathcal{F}_1\big] \circ \dots \circ \big[\mathcal{F}_t\big]$, where $m = \sum_{i<l} \langle \overline{\mathcal{F}_i}, \overline{\mathcal{F}_l} \rangle$. It shows that the map mult is surjective. It is also not difficult to see that mult is injective, hence isomorphism. □

Theorem 7.21. *Let $\mathbb{X}$ be a tubular weighted projective line. Then the group of exact auto-equivalences $\mathsf{Aut}\big(D^b(\mathsf{Coh}(\mathbb{X})\big)$ acts on the reduced Drinfeld double $DH(\mathbb{X})$ by algebra automorphisms. For any $\mathbb{F} \in \mathsf{Aut}\big(D^b(\mathsf{Coh}(\mathbb{X})\big)$ its action on $DH(\mathbb{X})$ is uniquely determined by the following rules: for $\alpha \in K_0(\mathbb{X})$ we have $\mathbb{F}(K_\alpha) = K_{\mathbb{F}(\alpha)}$ and for any semi-stable sheaf $\mathcal{F}$ such that $\mathbb{F}(\mathcal{F}) \cong \widehat{\mathcal{F}}[-n_{\mathbb{F}}(\mathcal{F})]$ with $\widehat{\mathcal{F}} \in \mathsf{Coh}(\mathbb{X})$ and $n_{\mathbb{F}}(X) \in \mathbb{Z}$ we have:*

$$\mathbb{F}\big([\mathcal{F}]^{\pm}\big) = v^{n_{\mathbb{F}}(\mathcal{F})\langle \bar{\mathcal{F}}, \bar{\mathcal{F}} \rangle} \big[\widehat{\mathcal{F}}\big]^{\overline{n_{\mathbb{F}}(\mathcal{F})}} K_{\widehat{\mathbb{F}}(\mathcal{F}^{\pm})}^{n_{\mathbb{F}}(\mathcal{F})},$$

where $\overline{n_{\mathbb{F}}(\mathcal{F})} = +$ if $n_{\mathbb{F}}(\mathcal{F})$ is even and $-$ if $n_{\mathbb{F}}(\mathcal{F})$ is odd. Moreover, any element $\mathbb{F} \in \mathsf{Aut}\big(D^b\big(\mathsf{Coh}(\mathbb{X})\big)\big)$ maps the reduced Drinfeld double $DU(\mathbb{X})$ to itself. In other words, the group of exact auto-equivalences of the derived category $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$ acts on the reduced Drinfeld double of the composition algebra $U(\mathbb{X})$ by algebra automorphisms.

Proof. The fact that $\mathbb{F}$ is an algebra automorphism of $DH(\mathbb{X})$ can be proven along the same lines as [5, Theorem 3.8]. It is also a special case of a general Theorem 2.7 proven by Cramer [8].

Let $\mathcal{F}$ be an exceptional coherent sheaf on $\mathbb{X}$, i.e. $\mathsf{End}(\mathcal{F}) = k$ and $\mathsf{Ext}^1(\mathcal{F}, \mathcal{F}) = 0$. By [38, Proposition 8.7] it is known that the element $[\mathcal{F}]$ belongs to the composition algebra

$U(\mathbb{X})$. By Corollary 5.19, the reduced Drinfeld double $DU(\mathbb{X})$ is generated by the Cartan part $\widetilde{\mathbb{Q}}[K]$ and classes of certain exceptional objects (e.g. by the line bundles). But this implies that the algebra $DU(\mathbb{X})$ is invariant under the action of $\mathsf{Aut}\big(D^b\big(\mathsf{Coh}(\mathbb{X})\big)\big)$. $\square$

Remark 7.22. In a work of Lin and Peng [26], the Hall–Lie algebra $\mathfrak{g}(A)$ of the *root category* $D^b(A-\mathsf{mod})/[2]$ of a tubular algebra A was studied. They have shown that $\mathfrak{g}(A)$ is isomorphic to the toroidal Lie algebra of the corresponding Dynkin type.

