VECTOR BUNDLES ON DEGENERATIONS OF ELLIPTIC CURVES AND YANG–BAXTER EQUATIONS

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Abstract. In this paper we introduce the notion of a geometric associative r-matrix attached to a genus one fibration with a section and irreducible fibres. It allows us to study degenerations of solutions of the classical Yang–Baxter equation using the approach of Polishchuk. We also calculate certain solutions of the classical, quantum and associative Yang–Baxter equations obtained from moduli spaces of (semi-)stable vector bundles on Weierstraß cubic curves.

1. Introduction

There are many indications (for example from homological mirror symmetry) that the formalism of derived categories provides a compact way to formulate and solve complicated non-linear analytical problems. However, one would like to have more concrete examples, in which one can follow the full path starting from a categorical set-up and ending with an analytical output. In this article we study the interplay between the theory of the associative, classical and quantum Yang–Baxter equations and properties of vector bundles on projective curves of arithmetic genus one, following the approach of Polishchuk [56].

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and $A = U(\mathfrak{g})$ its universal enveloping algebra. The classical Yang–Baxter equation (CYBE) is

$$[r^{12}(x), r^{13}(x+y)] + [r^{13}(x+y), r^{23}(y)] + [r^{12}(x), r^{23}(y)] = 0,$$

where $r(z)$ is the germ of a meromorphic function of one variable in a neighbourhood of 0 taking values in $\mathfrak{g} \otimes \mathfrak{g}$. The upper indices in this equation indicate various embeddings of $\mathfrak{g} \otimes \mathfrak{g}$ into $A \otimes A \otimes A$. For example, the function r^{13} is defined as

$$r^{13} : \mathbb{C} \xrightarrow{r} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\tau_{13}} A \otimes A \otimes A,$$

where $\tau_{13}(x \otimes y) = x \otimes 1 \otimes y$. Two other maps r^{12} and r^{23} have a similar meaning.

In the physical literature, solutions of (CYBE) are frequently called *r–matrices.* They play an important role in mathematical physics, representation theory, integrable systems and statistical mechanics.

By a famous result of Belavin and Drinfeld [8], there exist exactly three types of non-degenerate solutions of the classical Yang–Baxter equation: elliptic (two-periodic), trigonometric (one-periodic) and rational. This trichotomy corresponds to three models in statistical mechanics: XYZ (elliptic), XXZ (trigonometric) and XXX (rational), see [7].

Belavin and Drinfeld have also obtained a complete classification of elliptic and trigonometric solutions, see [8, Proposition 5.1 and Theorem 6.1]. A certain classification of rational solutions was given by Stolin [62, Theorem 1.1].

This article is devoted to a study of degenerations of elliptic r-matrices into trigonometric and then into rational ones. We hope that this sort of questions will be interesting from the point of view of applications in mathematical physics. In order to attack this problem we use a construction of Polishchuk [56]. After certain modifications of his original presentation, the core of this method can be described as follows.

Let E be a Weierstraß cubic curve, $\breve{E} \subset E$ the open subset of smooth points, $M = M_E^{(n,d)}$ the moduli space of stable bundles of rank n and degree d, assumed to be coprime. Let $\mathcal{P} = \mathcal{P}(n,d) \in \mathsf{VB}(E \times M)$ be a universal family of the moduli functor $\underline{\mathsf{M}}_E^{(n,d)}$. For a point $v \in M$ we denote by $\mathcal{V} = \mathcal{P}|_{E \times v}$ the corresponding vector bundle on E. Consider the following data:

- two distinct points $v_1, v_2 \in M$ in the moduli space;
- two distinct points $y_1, y_2 \in \breve{E}$ such that $\mathcal{V}_1(y_2) \not\cong \mathcal{V}_2(y_1)$.

Using Serre Duality, the triple Massey product

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2}),$$

induces a linear map

$$r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})$$

and satisfying the so-called *associative Yang–Baxter equation* (AYBE)

$$\big(r^{\mathcal{V}_3,\mathcal{V}_2}_{y_1,y_2}\big)^{12} \big(r^{\mathcal{V}_1,\mathcal{V}_3}_{y_1,y_3}\big)^{13} - \big(r^{\mathcal{V}_1,\mathcal{V}_3}_{y_2,y_3}\big)^{23} \big(r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}\big)^{12} + \big(r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_3}\big)^{13} \big(r^{\mathcal{V}_2,\mathcal{V}_3}_{y_2,y_3}\big)^{23} = 0$$

viewed as a map

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_3}) \longrightarrow$$

$$\longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_3}).$$

This map can be rewritten as the germ of a tensor-valued meromorphic function in four variables, defined in a neighbourhood of a smooth point o of the moduli space $M \times M \times E \times E$ (the choice of o will be explained in Corollary 6.13)

$$r(\mathcal{V}_1, \mathcal{V}_2; y_1, y_2) : (\mathbb{C}^2 \times \mathbb{C}^2, 0) \cong \big((M \times M) \times (E \times E), o\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}).$$

Since the complex manifold $M_E^{(n,d)}$ is a homogeneous space over the algebraic group $J = \mathsf{Pic}^0(E)$, it turns out that

$$r(v_1, v_2; y_1, y_2) \sim r(v_1 - v_2; y_1, y_2) = r(v; y_1, y_2),$$

with respect to a certain equivalence relation on the set of solutions. We show that this equivalence relation corresponds to a change of a trivialization of the universal family $\mathcal{P}$.

Let e be the neutral element of J. It was shown by Polishchuk [56, Lemma 1.2] that the function of two variables

$$\bar{r}(y_1, y_2) = \lim_{v \to e} (\mathrm{pr} \otimes \mathrm{pr}) r(v; y_1, y_2) \in \mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$$

is a non-degenerate unitary solution of the classical Yang–Baxter equation. Moreover, under certain restrictions (which are always fulfilled at least for elliptic curves and Kodaira cycles of projective lines), for any fixed value $g \neq e$ from a small neighbourhood $U_e \subseteq J$ of e, the tensor-valued function

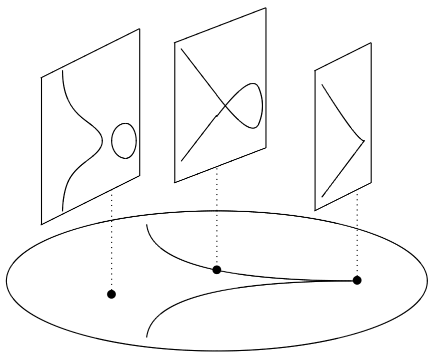
$$r : \bigl(\{g\} \times E \times E, e\bigr) \longrightarrow \mathsf{Mat}_{n \times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n \times n}(\mathbb{C})$$

satisfies the *quantum* Yang–Baxter equation, see [57, Theorem 1.4]. Hence, this approach gives an explicit method to quantize some known solutions of the classical Yang–Baxter equation.

Moreover, as was pointed out by Kirillov [38], a solution $r(v; y_1, y_2)$ of the associative Yang–Baxter equation defines an interesting family of pairwise commuting first order differential operators, generalizing Dunkl operators studied by Buchstaber, Felder and Veselov [15], see Proposition 2.9.

The aim of our article is to study a *relative* version of Polishchuk's construction. Although most of the results can be generalized to the case of arbitrary reduced projective curves of arithmetic genus one having trivial dualizing sheaf, in this article we shall concentrate mainly on the case of irreducible curves.

Let E be a Weierstraß cubic curve, i.e. a plane projective curve given by the equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$. It is singular if any only if $\Delta := g_2^3 - 27g_3^2 = 0$.



Unless $g_2 = g_3 = 0$, the singularity is a node, whereas for $g_2 = g_3 = 0$ it is a cusp.

A connection between the theory of vector bundles on cubic curves and exactly solvable models of mathematical physics was observed a long time ago, see for example [45, Chapter 13] and [48] for a link with KdV equation, [23] for applications to integrable systems and [10] for an interplay with Calogero-Moser systems. In particular, the correspondence

elliptic	elliptic
trigonometric	nodal
rational	cuspidal

was discovered at the very beginning of the algebraic theory of completely integrable systems.

In this article we follow another strategy. Instead of looking at each curve of arithmetic genus one individually, we consider the relative case, so that all solutions will be considered as specializations of one *universal* solution. Our main result can be stated as follows.

Let $E \to T$ be a genus one fibration with a section having reduced and irreducible fibres, $M = M_{E/T}^{(n,d)}$ the moduli space of relatively stable vector bundles of rank n and degree d. We construct a meromorphic function

$$r : (M \times_T \times M \times_T E \times_T E, o) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

in a neighbourhood of a smooth point o of $M \times_T \times M \times_T E \times_T E$, which satisfies the associative Yang–Baxter equation for each fixed value $t \in T$ and $(v_1, v_2, y_1, y_2) \in \big((M_{E_t} \times M_{E_t}) \times (E_t \times E_t), o\big)$. Moreover, $r_t(v_1, v_2, y_1, y_2)$ depends analytically on t, is compatible with base change of the given family $E \to T$ and the corresponding solution of the classical Yang–Baxter equation $\bar{r}_t(y)$ is

- *elliptic if E_t is smooth;*
- *trigonometric if E_t is nodal;*
- *rational if E_t is cuspidal.*

We also carry out explicit calculations for vector bundles of rank two and degree one on irreducible Weierstraß cubic curves. In the case of an elliptic curve $E = E_\tau$ the corresponding solution is

$$\begin{aligned} r_{\mathrm{ell}}(v; y) = \frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)} \Bigg[& \frac{\theta_1(y+v|\tau)}{\theta_1(v|\tau)} \mathbb{1} \otimes \mathbb{1} + \frac{\theta_2(y+v|\tau)}{\theta_2(v|\tau)} h \otimes h + \\ & + \frac{\theta_3(y+v|\tau)}{\theta_3(v|\tau)} \sigma \otimes \sigma + \frac{\theta_4(y+v|\tau)}{\theta_4(v|\tau)} \gamma \otimes \gamma \Bigg], \end{aligned}$$

where $\mathbb{1} = e_{11} + e_{22}$, $h = e_{11} - e_{22}, \sigma = i(e_{21} - e_{12})$ and $\gamma = e_{21} + e_{12}$.

In the case of a nodal cubic curve we get

$$\begin{aligned} r_{\mathrm{trg}}(v; y) = & \frac{\sin(y+v)}{\sin(y)\sin(v)} (e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) + \frac{1}{\sin(v)} (e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) + \\ & + \frac{1}{\sin(y)} (e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \sin(y+v) e_{21} \otimes e_{21} \end{aligned}$$

and in the case of a cuspidal cubic curve, the associative r–matrix is

$$r_{\mathrm{rat}}(v; y_1, y_2) = \frac{1}{v}\mathbb{1} \otimes \mathbb{1} + \frac{2}{y_2 - y_1}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22} + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \\ + (v - y_1)e_{21} \otimes h + (v + y_2)h \otimes e_{21} - v(v - y_1)(v + y_2)e_{21} \otimes e_{21}.$$

Our results imply that up to a gauge transformation the trigonometric and rational solutions $r_{\mathrm{trg}}(v; y)$ and $r_{\mathrm{rat}}(v; y_1, y_2)$ are degenerations of $r_{\mathrm{ell}}(v; y)$, which seems to be difficult to show by a direct computation.

Moreover, for a generic v the tensors $r_{\mathrm{ell}}(v; y)$, $r_{\mathrm{trg}}(v; y)$ and $r_{\mathrm{rat}}(v; y_1, y_2)$ satisfy the quantum Yang–Baxter equation and are quantizations of the following classical r–matrices:

- Elliptic solution found and studied by Baxter, Belavin and Sklyanin:

$$\bar{r}_{\mathrm{ell}}(y) = \frac{\mathrm{cn}(y)}{\mathrm{sn}(y)} h \otimes h + \frac{1 + \mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \frac{1 - \mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12} \otimes e_{12} + e_{21} \otimes e_{21}).$$

- Trigonometric solution of Cherednik:

$$\bar{r}_{\mathrm{trg}}(y) = \frac{1}{2}\cot(y) h \otimes h + \frac{1}{\sin(y)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \sin(y) e_{21} \otimes e_{21}.$$

- Rational solution

$$\bar{r}_{\mathrm{rat}}(y) = \frac{1}{y}\Big(\frac{1}{2} h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\Big) + y(e_{21} \otimes h + h \otimes e_{21}) - y^3 e_{21} \otimes e_{21},$$

which is gauge equivalent to a solution found by Stolin [62].

This paper is organized as follows. In Section 2 we collect some results about the associative Yang–Baxter equation and its relations with Dunkl operators as well as with the classical and quantum Yang–Baxter equations. Section 3 gives a short introduction into a construction of Polishchuk which provides a method to obtain solutions of Yang–Baxter equations from triple Massey products in a derived category.

In order to be able to calculate solutions explicitly, this construction has to be translated into another language, involving residue maps. In Section 4 we explain the corresponding result of Polishchuk whereby we provide some details which are only implicit in [56]. The understanding of these details is crucial for the study of the relative case, which is carried out in Sections 5 and 6. Theorem 6.13 is the main result of this article. It states that for any genus one fibration $E \to T$ satisfying certain restrictions and any pair of coprime integers $0 < d < r$ one can attach a family of solutions of the associative Yang-Baxter equation $r^{\xi}(v_1, v_2; y_1, y_2)$ depending *analytically* on the parameter of the basis and functorial with respect to the base change. This solutions actually depend on the choice of a trivialization ξ of the universal family $\mathcal{P}(n, d)$ of stable vector bundles of rank n and degree d.

However, in Proposition 6.12 we show that a choice of another trivialization ζ leads to a *gauge equivalent* solution $r^{\zeta}(v_1, v_2; y_1, y_2)$

In Section 7 we prove that in the case of a Weierstraß cubic curve E there exists a trivialization ξ of the universal family $\mathcal{P}(n, d)$ such that the corresponding solution $r^{\xi}(v_1, v_2; y_1, y_2)$ is invariant under simultaneous shifts $v_1 \mapsto v_1 + v, v_2 \mapsto v_2 + v$. In other words, the solution $r^{\xi}(v_1, v_2; y_1, y_2)$, also called *geometric associative r-matrix*, depends on the difference $v_2 - v_1$ of the first pair of spectral parameters. Hence, the obtained solution $r^{\xi}(v; y_1, y_2)$ also satisfies the quantum Yang–Baxter equation and defines an interesting quantum integrable system. The key point of the proof is to show equivariance of triple Massey products with respect to the action of the Jacobian J on the moduli space $M_E^{(n,d)}$.

Since it is indispensable for carrying out explicit calculations of r–matrices, in the following sections we elaborate foundations of the theory of vector bundles on genus one curves. In Section 8 we recall some classical results about holomorphic vector bundles on a smooth elliptic curve. Using the methods described before, we explicitly compute the solution of the associative Yang–Baxter equation and the classical r–matrix corresponding to a universal family of stable vector bundles of rank two and degree one. These solutions were obtained by Polishchuk in [56, Section 2] using homological mirror symmetry and formulas for higher products in the Fukaya category of an elliptic curve. Our direct computation, however, is independent of homological mirror symmetry. We are lead directly to express the resulting associative r–matrix in terms of Jacobi's theta-functions and the corresponding classical r–matrix in terms of the elliptic functions $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$.

Sections 9 and 10 are devoted to similar calculations for nodal and cuspidal Weierstraß curves. Our computations are based on the description of vector bundles on singular genus one curves in terms of so-called matrix problems, which was given by Drozd and Greuel [25] and Burban [16]. We show that their description of canonical forms of matrix problems corresponds precisely to a very explicit presentation of universal families of stable vector bundles. We explicitly compute geometric r–matrices coming from universal families of stable vector bundles of rank two and degree one on a nodal and cuspidal cubic curves and the r–matrix coming from the universal family of semi–stable vector bundles of rank two and degree zero on a nodal cubic curve.

This article is concluded with a brief summary of analytical results in Section 11.

Notation. Throughout this paper we work in the category of analytic spaces over the field of complex numbers $\mathbb{C}$, see [54]. However, most of the results remain valid in the category of algebraic varieties over an algebraically closed field $\boldsymbol{k}$ of characteristic zero. If V, W are two complex vector spaces, $\mathsf{Lin}(V, W)$ denotes the vector space of complex linear maps from V to W. For an additive category C, a pair of objects $X, Y \in \mathsf{C}$ and a pair of isomorphisms $X \xrightarrow{f} X'$ and $Y \xrightarrow{g} Y'$

we denote $\mathrm{cnj}(f, g)$ the morphism of abelian groups $\mathsf{Hom}_{\mathsf{C}}(X, Y) \longrightarrow \mathsf{Hom}_{\mathsf{C}}(X', Y')$ mapping a morphism h to the composition $g \circ h \circ f^{-1}$.

If X is a complex projective variety, we denote by $\mathsf{Coh}(X)$ the category of coherent $\mathcal{O}_X$-modules and by $\mathsf{VB}(X)$ its full subcategory of locally free sheaves (holomorphic vector bundles). The torsion sheaf of length one, supported at a closed point $y \in X$, is always denoted by $\mathbb{C}_y$. By $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ we denote the full subcategory of the derived category of the abelian category of all $\mathcal{O}_X$-modules whose objects are those complexes which have bounded and coherent cohomology. The notation $\mathsf{Perf}(X)$ is used for the full subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ whose objects are isomorphic to bounded complexes of locally free sheaves. For a morphism of reduced complex spaces $E \xrightarrow{p} T$ we denote by $\breve{E}$ the regular locus of p.

A Weierstraß curve is a plane cubic curve given in homogeneous coordinates by an equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$, where $g_1, g_2 \in \mathbb{C}$. Such a curve is always irreducible. It is a smooth elliptic curve if and only if $\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \neq 0$.

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Contents

2. Yang–Baxter equations

In this section we are going to recall some standard results about Yang–Baxter equations. Let $\mathfrak{g}$ be a simple complex Lie algebra (throughout this paper $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$), $\langle \quad , \quad \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Killing form. The classical Yang–Baxter equation is

$$[r^{12}(y_1, y_2), r^{23}(y_2, y_3)] + [r^{12}(y_1, y_2), r^{13}(y_1, y_3)] + [r^{13}(y_1, y_3), r^{23}(y_2, y_3)] = 0, \tag{1}$$

where $r(x, y)$ is the germ of a meromorphic function of two complex variables in a neighbourhood of 0, taking values in $\mathfrak{g} \otimes \mathfrak{g}$. A solution of (1) is called *unitary* if

$$r^{12}(y_1, y_2) = -r^{21}(y_2, y_1)$$

and *non-degenerate* if $r(y_1, y_2) \in \mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g}^* \otimes \mathfrak{g} \cong \mathsf{End}(\mathfrak{g})$ is invertible for generic (y_1, y_2). On the set of solutions of (1) there exists a natural action of the group of holomorphic function germs $\phi : (\mathbb{C}, 0) \longrightarrow \mathrm{Aut}(\mathfrak{g})$ given by the rule

$$r(y_1, y_2) \mapsto \big(\phi(y_1) \otimes \phi(y_2)\big) r(y_1, y_2). \tag{2}$$

Proposition 2.1 (see [9]). *Modulo the equivalence relation (2) any non-degenerate unitary solution of the equation (1) is equivalent to a solution $r(u, v) = r(u - v)$ depending on the difference (or the quotient) of spectral parameters only.*

This means that equation (1) is essentially equivalent to the equation

$$[r^{12}(x), r^{13}(x + y)] + [r^{13}(x + y), r^{23}(y)] + [r^{12}(x), r^{23}(y)] = 0. \tag{3}$$

Although the classical Yang–Baxter equation with one spectral parameter is better adapted for applications in mathematical physics, it seems that from a geometric point of view equation (1) is more natural.

Let $m = \dim(\mathfrak{g})$, $e_1, e_2, \dots, e_m$ be a basis of $\mathfrak{g}$ and $e^1, e^2, \dots, e^m$ be the dual basis of $\mathfrak{g}$ with respect to the Killing form $\langle \quad , \quad \rangle$. Then $\Omega = \sum_{i=1}^m e^i \otimes e_i \in \mathfrak{g} \otimes \mathfrak{g}$ is independent of the choice of a basis and is called the *Casimir element*.

Theorem 2.2 (see Proposition 2.1 and Proposition 4.1 in [8]). *Let $r(y)$ be a non-constant non-degenerate solution of (3). Then the tensor $r(y)$*

- *has a simple pole at* 0 *and* $\mathrm{res}_{y=0}\big(r(y)\big) = \alpha\Omega \in \mathfrak{g} \otimes \mathfrak{g}, \alpha \in \mathbb{C}^*$;
- *is automatically unitary, i.e.* $r^{12}(y) = -r^{21}(-y)$.

As it was already mentioned in the introduction, there is the following classification of non-degenerate solutions of (CYBE) due to Belavin and Drinfeld.

Theorem 2.3 (see Proposition 4.5 and Proposition 4.7 in [8]). *There are three types of non-degenerate solutions of the classical Yang–Baxter equation (3): elliptic, trigonometric and rational.*

Let us now consider some examples. Fix the following basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Note that $\Omega = \frac{1}{2}h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}$ is the Casimir element of $\mathfrak{sl}_2(\mathbb{C})$.

• Historically, the first solution ever found was the rational solution of Yang

$$r_{\mathrm{rat}}(y) = \frac{1}{y}\left(\frac{1}{2}h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\right).$$

• A few years later, Baxter discovered the trigonometric solution

$$r_{\mathrm{trg}}(y) = \frac{1}{2}\cot(y) h \otimes h + \frac{1}{\sin(y)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}).$$

• The following solution of elliptic type was found and studied by Baxter, Belavin and Sklyanin:

$$r_{\mathrm{ell}}(y) = \frac{\mathrm{cn}(y)}{\mathrm{sn}(y)} h \otimes h + \frac{1 + \mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \frac{1 - \mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12} \otimes e_{12} + e_{21} \otimes e_{21}),$$

where $\mathrm{cn}(y)$, $\mathrm{sn}(y)$ and $\mathrm{dn}(y)$ are doubly periodic meromorphic functions on $\mathbb{C}$ with periods 2 and 2τ. These functions also satisfy identities of the form $f(y+1) = \varepsilon f(y)$ and $f(y+\tau) = \varepsilon f(y)$ with $\varepsilon = \pm 1$.

At first glance, all these solutions seem to be completely different. However, it is easy to see that

$$\lim_{t \to \infty} \frac{1}{t} r_{\mathrm{trg}}\left(\frac{y}{t}\right) = r_{\mathrm{rat}}(y),$$

hence the solution of Yang is a degeneration of Baxter's solution. Moreover, there exist degenerations $\mathrm{dn}(y) \to 1$, $\mathrm{cn}(y) \to \cos(y)$ and $\mathrm{sn}(y) \to \sin(y)$, when the imaginary period τ tends to infinity, see for example [42, Section 2.6]. Hence, both solutions of Baxter and Yang are degenerations of the elliptic solution. However, as we shall see later, the theory of degenerations of r–matrices is more complicated as it might look like at first sight.

In this article we deal with a new type of Yang–Baxter equation, called *associative Yang–Baxter equation* (AYBE). It appeared for the first time in a paper of Fomin and Kirillov [28]. Later, it was studied by Aguiar [1] in the framework of the theory of infinitesimal Hopf algebras. The following version of the associative Yang–Baxter equation with *spectral parameters* is due to Polishchuk [56]. A special case of this equation was also considered by Odesski and Sokolov [53].

Definition 2.4. An associative r-matrix is the germ of a meromorphic function in four variables

$$r : \left(\mathbb{C}^4_{(v_1,v_2;y_1,y_2)}, 0\right) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

holomorphic on $\Big(\mathbb{C}^4 \setminus V\big((y_1 - y_2)(v_1 - v_2)\big), 0\Big)$ and satisfying the equation

$$r(v_1,v_2;y_1,y_2)^{12}r(v_1,v_3;y_2,y_3)^{23} = r(v_1,v_3;y_1,y_3)^{13}r(v_3,v_2;y_1,y_2)^{12}+ \\ +r(v_2,v_3;y_2,y_3)^{23}r(v_1,v_2;y_1,y_3)^{13}. \tag{4}$$

Such a matrix is called unitary if

$$r(v_1,v_2;y_1,y_2)^{12} = -r(v_2,v_1;y_2,y_1)^{21}. \tag{5}$$

On the set of solutions of the equation (4) there exists a natural equivalence relation.

Definition 2.5 (see section 1.2 in [56])**.** Let $\phi : (\mathbb{C}^2,0) \to \mathsf{GL}_n(\mathbb{C})$ be the germ of a holomorphic function and $r(v_1,v_2;y_1,y_2)$ be a solution of (AYBE) then

$$r'(v_1,v_2;y_1,y_2) = \big(\phi(v_1;y_1)\otimes\phi(v_2;y_2)\big)r(v_1,v_2;y_1,y_2)\big(\phi(v_2;y_1)^{-1}\otimes\phi(v_1;y_2)^{-1}\big) \tag{6}$$

is again a solution of (4). Two such tensors r and r' are called gauge equivalent. Note that if the matrix r is unitary then r' is unitary, too.

Example 2.6. Let $r(v_1,v_2;y_1,y_2) \in \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ be a solution of (4), $c \in \mathbb{C}$ and $\phi = \exp(cvy)\cdot \mathbb{1} : (\mathbb{C}^2,0) \longrightarrow \mathsf{GL}_n(\mathbb{C})$ be a gauge transformation. Then $\exp\big(c(v_2-v_1)(y_2-y_1)\big)r(v_1,v_2;y_1,y_2)$ is a gauge equivalent solution of (AYBE).

Lemma 2.7. *Let $r(v_1,v_2;y_1,y_2)$ be a unitary solution of the associative Yang–Baxter equation (4). Then r also satisfies the "dual" equation*

$$r(v_2,v_3;y_2,y_3)^{23}r(v_1,v_3;y_1,y_2)^{12} = r(v_1,v_2;y_1,y_2)^{12}r(v_2,v_3;y_1,y_3)^{13}+ \\ +r(v_1,v_3;y_1,y_3)^{13}r(v_2,v_1;y_2,y_3)^{23}. \tag{7}$$

Proof. Let τ be the linear automorphism of $\mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ defined by $\tau(a\otimes b) = b\otimes a$. Applying the operator $\tau\otimes\mathbb{1}$ to the equation (4), we obtain:

$$r(v_1,v_2;y_1,y_2)^{21}r(v_1,v_3;y_2,y_3)^{13} = r(v_1,v_3;y_1,y_3)^{23}r(v_3,v_2;y_1,y_2)^{21}+ \\ +r(v_2,v_3;y_2,y_3)^{13}r(v_1,v_2;y_1,y_3)^{23}.$$

Using the unitarity condition (5) we get:

$$-r(v_2,v_1;y_2,y_1)^{12}r(v_1,v_3;y_2,y_3)^{13} = -r(v_1,v_3;y_1,y_3)^{23}r(v_2,v_3;y_2,y_1)^{12}+ \\ +r(v_2,v_3;y_2,y_3)^{13}r(v_1,v_2;y_1,y_3)^{23}.$$

After the change of variables $v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_3$ and $y_1 \leftrightarrow y_2, y_3 \leftrightarrow y_3$, we obtain the equation (7). □

Assume a unitary solution $r(v_1,v_2;y_1,y_2)$ of the associative Yang–Baxter equation (4) depends on the difference $v = v_1 - v_2$ of the first pair of spectral parameters only. For the sake of simplicity, we shall use the notation: $r(v_1,v_2;y_1,y_2) = r(v_1 - v_2;y_1,y_2) = r(v;y_1,y_2)$. Then the equation (4) can be rewritten as

$$r(u;y_1,y_2)^{12}r(u+v;y_2,y_3)^{23} = r(u+v;y_1,y_3)^{13}r(-v;y_1,y_2)^{12} + r(v;y_2,y_3)^{23}r(u;y_1,y_3)^{13}. \tag{8}$$

Remark 2.8. It will be shown in Theorem 7.5 that any solution r of the associative Yang–Baxter equation (4) obtained from a universal family of stable vector bundles on an irreducible genus one curve, is gauge equivalent to a solution r' depending on the difference $v_1 - v_2$ only.

Let A be the algebra of germs of meromorphic functions $f : \big(\mathbb{C}^4_{(v_1,v_2;w_1,w_2)}, 0\big) \longrightarrow \mathbb{C}$ holomorphic on $\Big(\mathbb{C}^4 \setminus V\big((v_1-v_2)(w_1-w_2)\big), 0\Big)$. A solution of the equation (8) defines an element

$$r \in \mathsf{Mat}_{n\times n}(A) \otimes_A \mathsf{Mat}_{n\times n}(A) \cong A \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C}).$$

In a similar way, for any integer $m \ge 3$ denote by B the algebra of germs of meromorphic functions $f : \big(\mathbb{C}^{2m}_{(x_1,\dots,x_m;y_1,\dots,y_m)}, 0\big) \longrightarrow \mathbb{C}$ holomorphic on $\big(\mathbb{C}^{2m} \setminus D, 0\big)$, where D is the divisor

$$D = V\left(\prod_{i\ne j}(x_i - x_j)(y_i - y_j)\right).$$

Next, for any pair of indices $1 \le i \ne j \le m$ we have

- a ring homomorphism $\psi^{ij} : A \longrightarrow B$ which sends a function $f(v_1,v_2;w_1,w_2)$ to $f(x_i,x_j;y_i,y_j)$;
- a ring homomorphism $k^{ij} : B \longrightarrow B$ defined as
$$f(\dots,x_i,\dots,x_j,\dots;\underline{y}) \mapsto f(\dots,x_j,\dots,x_i,\dots;\underline{y});$$
- a morphism $\varrho_{ij} : \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes 2} \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}$ mapping a simple tensor $a\otimes b$ to $1\otimes\dots 1\otimes a\otimes 1\otimes\dots 1\otimes b\otimes 1\otimes\dots\otimes 1$, where a and b belong to the i-th and j-th components respectively.

In these notations, consider

$$\Psi^{ij} := \psi_{ij}\otimes\varrho_{ij} \;:\; A\otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes 2} \longrightarrow B\otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}.$$

For example $\Psi^{13}\big(f(v_1,v_2;w_1,w_2)\otimes a\otimes b\big) = f(x_1,x_3;y_1,y_3)\otimes a\otimes 1\otimes b$. Next, we set $r^{ij} := \Psi^{ij}(r) \in B\otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}$ and

$$K^{ij} = k^{ij}\otimes \mathbb{1} \;:\; B\otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B\otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}.$$

Consider the linear operator

$$\tilde{r}^{ij} = r^{ij}\circ K^{ij} \;:\; B\otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B\otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m},$$

which is the composition of K^{ij} and the multiplication with the element r^{ij}. For any $1 \le i \le m$ consider the differential operator

$$\partial_i = \frac{\partial}{\partial x_i} \otimes \mathbb{1} \;:\; B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}.$$

Next, for any $\kappa \in \mathbb{C}$ let

$$\theta_i := \kappa\partial_i + \sum_{j\neq i} \tilde{r}^{ij} \;:\; B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}$$

be the *Dunkl operator* of level κ. The following result was explained to the first-named author by Anatoly Kirillov, see also [38].

Proposition 2.9. *Let $r(v; y_1, y_2) \in \mathsf{Mat}_{n\times n}(A) \otimes_A \mathsf{Mat}_{n\times n}(A)$ be a unitary solution of the equation (8), $\kappa \in \mathbb{C}$ be a scalar and θ_i be the Dunkl operator of level κ defined above. Then for all $1 \le i, j \le m$ we have: $\big[\theta_i, \theta_j\big] = 0$.*

Proof. First note that we have:

$$\Big(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\Big) r(x_i - x_j; y_i, y_j) = 0,$$

which implies the equality $\big[\partial_i + \partial_j, \tilde{r}^{ij}\big] = 0$. Next, the Yang–Baxter relations (4) and (7) yield that for any triple of mutually different indices $1 \le i < j < k \le m$ we have:

$$\tilde{r}^{ij}\tilde{r}^{jk} = \tilde{r}^{jk}\tilde{r}^{ik} + \tilde{r}^{ik}\tilde{r}^{ij} \quad \text{and} \quad \tilde{r}^{jk}\tilde{r}^{ij} = \tilde{r}^{ik}\tilde{r}^{jk} + \tilde{r}^{ij}\tilde{r}^{ik}.$$

From the unitarity of r it follows that $\tilde{r}^{ij} = -\tilde{r}^{ji}$ for all $1 \le i \ne j \le m$. Finally, the following two equalities are obvious:

$$\big[\tilde{r}^{ij}, \tilde{r}^{kl}\big] = 0, \quad \big[\partial_i, \tilde{r}^{kl}\big] = 0$$

where $1 \le i, j, k, l \le m$ are mutually distinct. Combining these equalities together, we obtain the claim. □

Remark 2.10. The above proposition means that to any unitary solution of the associative Yang–Baxter equation (8) one can attach a very interesting second order differential operator

$$H = \theta_1^2 + \theta_2^2 + \cdots + \theta_m^2 : \quad B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}.$$

These operators are "matrix versions" of the Hamiltonians considered in the work of Buchstaber, Felder and Veselov [15].

Another motivation to study solutions of the equation (8) is provided by their close connection with the theory of the classical Yang–Baxter equation.

Lemma 2.11 (see Lemma 1.2 in [56]). *Let $r(v; y_1, y_2)$ be a unitary solution of the associative Yang–Baxter equation (8) and* $\mathrm{pr} : \mathsf{Mat}_{n\times n}(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$ *be the projection along the scalar matrices, i.e.* $\mathrm{pr}(A) = A - \dfrac{\mathrm{tr}(A)}{n}\cdot \mathbb{1}$. *Assume that* $(\mathrm{pr}\otimes\mathrm{pr})r(v; y_1, y_2)$ *has a limit as $v \to 0$. Then*

$$\bar{r}(y_1, y_2) := \lim_{v\to 0}(\mathrm{pr}\otimes\mathrm{pr})r(v; y_1, y_2)$$

is a unitary solution of the classical Yang–Baxter equation (1).

Proof. First note that (7) implies the equality

$$r(v; y_2, y_3)^{23} r(u+v; y_1, y_2)^{12} = r(u; y_1, y_2)^{12} r(v; y_1, y_3)^{13} + \\ + r(u+v; y_1, y_3)^{13} r(-u; y_2, y_3)^{23}.$$

Using the change of variables $u \mapsto -v$ and $v \mapsto u+v$, we obtain the relation

$$r(u+v; y_2, y_3)^{23} r(u; y_1, y_2)^{12} = r(-v; y_1, y_2)^{12} r(u+v; y_1, y_3)^{13} + \\ + r(u; y_1, y_3)^{13} r(v; y_2, y_3)^{23}.$$

Subtracting this equation from (8) we get

$$\begin{aligned} (9)\qquad & [r(-v; y_1, y_2)^{12}, r(u+v; y_1, y_3)^{13}] + [r(u; y_1, y_2)^{12}, r(u+v; y_2, y_3)^{23}] + \\ & + [r(u; y_1, y_3)^{13}, r(v; y_2, y_3)^{23}] = 0. \end{aligned}$$

By definition, the function $r(v; y_1, y_2)$ is meromorphic, hence we can write its Laurent expansion: $r(v; y_1, y_2) = \sum_{\alpha\in\mathbb{Z}} r_\alpha(y_1, y_2)v^\alpha$, where $r_\alpha(y_1, y_2)$ are meromorphic and $r_\alpha = 0$ for $\alpha \ll 0$. Since we have assumed that $(\mathrm{pr}\otimes\mathrm{pr})r(v; y_1, y_2)$ is regular with respect to v in a neighbourhood of $v = 0$, we have: $(\mathrm{pr}\otimes\mathrm{pr})r_\alpha(y_1, y_2) = 0$ for all $\alpha \le -1$. This implies, if $\alpha \le -1$, that

$$r_\alpha(y_1, y_2) = s_\alpha(y_1, y_2) \otimes \mathbb{1} + \mathbb{1} \otimes t_\alpha(y_1, y_2)$$

for some matrix-valued functions $s_\alpha(y_1, y_2)$ and $t_\alpha(y_1, y_2)$. Hence,

$$(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\left[r_\alpha^{ij}, r_\beta^{lk}\right] = 0$$

for arbitrary permutations $(ij) \ne (lk)$ and all indices $\alpha \le -1$ and $\beta \in \mathbb{Z}$. The claim of Lemma 2.11 follows by applying $\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr}$ to the equation (9) and taking the limit $u, v \to 0$. □

The statement of Lemma 2.11 leads to the following question. Let $r = r(v; y_1, y_2)$ be a unitary solution of the associative Yang–Baxter equation (8) satisfying the conditions of Lemma 2.11 and $s = s(v; y_1, y_2)$ be an equivalent solution in the sense of Definition 2.5. Are the corresponding solutions $\bar{r}(y_1, y_2)$ and $\bar{s}(y_1, y_2)$ of the classical Yang–Baxter equation also gauge equivalent?

The answer on this question is affirmative, if one imposes an additional restriction on the function r. Namely, we assume that the Laurent expansion of r has the form:

$$(10)\qquad r(v; y_1, y_2) = \frac{\mathbb{1}\otimes\mathbb{1}}{v} + r_0(y_1, y_2) + v r_1(y_1, y_2) + v^2 r_2(y_1, y_2) + \ldots$$

Then the following proposition is true.

Proposition 2.12. *Let* $r : (\mathbb{C}^3_{(v;y_1,y_2)}, 0) \to \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ *be a unitary solution of the associative Yang–Baxter equation (8) having a Laurent expansion of the form (10) and* $\bar{r}_0(y_1, y_2)$ *be the corresponding solution of the classical Yang–Baxter equation. If* $\phi : (\mathbb{C}^2_{(v;y)}, 0) \to \mathsf{GL}_n(\mathbb{C})$ *is the germ of a holomorphic function such that*

$$s(v_1, v_2; y_1, y_2) := \bigl(\phi(v_1; y_1) \otimes \phi(v_2; y_2)\bigr) r(v; y_1, y_2) \bigl(\phi(v_2; y_1)^{-1} \otimes \phi(v_1; y_2)^{-1}\bigr) \tag{11}$$

is again a function of $v = v_2 - v_1$*, then we have*

$$s(v; y_1, y_2) = \frac{1}{v}\mathbb{1} \otimes \mathbb{1} + s_0(y_1, y_2) + v s_1(y_1, y_2) + v^2 s_2(y_1, y_2) + \dots$$

and moreover, $\bar{r}_0(y_1, y_2)$ *and* $\bar{s}_0(y_1, y_2)$ *are equivalent in the sense of the relation (2).*

Proof. We denote $v = v_1 - v_2$ and $h = v_2$. Then $v_1 = v + h$ and using the Taylor expansion of ϕ with respect to v, we may rewrite (11) in the form

$$\begin{aligned} &\left(\phi(h; y_1) + v\phi'(h; y_1) + \frac{v^2}{2}\phi''(h; y_1) + \dots\right) \otimes \phi(h; y_2)\cdot \\ &\qquad\qquad\qquad\qquad \cdot \left(\frac{\mathbb{1} \otimes \mathbb{1}}{v} + r_0(y_1, y_2) + v r_1(y_1, y_2) + \dots\right) \\ &= \left(\sum_{i\in\mathbb{Z}} s_i(y_1, y_2) v^i\right) \cdot \left(\phi(h; y_1) \otimes \left(\phi(h; y_2) + v\phi'(h; y_2) + \frac{v^2}{2}\phi''(h; y_2) + \dots\right)\right), \end{aligned}$$

where $\phi'(v; y)$ and $\phi''(v; y)$ are the partial derivatives of $\varphi(v; y)$ of the first and the second order with respect to v. This equality implies that $s_i(y_1, y_2) = 0$ for $i \le -2$ and $s_{-1}(y_1, y_2) = \mathbb{1} \otimes \mathbb{1}$. Moreover, we have:

$$\begin{aligned} &\bigl(\phi(h; y_1) \otimes \phi(h; y_2)\bigr) \cdot r_0(y_1, y_2) \cdot \bigl(\phi(h; y_1)^{-1} \otimes \phi(h; y_2)^{-1}\bigr) = \\ &= s_0(y_1, y_2) + \mathbb{1} \otimes \phi'(h; y_2)\phi(h; y_2)^{-1} - \phi'(h; y_1)\phi(h; y_1)^{-1} \otimes \mathbb{1}. \end{aligned}$$

Let $\bar{\phi}(y) := \phi(0; y)$. Applying the operator $\mathrm{pr} \otimes \mathrm{pr}$ to the last equality and putting $h = 0$ we obtain:

$$\bar{s}_0(y_1, y_2) = \bigl(\bar{\phi}(y_1) \otimes \bar{\phi}(y_2)\bigr) \cdot \bar{r}_0(y_1, y_2) \cdot \bigl(\bar{\phi}(y_1)^{-1} \otimes \bar{\phi}(y_2)^{-1}\bigr).$$

This implies the claim. □

Remark 2.13. It was proven by Polishchuk [56, Theorem 2] that all solutions of the associative Yang–Baxter equation (8) arising from a universal family of *stable* vector bundles on an (irreducible) projective cubic curve satisfy the conditions of Lemma 2.11. However, we do not know a conceptual explanation of the fact that all these solutions have a Laurent expansion of the form (10), although in turns out to be so in all the examples known so far. Later, we shall see that the equation (8) has many solutions $r(v; y_1, y_2)$ with higher order poles with respect to v. Some of them can be obtained by the same geometric method, applied to certain families of *semi-stable* vector bundles, see Subsection 10.4. However, they do not project to a solution of the classical Yang–Baxter equation.