Lemma 7.23. *For any $\alpha \in K_0(\mathbb{X})$, let $\mathbb{1}^{\mathrm{ss}}_\alpha := \sum\limits_{[\mathcal{F}]\in \mathsf{J}:\, \mathcal{F}\in \mathsf{SS}_\alpha} [\mathcal{F}] \in \overline{H}(\mathbb{X})[\alpha]$, where SS_α is the category of semi-stable objects of $\mathsf{Coh}(\mathbb{X})$ of class α with respect to the standard stability condition $Z = (\mathrm{rk}, -\deg)$. Then $\mathbb{1}^{\mathrm{ss}}_\alpha$ belongs to the composition algebra $U(\mathbb{X})$ (note that the category SS_α can be empty).*

Proof. By Proposition 5.11 for any $\alpha \in K_0(\mathbb{X})$ such that $\mathrm{rk}(\alpha) = 0$, the element $\mathbb{1}_\alpha = \mathbb{1}^{\mathrm{ss}}_\alpha$ belongs to the algebra $U(\mathbb{X})_{\mathsf{tor}}$. Moreover, there exists an auto-equivalence $\mathbb{F} \in \mathsf{Aut}\big(D^b\big(\mathsf{Coh}(\mathbb{X})\big)\big)$ such that $\mathrm{rk}\big(\mathbb{F}(\alpha)\big) = 0$. Moreover, up to a shift $\mathbb{F}$ maps the category SS^μ to $\mathsf{Tor}(\mathbb{X})$, where μ is the slope corresponding to the class α. Let $\gamma := \mathbb{F}(\alpha)$, then by Theorem 7.21 the element $\mathbb{1}^{\mathrm{ss}}_\alpha = \mathbb{F}^{-1}\big(\mathbb{1}^{\mathrm{ss}}_\gamma\big)$ belongs to the algebra $DU(\mathbb{X}) = \overline{U}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})^-$. Since $\mathbb{1}^{\mathrm{ss}}_\alpha$ lies in $\overline{H}(\mathbb{X})^+$, it is an element of $\overline{U}(\mathbb{X})^+$, too. $\square$

Theorem 7.24. *Let $\mathbb{X}$ be a weighted projective line of tubular type $\underline{p} = (p_1, \dots, p_n)$. For any slope $\mu \in \mathbb{Q}\cup\{\infty\}$, let $U^\mu = U(\mathbb{X})^\mu$ be the subalgebra of the composition algebra $\overline{U}(\mathbb{X})$ generated by the set $\big\{\mathbb{1}^{\mathrm{ss}}_\alpha\big\}_{\alpha\in K_0(\mathbb{X}):\mu(\alpha)=\mu}$. Then we have:*

1. *U^μ is isomorphic to the algebra $\overline{U}(\mathbb{X})_{\mathsf{tor}} \cong \mathcal{Z} \otimes_{\widetilde{\mathbb{Q}}} U_q^+(\widehat{\mathfrak{sl}}_{p_1}) \otimes_{\widetilde{\mathbb{Q}}} \cdots \otimes_{\widetilde{\mathbb{Q}}} U_q^+(\widehat{\mathfrak{sl}}_{p_n})$.*
2. *The canonical map $\overrightarrow{\otimes}_{\mu\in\mathbb{Q}\cup\{\infty\}} U^\mu \xrightarrow{\mathsf{mult}} \overline{U}(\mathbb{X})$ is an isomorphism of vector spaces.*