Finally, assume that a solution of (4) has the form $r(v_1, v_2; y_1, y_2) = r(v_2-v_1; y_2-y_1)$. Then the associative Yang–Baxter equation can be rewritten as

$$r(u;x)^{12}r(u+v;y)^{23} = r(u+v;x+y)^{13}r(-v;x)^{12} + r(v;y)^{23}r(u;x+y)^{13}. \tag{12}$$

This is the form of the associative Yang–Baxter equation introduced and studied by Polishchuk in [56] and [57].

Definition 2.14. A solution $r(y)$ of the classical Yang–Baxter equation (3) has an infinitesimal symmetry, if there exists an element $a \in \mathfrak{g}$ such that

$$\bigl[r(y), a \otimes \mathbb{1} + \mathbb{1} \otimes a\bigr] = 0.$$

For example, let $r(y) = r_{\mathrm{rat}}(y) = \dfrac{1}{y}\Omega$ be Yang's solution for $\mathfrak{sl}_2(\mathbb{C})$, then

$$[r(y), a \otimes \mathbb{1} + \mathbb{1} \otimes a] = 0$$

for any $a \in \mathfrak{g}$.

An important reason to study unitary solutions of the equation (12) satisfying (10) is explained by the following theorem.

Theorem 2.15 (see Theorem 1.4 of [57] and Theorem 6 of [56])**.** *Let $r(v;y)$ be a non-degenerate unitary solution of the associative Yang–Baxter equation (12) with Laurent expansion of the form (10). Then we have:*

- *The function $\bar{r}_0(y) := (\mathrm{pr} \otimes \mathrm{pr})\bigl(r_0(y)\bigr)$ is a non-degenerate unitary solution of the classical Yang–Baxter equation (3).*
- *If $\bar{r}_0(y)$ is either periodic (elliptic or trigonometric), or without infinitesimal symmetries, then for a fixed $v = v_0$ the tensor $r(v_0;y)$ satisfies the* quantum Yang–Baxter *equation:*

$$r(v_0;x)^{12}r(v_0;x+y)^{13}r(v_0;y)^{23} = r(v_0;y)^{23}r(v_0;x+y)^{13}r(v_0;x)^{12} \tag{13}$$

- *If $\bar{r}_0(y)$ does not have infinitesimal symmetries and if $s(v;y)$ is another solution of (12) of the form (10) and such that $\bar{r}_0(y) = \bigl(\mathrm{pr} \otimes \mathrm{pr}\bigr)(s_0(y))$, then there exist $\alpha_1 \in \mathbb{C}^*$ and $\alpha_2 \in \mathbb{C}$ such that $s(v;y) = \alpha_1 \exp(\alpha_2 vy) r(v;y)$. In other words, under these conditions, a solution of the associative Yang–Baxter equation $r(v;y)$ is uniquely determined by the corresponding solution of the classical Yang–Baxter equation $\bar{r}_0(y)$ up a gauge transformation described in Example 2.6 and rescaling.*

Remark 2.16. Theorem 2.15 gives an explicit recipe to lift a non-degenerate solution of the classical Yang–Baxter equation to a solution of the quantum Yang–Baxter equation. Existence of such quantization is known due to a result of Etingof and Kazhdan [27]. Moreover, it was proven by Polishchuk in [56] that any elliptic solution of the classical Yang–Baxter equation (3) can be lifted to a solution of (12) having a Laurent expansion of the form (10). However, Schedler showed in [59] that there exist trigonometric solutions of (CYBE), which *can not* be lifted to a solution of the associative Yang–Baxter equation (12) of the form (10).

3. Polishchuk's construction

Let X be a connected projective Gorenstein variety of dimension n over a field $\boldsymbol{k}$ and $\mathsf{Perf}(X)$ be the triangulated category of perfect complexes, i.e. a full subcategory of the derived category $D(X) = \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ consisting of complexes quasi-isomorphic to bounded complexes of locally free $\mathcal{O}_X$–modules.

We denote by ω_X the dualizing sheaf on X. This means (see for example [34, Section III.7]) that we have an isomorphism $t : H^n(\omega_X) \to \boldsymbol{k}$, also called a *trace map*, such that for any coherent sheaf $\mathcal{F} \in \mathsf{Coh}(X)$ the pairing

$$H^n(\mathcal{F}) \times \mathsf{Hom}_X(\mathcal{F}, \omega_X) \longrightarrow H^n(\omega_X) \xrightarrow{t} \boldsymbol{k}$$

is non-degenerate.

Remark 3.1. Such a map t is defined only up to a non-zero constant. However, it will be explained later that in the case of reduced projective Gorenstein curves with trivial canonical sheaf there exists a "canonical" choice for t, see subsection 4.3, directly after Theorem 4.16.

Let $\mathbb{S} : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X) \longrightarrow \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ be the functor given by the rule $\mathbb{S}(\mathcal{F}) = \mathcal{F} \overset{\mathbb{L}}{\otimes} \omega_X[n]$. For a perfect complex $\mathcal{F}$, let $\mathrm{tr}_{\mathcal{F}} : \mathsf{Hom}_{D(X)}\big(\mathcal{F}, \mathbb{S}(\mathcal{F})\big) \to \boldsymbol{k}$ be the morphism

$$\mathsf{Hom}_{D(X)}(\mathcal{F}, \mathcal{F} \overset{\mathbb{L}}{\otimes} \omega_X[n]) \xrightarrow{\cong} \mathsf{Hom}_{D(X)}(\mathcal{O}, \mathcal{F}^\vee \overset{\mathbb{L}}{\otimes} \mathcal{F} \overset{\mathbb{L}}{\otimes} \omega_X[n]) \longrightarrow H^n(\omega_X) \xrightarrow{t} \boldsymbol{k},$$

where the first arrow is a canonical isomorphism and the second is induced by the canonical evaluation morphism $\mathcal{F}^\vee \overset{\mathbb{L}}{\otimes} \mathcal{F} \longrightarrow \mathcal{O}$.

The following theorem seems to be well-known, however we were not able to find its proof in the literature and therefore sketch it here.

Theorem 3.2. *Let $p : X \to Y = \mathsf{Spec}(\boldsymbol{k})$ be a connected projective Gorenstein variety of dimension n over a field $\boldsymbol{k}$. Then for any $\mathcal{F} \in \mathsf{Perf}(X)$ and $\mathcal{G} \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ the pairing*

$$\langle\ ,\ \rangle_{\mathcal{F},\mathcal{G}} : \mathsf{Hom}_{D(X)}(\mathcal{F}, \mathcal{G}) \times \mathsf{Hom}_{D(X)}\big(\mathcal{G}, \mathbb{S}(\mathcal{F})\big) \longrightarrow \boldsymbol{k},$$

given by the formula $\langle f, g\rangle_{\mathcal{F},\mathcal{G}} := \mathrm{tr}_{\mathcal{F}}(gf)$, is non-degenerate.[1] *Moreover, if the complex $\mathcal{G}$ is also perfect, we have: $\mathrm{tr}_{\mathcal{F}}(gf) = \mathrm{tr}_{\mathcal{G}}\big(\mathbb{S}(f)g\big)$.*

Proof. Recall that by the Grothendieck duality the functor $\mathbb{R}p_* : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y)$ has a right adjoint $p^! : \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(Y) \to \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$, see [33, 51, 24]. Moreover, $p^!(\boldsymbol{k}) \cong \omega_X[n]$

[1] We thank Amnon Neeman for helping us at this place.

and the adjunction morphism $\tilde{t} : \mathbb{R}p_* p^!(\boldsymbol{k}) \to \boldsymbol{k}$ coincides up to a non-zero constant with the morphism

$$\mathbb{R}p_* p^!(\boldsymbol{k}) \xrightarrow{\text{can}} H^0\big(\mathbb{R}p_* p^!(\boldsymbol{k})\big) \cong H^n(\omega_X) \xrightarrow{t} \boldsymbol{k},$$

see the proof of [24, Theorem 5.12]). Next, we have the following canonical isomorphisms:

$$\begin{aligned} \mathsf{Hom}_{D(X)}\big(\mathcal{G}, \mathcal{F} \overset{\mathbb{L}}{\otimes} p^!(\boldsymbol{k})\big) &\cong \mathsf{Hom}_{D(X)}\big(\mathcal{F}^\vee \overset{\mathbb{L}}{\otimes} \mathcal{G}, p^!(\boldsymbol{k})\big) \cong \\ \cong \mathsf{Hom}_{D(X)}\big(\mathcal{RH}om_X(\mathcal{F}, \mathcal{G}), p^!(\boldsymbol{k})\big) &\cong \mathsf{Hom}_{D(Y)}\big(\mathbb{R}p_*(\mathcal{RH}om_X(\mathcal{F}, \mathcal{G})), \boldsymbol{k}\big) \cong \\ \cong \mathsf{Hom}_Y\Big(H^0\big(\mathbb{R}p_*(\mathcal{RH}om_X(\mathcal{F}, \mathcal{G}))\big), \boldsymbol{k}\Big) &\cong \mathsf{Hom}_{D(X)}(\mathcal{F}, \mathcal{G})^*. \end{aligned}$$

One can check that the image of a morphism $g \in \mathsf{Hom}_{D(X)}\big(\mathcal{G}, \mathbb{S}(\mathcal{F})\big)$ under this chain of isomorphisms is the functional $\mathrm{tr}_{\mathcal{F}}(g \circ -) : \mathsf{Hom}_{D(X)}\big(\mathcal{F}, \mathcal{G}\big) \to \boldsymbol{k}$ up to a constant not depending on f. This implies that the bilinear form $\langle \;,\; \rangle_{\mathcal{F},\mathcal{G}}$ is non-degenerate. In particular, $\mathbb{S}$ is a Serre functor of the category $\mathsf{Perf}(X)$ and the equality $\mathrm{tr}_{\mathcal{F}}(gf) = \mathrm{tr}_{\mathcal{G}}\big(\mathbb{S}(f)g\big)$ is automatically true, see [58, Lemma I.1.1]. $\square$

Corollary 3.3. *Let X be a connected projective Gorenstein variety over $\boldsymbol{k}$ of dimension n such that its dualizing sheaf is trivial. Let $\omega \in H^0(\omega_X)$ be a non-zero global section of ω_X. Then for any pair of perfect complexes $\mathcal{F}, \mathcal{G}$ on X the pairing $\langle \;,\; \rangle^\omega_{\mathcal{F},\mathcal{G}} : \mathsf{Hom}_{D(X)}(\mathcal{F}, \mathcal{G}) \times \mathsf{Hom}_{D(X)}(\mathcal{G}, \mathcal{F}[n]) \longrightarrow \mathsf{Hom}_{D(X)}(\mathcal{F}, \mathcal{F}[n]) \xrightarrow{\omega_*} \mathsf{Hom}_{D(X)}\big(\mathcal{F}, \mathbb{S}(\mathcal{F})\big) \xrightarrow{\mathrm{tr}_{\mathcal{F}}} \boldsymbol{k}$ is non-degenerate.*

Let X be a reduced Gorenstein curve over the field $\mathbb{C}$. By [3, Chapter VIII] or by [24, Appendix B] the dualising sheaf ω_X is isomorphic to the sheaf of *regular* or *Rosenlicht's* differential 1-forms $\Omega_X = \Omega_X^{1,R}$. If X is smooth, then Ω_X coincides with the sheaf of holomorphic 1-forms. For X singular the definition is as follows.

Definition 3.4. Let X be a reduced projective Gorenstein curve, $n : \widetilde{X} \to X$ its normalization. Denote by Ω_X^M and $\Omega_{\widetilde{X}}^M$ the sheaves of meromorphic differential 1-forms on X and $\widetilde{X}$ respectively. Observe that $\Omega_X^M = n_*(\Omega_{\widetilde{X}}^M)$. Then Ω_X is defined to be the subsheaf of Ω_X^M such that for any open subset $U \subseteq X$ one has

$$\Omega_X(U) = \left\{ \omega \in \Omega_{\widetilde{X}}^M\big(n^{-1}(U)\big) \;\middle|\; \forall p \in U, \forall f \in \mathcal{O}_X(U) : \;\sum_{i=1}^t \mathrm{res}_{p_i}\big((f \circ n)\omega\big) = 0 \right\},$$

where $\{p_1, p_2, \dots, p_t\} = n^{-1}(p)$.

A reduced projective curve E whose dualizing sheaf ω_E is isomorphic to the structure sheaf is Gorenstein and has arithmetic genus one. For example, reduced plane cubics, Kodaira cycles and generic configurations of $n+1$ lines in $\mathbb{P}^n$ passing through a given point are of this type. In what follows, for such a curve E, we identify ω_E wiht Ω_E and fix a global section $\omega \in H^0(\Omega_E)$ giving an isomorphism $\omega : \mathcal{O} \to \Omega_E$ and a trace map $t^\omega : H^1(\mathcal{O}) \xrightarrow{\omega_*} H^1(\Omega_E) \xrightarrow{t} \mathbb{C}$.

A characteristic property of reduced projective curves with trivial dualizing sheaf is a very special form of the Serre duality. By Theorem 3.2 we have the following result.

Proposition 3.5. *Let E be a reduced projective curve with trivial dualizing sheaf and $\mathcal{E}, \mathcal{F} \in \mathsf{Perf}(E)$. Then the map*

$$\langle\ ,\ \rangle^{\omega}_{\mathcal{E},\mathcal{F}} : \mathsf{Hom}(\mathcal{E},\mathcal{F}) \otimes \mathsf{Hom}(\mathcal{F},\mathcal{E}[1]) \xrightarrow{\circ} \mathsf{Hom}(\mathcal{E},\mathcal{E}[1]) \xrightarrow{\mathrm{Tr}_{\mathcal{E}}} H^1(\mathcal{O}) \xrightarrow{t^{\omega}} \mathbb{C}$$

where $\mathrm{Tr}_{\mathcal{E}} : \mathsf{Hom}(\mathcal{E},\mathcal{E}[1]) \xrightarrow{\cong} \mathsf{Hom}(\mathcal{O},\mathcal{E}^{\vee} \overset{\mathbb{L}}{\otimes} \mathcal{E}[1]) \longrightarrow \mathsf{Hom}(\mathcal{O},\mathcal{O}[1]) = H^1(\mathcal{O})$, *is a non-degenerate pairing. This pairing coincides with the composition*

$$\langle\ ,\ \rangle^{\omega}_{\mathcal{F},\mathcal{E}} : \mathsf{Hom}(\mathcal{E},\mathcal{F}) \otimes \mathsf{Hom}(\mathcal{F},\mathcal{E}[1]) \xrightarrow{\cong} \mathsf{Hom}(\mathcal{F},\mathcal{E}[1]) \otimes \mathsf{Hom}(\mathcal{E}[1],\mathcal{F}[1]) \xrightarrow{\circ}$$

$$\longrightarrow \mathsf{Hom}(\mathcal{F},\mathcal{F}[1]) \xrightarrow{\mathrm{Tr}_{\mathcal{F}}} H^1(\mathcal{O}) \xrightarrow{t^{\omega}} \mathbb{C}.$$

Remark 3.6. The choice of non-degenerate pairings $\langle\ ,\ \rangle_{\mathcal{E},\mathcal{F}}$ is actually not unique, see the proof of Proposition I.2.3 in [58]. In particular, $\langle\ ,\ \rangle_{\mathcal{E},\mathcal{F}}$ depends on the choice of a global section of the dualizing sheaf ω_E. If $\omega_* : \mathsf{Hom}(\mathcal{F},\mathcal{E}[1]) \longrightarrow \mathsf{Hom}(\mathcal{F},\mathcal{E} \otimes \omega_E[1])$ denotes the isomorphism induced by $\omega : \mathcal{O}_E \xrightarrow{\sim} \omega_E$, we obtain $\langle\_\,,\,\_\rangle^{\omega}_{\mathcal{E},\mathcal{F}} = \langle\_\,,\omega_*(\_)\rangle_{\mathcal{E},\mathcal{F}}$.

In [55] Polishchuk showed how a construction of Merkulov [46], which fairly explicitly provides an A_∞-structure, applied to the category of Hermitian vector bundles on a complex manifold with a hermitian metric, gives an A_∞-structure on the bounded derived category of this manifold. He shows that this construction provides a cyclic A_∞-structure [55, Theorem 1.1], which is crucial for the proof of the Yang-Baxter equations in the situation considered here. Therefore, we formulate his result explicitly in the following proposition.

Proposition 3.7. *If E is a smooth elliptic curve, there exits an A_∞-structure on the category $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(E)$, which is cyclic with respect to the pairing described in Proposition 3.5. In particular, this means*

$$\big\langle m_3(f_1,g_1,f_2),g_2\big\rangle = -\big\langle f_1, m_3(g_1,f_2,g_2)\big\rangle = -\big\langle m_3(f_2,g_2,f_1),g_1\big\rangle$$

for any objects $\mathcal{E}_1,\mathcal{E}_2$ and $\mathcal{F}_1,\mathcal{F}_2 \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(E)$, and any morphisms

$$f_i \in \mathsf{Hom}(\mathcal{E}_i,\mathcal{F}_i) \text{ and } g_i \in \mathsf{Hom}(\mathcal{F}_i,\mathcal{E}_{3-i}[1]), i = 1,2.$$

Because the proof of Polishchuk uses harmonic forms and Hodge theory, it heavily depends on the smoothness assumption for E. If E is singular, the same result can be derived with some effort using more recent methods from non-commutative symplectic geometry [40, Theorem 10.2.2].

Now we recall the main construction of [56]. Take a reduced projective curve E with trivial dualising sheaf and fix the following data:

- Two vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ on E having the same rank n and such that $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = 0 = \mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2)$.
- Two distinct smooth points $y_1, y_2 \in \breve{E}$ lying on the same irreducible component of E and such that $\mathsf{Hom}_E\bigl(\mathcal{V}_1(y_2), \mathcal{V}_2(y_1)\bigr) = 0 = \mathsf{Ext}^1_E\bigl(\mathcal{V}_1(y_2), \mathcal{V}_2(y_1)\bigr)$.

Remark 3.8. This "orthogonality" assumption on vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ might seem to be quite artificial. The natural example of such data is the following. Let $(n, d) \in \mathbb{N} \times \mathbb{Z}$ be a pair of coprime integers, $M_E^{(n,d)}$ the moduli space of stable vector bundles of rank n and degree d on a Weierstraß curve E. Let $\mathcal{P}(n, d)$ be a universal family on $E \times M_E^{(n,d)}$. For a point $v_i \in M_E^{(n,d)}$ denote by $\mathcal{V}_i$ the corresponding stable vector bundle $\mathcal{P}(n, d)|_{E\times v_i}$ on the curve E. Then for any two distinct points $v_1, v_2 \in M_E^{(n,d)}$ we have $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = 0 = \mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2)$.

Actually, one can also consider a more general situation. Namely, for any pair $(n, d) \in \mathbb{N} \times \mathbb{Z}$, not necessarily coprime, one can take indecomposable semi-stable vector bundles of rank n and degree d having *locally free* Jordan-Hölder factors. The orthogonality condition between non-isomorphic bundles of this type follows from the following lemma.

Lemma 3.9 (see [20])**.** *Let $(n, d) \in \mathbb{N} \times \mathbb{Z}$, $m = \gcd(n, d)$ and $n = mn', d = md'$. Let $\mathcal{V}$ be an indecomposable semi-stable vector bundle of rank n and degree d on a Weierstraß curve E with locally free Jordan-Hölder factors. Then all these factors are isomorphic to a single stable vector bundle $\mathcal{V}' \in M_E^{(n',d')}$. Moreover, we have $\mathcal{V} \cong \mathcal{V}' \otimes \mathcal{A}_m$, where $\mathcal{A}_m$ is the indecomposable vector bundle of rank m and degree 0 defined recursively by the non-split extension sequences*

$$0 \longrightarrow \mathcal{A}_m \longrightarrow \mathcal{A}_{m+1} \longrightarrow \mathcal{O} \longrightarrow 0 \quad m \geq 1,$$

where $\mathcal{A}_1 = \mathcal{O}$.