Proof. We know that there exists an equivalence of categories $\mathsf{SS}^\mu(\mathbb{X}) \xrightarrow{\mathbb{F}} \mathsf{Tor}(\mathbb{X})$. Hence, $\mathbb{F}$ induces an isomorphism of non-extended Hall algebras $\overline{H}\big(\mathsf{SS}^\mu(\mathbb{X})\big) \xrightarrow{\mathbb{F}} \overline{H}\big(\mathsf{Tor}(\mathbb{X})\big)$. Moreover, $\mathbb{F}$ induces a bijection between the sets $\big\{\mathbb{1}_\gamma | \,\mathrm{rk}(\gamma) = 0\big\}$ and $\big\{\mathbb{1}^{\mathrm{ss}}_\alpha | \mu(\alpha) = \mu\big\}$. Hence, the algebras U^μ and $U^\infty = \overline{U}(\mathbb{X})_{\mathsf{tor}}$ are isomorphic. By Proposition 6.9 we know that $\overline{U}(\mathbb{X})_{\mathsf{tor}} \cong \mathcal{Z} \otimes_{\widetilde{\mathbb{Q}}} U_q^+(\widehat{\mathfrak{sl}}_{p_1}) \otimes_{\widetilde{\mathbb{Q}}} \cdots \otimes_{\widetilde{\mathbb{Q}}} U_q^+(\widehat{\mathfrak{sl}}_{p_n})$. This implies the first part of the claim.

Since the map $\overrightarrow{\otimes}_{\mu\in\mathbb{Q}\cup\{\infty\}} U^\mu \xrightarrow{\mathsf{mult}} \overline{U}(\mathbb{X})$ is a restriction of the isomorphism (26), it is at least injective. Hence, we only have to prove the surjectivity of mult. For any slope ν denote $H^\nu = \overline{H}(\mathsf{SS}^\nu)$. It is sufficient to show that for any $\mu \in \mathbb{Q}\cup\{\infty\}$ the algebra $\overline{U}(\mathbb{X})$ is contained in the image of the map $\big(\overrightarrow{\otimes}_{\nu<\mu} H^\nu\big) \otimes U^\mu \otimes \big(\overrightarrow{\otimes}_{\nu>\mu} H^\nu\big) \xrightarrow{\mathsf{mult}} \overline{H}(\mathbb{X})$. By Proposition 6.11, it is true for $\mu = \infty$. Let $\mathbb{F}$ be an auto-equivalence of $D^b\big(\mathsf{Coh}(\mathbb{X})\big)$ mapping the given slope $\mu \in \mathbb{Q}$ to ∞. If κ is the image of the slope ∞ under $\mathbb{F}$ then we have a commutative diagram

$$\begin{tikzcd}
\mathbb{F}\big(\overline{U}(\mathbb{X})^+\big) \arrow[r, hook] \arrow[d, hook] & \overline{U}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})^- \arrow[d, hook] \\
\big(\overrightarrow{\otimes}_{\nu>-\kappa} H^{\nu,+}\big) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \big(\overrightarrow{\otimes}_{\nu\le\kappa} H^{\nu,-}\big) \arrow[r, hook] & \overline{H}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \overline{H}(\mathbb{X})^-.
\end{tikzcd}$$

Let $a \in \overline{U}(\mathbb{X}) = \overline{U}(\mathbb{X})^+$ be an arbitrary element. Since its image $b := \mathbb{F}(a)$ belongs both to $DU(\mathbb{X}) = \overline{U}(\mathbb{X})^+ \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \overline{U}(\mathbb{X})^-$ and to $\big(\overrightarrow{\otimes}_{\nu > -\kappa}\, H^{\nu,+}\big) \otimes_{\widetilde{\mathbb{Q}}} \widetilde{\mathbb{Q}}[K] \otimes_{\widetilde{\mathbb{Q}}} \big(\overrightarrow{\otimes}_{\nu \le \kappa}\, H^{\nu,-}\big)$, it can be written as a sum of monomials of the form $c_1 \dots c_t d e_1 \dots e_r$, where $c_i \in H^{\mu_i,+}$, $d \in \widetilde{\mathbb{Q}}[K]$ and $d_l \in H^{\nu_l,-}$ are such that $-\kappa < \mu_1 < \dots < \mu_t$, $\nu_1 < \dots < \nu_r \le \kappa$ and $c_t \in U^{\infty,+}$ (it is also possible that $c_t = 1$). Hence, $a = \mathbb{F}^{-1}(b) \in \big(\overrightarrow{\otimes}_{\nu < \mu}\, H^{\nu}\big) \otimes U^{\mu} \otimes \big(\overrightarrow{\otimes}_{\nu > \mu}\, H^{\nu}\big)$. $\square$