Let us return to Polishchuk's construction. Since $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = 0 = \mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2)$, we have a linear map

$$m_3 : \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})$$

called the *triple Massey product* and defined as follows. Let $a \in \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2)$, $g \in \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1})$, $h \in \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2})$ and let

$$0 \longrightarrow \mathcal{V}_2 \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathbb{C}_{y_1} \longrightarrow 0$$

be an exact sequence representing the element a. The vanishing of $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2)$ and $\mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2)$ implies that we can uniquely lift the morphisms g and h to morphisms

$\tilde{g} : \mathcal{V}_1 \longrightarrow \mathcal{A}$ and $\tilde{h} : \mathcal{A} \longrightarrow \mathbb{C}_{y_2}$. So, we obtained a diagram

$$\begin{tikzcd}
 & & & \mathcal{V}_1 \arrow[dl, "\tilde{g}"'] \arrow[d, "g"] & \\
a : 0 \arrow[r] & \mathcal{V}_2 \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathcal{A} \arrow[r, "\beta"] \arrow[dl, "\tilde{h}"] & \mathbb{C}_{y_1} \arrow[r] & 0 \\
 & \mathbb{C}_{y_2} & & &
\end{tikzcd}$$

and the triple Massey product is defined as $m_3(g \otimes a \otimes h) = \tilde{h}\tilde{g}$. Note that a determines an extension only up to an automorphism of the middle term, but the action of $\mathsf{Aut}(\mathcal{A})$ leads to the same answer for $m_3(g \otimes a \otimes h) =: m_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}(g \otimes a \otimes h)$.

Now one can use a sequence of canonical isomorphisms in order to rewrite $m_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ in another form:

$$\mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}), \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})\big) \cong$$

$$\mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}), \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2)^* \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})\big) \cong$$

$$\mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}), \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})\big),$$

where we use the Serre duality formula $\mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2)^* \cong \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1})$ given by the bilinear form $\langle\ ,\ \rangle^{\omega}_{\mathcal{V}_2,\mathbb{C}_{y_1}}$ from Proposition 3.5. Let $\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ be the image of $m_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ under this chain of isomorphisms.

Theorem 3.10 (see Theorem 1 in [56])**.** *If E is smooth, then $\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ satisfies the following "triangle equation" (associative Yang–Baxter equation)*

$$(\widetilde{m}_{y_1,y_2}^{\mathcal{V}_3,\mathcal{V}_2})^{12}(\widetilde{m}_{y_1,y_3}^{\mathcal{V}_1,\mathcal{V}_3})^{13} - (\widetilde{m}_{y_2,y_3}^{\mathcal{V}_1,\mathcal{V}_3})^{23}(\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2})^{12} + (\widetilde{m}_{y_1,y_3}^{\mathcal{V}_1,\mathcal{V}_2})^{13}(\widetilde{m}_{y_2,y_3}^{\mathcal{V}_2,\mathcal{V}_3})^{23} = 0. \tag{14}$$

The left-hand side of this equation is a linear map

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_3}) \longrightarrow$$

$$\longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_3}).$$

Moreover, the tensor $\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ is non-degenerate and skew-symmetric:

$$\tau(\widetilde{m}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}) = -\widetilde{m}_{y_2,y_1}^{\mathcal{V}_2,\mathcal{V}_1},$$

where τ is the isomorphism

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1})$$

given by $\tau(f \otimes g) = g \otimes f$.

Idea of the proof. This equality is a consequence of the A_∞ - constraint

$$m_3 \circ (m_3 \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes m_3 \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes m_3) = 0,$$

and skew-symmetry of $\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ follows from the cyclicity of the A_∞ - structure. $\square$

Note that for a vector bundle $\mathcal{V}$ and a smooth point $y \in E$ we have canonical isomorphisms

$$\mathsf{Hom}_E(\mathcal{V}, \mathbb{C}_y) \cong \mathsf{Hom}_E(\mathcal{V} \otimes \mathbb{C}_y, \mathbb{C}_y) = \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}|_y, \mathbb{C}) = \mathcal{V}|_y^*.$$

In these terms, $\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ is a linear map

$$\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : \mathcal{V}_1|_{y_1}^* \otimes \mathcal{V}_2|_{y_2}^* \longrightarrow \mathcal{V}_2|_{y_1}^* \otimes \mathcal{V}_1|_{y_2}^*.$$

Now we use the canonical isomorphism

$$\alpha : \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_1}, \mathcal{V}_1|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}^* \otimes \mathcal{V}_2|_{y_2}^*, \mathcal{V}_2|_{y_1}^* \otimes \mathcal{V}_1|_{y_2}^*)$$

mapping a simple tensor $f_1 \otimes f_2$ to $f_1^t \otimes f_2^t$. Then the tensor

$$r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} := \alpha^{-1}(\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}) \in \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_1}, \mathcal{V}_1|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})$$

satisfies the equation

$$(r^{\mathcal{V}_1,\mathcal{V}_3}_{y_1,y_3})^{13}(r^{\mathcal{V}_3,\mathcal{V}_2}_{y_1,y_2})^{12} - (r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2})^{12}(r^{\mathcal{V}_1,\mathcal{V}_3}_{y_2,y_3})^{23} + (r^{\mathcal{V}_2,\mathcal{V}_3}_{y_2,y_3})^{23}(r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_3})^{13} = 0 \tag{15}$$

and the unitarity condition

$$\tau(r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}) = -r^{\mathcal{V}_2,\mathcal{V}_1}_{y_2,y_1}. \tag{16}$$

Remark 3.11. Since the functorial isomorphism of vector spaces $\mathsf{Hom}_{\mathbb{C}}(U,V) \to \mathsf{Hom}_{\mathbb{C}}(V^*,U^*)$ is contravariant, the tensors $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ and $\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ appear in inverse order in Equations (14) and (15).

Note that the bilinear map

$$\mathrm{tr} : \mathsf{Hom}_{\mathbb{C}}(U,V) \times \mathsf{Hom}_{\mathbb{C}}(V,U) \longrightarrow \mathbb{C}, \quad (f,g) \mapsto \mathrm{tr}(f \circ g)$$

is non-degenerate and induces an isomorphism $\mathsf{Hom}_{\mathbb{C}}(U,V)^* \cong \mathsf{Hom}_{\mathbb{C}}(V,U)$. Using this, we get a chain of canonical isomorphisms

$$\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_1}, \mathcal{V}_1|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \cong \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1})^* \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \cong$$
$$\cong \mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}), \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})\big).$$

We let $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} \in \mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}), \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})\big)$ be the image of $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$.

Remark 3.12. Note that the triple Massey product m_3 is canonical, however the tensor $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ and the linear map $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ *depend* on the choice of a global section $\omega \in H^0(\Omega_E)$. Indeed, passing from m_3 to $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ and $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ we use the bilinear form $\langle\ ,\ \rangle^{\omega}_{\mathcal{V}_2,\mathbb{C}_{y_1}}$, which depends on the choice of ω.

Our next aim is to answer the following questions:

Q1 What is a geometrical interpretation of the equivalence relation given in Definition 2.5?
Q2 How can we view $\mathcal{V}_1$ and $\mathcal{V}_2$ as variables?
Q3 Is there a practical way to compute $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$?

4. Geometric description of triple Massey products

Let E be a reduced projective curve with trivial dualizing sheaf and $\breve{E}$ be the open subset of the smooth points of E. As in the previous section, we fix the following data:

- two vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ on E of rank n such that $\mathsf{Hom}_E(\mathcal{V}_1, \mathcal{V}_2) = 0$ and $\mathsf{Ext}^1_E(\mathcal{V}_1, \mathcal{V}_2) = 0$.
- two distinct points $y_1, y_2 \in \breve{E}$ lying on the same irreducible component of E such that $\mathsf{Hom}_E\big(\mathcal{V}_1(y_2), \mathcal{V}_2(y_1)\big) = \mathsf{Ext}^1_E\big(\mathcal{V}_1(y_2), \mathcal{V}_2(y_1)\big) = 0$.
- A non-zero global regular differential one-form $\omega \in H^0(\Omega_E)$.

The main goal of this section is to get an alternative description of the linear map

$$\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}(\omega) : \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}) \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})$$

introduced in Section 3. To do this, we first elaborate the theory of residues and evaluation morphisms on reduced complex Gorenstein curves.

4.1. Residue map for vector bundles.

In this subsection, our set-up is rather general. We fix the following data:

- a reduced Gorenstein analytic curve X (not necessarily compact);
- a subset $D \subset \breve{X}$, where $\breve{X}$ denotes the open subset of smooth points of X, such that $D \subset X$ is locally finite[2];
- a point $x \in D$; the set $D' = D \setminus \{x\}$ may be empty;
- a pair of vector bundles $\mathcal{V}$ and $\mathcal{W}$ on X;
- a germ of a differential 1-form $\omega \in \Omega_x$, not vanishing at x (i.e. $\omega \notin \Omega(-x)_x$), where $\Omega = \Omega_X$ is the sheaf of regular differential 1-forms on X.

The only application of this general set-up with $D' \neq \emptyset$ occurs in Section 8, where $Y = \mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Lambda = X$ (cf. Prop. 4.8, Prop. 4.12 and Thm. 4.23) is the universal cover of an elliptic curve and $D = y + \Lambda$ is an infinite Λ-invariant subset of $\mathbb{C}$. Therefore, we resist the temptation to include D or D' into the decorations of the residue maps below. In all other applictions of Theorem 4.23, $\pi : Y \to X$ is the normalisation, hence restricts to an isomorphism $\pi : \breve{Y} \to \breve{X}$ and D consists of a single point.

Consider the canonical short exact sequence

$$0 \to \Omega_X \to \Omega_X(x) \xrightarrow{\mathrm{res}_x} \mathbb{C}_x \to 0. \tag{17}$$

[2]This means that for any $p \in X$, there exists an open neighbourhood of p in X which contains only a finite number of points from D. According to [30, Ch. 1.1], D defines a divisor and a corresponding line bundle $\mathcal{O}_X(D)$ on X.

Proposition 4.1. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{H}om_X\big(\mathcal{V}\otimes\Omega, \mathcal{W}\otimes\Omega(D)\big) \arrow[r, "(1\otimes\underline{\mathrm{res}}_x)_*"] & \mathcal{H}om_X\big(\mathcal{V}\otimes\Omega, \mathcal{W}\otimes\mathcal{O}(D')\otimes\mathbb{C}_x\big) \\
\mathcal{H}om_X\big(\mathcal{V}, \mathcal{W}\otimes\mathcal{O}(D)\big) \arrow[u, "-\otimes\Omega"] & \mathcal{H}om_X\big(\mathcal{V}\otimes\Omega, \mathcal{W}\otimes\mathbb{C}_x\big) \arrow[u, "(r_{D'})_*"'] \\
\mathcal{H}om_X(\mathcal{V}, \mathcal{W})\otimes\mathcal{O}(D) \arrow[u, "\mathrm{can}"] \arrow[r, "\underline{\mathrm{res}}'_x"] & \mathcal{H}om_X\big(\mathcal{V}\otimes\Omega\otimes\mathbb{C}_x, \mathcal{W}\otimes\mathbb{C}_x\big). \arrow[u, "\mathrm{can}"']
\end{tikzcd}$$

The map $(r_{D'})_*$ is induced by the inclusion $r_{D'} : \mathcal{O} \to \mathcal{O}(D')$ of subsheaves of the sheaf of meromorphic functions on X and $\underline{\mathrm{res}}'_x$ is determined by the following morphism of presheaves. Let $x \in U \subseteq X$ be an open subset, $s \in \mathsf{Hom}_{\mathcal{O}(U)}\big(\mathcal{V}(U), \mathcal{W}(U)\big)$, $f \in \Gamma\big(U, \mathcal{O}(D)\big), v \in \mathcal{V}(U), \delta \in \Omega(U)$. Then

$$\underline{\mathrm{res}}'_x(s\otimes f)\big[v\otimes\delta\big] = \mathrm{res}_x(f\delta)[s(v)],$$

where $[v\otimes\delta] := v\otimes\delta\otimes 1$. If $x \notin U$, $\underline{\mathrm{res}}'_x|_U$ is the zero map. The upper horizontal morphism is induced by the short exact sequence (17) *and all other morphisms are standard canonical isomorphisms.*

Proof. After observing that $r_{D'}$ is the identity map over any open set U which does not intersect D', the proof is an easy diagram chase. □

Note that a germ $\omega \in \Omega_x$, which is not in $\Omega(-x)_x$, i.e. not equal to zero in $\Omega\otimes\mathbb{C}_x$, induces an isomorphism of sheaves

$$\mathcal{H}om_X\big(\mathcal{V}\otimes\Omega\otimes\mathbb{C}_x, \mathcal{W}\otimes\mathbb{C}_x\big) \xrightarrow{\omega^*} \mathcal{H}om_X\big(\mathcal{V}\otimes\mathbb{C}_x, \mathcal{W}\otimes\mathbb{C}_x\big).$$

Definition 4.2. Consider the morphism of sheaves of $\mathcal{O}_X$–modules

$$\underline{\mathrm{res}}_x^{\mathcal{V},\mathcal{W}}(\omega) : \mathcal{H}om_X\big(\mathcal{V}, \mathcal{W}\otimes\mathcal{O}(D)\big) \longrightarrow \mathcal{H}om_X\big(\mathcal{V}\otimes\mathbb{C}_x, \mathcal{W}\otimes\mathbb{C}_x\big)$$

defined as the composition of morphisms

$$\mathcal{H}om_X\big(\mathcal{V}, \mathcal{W}\otimes\mathcal{O}(D)\big) \xrightarrow{-\otimes\Omega} \mathcal{H}om_X\big(\mathcal{V}\otimes\Omega, \mathcal{W}\otimes\Omega(D)\big)$$

$$\xrightarrow{(1\otimes\underline{\mathrm{res}}_x)_*} \mathcal{H}om_X\big(\mathcal{V}\otimes\Omega, \mathcal{W}\otimes\mathcal{O}(D')\otimes\mathbb{C}_x\big)$$

$$\xrightarrow{\left((r_{D'})_*\circ\mathrm{can}\right)^{-1}} \mathcal{H}om_X\big(\mathcal{V}\otimes\Omega\otimes\mathbb{C}_x, \mathcal{W}\otimes\mathbb{C}_x\big) \xrightarrow{\omega^*} \mathcal{H}om_X\big(\mathcal{V}\otimes\mathbb{C}_x, \mathcal{W}\otimes\mathbb{C}_x\big),$$

coinciding with the composition $\mathcal{H}om_X\big(\mathcal{V}, \mathcal{W}\otimes\mathcal{O}(D)\big) \xrightarrow{\mathrm{can}} \mathcal{H}om_X(\mathcal{V}, \mathcal{W})\otimes\mathcal{O}(D)$ $\xrightarrow{\underline{\mathrm{res}}'_x} \mathcal{H}om_X\big(\mathcal{V}\otimes\Omega\otimes\mathbb{C}_x, \mathcal{W}\otimes\mathbb{C}_x\big) \xrightarrow{\omega^*} \mathcal{H}om_X\big(\mathcal{V}\otimes\mathbb{C}_x, \mathcal{W}\otimes\mathbb{C}_x\big)$.

Remark 4.3. The fact that the composition

$$\mathcal{H}om(\mathcal{V},\mathcal{W}(D)) \xrightarrow{-\otimes\Omega} \mathcal{H}om(\mathcal{V}\otimes\Omega,\mathcal{W}\otimes\Omega(D)) \xrightarrow{\omega^*} \mathcal{H}om(\mathcal{V},\mathcal{W}\otimes\Omega(D))$$

coincides with the map ω_* implies that there is a third description for $\underline{\mathrm{res}}_x^{\mathcal{V},\mathcal{W}}(\omega)$ as: $\mathcal{H}om(\mathcal{V},\mathcal{W}(D)) \xrightarrow{\omega_*} \mathcal{H}om(\mathcal{V},\mathcal{W}\otimes\Omega(D)) \xrightarrow{(1\otimes\underline{\mathrm{res}}_x)_*} \mathcal{H}om(\mathcal{V},\mathcal{W}\otimes\mathbb{C}_x) \xrightarrow{\left((r_{D'})_*\circ\mathrm{can}\right)^{-1}}$ $\mathcal{H}om(\mathcal{V}\otimes\mathbb{C}_x,\mathcal{W}\otimes\mathbb{C}_x)$. This will be used in the proof of Proposition 4.22.

Definition 4.4. In the above notation we define:

$$\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega) := H^0\big(\underline{\mathrm{res}}_x^{\mathcal{V},\mathcal{W}}(\omega)\big) : \mathsf{Hom}_X\big(\mathcal{V},\mathcal{W}(D)\big) \longrightarrow \mathsf{Hom}_X(\mathcal{V}\otimes\mathbb{C}_x,\mathcal{W}\otimes\mathbb{C}_x).$$

Proposition 4.1 yields an explicit formula for the morphisms $\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)$ in the following simple situation.

Lemma 4.5. *Let $U \subseteq \mathbb{C}$ be open, $O = H^0(U,\mathcal{O}_{\mathbb{C}})$, $D \subset U$ a locally finite subset, $O(D) = H^0\big(U,\mathcal{O}_{\mathbb{C}}(D)\big)$ and $x \in D$. Let $\mathcal{V} = \mathcal{O}_U^n, \mathcal{W} = \mathcal{O}_U^m$ and $\omega = f(z)dz$ be a meromorphic one-form on U, holomorphic at x with $f(x) \neq 0$. Then the morphism* res_x*, which is defined as the composition of canonical morphisms*

$$\begin{tikzcd}
\mathsf{Hom}_U\big(\mathcal{V},\mathcal{W}(D)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)"] & \mathsf{Hom}_U(\mathcal{V}\otimes\mathbb{C}_x,\mathcal{W}\otimes\mathbb{C}_x) \arrow[d, "\mathrm{can}"] \\
\mathsf{Mat}_{m\times n}\big(O(D)\big) \arrow[u, "\mathrm{can}"] \arrow[r, "\mathrm{res}_x"] & \mathsf{Mat}_{m\times n}(\mathbb{C})
\end{tikzcd}$$

is given by the formula $\mathrm{res}_x(F) = \mathrm{res}_x(F\cdot\omega)$.

Proof. Let $F(z) = \big(f_{ij}(z)\big) \in \mathsf{Mat}_{m\times n}\big(O(D)\big)$ be a matrix whose entries are meromorphic functions on U having at most simple poles along D. Then we can write $F(z) = \dfrac{G(z)}{z-x}$, where the entries of the matrix $G(z)$ are meromorphic functions which are holomorphic at x. Using the definition of res_x in terms of $\underline{\mathrm{res}}'_x$ we obtain

$$\mathrm{res}_x(F)(a) = \left(\mathrm{res}_x\,\frac{\omega}{z-x}\right)G(x)a = \mathrm{res}_x(F\omega)a$$

for all $a \in \mathbb{C}^n$. □

In general, the morphism $\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)$ is neither surjective nor injective. However, there is an important special case where it is an isomorphism.

Proposition 4.6. *Let E be a reduced projective curve with trivial dualizing sheaf. Let $\mathcal{V}$ and $\mathcal{W}$ be a pair of vector bundles on E such that $\mathsf{Hom}_E(\mathcal{V},\mathcal{W}) = 0$ and $\mathsf{Hom}_E(\mathcal{W},\mathcal{V}) = 0$ and let $x \in \breve{E}$, $\omega \in \Omega_x$ be as above. Then, the morphism*

$$\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega) : \mathsf{Hom}_E\big(\mathcal{V},\mathcal{W}(x)\big) \longrightarrow \mathsf{Hom}_E(\mathcal{V}\otimes\mathbb{C}_x,\mathcal{W}\otimes\mathbb{C}_x),$$

defined with $D = \{x\}$, is an isomorphism.

Proof. First note that by Serre duality we have: $\mathsf{Ext}^1_E(\mathcal{V},\mathcal{W}) \cong \mathsf{Hom}_E(\mathcal{W},\mathcal{V})^* = 0$. Hence, then short exact sequence $0 \to \mathcal{W}\otimes\Omega \to \mathcal{W}\otimes\Omega(x) \to \mathcal{W}\otimes\mathbb{C}_x \to 0$ induces an isomorphism $H^0\big((1\otimes \underline{\mathrm{res}}_x)_*\big) : \mathsf{Hom}_E\big(\mathcal{V}\otimes\Omega, \mathcal{W}\otimes\Omega(x)\big) \xrightarrow{\cong} \mathsf{Hom}_E(\mathcal{V}\otimes\Omega,\mathcal{W}\otimes\mathbb{C}_x)$. As a result, the morphism $\mathsf{Hom}_E\big(\mathcal{V},\mathcal{W}(x)\big) \xrightarrow{\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)} \mathsf{Hom}_E(\mathcal{V}\otimes\mathbb{C}_x,\mathcal{W}\otimes\mathbb{C}_x)$ is an isomorphism, too. $\square$

Remark 4.7. The vanishing conditions of Proposition 4.6 are satisfied if E is an irreducible projective Weierstraß curve and $\mathcal{V}$ and $\mathcal{W}$ are two stable vector bundles of the same rank and degree and such that $\mathcal{V} \ncong \mathcal{W}$.

The next goal is to show that the morphism $\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)$ has nice functorial properties.

Proposition 4.8. *Let $Y \xrightarrow{\pi} X$ be a morphism of reduced Gorenstein curves, $y \in \breve{Y}$ and $x = \pi(y) \in \breve{X}$ be such that f is unramified over x. Let $D = \pi^{-1}(x)$ and $\mathcal{V},\mathcal{W}$ be a pair of vector bundles on X, $\widetilde{\mathcal{V}} = \pi^*\mathcal{V}$ and $\widetilde{\mathcal{W}} = \pi^*\mathcal{W}$. Finally, let $\omega \in \Omega_{X,x}$ be the germ of a regular differential one-form, not vanishing at x, and $\tilde{\omega} = \pi^*(\omega) \in \Omega_{Y,y}$ be the corresponding germ on Y. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathsf{Hom}_X\big(\mathcal{V},\mathcal{W}(x)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)"] \arrow[d, "\pi^*"'] & \mathsf{Hom}_X\big(\mathcal{V}\otimes\mathbb{C}_x,\mathcal{W}\otimes\mathbb{C}_x\big) \arrow[d, "\pi^*"] \\
\mathsf{Hom}_Y\big(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}}(D)\big) \arrow[r, "\mathrm{res}_y^{\widetilde{\mathcal{V}},\widetilde{\mathcal{W}}}(\tilde{\omega})"] & \mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}\otimes\mathbb{C}_y,\widetilde{\mathcal{W}}\otimes\mathbb{C}_y\big).
\end{tikzcd}$$

Proof. Let $i : U \hookrightarrow X$ be an open embedding containing the point x, $\mathcal{V}' = \mathcal{V}|_U$ and $\mathcal{W}' = \mathcal{W}|_U$. Then the diagram

$$\begin{tikzcd}
\mathsf{Hom}_X\big(\mathcal{V},\mathcal{W}(x)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)"] \arrow[d, "i^*"'] & \mathsf{Hom}_X\big(\mathcal{V}\otimes\mathbb{C}_x,\mathcal{W}\otimes\mathbb{C}_x\big) \arrow[d, "i^*"] \\
\mathsf{Hom}_U\big(\mathcal{V}',\mathcal{W}'(x)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V}',\mathcal{W}'}(\omega)"] & \mathsf{Hom}_U\big(\mathcal{V}'\otimes\mathbb{C}_x,\mathcal{W}'\otimes\mathbb{C}_x\big).
\end{tikzcd}$$

is commutative: this is a consequence of the "local" definition of the morphism $\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)$ as $H^0\big(\underline{\mathrm{res}}_x^{\mathcal{V},\mathcal{W}}(\omega)\big)$.

In order to pass to the general case, recall that any unramified morphism of smooth Riemann surfaces is locally biholomorphic. The assumptions imply $\pi^*\mathcal{O}(x) = \mathcal{O}(D)$. Since both points $x \in X$ and $y \in Y$ are smooth, there exist open neighbourhoods $x \in U \overset{i}{\hookrightarrow} X$ and $y \in V \overset{j}{\hookrightarrow} Y$ such that $\pi : V \to U$ is an isomorphism. In particular,

$V \cap D = \{y\}$ and we have a diagram

$$\begin{tikzcd}
\mathsf{Hom}_X\big(\mathcal{V}, \mathcal{W}(x)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)"] \arrow[d, "i^*"'] \arrow[ddd, bend right=60, "\pi^*"'] & \mathsf{Hom}_X\big(\mathcal{V} \otimes \mathbb{C}_x, \mathcal{W} \otimes \mathbb{C}_x\big) \arrow[d, "i^*"] \arrow[ddd, bend left=60, "\pi^*"] \\
\mathsf{Hom}_U\big(\mathcal{V}', \mathcal{W}'(x)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V}',\mathcal{W}'}(\omega)"] \arrow[d, "\pi^*"'] & \mathsf{Hom}_U\big(\mathcal{V}' \otimes \mathbb{C}_x, \mathcal{W}' \otimes \mathbb{C}_x\big) \arrow[d, "\pi^*"] \\
\mathsf{Hom}_V\big(\widetilde{\mathcal{V}}', \widetilde{\mathcal{W}}'(y)\big) \arrow[r, "\mathrm{res}_y^{\widetilde{\mathcal{V}}',\widetilde{\mathcal{W}}'}(\tilde{\omega})"] & \mathsf{Hom}_V\big(\widetilde{\mathcal{V}}' \otimes \mathbb{C}_y, \widetilde{\mathcal{W}}' \otimes \mathbb{C}_y\big) \\
\mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}(D)\big) \arrow[r, "\mathrm{res}_y^{\widetilde{\mathcal{V}},\widetilde{\mathcal{W}}}(\tilde{\omega})"] \arrow[u, "j^*"] & \mathsf{Hom}_Y\big(\widetilde{\mathcal{V}} \otimes \mathbb{C}_y, \widetilde{\mathcal{W}} \otimes \mathbb{C}_y\big), \arrow[u, "j^*"']
\end{tikzcd}$$

in which all three cental squares and both exterior squares are commutative. To conclude the claim, it remains to note that all vertical morphisms on the right hand side are isomorphisms. □

Note that the constructed morphism $\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)$ is functorial in $\mathcal{V}$ and $\mathcal{W}$. In what follows, we shall use the following bifunctoriality.

Definition 4.9. Let $\mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{W}_1, \mathcal{W}_2$ be coherent sheaves on X, $f : \mathcal{V}_1 \to \mathcal{V}_2$ and $g : \mathcal{W}_1 \to \mathcal{W}_2$ be isomorphisms in $\mathsf{Coh}(X)$. Then we have an morphism

$$\mathrm{cnj}(f, g) : \mathsf{Hom}_X(\mathcal{V}_1, \mathcal{W}_1) \longrightarrow \mathsf{Hom}_X(\mathcal{V}_2, \mathcal{W}_2)$$

given by the rule $\mathsf{Hom}_X(\mathcal{V}_1, \mathcal{W}_1) \ni h \mapsto g \circ h \circ f^{-1} \in \mathsf{Hom}_X(\mathcal{V}_2, \mathcal{W}_2)$.

Having this notation, the proof of the following lemma is straightforward.

Lemma 4.10. *Let X be a reduced Gorenstein curve, $D \subset \breve{X}$, $x \in D$ and $\omega \in \Omega_x$ as before. Let $f : \mathcal{V}_1 \to \mathcal{V}_2$ and $g : \mathcal{W}_1 \to \mathcal{W}_2$ be isomorphisms of vector bundles on X. Then the following diagram, in which $g(D)$ denotes $g \otimes 1$, is commutative:*

$$\begin{tikzcd}
\mathsf{Hom}_X\big(\mathcal{V}_1, \mathcal{W}_1(D)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V}_1,\mathcal{W}_1}(\omega)"] \arrow[d, "\mathrm{cnj}(f, g(D))"'] & \mathsf{Hom}_X\big(\mathcal{V}_1 \otimes \mathbb{C}_x, \mathcal{W}_1 \otimes \mathbb{C}_x\big) \arrow[d, "\mathrm{cnj}(\bar{f}, \bar{g})"] \\
\mathsf{Hom}_X\big(\mathcal{V}_2, \mathcal{W}_2(D)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V}_2,\mathcal{W}_2}(\omega)"] & \mathsf{Hom}_X\big(\mathcal{V}_2 \otimes \mathbb{C}_x, \mathcal{W}_2 \otimes \mathbb{C}_x\big).
\end{tikzcd}$$

4.2. **Evaluation map for vector bundles.** As in the previous subsection, we fix the following notation:

- a reduced Gorenstein analytic curve X (not necessarily compact);
- a subset $D \subset \breve{X}$, locally finite in X, and a smooth point $y \in \breve{X}$, $y \notin D$;
- a pair of vector bundles $\mathcal{V}$ and $\mathcal{W}$ on X.

Consider the short exact sequence

$$0 \to \mathcal{O}(-y) \to \mathcal{O} \xrightarrow{\mathrm{ev}_y} \mathbb{C}_y \to 0. \tag{18}$$

It induces a short exact sequence of coherent sheaves

$$0 \to \mathcal{W}(D) \otimes \mathcal{O}(-y) \to \mathcal{W}(D) \xrightarrow{1\otimes \underline{\mathrm{ev}}_y} \mathcal{W}(D) \otimes \mathbb{C}_y \to 0$$

and a morphism of sheaves $\underline{\mathrm{ev}}_y^{\mathcal{V},\mathcal{W}(D)}$ making the following diagram commutative

$$\begin{tikzcd}
\mathcal{H}om_X\big(\mathcal{V}, \mathcal{W}(D)\big) \arrow[r, "(1\otimes\underline{\mathrm{ev}}_y)_*"] \arrow[d, "\underline{\mathrm{ev}}_y^{\mathcal{V},\mathcal{W}(D)}"'] & \mathcal{H}om_X\big(\mathcal{V}, \mathcal{W}(D)\otimes\mathbb{C}_y\big) \\
\mathcal{H}om_X\big(\mathcal{V}\otimes\mathbb{C}_y, \mathcal{W}\otimes\mathbb{C}_y\big) \arrow[r, "(r_D)_*"] & \mathcal{H}om_X\big(\mathcal{V}\otimes\mathbb{C}_y, \mathcal{W}(D)\otimes\mathbb{C}_y\big) \arrow[u, "\cong"']
\end{tikzcd}$$

where the lower horizontal morphism $(r_D)_*$ is induced by the canonical inclusion $r_D : \mathcal{O} \to \mathcal{O}(D)$ (here we view both sheaves as subsheaves of the sheaf of meromorphic functions on X). Note that $(r_D)_*$ is an isomorphism because $y \notin D$ by assumption.

Definition 4.11. In the notation as above, we set:

$$\mathrm{ev}_y^{\mathcal{V},\mathcal{W}(D)} := H^0\big(\underline{\mathrm{ev}}_y^{\mathcal{V},\mathcal{W}(D)}\big) : \mathsf{Hom}_X\big(\mathcal{V}, \mathcal{W}(D)\big) \longrightarrow \mathsf{Hom}_X(\mathcal{V}\otimes\mathbb{C}_y, \mathcal{W}\otimes\mathbb{C}_y).$$

Similar to the case of the residue map, the following statements are true.

Proposition 4.12. *Let $\pi : Y \to X$ be a morphism of reduced Gorenstein curves, $x_1 \in \breve{X}$ a point over which π is unramified, $D_1 = \pi^{-1}(x_1)$ and $y_2 \in \breve{Y}$ a smooth point, such that $x_1 \neq x_2 = \pi(y_2) \in \breve{X}$. For a pair of vector bundles $\mathcal{V}$ and $\mathcal{W}$ on X we denote $\widetilde{\mathcal{V}} = \pi^*\mathcal{V}$ and $\widetilde{\mathcal{W}} = \pi^*\mathcal{W}$. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathsf{Hom}_X\big(\mathcal{V}, \mathcal{W}(x_1)\big) \arrow[r, "\mathrm{ev}_{x_2}^{\mathcal{V},\mathcal{W}(x_1)}"] \arrow[d, "\pi^*"'] & \mathsf{Hom}_X\big(\mathcal{V}\otimes\mathbb{C}_{x_2}, \mathcal{W}\otimes\mathbb{C}_{x_2}\big) \arrow[d, "\pi^*"] \\
\mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}(D_1)\big) \arrow[r, "\mathrm{ev}_{y_2}^{\widetilde{\mathcal{V}},\widetilde{\mathcal{W}}(D_1)}"] & \mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}\otimes\mathbb{C}_{y_2}, \widetilde{\mathcal{W}}\otimes\mathbb{C}_{y_2}\big).
\end{tikzcd}$$

Moreover, the constructed morphism of vector spaces $\mathrm{ev}_{y_2}^{\widetilde{\mathcal{V}},\widetilde{\mathcal{W}}(D_1)}$ is natural in $\widetilde{\mathcal{V}}$ and $\widetilde{\mathcal{W}}$. In particular, if $f : \widetilde{\mathcal{V}}_1 \to \widetilde{\mathcal{V}}_2$ and $g : \widetilde{\mathcal{W}}_1 \to \widetilde{\mathcal{W}}_2$ are isomorphisms of vector bundles on Y then the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}_1, \widetilde{\mathcal{W}}_1(D_1)\big) \arrow[r, "\mathrm{ev}_{y_2}^{\widetilde{\mathcal{V}}_1,\widetilde{\mathcal{W}}_1(D_1)}"] \arrow[d, "\mathrm{cnj}(f, g(D_1))"'] & \mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}_1\otimes\mathbb{C}_{y_2}, \widetilde{\mathcal{W}}_1\otimes\mathbb{C}_{y_2}\big) \arrow[d, "\mathrm{cnj}(\bar{f}, \bar{g})"] \\
\mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}_2, \widetilde{\mathcal{W}}_2(D_1)\big) \arrow[r, "\mathrm{ev}_{y_2}^{\widetilde{\mathcal{V}}_2,\widetilde{\mathcal{W}}_2(D_1)}"] & \mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}_2\otimes\mathbb{C}_{y_2}, \widetilde{\mathcal{W}}_2\otimes\mathbb{C}_{y_2}\big).
\end{tikzcd}$$

The proof of the following formula for the evaluation morphism ev_y is straightforward.

Lemma 4.13. *Let $U \subseteq \mathbb{C}$ be open, $D \subset U$ locally finite and $y \in U \setminus D$. Define $O(D) = H^0\big(U, \mathcal{O}_{\mathbb{C}}(D)\big)$ and let $\mathcal{V} = \mathcal{O}_U^n, \mathcal{W} = \mathcal{O}_U^m$. The formula $\mathrm{ev}_y(F) = F(y)$ defines a morphism $\mathrm{ev}_y : \mathsf{Mat}_{n\times n}\big(O(D)\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})$ which fits into the following commutative diagram*

$$\begin{tikzcd}
\mathsf{Hom}_U\big(\mathcal{V}, \mathcal{W}(D)\big) \arrow[r, "\mathrm{ev}_y^{\mathcal{V},\mathcal{W}(D)}"] & \mathsf{Hom}_U(\mathcal{V}\otimes\mathbb{C}_y, \mathcal{W}\otimes\mathbb{C}_y) \arrow[d, "\mathrm{can}"] \\
\mathsf{Mat}_{m\times n}\big(O(D)\big) \arrow[u, "\mathrm{can}"] \arrow[r, "\mathrm{ev}_y"] & \mathsf{Mat}_{m\times n}(\mathbb{C}).
\end{tikzcd}$$

In general, the evaluation morphism $\mathrm{ev}_{x_2}^{\mathcal{V},\mathcal{W}(x_1)}$ is neither injective nor surjective. However, there is one particular case, when it is an isomorphism.

Proposition 4.14. *Let E be a reduced projective curve with trivial dualizing sheaf. Let $\mathcal{V}, \mathcal{W}$ be vector bundles on E and $x, y \in \breve{E}$ be such that $x \neq y$ and*

$$\mathsf{Hom}_E\big(\mathcal{V}, \mathcal{W}(x-y)\big) = 0 = \mathsf{Ext}^1_E\big(\mathcal{V}, \mathcal{W}(x-y)\big).$$

Then the morphism $\mathrm{ev}_y^{\mathcal{V},\mathcal{W}(x)} : \mathsf{Hom}_E\big(\mathcal{V}, \mathcal{W}(x)\big) \longrightarrow \mathsf{Hom}_E(\mathcal{V}\otimes\mathbb{C}_y, \mathcal{W}\otimes\mathbb{C}_y)$ is an isomorphism of vector spaces.

Proof. Applying the functor $\mathsf{Hom}_E(\mathcal{V}, -)$ to the short exact sequence

$$0 \to \mathcal{W}(x-y) \to \mathcal{W}(x) \xrightarrow{1\otimes\underline{\mathrm{ev}}_y} \mathcal{W}(x)\otimes\mathbb{C}_y \to 0$$

we see that the morphism defined as the composition

$$\mathsf{Hom}_E\big(\mathcal{V}, \mathcal{W}(x)\big) \xrightarrow{(1\otimes\underline{\mathrm{ev}}_y)_*} \mathsf{Hom}_E\big(\mathcal{V}\otimes\mathbb{C}_y, \mathcal{W}(x)\otimes\mathbb{C}_y\big) \xrightarrow{(r_x)_*^{-1}} \mathsf{Hom}_E\big(\mathcal{V}\otimes\mathbb{C}_y, \mathcal{W}\otimes\mathbb{C}_y\big)$$

coincides with $\mathrm{ev}_y^{\mathcal{V},\mathcal{W}(x)}$ and is an isomorphism of vector spaces. $\square$

Remark 4.15. The vanishing conditions of Proposition 4.14 are satisfied if E is an irreducible Weierstraß projective curve and $\mathcal{V}$ and $\mathcal{W}$ are two stable vector bundles of the same rank and degree and such that $\mathcal{V} \not\cong \mathcal{W}(x-y)$.

4.3. **Geometric description of triple Massey products on genus one curves.** In this subsection, E is a reduced projective Gorenstein curve with trivial dualizing sheaf. In particular, the sheaf Ω_E of regular differential 1-forms on E is trivial. For any smooth point $x \in E$ consider the coboundary map $\delta_x : H^0(\mathbb{C}_x) \to H^1(\Omega_E)$ of the short exact sequence (17). This is an isomorphism. Define $w_x := \delta_x(1_x) \in H^1(\Omega_E)$. By a result of Kunz, which is also true without the assumption $\Omega_E \cong \mathcal{O}_E$, we get:

Theorem 4.16 (see Satz 4.1 in [41])**.** *The element w_x does not depend on x.*

Let $w = w_x$ and $t : H^1(\Omega_E) \to \mathbb{C}$ be the isomorphism which maps w to 1. We fix a global regular differential form $\omega : \mathcal{O}_E \to \Omega_E$. For any two perfect complexes $\mathcal{E}, \mathcal{F} \in \mathsf{Perf}(E)$, ω induces a non-degenerate pairing (see Proposition 3.5)

$$\langle\ ,\ \rangle^{\omega}_{\mathcal{E},\mathcal{F}} : \mathsf{Hom}_{D(E)}(\mathcal{E},\mathcal{F}) \otimes \mathsf{Hom}_{D(E)}(\mathcal{F},\mathcal{E}[1]) \longrightarrow \mathbb{C}.$$

Recall that, when passing from the triple Massey product $m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ to the tensor $\widetilde{m}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$, we have already used these bilinear forms.

The alternative description of $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ involves the two isomorphisms $\mathrm{res}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1}(\omega)$ and $\mathrm{ev}^{\mathcal{V}_1,\mathcal{V}_2(y_1)}_{y_2}$ constructed in Subsections 4.1 and 4.2 respectively. The following theorem (see also [56, Theorem 4]) is the key statement to explicitly compute the tensor $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ describing triple Massey products.

Theorem 4.17. *Let E be a reduced projective curve with trivial dualizing sheaf, $\mathcal{V}_1, \mathcal{V}_2 \in \mathsf{VB}(E)$, $y_1, y_2 \in \breve{E}$ and $\omega \in H^0(\Omega_E)$ be as at the beginning of Section 4. By ω_{y_1} we denote the germ of ω at $y_1 \in E$. Then the diagram*

$$\begin{tikzcd}[column sep=small] & \mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \arrow[dl, "\mathrm{res}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1}(\omega_{y_1})"'] \arrow[dr, "\mathrm{ev}^{\mathcal{V}_1,\mathcal{V}_2(y_1)}_{y_2}"] & \\ \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}) \arrow[rr, "\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}(\omega)"] & & \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}) \end{tikzcd}$$

is commutative.

An important message from this theorem is: *only* if ω_{y_1} is the germ of a *global* holomorphic 1-form $\omega \in H^0(\Omega_E)$, we can guarantee that $\tilde{r} = \mathrm{ev}_{y_2} \circ \big(\mathrm{res}_{y_1}(\omega_{y_1})\big)^{-1}$.