Corollary 7.25. *Taking some basis in the algebra $\overline{U}(\mathbb{X})^{\mathsf{exc}}_{\mathsf{tor}} \cong U_q^+(\widehat{\mathfrak{sl}}_{p_1}) \otimes_{\widetilde{\mathbb{Q}}} \dots \otimes_{\widetilde{\mathbb{Q}}} U_q^+(\widehat{\mathfrak{sl}}_{p_n})$, which is orthonormal with respect to the Green's form, Theorem 7.24 gives a construction of a PBW-type basis of the algebra $U(\mathbb{X})$, which is orthonormal with respect to the Green's form. In a similar way, we get a PBW-type basis of the reduced Drinfeld double $DU(\mathbb{X})$.*

Remark 7.26. Let $\mathbb{X}$ be a weighted projective line of tubular type $\widehat{\widehat{\Delta}}$, where $\Delta \in \big\{D_4, E_6, E_7, E_8\big\}$ and $\mathfrak{g}$ be the simple Lie algebra of the Dynkin type Δ. In a work of the second-named author [38] it was shown that the composition algebra $U(\mathbb{X})$ can be identified with a certain (quite non-standard) Borel subalgebra $U_q(\mathfrak{b})$ of the quantized double loop algebra $U_q\big(\mathfrak{L}\mathfrak{g}_{\widehat{\Delta}}\big)$, where $\mathfrak{g}_{\widehat{\Delta}}$ is the affine Lie algebra of type $\widehat{\Delta}$ (see the survey article [18] for a definition and applications of quantized double loop algebras). In [39] it was shown how a Lusztig-type approach [27] leads to a construction of a *canonical basis* of $U_q(\mathfrak{b})$. Therefore, it is natural to conjecture that the reduced Drinfeld double $DU(\mathbb{X})$ is isomorphic to the whole quantized double loop algebra $U_q\big(\mathfrak{L}\mathfrak{g}_{\widehat{\Delta}}\big)$.

Remark 7.27. The general results on the structure of the reduced Drinfeld double of a weighted projective line $\mathbb{X}$, listed in Subsection 6.3, lead to very interesting consequences for the theory of the quantized toroidal (or double loop) enveloping algebras of types $\widehat{\widehat{D}}_4$, $\widehat{\widehat{E}}_6$, $\widehat{\widehat{E}}_7$ and $\widehat{\widehat{E}}_8$. For example, let $\underline{p} = (2, 3, 6)$ and $\mathbb{X} = \mathbb{X}(\underline{p})$ be the corresponding tubular weighted projective line of type $\widehat{\widehat{E}}_8$. Then we know that

(1) The algebra $DU(\mathbb{X})$ contains a subalgebra isomorphic to
$$\mathcal{H} \otimes_{\mathcal{A}} U_q\big(\widehat{\mathfrak{sl}}_2\big) \otimes_{\mathcal{A}} U_q\big(\widehat{\mathfrak{sl}}_3\big) \otimes_{\mathcal{A}} U_q\big(\widehat{\mathfrak{sl}}_6\big),$$
where $\mathcal{H}$ is the Heisenberg algebra and $\mathcal{A}$ is the ring of Laurent polynomials in the "common" central element C.

(2) The algebra $DU(\mathbb{X})$ contains a subalgebra isomorphic to $DU(\mathbb{P}^1) \cong U_q\big(\mathfrak{L}\mathfrak{sl}_2\big)$.

(3) More generally, let $\underline{q} = (2, 3, 5)$ and $\mathbb{Y} = \mathbb{Y}(\underline{q})$ be the corresponding weighted projective line of domestic type. Then there exists an embedding $DU(\mathbb{Y}) \to DU(\mathbb{X})$. This should correspond to some rather non-trivial embedding of the quantized enveloping algebra $U_q\big(\widehat{E}_8\big)$ into the algebra $U_q\big(\widehat{\widehat{E}}_8\big)$.

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