Since this result plays a crucial role in our approach to degeneration problems, we decided to give a detailed proof of this statement, stressing those points which are implicit in [56]. As a preparation, several technical lemmas have to be proven.

Lemma 4.18. *Let E be a reduced projective curve with trivial dualizing sheaf and $x \in \breve{E}$. Then we have an isomorphism of functors $\mathsf{VB}(E) \longrightarrow \mathsf{Vect}_{\mathbb{C}}$:*

$$T_x : \mathsf{Ext}^1_E(\mathbb{C}_x, -\,) \longrightarrow \mathsf{Hom}_E(\mathbb{C}_x, -\otimes \mathbb{C}_x)$$

Proof. Let $\mathcal{V}$ be a vector bundle on E of rank n. The short exact sequence (17) and the isomorphism $\omega : \mathcal{O}_C \xrightarrow{\sim} \Omega_E$ yield the short exact sequence $0 \to \mathcal{V} \to \mathcal{V}(x) \to \mathcal{V} \otimes \mathbb{C}_x \to 0$, which induces the long exact sequence

$$0 \to \mathsf{Hom}_E(\mathbb{C}_x, \mathcal{V}\otimes\mathbb{C}_x) \xrightarrow{\delta_x} \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}) \to \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}(x)) \to \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}\otimes\mathbb{C}_x) \to 0.$$

Because $\mathsf{Ext}^1_E\big(\mathbb{C}_x, \mathcal{V}(x)\big) \cong H^0\big(\mathcal{E}xt^1(\mathbb{C}_x, \mathcal{V}(x))\big)$ and $\mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V} \otimes \mathbb{C}_x)$ are both of dimension $n = \mathrm{rank}(\mathcal{V})$, we conclude that δ_x is an isomorphism. Moreover, this map is functorial and we can put $T_x = \delta_x^{-1}$. $\square$

Remark 4.19. Due to the construction of the functor T_x we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V} \arrow[r] \arrow[d, equal] & \mathcal{A} \arrow[r] \arrow[d] & \mathbb{C}_x \arrow[r] \arrow[d, "T_x(a)"] & 0 \\
0 \arrow[r] & \mathcal{V} \arrow[r] & \mathcal{V}(x) \arrow[r, "\mathrm{res}_x^{\mathcal{V}}"] & \mathcal{V} \otimes \mathbb{C}_x \arrow[r] & 0,
\end{tikzcd}$$

where the upper exact sequence corresponds to the element $a \in \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V})$.

In order to justify our calculations in Sections 8 and 10 we need to establish an explicit link between the "categorical trace map" of Proposition 3.5 and the usual trace from linear algebra.

Let X be a reduced projective Gorenstein curve, $x \in \breve{X}$ a smooth point, $\mathcal{V}$ a vector bundle on X. From the exact sequence (17) we get a commutative diagram

$$\begin{tikzcd}
\mathsf{Ext}^1_X(\mathcal{V}, \mathcal{V} \otimes \Omega_X) \arrow[r, "\mathrm{Tr}^1_{\mathcal{V}}"] & H^1(\Omega_X) \arrow[r, "t"] & \mathbb{C} \\
\mathsf{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathbb{C}_x) \arrow[u, "\delta_x"] \arrow[r, "\mathrm{Tr}^0_{\mathcal{V}}"] & H^0(\mathbb{C}_x) \arrow[u, "\delta_x"'] \arrow[ur, "\mathrm{can}"'] & \\
\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}|_x, \mathcal{V}|_x) \arrow[u] \arrow[r, "\mathrm{tr}"] & \mathbb{C} \arrow[u, "\mathrm{can}"'] \arrow[uur, bend right, "="'] &
\end{tikzcd}$$

Here t is the trace map from Theorem 4.16 and tr is the ordinary trace of an endomorphism of the vector space $\mathcal{V}|_x$. The morphism $\mathrm{Tr}^0_{\mathcal{V}}$ is the composition $\mathsf{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathbb{C}_x) \to H^0\big(\mathcal{V}^\vee \otimes \mathcal{V} \otimes \mathbb{C}_x\big) \to H^0(\mathbb{C}_x)$ and $\mathrm{Tr}^1_{\mathcal{V}}$ is defined in a similar way, so that $t \circ \mathrm{Tr}^1_{\mathcal{V}} = \mathrm{tr}_{\mathcal{V}}$, see Theorem 3.2. The commutativity of this diagram gives us the following result.

Lemma 4.20. *For an element $f \in \mathsf{Hom}_X(\mathcal{V}, \mathcal{V} \otimes \mathbb{C}_x)$ we have:*

$$t\Big(\mathrm{Tr}^1_{\mathcal{V}}\big(\delta_x(f)\big)\Big) = \mathrm{tr}(f_x),$$

which is the required link between the categorical trace and the usual trace for vector spaces.

Lemma 4.21. *Let E be a reduced projective curve with trivial dualizing sheaf, $x \in E$ a smooth point, $\mathcal{V} \in \mathsf{VB}(E)$ a vector bundle and $S : \mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}, \mathbb{C}_x)^*$ the isomorphism induced by the bilinear form $\langle\ ,\ \rangle^{\omega}_{\mathcal{V}, \mathbb{C}_x}$, defined in Proposition 3.5. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathsf{Ext}^1_E(\mathbb{C}_x, \mathcal{V}) \arrow[r, "S"] \arrow[d, "T_x"'] & \mathsf{Hom}_E(\mathcal{V}, \mathbb{C}_x)^* \arrow[d] \\
\mathsf{Hom}_E(\mathbb{C}_x, \mathcal{V} \otimes \mathbb{C}_x) \arrow[r, "\mathrm{tr}"] & \mathsf{Hom}_E(\mathcal{V} \otimes \mathbb{C}_x, \mathbb{C}_x)^*,
\end{tikzcd}$$

where tr *is induced by the canonical isomorphism of vector spaces* $\mathsf{Hom}_{\mathbb{C}}(U,V)^* \cong \mathsf{Hom}_{\mathbb{C}}(V,U)$*, which is given by the usual trace of endomorphisms.*

Proof. Let $\xi \in \mathsf{Hom}_E(\mathcal{V},\mathbb{C}_x)$ and $a \in \mathsf{Ext}^1_E(\mathbb{C}_x,\mathcal{V})$. Then $T_x(a) : \mathbb{C}_x \to \mathcal{V} \otimes \mathbb{C}_x$ and $\xi_x : \mathcal{V} \otimes \mathbb{C}_x \to \mathbb{C}_x$ satisfy:

$$\begin{aligned}\mathrm{tr}\big(\xi_x \circ T_x(a)\big) &= \mathrm{tr}\big(T_x(a) \circ \xi_x\big) = t\big(\mathrm{Tr}^1_{\mathcal{V}}\big(\delta_x(T_x(a)\circ\xi)\big)\big) = t\big(\mathrm{Tr}^1_{\mathcal{V}}\big(\omega_*(a\circ\xi)\big)\big)\\ &= t\big(\omega_*\big(\mathrm{Tr}_{\mathcal{V}}(a\circ\xi)\big)\big) = S(a)(\xi),\end{aligned}$$

where the second equality holds by Lemma 4.20 and the others come from straightforward commutative diagrams. □

Now, after proving these preliminary statements we are ready to prove Theorem 4.17. Let $(\mathcal{V}_1,\mathcal{V}_2,y_1,y_2,\omega)$ be the data fixed at the beginning of Section 4. Recall that we have to compare the triple Massey product

$$m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : \mathsf{Hom}_E(\mathcal{V}_1,\mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1},\mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2,\mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1,\mathbb{C}_{y_2})$$

with the map

$$\tilde{\tilde{r}}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} := \mathrm{ev}^{\mathcal{V}_1,\mathcal{V}_2(y_1)}_{y_2} \circ \big(\mathrm{res}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1}(\omega_{y_1})\big)^{-1} : \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1},\mathcal{V}_2|_{y_1}) \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2},\mathcal{V}_2|_{y_2}).$$

Proposition 4.22. *If* $g \in \mathsf{Hom}_E(\mathcal{V}_1,\mathbb{C}_{y_1})$*,* $h \in \mathsf{Hom}_E(\mathcal{V}_2,\mathbb{C}_{y_2})$ *and* $a \in \mathsf{Ext}^1_E(\mathbb{C}_{y_1},\mathcal{V}_2)$*, then*

$$h_{y_2} \circ \tilde{\tilde{r}}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}\big(T_{y_1}(a)g_{y_1}\big) = \big(m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}(g\otimes a\otimes h)\big)_{y_2}.$$

Proof. Let us first explain our notation. We have a composition map

$$\mathcal{V}_1|_{y_1} \xrightarrow{g_{y_1}} \mathbb{C} \xrightarrow{T_{y_1}(a)} \mathcal{V}_2|_{y_1},$$

hence we may consider

$$\mathcal{V}_1|_{y_2} \xrightarrow{\tilde{\tilde{r}}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}\left(T_{y_1}(a)g_{y_1}\right)} \mathcal{V}_2|_{y_2}.$$

Let $0 \to \mathcal{V}_2 \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathbb{C}_{y_1} \to 0$ be an exact sequence representing $a \in \mathsf{Ext}^1_E(\mathbb{C}_{y_1},\mathcal{V}_2)$. Then we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \mathcal{V}_2 \arrow[r, equal] \arrow[d, "\alpha"'] & \mathcal{V}_2 \arrow[d, "r"] \\
\mathcal{V}_1 \arrow[r, "\tilde{g}"] \arrow[d] \arrow[dr, "g"'] & \mathcal{A} \arrow[r, "\varepsilon"] \arrow[d, "\beta"'] & \mathcal{V}_2(y_1) \arrow[d, "\underline{\mathrm{res}}^{\mathcal{V}_2}_{y_1}"] \\
\mathcal{V}_1 \otimes \mathbb{C}_{y_1} \arrow[r, "g_{y_1}"'] & \mathbb{C}_{y_1} \arrow[r, "T_{y_1}(a)"'] \arrow[d] & \mathcal{V}_2 \otimes \mathbb{C}_{y_1} \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

where $\tilde{g}$ is the unique lift of g and the two columns on the right form a transposed version of the diagram from Remark 4.19. Since

$$\operatorname{res}_{y_1}^{\mathcal{V}_1,\mathcal{V}_2}(\omega_{y_1}) : \mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(y_1)\big) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1 \otimes \mathbb{C}_{y_1}, \mathcal{V}_2 \otimes \mathbb{C}_{y_1})$$

is an isomorphism, by definition we have (see Remark 4.3)

$$\big(\operatorname{res}_{y_1}^{\mathcal{V}_1,\mathcal{V}_2}(\omega_{y_1})\big)^{-1}\big(T_{y_1}(a)g_{y_1}\big) = \varepsilon\tilde{g}.$$

Moreover, tensoring the whole diagram with $\mathbb{C}_{y_2}$ we obtain a new commutative diagram

$$\begin{tikzcd}
 & \mathcal{V}_2 \otimes \mathbb{C}_{y_2} \arrow[r, "="] \arrow[d, "\alpha_{y_2}"'] & \mathcal{V}_2 \otimes \mathbb{C}_{y_2} \arrow[d, "r_{y_2}"] \\
\mathcal{V}_1 \otimes \mathbb{C}_{y_2} \arrow[r, "\tilde{g}_{y_2}"] & \mathcal{A} \otimes \mathbb{C}_{y_2} \arrow[r, "\varepsilon_{y_2}"] & \mathcal{V}_2(y_1) \otimes \mathbb{C}_{y_2},
\end{tikzcd}$$

from which the identity $\operatorname{ev}_{y_2}^{\mathcal{V}_1,\mathcal{V}_2(y_1)}(\varepsilon\tilde{g}) = \alpha_{y_2}^{-1}\tilde{g}_{y_2}$ follows. By the definition of Massey products we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V}_2 \arrow[r, "\alpha"] \arrow[d, "h"'] & \mathcal{A} \arrow[r] \arrow[dl, "\tilde{h}"'] & \mathbb{C}_{y_1} \arrow[r] & 0 \\
 & \mathbb{C}_{y_2} & \mathcal{V}_1 \arrow[l, "m_3(g\otimes a\otimes h)"] \arrow[u, "\tilde{g}"'] & &
\end{tikzcd}$$

which finally implies

$$\big(m_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}(g \otimes a \otimes h)\big)_{y_2} = h_{y_2} \circ \alpha_{y_2}^{-1} \circ \tilde{g}_{y_2} = h_{y_2} \circ \tilde{\tilde{r}}_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}(T_{y_1}(a)g_{y_1}).$$

□

Now we are ready to finish the proof of Theorem 4.17. Our goal is to keep track of the linear map $m_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ under a long chain of canonical isomorphisms. Let us do it step by step. Each linear map

$$m \in \mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}_E^1(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}), \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})\big)$$

corresponds to an element

$$n \in \mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1})^*, \mathsf{Lin}\big(\mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}), \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})\big)\big)$$

which is related to m by the formula

$$n\big(g \otimes S(a)\big)(h) = m(g \otimes a \otimes h),$$

where $S : \mathsf{Ext}_E^1(\mathbb{C}_{y_1}, \mathcal{V}_2) \to \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1})^*$ is given by the bilinear form $\langle\ ,\ \rangle_{\mathcal{V}_2,\mathbb{C}_{y_1}}^{\omega}$ from Proposition 3.5. By Lemma 4.21, the element $S(a) \in \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1})^*$ is mapped to $T_{y_1}(a) \in \mathsf{Hom}_{\mathbb{C}}(\mathbb{C}, \mathcal{V}_2|_{y_1})$ under the chain of isomorphisms

$$\mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1})^* \longrightarrow \mathsf{Hom}_E(\mathcal{V}_2 \otimes \mathbb{C}_{y_1}, \mathbb{C}_{y_1})^* \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_1}, \mathbb{C})^* \longrightarrow \mathsf{Hom}_{\mathbb{C}}(\mathbb{C}, \mathcal{V}_2|_{y_1}).$$

This implies that the linear map n corresponds to

$$l \in \mathsf{Lin}\Big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathbb{C}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathbb{C}, \mathcal{V}_2|_{y_1}), \mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_2}, \mathbb{C}), \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathbb{C})\big)\Big)$$

given by $l(g_{y_1} \otimes T_{y_1}(a))(h_{y_2}) = m(g \otimes a \otimes h)_{y_2}$. But since

$$\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathbb{C}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathbb{C}, \mathcal{V}_2|_{y_1}) \xrightarrow{\circ} \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1})$$

is an isomorphism and

$$\mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_2|_{y_2}, \mathbb{C}), \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathbb{C})\big) \cong \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2}),$$

we obtain a linear map

$$k \in \mathsf{Lin}\big(\mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_1}, \mathcal{V}_2|_{y_1}), \mathsf{Hom}_{\mathbb{C}}(\mathcal{V}_1|_{y_2}, \mathcal{V}_2|_{y_2})\big)$$

such that for any elements g, a and h the following diagram commutes

$$\begin{tikzcd}
\mathcal{V}_1|_{y_2} \arrow[r, "k\left(T_{y_1}(a) g_{y_1}\right)"] \arrow[dr, "m(g\otimes a\otimes h)_{y_2}"'] & \mathcal{V}_2|_{y_2} \arrow[d, "h_{y_2}"] \\
 & \mathbb{C}.
\end{tikzcd}$$

Using Proposition 4.22, we obtain with $m = m^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$ the identity

$$k = \tilde{\tilde{r}}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} = \mathrm{ev}_{y_2} \circ \mathrm{res}^{-1}_{y_1},$$

A tedious diagram chase shows that k is equal to $\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}$, which was defined directly after Remark 3.11. This completes the proof. $\square$

The following Theorem explains how triple Massey products on a genus one curve can be computed in a practical way.

Theorem 4.23. *Let E be a reduced projective curve with trivial dualizing sheaf, $x_1, x_2 \in \breve{E}$ be a pair of distinct smooth points lying on the same irreducible component of E. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be a pair of vector bundles on E satisfying both vanishing conditions from the beginning of Section 4. Let $\pi : Y \to E$ be the normalization morphism if E is singular or the universal covering $\mathbb{C} \to E = \mathbb{C}/\langle 1, \tau\rangle$ if E is smooth. Take a point y_2 on Y such that $\pi(y_2) = x_2$, let $y_1 \in D_1 = \pi^{-1}(x_1)$ and denote $\widetilde{\mathcal{V}}_i = \pi^*\mathcal{V}_i$ for $i = 1, 2$. Let $\omega \in H^0(\Omega_E)$ be a global regular differential form* *on E and $\tilde{\omega}$ be its (possibly meromorphic) lift on Y. Then the diagram*

$$\begin{tikzcd}
\mathsf{Hom}_E\big(\mathcal{V}_1 \otimes \mathbb{C}_{x_1}, \mathcal{V}_2 \otimes \mathbb{C}_{x_1}\big) \arrow[rr, "\pi^*"] \arrow[dd, "\tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{x_1,x_2}(\omega)"'] & & \mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}_1 \otimes \mathbb{C}_{y_1}, \widetilde{\mathcal{V}}_2 \otimes \mathbb{C}_{y_1}\big) \\
 & \mathsf{Hom}_E\big(\mathcal{V}_1, \mathcal{V}_2(x_1)\big) \arrow[r, "\pi^*"] \arrow[ul, "\mathrm{res}^{\mathcal{V}_1,\mathcal{V}_2}_{x_1}(\omega_{x_1})"'] \arrow[dl, "\mathrm{ev}^{\mathcal{V}_1,\mathcal{V}_2(x_1)}_{x_2}"] & \mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}_1, \widetilde{\mathcal{V}}_2(D_1)\big) \arrow[u, "\mathrm{res}^{\widetilde{\mathcal{V}}_1,\widetilde{\mathcal{V}}_2}_{y_1}(\tilde{\omega}_{y_1})"'] \arrow[d, "\mathrm{ev}^{\widetilde{\mathcal{V}}_1,\widetilde{\mathcal{V}}_2(D_1)}_{y_2}"] \\
\mathsf{Hom}_E\big(\mathcal{V}_1 \otimes \mathbb{C}_{x_2}, \mathcal{V}_2 \otimes \mathbb{C}_{x_2}\big) \arrow[rr, "\pi^*"] & & \mathsf{Hom}_Y\big(\widetilde{\mathcal{V}}_1 \otimes \mathbb{C}_{y_2}, \widetilde{\mathcal{V}}_2 \otimes \mathbb{C}_{y_2}\big)
\end{tikzcd}$$

is commutative. This shows in particular that the computation of triple Massey products on elliptic curves (resp. on singular genus one curves) can be expressed by computations on the universal covering (resp. on the normalization).

Proof. The left triangle is commutative by Theorem 4.17. The two squares are commutative by Propositions 4.8 and 4.12. □

5. A relative construction of geometric triple Massey products

Our next goal is to extend the definition of the morphism $r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}(\omega)$, constructed in the previous section, to genus one fibrations. We achieve this by generalizing the construction of Theorem 4.17 to the relative case. Throughout this section we work either in the category of locally Noetherian algebraic schemes over an algebraically closed field $\boldsymbol{k}$ of characteristic zero or in the category of complex analytic spaces.

5.1. **The relative residue map.** Let $p : X \longrightarrow S$ be a *smooth* map of complex analytic spaces or of algebraic schemes. Assume p has a section $\imath : S \longrightarrow X$, let D be the image of $\imath$ equipped with the ringed space structure induced from S. Recall that the sheaf of relative differentials $\Omega^1_{X/S}$ is defined via the exact sequence

$$p^*\Omega^1_S \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/S} \longrightarrow 0,$$

see [3, Chapter 7], [34, Section II.8 and Section III.10] and [54] for definitions and basic properties of smooth morphisms and Kähler differential forms. In particular, for any closed point $s \in S$ we have: $\Omega^1_{X/S}|_{X_s} \cong \Omega^1_{X_s}$ and $\Omega^1_{X/S}$ is locally free.

Assume additionally that p has relative dimension one and X itself is smooth. Our aim is to define a canonical epimorphism of $\mathcal{O}_X$–modules

$$\underline{\mathrm{res}}_D : \Omega^1_{X/S}(D) \longrightarrow \mathcal{O}_D,$$

later called *the residue map.* We shall explain our construction in the case of algebraic schemes, whereas its generalisation on the case of complex analytic spaces is straightforward.

Let $x \in D \subset X$ be a closed point, then we can find affine neighbourhoods $U = \mathsf{Spec}(B)$ of $x \in X$ and $V = \mathsf{Spec}(A)$ of $f(x) \in S$ such that the map $p|_U : U \to V$ is induced by a ring homomorphism $p^* : A \to B$:

$$\begin{tikzcd} U = \mathsf{Spec}(B) \arrow[r, hook] \arrow[d, "p|_U"'] & X \arrow[d, "p"] \\ V = \mathsf{Spec}(A) \arrow[r, hook] & S. \end{tikzcd}$$

Then the sheaf $\Omega^1_{X/S}|_U$ is isomorphic to the sheafification of the B–module of Kähler differentials $\Omega_{B/A}$.

Let $\imath^* : B \to A$ be the ring homomorphism corresponding to the section $\imath$ and $I = \ker(\imath^*)$. Then the map $C := B/I \xrightarrow{\imath^*} A$ is an isomorphism and I is the ideal, locally defining the subscheme D. By Krull's Hauptidealsatz, since U is smooth and $V(I) \subset U$ has codimension one, shrinking the open sets U and V if necessary, we can achieve that I is generated by a single element $a \in B$. From the exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0$$

where $\delta([b]) = d(b) \otimes 1$ and the fact that $\Omega_{C/A} = 0$ it follows that the C–module $\Omega_{B/A} \otimes_B C$ is generated by a single element, namely $d(a) \otimes 1$. The smoothness of p implies that $\Omega_{B/A} \otimes_B C$ is a free C–module with this generator.

Definition 5.1. Let $p : X \longrightarrow S$ be a smooth map of relative dimension one, $\imath : S \longrightarrow X$ a section of p and $D = \imath(S)$. We define the sheaf homomorphism

$$\underline{\mathrm{res}}_D : \Omega^1_{X/S}(D) \longrightarrow \mathcal{O}_D$$

to be the composition of the canonical map $\Omega^1_{X/S}(D) \longrightarrow \Omega^1_{X/S}|_D \otimes \mathcal{O}(D)|_D$ and the morphism $\Omega^1_{X/S}|_D \otimes \mathcal{O}(D)|_D \to \mathcal{O}_D$ locally defined as follows. In the above notation let $M = \left\{ \frac{u}{a} \;\middle|\; u \in B \right\} = \Gamma\big(U, \mathcal{O}_X(D)\big) \subset Q(B)$, where $Q(B)$ is the field of fractions of the integral domain B. The map

$$\rho : (\Omega_{B/A} \otimes_B C) \otimes (M \otimes_B C) \longrightarrow C$$

is given by the formula $\big(d(a) \otimes 1\big) \otimes \big(\frac{u}{a} \otimes 1\big) \mapsto u \otimes 1 = \overline{u} := u \mod I$.

It is easy to see that the morphism ρ is C-linear, surjective and does not depend on the choice of a generator of the ideal I.

Remark 5.2. If S is a point and X a smooth complex curve, the residue map in Definition 5.1 coincides with the classical residue map, which was used in sequence (17) at the beginning of subsection 4.1. To see this, we let $D = \{x\}$ and U a neighbourhood of x in X with a coordinate z centred at x. Then, in the notation of Definition 5.1, $a = z$ and $\frac{f(z)}{z}dz \in \Omega^1_X(x)$ is first sent to $\big(dz \otimes 1\big) \otimes \big(\frac{f(z)}{z} \otimes 1\big)$ and then to $f \mod I_x = f(0)$, which is equal to the ordinary residue of $\frac{f(z)}{z}dz$.

Proposition 5.3. *Let $p : X \longrightarrow S$ be a smooth map as above, $\imath : S \longrightarrow X$ a section of p and $g : S' \longrightarrow S$ any morphism. Let $X' = X \times_S S'$ and $\imath' : S' \longrightarrow X'$ be the section defined by the universal property of pull-backs:*

$$
\begin{tikzcd}
S' \arrow[drr, bend left, "\imath g"] \arrow[dr, dotted, "\imath'"] \arrow[ddr, bend right, "="'] & & \\
& X' \arrow[r, "f"] \arrow[d, "p'"] & X \arrow[d, "p"'] \\
& S' \arrow[r, "g"] & S \arrow[u, bend right, "\imath"']
\end{tikzcd}
$$

and $D' = \imath'(S')$. Then the following diagram is commutative:

$$
\begin{tikzcd}
f^*\big(\Omega^1_{X/S}(D)\big) \arrow[r, "f^*(\underline{\mathrm{res}}_D)"] \arrow[d, "\cong"'] & f^*(\mathcal{O}_D) \arrow[d, "\cong"] \\
\Omega^1_{X'/S'}(D') \arrow[r, "\underline{\mathrm{res}}_{D'}"] & \mathcal{O}_{D'}
\end{tikzcd}
$$

where the vertical arrows are canonical isomorphisms.

Proof. The problem is local, so we can assume, without loss of generality, $X = \mathsf{Spec}(B), X' = \mathsf{Spec}(B'), S = \mathsf{Spec}(A)$ and $S' = \mathsf{Spec}(A')$. Then, we have a Cartesian diagram of rings and ring homomorphisms

$$
\begin{tikzcd}
B' \arrow[d, bend right, "\imath'^*"'] & B \arrow[l, "f^*"'] \arrow[d, bend left, "\imath^*"] \\
A' \arrow[u, "p'^*"'] & A \arrow[l, "g^*"] \arrow[u, "p^*"]
\end{tikzcd}
$$

where $B' = B \otimes_A A'$, $p'^*(a') = 1 \otimes a'$ and $f^*(b) = b \otimes 1$. Denote $C := B/\ker(\imath^*)$ and $C' := B'/\ker(\imath'^*)$ then we have an isomorphism of C'–modules $C \otimes_B B' \longrightarrow C'$.

Let $d : B \to \Omega_{B/A}$ and $d' : B' \to \Omega_{B'/A'}$ be the universal derivations from the definition of Kähler differentials. By the universal property we obtain a uniquely determined B–module homomorphism $\Omega_{B/A} \longrightarrow \Omega_{B'/A'}$ and an induced B'–module isomorphism $\tilde{f}^* : \Omega_{B/A} \otimes_B B' \longrightarrow \Omega_{B'/A'}$ making the following diagram

$$
\begin{tikzcd}
B \arrow[r, "d"] \arrow[d, "f^*"'] & \Omega_{B/A} \arrow[r] \arrow[d, dotted] & \Omega_{B/A} \otimes_B B' \arrow[dl, dotted, "\tilde{f}^*"] \\
B' \arrow[r, "d'"] & \Omega_{B'/A'} &
\end{tikzcd}
$$

commutative, in particular $\tilde{f}^*\big(d(b)\otimes 1\big) = d'\big(f^*(b)\big)$. Moreover, we have a canonical homomorphism $f_M^* : M\otimes_B B' \longrightarrow M'$, given by $f_M^*\Big(\dfrac{u}{a}\otimes 1\Big) = \dfrac{f^*(u)}{f^*(a)}$, where

$$M = \left\{ \frac{u}{a} \,\middle|\, u \in B \right\} \subset Q(B) \quad \text{and} \quad M' = \left\{ \frac{v}{f^*(a)} \,\middle|\, v \in B' \right\} \subset Q(B').$$

We know that the C–module $\Omega_{B/A}\otimes C$ is generated by the single element $d(a)\otimes 1$. Hence the commutativity of the diagram

$$\begin{tikzcd}
\big(\Omega_{B/A}\otimes_B B'\big)\otimes\big(M\otimes_B B'\big) \arrow[r, "f^*(\underline{\mathrm{res}}_D)"] \arrow[d, "\tilde{f}^*\otimes f_M^*"'] & C\otimes_B B' \arrow[d] \\
\Omega_{B'/A'}\otimes M' \arrow[r, "\underline{\mathrm{res}}_{D'}"] & C'
\end{tikzcd}$$

can be checked on elements of the following form:

$$\begin{tikzcd}
\big(d(a)\otimes 1\big)\otimes\big(\frac{u}{a}\otimes 1\big) \arrow[r, mapsto, "f^*(\underline{\mathrm{res}}_D)"] \arrow[d, mapsto, "\tilde{f}^*\otimes f_M^*"'] & \bar{u}\otimes 1 \arrow[d, mapsto] \\
d'\big(f^*(a)\big)\otimes\dfrac{f^*(u)}{f^*(a)} \arrow[r, mapsto, "\underline{\mathrm{res}}_{D'}"] & \overline{f^*(u)}.
\end{tikzcd}$$

and the proposition is proven. □

5.2. **On the sheaf of relative differential forms of a Gorenstein fibration.** Let $p: X \longrightarrow S$ be a *proper* and *flat* morphism of relative dimension *one*, either in the category of complex analytic spaces or of algebraic schemes over an algebraically closed field $\boldsymbol{k}$ of characteristic zero. Assume additionally that for all closed points $s \in S$ the fibres X_s are reduced and we have an embedding

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

where $q: Y \longrightarrow S$ is a *proper* and *smooth* morphism of the relative dimension *two*.

Remark 5.4. Since for any $s \in S$ the surface Y_s is smooth and $X_s \subset Y_s$ has codimension one, the curve X_s has hypersurface singularities and is in particular Gorenstein.

Recall that for any morphism q we have an exact sequence

$$q^*\Omega^1_S \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_{Y/S} \longrightarrow 0$$

and that $\Omega^1_{Y/S}$ is a locally free $\mathcal{O}_Y$–module of rank two, because q is a smooth morphism.

Definition 5.5. The relative dualizing sheaf is defined by the formula

$$\omega_{X/S} := \bigl(\bigwedge^2 \Omega^1_{Y/S} \otimes \mathcal{O}_Y(X)\bigr)\big|_X .$$

Proposition 5.6 (see Chapter II in [6])**.** *For any $s \in S$ the sheaf $\omega_{X/S}|_{X_s}$ is the dualising sheaf of the projective curve X_s.*

Remark 5.7. It can be shown that up to the pull-back of a line bundle on S this definition of $\omega_{X/S}$ does not depend on the embedding $X \hookrightarrow Y$.

Let $\breve{X}$ be the regular locus of p, then $\jmath : \breve{X} \longrightarrow X$ is an open embedding and the morphism $\breve{X} \xrightarrow{p} S$ is flat but in general not proper. Our aim is to define an injective map of $\mathcal{O}_X$–modules $\underline{\mathrm{cl}}_S : \omega_{X/S} \longrightarrow \jmath_*(\Omega^1_{\breve{X}/S})$.

For a closed point $x \in \breve{X}$ let $U \subset Y$ be an open neighbourhood of x and S_0 an open neighbourhood of $f(x)$ in S. Choose local coordinates (u, v, s) on U such that we have a commutative diagram

$$\begin{array}{ccc} \bigl(\mathbb{C}^2 \times S_0\bigr) = U & \longrightarrow & Y \\ \Big\downarrow{\scriptstyle \mathrm{pr}} & & \Big\downarrow{\scriptstyle q} \\ S_0 & \longrightarrow & S \end{array}$$

where $\mathrm{pr}(u, v, s) = s$ and $du|_{\breve{X}} \not\equiv 0$, $dv|_{\breve{X}} \not\equiv 0$. Assume that the closed subset $X \cap U$ is given in U by an equation $f(u, v, s) = 0$. Then

$$\left(\frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv\right)\bigg|_{\breve{X}} = 0.$$

where the left-hand side of this equality is viewed as a local section of $\Omega^1_{\breve{X}/S}$.

Consider the composition map $\ell : \breve{X} \xrightarrow{\jmath} X \longrightarrow Y$.

Definition 5.8 (see Section II.1 in [6])**.** The Poincaré residue map is the morphism of $\mathcal{O}_Y$–modules

$$\underline{\mathrm{res}}^P : \wedge^2 \Omega^1_{Y/S}(X) \longrightarrow \ell_* \Omega^1_{\breve{X}/S}$$

locally defined as follows. Let $U \subseteq Y$ be an open neighbourhood of $x \in \breve{X}$ as above and $V := U \cap \breve{X}$, then the map

$$\mathrm{res}^P : \Gamma\bigl(U, \wedge^2 \Omega^1_{Y/S}(X)\bigr) \longrightarrow \Gamma\bigl(V, \Omega^1_{\breve{X}/S}\bigr) = \Gamma\bigl(U, \ell_* \Omega^1_{\breve{X}/S}\bigr)$$

is given by the formula

$$\frac{h\,du \wedge dv}{f} \mapsto \begin{cases} \left.\dfrac{hdu}{\partial_v f}\right|_V & \text{if } \dfrac{\partial f}{\partial v}(u,v,s) \neq 0, \\ -\left.\dfrac{hdv}{\partial_u f}\right|_V & \text{if } \dfrac{\partial f}{\partial u}(u,v,s) \neq 0. \end{cases}$$

Remark 5.9. Since for any point $s \in S$ the fibre $\breve{X}_s$ is a smooth curve, the set $V(f, \partial_u f, \partial_v f) \subseteq \breve{X}_s$ is empty and the map res^P is well-defined. Moreover, $\underline{\mathrm{res}}^P$ is independent of the choice of a local equation $f \in \mathcal{O}_Y(U)$ for $X \subset Y$ and also of the choice of local coordinates (u, v, s) on Y, see for example [6, Section II.1].

From what was said above it follows:

Corollary 5.10. *The commutative diagram of $\mathcal{O}_Y$–modules*

$$\begin{tikzcd} 0 \arrow[r] & \wedge^2\Omega^1_{Y/S} \arrow[r] \arrow[dr, "0"'] & \wedge^2\Omega^1_{Y/S}(X) \arrow[r] \arrow[d, "\underline{\mathrm{res}}^P"] & \wedge^2\Omega^1_{Y/S}(X)\big|_X \arrow[r] \arrow[dl, dotted] & 0 \\ & & \ell_*\Omega^1_{\breve{X}/S} & & \end{tikzcd}$$

induces an injective morphism of $\mathcal{O}_X$–modules

$$\underline{\mathrm{cl}}_S : \omega_{X/S} = \wedge^2\Omega^1_{Y/S}(X)\big|_X \longrightarrow \jmath_*\Omega^1_{\breve{X}/S}.$$

Remark 5.11. In what follows the morphism $\underline{\mathrm{cl}}_S$ will be called the *class map.* For a Gorenstein projective variety X of dimension n let $\mathcal{M}_X$ denote the sheaf of meromorphic functions on X. Angéniol and Lejeune-Jalabert construct a morphism $\Omega^n_X \longrightarrow \omega_X$ which induces an isomorphism $\Omega^n_X \otimes \mathcal{M}_X \xrightarrow{\cong} \omega_X \otimes \mathcal{M}_X$, also called "class map", see [4]. The relationship between this class map and the class map from Corollary 5.10 will be discussed elsewhere.

The following proposition can be shown on the lines of [6, Section II.1].

Proposition 5.12. *Let $p : X \longrightarrow S$ be a Gorenstein fibration of relative dimension one satisfying the conditions from the beginning of this subsection. If $g : S' \longrightarrow S$ is any base change, we obtain the Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ S' \arrow[r, "g"] & S. \end{tikzcd}$$

Then, the following diagram is commutative

$$\begin{tikzcd}
f^*\left(\wedge^2\Omega^1_{Y/S}(X)\Big|_X\right) \arrow[r, "\cong"] \arrow[d, "f^*(\underline{\mathrm{cl}}_S)"'] & \wedge^2\Omega^1_{Y'/S'}(X')\Big|_{X'} \arrow[d, "\underline{\mathrm{cl}}_{S'}"] \\
f^*(\jmath_*\Omega^1_{\breve{X}/S}) \arrow[r, "\cong"] & \jmath'_*\Omega^1_{\breve{X}'/S'}
\end{tikzcd}$$

where the upper horizontal isomorphism is canonical and the lower one is induced by the base-change property.

The reason to introduce the map $\underline{\mathrm{cl}}_S$ is explained by the following proposition.

Proposition 5.13 (see Proposition 6.2 in [6]). *Let $p : X \longrightarrow S$ be as in Proposition 5.12, $t \in S$ a closed point and $\underline{\mathrm{cl}}_t : \omega_{X_t} \longrightarrow \jmath_{t*}\Omega^1_{\breve{X}_t}$ the class map constructed in Corollary 5.10. Then we have:*

(1) *If the fibre X_t is smooth, then the image of $\underline{\mathrm{cl}}_t$ is the sheaf $\Omega^1_{X_t}$ of holomorphic differential one-forms on X_t.*
(2) *In the case X_t is singular, the image of $\underline{\mathrm{cl}}_t$ is the sheaf of Rosenlicht's differential forms, see Definition 3.4. In particular,* $\mathrm{im}(\underline{\mathrm{cl}}_t)$ *is a subsheaf of the sheaf of* meromorphic *differential one-forms on X_t regular at smooth points of X_t.*

The following definition is central for our construction of associative geometric r–matrices. Let $p : X \longrightarrow S$ be flat and proper morphism such that
• All fibres X_t, $t \in S$ are reduced projective Gorenstein curves.
• There exists an embedding

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

where $q : Y \longrightarrow S$ is a proper and smooth morphism of relative dimension two.

Definition 5.14. Let $\jmath : \breve{X} \longrightarrow X$ be the inclusion of the smooth locus of p, $\imath : S \longrightarrow \breve{X}$ a section of p and $D = \imath(S)$. Then the residue map

$$\underline{\mathrm{res}}_D : \omega_{X/S}(D) \xrightarrow{\underline{\mathrm{cl}}_S} \jmath_*\big(\Omega^1_{\breve{X}/S}(D)\big) \longrightarrow \mathcal{O}_D$$

is defined as the composition of the class map $\underline{\mathrm{cl}}_S$ from Corollary 5.10 and the residue map for smooth morphisms of relative dimension one from Definition 5.1.

Remark 5.15. If S is a point, X a complex curve and $D = \{x\}$, the residue map in Definition 5.14 fits as the top horizontal arrow into the following commutative diagram

$$\begin{tikzcd}
\omega_X(x) \arrow[r, "\mathrm{res}_x"] \arrow[d, "\mathrm{cl}_x"] & \mathbb{C}_x \arrow[d, equal] \\
\Omega_X(x) \arrow[r, "\mathrm{res}_x"] & \mathbb{C}_x
\end{tikzcd}$$

in which the lower horizontal map is the classical residue map used in the sequence (17) at the beginning of subsection 4.1. This follows from Remark 5.2 and Proposition 5.13.

Propositions 5.3 and 5.12 imply the following corollary.

Proposition 5.16. *Let $p : X \longrightarrow S$ and $\imath : S \longrightarrow X$ be as in Definition 5.14 and $g : S' \longrightarrow S$ be any base change. Denote $X' = X \times_S S'$, $f : X' \longrightarrow X$, $\imath' : S' \longrightarrow X'$ the pull-back of $\imath$ and $D' = \imath'(S')$. Then the following diagram is commutative*

$$\begin{tikzcd}
f^*\big(\omega_{X/S}(D)\big) \arrow[r, "f^*(\mathrm{res}_D)"] \arrow[d, "\cong"'] & f^*(\mathcal{O}_D) \arrow[d, "\cong"] \\
\omega_{X'/S'}(D') \arrow[r, "\mathrm{res}_{D'}"] & \mathcal{O}_{D'}
\end{tikzcd}$$

where the vertical maps are canonical isomorphisms.

5.3. **Geometric triple Massey products.** Let $E \xrightarrow{p} S$ be a genus one fibration embedded into a smooth fibration of surfaces, i.e we have a commutative diagram

$$\begin{tikzcd}
E \arrow[rr, hook] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\
& S &
\end{tikzcd}$$

where p is a proper and flat map, for all $t \in S$ the fibre E_t is a reduced projective curve with trivial dualizing sheaf and q is a smooth and proper map of relative dimension two.

Let $\breve{E}$ be the regular locus of p. Assume S is chosen sufficiently small, so that $\omega_{E/S} \cong \mathcal{O}_E$. Fix the following data:

- A nowhere vanishing global section $\omega \in H^0(\omega_{E/S})$.
- Two holomorphic vector bundles $\mathcal{V}$ and $\mathcal{W}$ on the total space E having the same rank and such that for all $t \in S$ we have:

$$\mathsf{Hom}_{E_t}\big(\mathcal{V}_t, \mathcal{W}_t\big) = 0 = \mathsf{Ext}^1_{E_t}\big(\mathcal{V}_t, \mathcal{W}_t\big).$$

  Here and in the sequel we denote $\mathcal{F}_t = \mathcal{F}|_{E_t}$ for any vector bundle $\mathcal{F}$ on E.

- Two sections $h_1, h_2 : S \longrightarrow \breve{E}$ of p such that for all $t \in S$ we have: $h_1(t) \neq h_2(t)$ and $h_1(t), h_2(t)$ belong to the same irreducible component of E_t. We additionally assume that
$$\mathsf{Hom}_{E_t}\big(\mathcal{V}_t(h_2(t)), \mathcal{W}_t(h_1(t))\big) = 0 = \mathsf{Ext}^1_{E_t}\big(\mathcal{V}_t(h_2(t)), \mathcal{W}_t(h_1(t))\big).$$

The main result of this section is the following theorem.

Theorem 5.17. *There exists an isomorphism of vector bundles on S*
$$\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}(\omega) = \tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2} : \quad h_1^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \longrightarrow h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W})$$
such that for any base-change diagram
$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$
the following diagram is commutative:
$$\begin{tikzcd} g^*h_1^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \arrow[r, "g^*\left(\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}(\omega)\right)"] \arrow[d, "\cong"'] & g^*h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \arrow[d, "\cong"] \\ h_1'^*\mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W}) \arrow[r, "\tilde{r}^{f^*\mathcal{V},f^*\mathcal{W}}_{h_1',h_2'}(\omega')"] & h_2'^*\mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W}) \end{tikzcd}$$
where $h_1', h_2' : S' \longrightarrow E'$ are sections of p' obtained as pull-backs of h_1 and h_2. The vertical arrows are canonical isomorphisms and the section $\omega' \in H^0(\omega_{E'/S'})$ is defined via the commutative diagram
$$\begin{tikzcd} f^*\mathcal{O}_E \arrow[r, "\cong"] \arrow[d, "f^*(\omega)"'] & \mathcal{O}_{E'} \arrow[d, "\omega'"] \\ f^*\omega_{E/S} \arrow[r, "\cong"] & \omega_{E'/S'}. \end{tikzcd}$$
Moreover, for any $s \in S$ the morphism $\tilde{r}^{\mathcal{V}_s,\mathcal{W}_s}_{h_1(s),h_2(s)}$ coincides with the morphism describing triple Massey products constructed in Section 3.

Proof. The construction of the morphism $\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}$ is the following. Let $D_i = h_i(S)$ and $D_i' = h_i'(S)$, then the exact sequence

$$0 \longrightarrow \omega_{E/S} \longrightarrow \omega_{E/S}(D_1) \xrightarrow{\mathrm{res}_{D_1}} \mathcal{O}_{D_1} \longrightarrow 0 \tag{19}$$

induces an exact sequence $0 \longrightarrow \mathcal{W}\otimes\omega_{E/S} \longrightarrow \mathcal{W}\otimes\omega_{E/S}(D_1) \longrightarrow \mathcal{W}\otimes\mathcal{O}_{D_1} \longrightarrow 0$. Since $\mathcal{E}xt^1_E(\mathcal{V},\mathcal{W}) = 0$ and $\omega_{E/S} \cong \mathcal{O}_E$, applying the functor $\mathcal{H}om_E(\mathcal{V}, -\,)$ we obtain the exact sequence

$$0 \to \mathcal{H}om_E(\mathcal{V},\mathcal{W}) \to \mathcal{H}om_E(\mathcal{V},\mathcal{W}\otimes\omega_{E/S}(D_1)) \to \mathcal{H}om_E(\mathcal{V},\mathcal{W}\otimes\mathcal{O}_{D_1}) \to 0. \tag{20}$$

Lemma 5.18. *In the notations of the theorem we have:*

$$p_*\big(\mathcal{H}om_E(\mathcal{V},\mathcal{W})\big) \cong R^1p_*\big(\mathcal{H}om_E(\mathcal{V},\mathcal{W})\big) = 0.$$

Proof of the lemma. It suffices to show that $\mathbb{R}p_*\big(\mathcal{H}om_E(\mathcal{V},\mathcal{W})\big) = 0$ viewed as an object of the derived category of coherent sheaves $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(S)$. Note that a complex $\mathcal{F} \in \mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(S)$ is zero if and only if for all points $t \in S$ we have: $\mathcal{F} \overset{\mathbb{L}}{\otimes} \mathbb{C}_t \cong 0$. Since the morphism p is flat, from a base-change isomorphism it follows that

$$\mathbb{R}p_*\big(\mathcal{H}om_E(\mathcal{V},\mathcal{W})\big) \overset{\mathbb{L}}{\otimes} \mathbb{C}_t \cong \mathsf{RHom}_{E_t}(\mathcal{V}_t,\mathcal{W}_t) \cong 0,$$

where the last equality follows from the assumption $\mathsf{Ext}^i_{E_t}(\mathcal{V}_t,\mathcal{W}_t) = 0$ for all $i \in \mathbb{Z}$ and $t \in S$. □

Hence, applying the left-exact functor p_* to the exact Sequence (20) we obtain an isomorphism $p_*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\otimes\omega_{E/S}(D_1)\big) \xrightarrow{\cong} p_*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\otimes\mathcal{O}_{D_1}\big)$. Combining it with the canonical isomorphisms

$$\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\otimes\mathcal{O}_{D_1}\big) \xrightarrow{\cong} h_{1*}\mathcal{H}om_S\big(h_1^*\mathcal{V},h_1^*\mathcal{W}\big) \xrightarrow{\cong} h_{1*}h_1^*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\big)$$

we obtain an isomorphism

$$\underline{\mathrm{res}}^{\mathcal{V},\mathcal{W}}_{h_1} :\ p_*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\otimes\omega_{E/S}(D_1)\big) \xrightarrow{\cong} h_1^*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\big).$$

Moreover, the choice of a global section $\mathcal{O}_E \xrightarrow{\omega} \omega_{E/S}$ induces an isomorphism

$$\underline{\mathrm{res}}^{\mathcal{V},\mathcal{W}}_{h_1}(\omega) :\ p_*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}(D_1)\big) \xrightarrow{\cong} h_1^*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\big),$$

which we shall frequently denote by $\underline{\mathrm{res}}^{\mathcal{V},\mathcal{W}}_{h_1}$.

Remark 5.19. If S is a point, E a curve and $D_1 = \{x\}$, this isomorphism coincides with the map $\mathrm{res}^{\mathcal{V},\mathcal{W}}_x(\omega)$ from Definition 4.4, if we identify $\mathsf{Hom}_E(\mathcal{V}\otimes\mathbb{C}_x,\mathcal{W}\otimes\mathbb{C}_x)$ with the vector space $\mathcal{H}om_E(\mathcal{V},\mathcal{W})|_x$. This follows from Remarks 4.3, 5.2 and 5.15 by comparing the two constructions.

The construction of another isomorphism

$$\underline{\mathrm{ev}}^{\mathcal{V},\mathcal{W}(D_1)}_{h_2} :\ p_*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}(D_1)\big) \xrightarrow{\cong} h_2^*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\big)$$

is similar. We start with the exact sequence

$$0 \longrightarrow \mathcal{O}_E(D_1-D_2) \longrightarrow \mathcal{O}_E(D_1) \longrightarrow \mathcal{O}_E(D_1)\otimes\mathcal{O}_{D_2} \longrightarrow 0. \tag{21}$$

For any Weil divisor $D \subset \breve{E}$ we view the line bundle $\mathcal{O}_E(D)$ as a subsheaf of the sheaf of meromorphic functions $\mathcal{M}_E$. There exists a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_E(D_1) \xrightarrow{\mathrm{ev}_{D_1}} \mathcal{O}_{D_1}(D_1) \longrightarrow 0$$

inducing an isomorphism $\mathcal{O}_{D_2} \longrightarrow \mathcal{O}_E(D_1) \otimes \mathcal{O}_{D_2}$. Tensoring the exact sequence (21) with the vector bundle $\mathcal{W}$ and applying $\mathcal{H}om_E(\mathcal{V}, -)$ we obtain an exact sequence

$$0 \to \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1 - D_2)\big) \to \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1)\big) \to \mathcal{H}om_E(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_2}) \to 0.$$

By the same argument as in Lemma 5.18 one can show that

$$p_* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1 - D_2)\big) \cong R^1 p_* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1 - D_2)\big) = 0$$

which implies that we obtain an isomorphism of vector bundles on S

$$\underline{\mathrm{ev}}_{h_2}^{\mathcal{V},\mathcal{W}(D_1)} : \; p_* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1)\big) \xrightarrow{\cong} h_2^* \mathcal{H}om_E(\mathcal{V}, \mathcal{W}).$$

Remark 5.20. If S is a point, E a curve, $D_1 = \{x\}$ and $D_2 = \{y\}$, this isomorphism coincides with the map $\mathrm{ev}_y^{\mathcal{V},\mathcal{W}(x)}$ from Definition 4.11. This follows easily by comparing the two constructions, using the canonical identification of $\mathsf{Hom}_E(\mathcal{V} \otimes \mathbb{C}_x, \mathcal{W} \otimes \mathbb{C}_x)$ with $\mathcal{H}om_E(\mathcal{V}, \mathcal{W})|_x$.

The isomorphism of vector bundles

$$\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}} : \; h_1^* \mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \longrightarrow h_2^* \mathcal{H}om_E(\mathcal{V}, \mathcal{W})$$

is defined by the commutative diagram of vector bundles on S

$$\begin{tikzcd}[column sep=small] & p_* \mathcal{H}om_E\big(\mathcal{V}, \mathcal{W}(D_1)\big) \arrow[dl, "\underline{\mathrm{res}}_{h_1}^{\mathcal{V},\mathcal{W}}(\omega)"'] \arrow[dr, "\underline{\mathrm{ev}}_{h_2}^{\mathcal{V},\mathcal{W}(D_1)}"] & \\ h_1^* \mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \arrow[rr, "\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}(\omega)"] & & h_2^* \mathcal{H}om_E(\mathcal{V}, \mathcal{W}). \end{tikzcd}$$

Remark 5.21. If we drop the assumption that

$$\mathsf{Hom}_{E_t}\big(\mathcal{V}_t(h_2(t)), \mathcal{W}_t(h_1(t))\big) = 0 = \mathsf{Ext}^1_{E_t}\big(\mathcal{V}_t(h_2(t)), \mathcal{W}_t(h_1(t))\big)$$

for all $t \in S$, we still get a morphism $\underline{\mathrm{ev}}_{h_2}^{\mathcal{V},\mathcal{W}(D_1)}$ but it may no longer be an isomorphism. This is the only change that occurs to the construction. Therefore, in this situation we still obtain a morphism of vector bundles

$$\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}} = \tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}(\omega) : \; h_1^* \mathcal{H}om_E(\mathcal{V}, \mathcal{W}) \longrightarrow h_2^* \mathcal{H}om_E(\mathcal{V}, \mathcal{W}).$$

Remark 5.22. Because $p : E \to S$ is proper, two nowhere vanishing global sections $\omega, \omega' \in H^0(\omega_{E|S})$ differ by a factor $\varphi = p^*(\psi)$ only, where $\psi \in H^0(\mathcal{O}_S^*)$. If $\omega' = \varphi\omega$, we obtain $\underline{\mathrm{res}}_{h_1}^{\mathcal{V},\mathcal{W}}(\omega') = \varphi \cdot \underline{\mathrm{res}}_{h_1}^{\mathcal{V},\mathcal{W}}(\omega)$ and $\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}(\omega) = \varphi \cdot \tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}(\omega')$. In particular, if S is a point, φ is a constant factor.

Now let us prove the compatibility of $\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}$ with respect to base-change. We start with the commutative diagram of coherent sheaves on E'

$$\begin{tikzcd} f^*\mathcal{O}_E \otimes f^*\mathcal{O}_E(D_1) \arrow[r, "f^*(\omega)\otimes \mathrm{id}"] \arrow[d, "\cong"', "\mathrm{can}"] & f^*\big(\omega_{E/S}(D_1)\big) \arrow[r, "f^*(\underline{\mathrm{res}}_{D_1})"] \arrow[d, "\cong"'] & f^*(\mathcal{O}_{D_1}) \arrow[d, "\cong"] \\ \mathcal{O}_{E'} \otimes \mathcal{O}_{E'}(D_1') \arrow[r, "\omega'\otimes \mathrm{id}"] & \omega_{E'/S'}(D_1') \arrow[r, "\underline{\mathrm{res}}_{D_1'}"] & \mathcal{O}_{D_1'} \end{tikzcd} \tag{22}$$

the right part of which was obtained in Proposition 5.16. Next apply the functor

$$p'_*\mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W} \otimes\, - \,) : \mathsf{Coh}(E') \longrightarrow \mathsf{Coh}(S')$$

to the right square, which yields the commutative diagram

$$\begin{tikzcd} p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes f^*(\omega_{E/S}(D_1))\big) \arrow[r] \arrow[d, "\cong"'] & p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes f^*\mathcal{O}_{D_1}\big) \arrow[d, "\cong"] \\ p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes \omega_{E'/S'}(D_1')\big) \arrow[r] & p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes \mathcal{O}_{D_1'}\big) \end{tikzcd}$$

in $\mathsf{Coh}(S')$. There is an isomorphism of functors

$$f^*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes\, -\,\big) \longrightarrow \mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes f^*(\,-\,)\big)$$

between the categories of coherent sheaves $\mathsf{Coh}(E)$ and $\mathsf{Coh}(E')$. Composing these functors with p'_* and then applying them to the residue map $\omega_{E/S}(D_1) \longrightarrow \mathcal{O}_{D_1}$ we obtain a commutative diagram

$$\begin{tikzcd} p'_*f^*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\big) \arrow[r] \arrow[d, "\cong"'] & p'_*f^*\mathcal{H}om_{E'}\big(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}\big) \arrow[d, "\cong"] \\ p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes f^*(\omega_{E/S}(D_1))\big) \arrow[r] & p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes f^*\mathcal{O}_{D_1}\big). \end{tikzcd}$$

Finally, there is a natural transformation of functors $g^*p_* \longrightarrow p'_*f^*$ (base-change), which is an isomorphism of functors on the category of S–flat coherent sheaves on E. Since both sheaves $\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\big)$ and $\mathcal{H}om_E(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}) \cong h_{1*}\mathcal{H}om_S(h_1^*\mathcal{V}, h_1^*\mathcal{W})$ are flat over S, we obtain a commutative diagram

$$\begin{tikzcd} g^*p_*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \omega_{E/S}(D_1)\big) \arrow[r] \arrow[d, "\cong"'] & g^*p_*\mathcal{H}om_E\big(\mathcal{V}, \mathcal{W} \otimes \mathcal{O}_{D_1}\big) \arrow[d, "\cong"] \\ p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes \omega_{E'/S'}(D_1')\big) \arrow[r] & p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W} \otimes \mathcal{O}_{D_1'}\big). \end{tikzcd}$$

Using similar arguments one can show that the following diagram is commutative:

$$\begin{tikzcd} g^*p_*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\otimes\mathcal{O}_{D_1}\big) \arrow[r, "\cong"] \arrow[d, "\cong"'] & g^*h_1^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \arrow[d, "\cong"] \\ p'_*\mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W}\otimes\mathcal{O}_{D'_1}) \arrow[r, "\cong"] & {h'_1}^*\mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W}), \end{tikzcd}$$

in which all arrows are canonical isomorphisms. Composinig the two previous diagrams, we obtain the compatibility of $\underline{\mathrm{res}}_{h_1}^{\mathcal{V},\mathcal{W}}$ with base change, i.e. the commutative diagram

$$\begin{tikzcd} g^*p_*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\otimes\omega_{E/S}(D_1)\big) \arrow[r, "g^*\left(\underline{\mathrm{res}}_{h_1}^{\mathcal{V},\mathcal{W}}\right)"] \arrow[d, "\cong"'] & g^*h_1^*\mathcal{H}om_E\big(\mathcal{V},\mathcal{W}\big) \arrow[d, "\cong"] \\ p'_*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W}\otimes\omega_{E'/S'}(D'_1)\big) \arrow[r, "\underline{\mathrm{res}}_{h'_1}^{f^*\mathcal{V},f^*\mathcal{W}}"] & {h'_1}^*\mathcal{H}om_{E'}\big(f^*\mathcal{V}, f^*\mathcal{W}\big) \end{tikzcd}$$

in which the vertical arrows are compositions of the natural isomorphisms constructed above. If we follow the same steps with the left square in diagram (22), we obtain the compatibility with base change for $\underline{\mathrm{res}}_{h_1}^{\mathcal{V},\mathcal{W}}(\omega)$.

In an analogue way, we can show that $\underline{\mathrm{ev}}_{h_2}^{\mathcal{V},\mathcal{W}}$ is compatible with base change. This proves the base-change property for $\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}(\omega)$.

It remains to show that, in case S is a single point, the relative construction yields the same result as the construction in Section 3. This follows from Theorem 4.17, Remark 5.19 and Remark 5.20. This finishes the proof of the theorem. $\square$

Let $r_{h_1,h_2}^{\mathcal{V},\mathcal{W}} = r_{h_1,h_2}^{\mathcal{V},\mathcal{W}}(\omega)$ denote the image of $\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}$ under the canonical isomorphism

$$\begin{tikzcd} \mathsf{Hom}_S\big(h_1^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}), h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W})\big) \arrow[d, "\cong"] \\ \Gamma\big(S, h_1^*\mathcal{H}om_E(\mathcal{W},\mathcal{V})\otimes h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W})\big). \end{tikzcd}$$

From Theorem 5.17 we immediately obtain the following corollary.

Corollary 5.23. *In the notation of Theorem 5.17 let* $\eta_{\mathcal{V},\mathcal{W}} : g^*\big(h_1^*\mathcal{H}om_E(\mathcal{W},\mathcal{V})\otimes h_2^*\mathcal{H}om(\mathcal{V},\mathcal{W})\big) \longrightarrow {h'_1}^*\mathcal{H}om_{E'}(f^*\mathcal{W}, f^*\mathcal{V})\otimes {h'_2}^*\mathcal{H}om_{E'}(f^*\mathcal{V}, f^*\mathcal{W})$ *be the canonical isomorphism of bifunctors. Then we have:*

$$\eta_{\mathcal{V},\mathcal{W}}\left(g^*(r_{h_1,h_2}^{\mathcal{V},\mathcal{W}})\right) = r_{h'_1,h'_2}^{f^*\mathcal{V},f^*\mathcal{W}}.$$

The following properties of the morphism $\tilde{r}_{h_1,h_2}^{\mathcal{V},\mathcal{W}}$ are straightforward consequences of the naturality of all the morphisms involved in the construction.

Proposition 5.24. *In the situation of Theorem 5.17, the isomorphism* $\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}$ *is functorial with respect to isomorphisms* $f : \mathcal{V} \longrightarrow \mathcal{V}'$ *and* $g : \mathcal{W} \longrightarrow \mathcal{W}'$*, this means that*

$$\begin{tikzcd} h_1^*\mathcal{H}om(\mathcal{V},\mathcal{W}) \arrow[r, "\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}"] \arrow[d, "h_1^*\left(\mathrm{cnj}(f,g)\right)"'] & h_2^*\mathcal{H}om(\mathcal{V},\mathcal{W}) \arrow[d, "h_2^*\left(\mathrm{cnj}(f,g)\right)"] \\ h_1^*\mathcal{H}om(\mathcal{V}',\mathcal{W}') \arrow[r, "\tilde{r}^{\mathcal{V}',\mathcal{W}'}_{h_1,h_2}"] & h_2^*\mathcal{H}om(\mathcal{V}',\mathcal{W}') \end{tikzcd}$$

is commutative. Moreover, for any line bundle $\mathcal{L}$ *on* E *the following diagram is commutative*

$$\begin{tikzcd} h_1^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \arrow[r, "\tilde{r}^{\mathcal{V},\mathcal{W}}_{h_1,h_2}"] \arrow[d, "\cong"'] & h_2^*\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \arrow[d, "\cong"] \\ h_1^*\mathcal{H}om_E(\mathcal{V}\otimes\mathcal{L},\mathcal{W}\otimes\mathcal{L}) \arrow[r, "\tilde{r}^{\mathcal{V}\otimes\mathcal{L},\mathcal{W}\otimes\mathcal{L}}_{h_1,h_2}"] & h_2^*\mathcal{H}om_E(\mathcal{V}\otimes\mathcal{L},\mathcal{W}\otimes\mathcal{L}), \end{tikzcd}$$

where the vertical arrows are induced by the canonical isomorphism

$$\mathcal{H}om_E(\mathcal{V},\mathcal{W}) \xrightarrow{\cong} \mathcal{H}om_E(\mathcal{V}\otimes\mathcal{L},\mathcal{W}\otimes\mathcal{L}).$$

6. Geometric associative r–matrix

The main goal of this section is to define the so-called geometric associative r-matrix attached to a genus one fibration. Throughout this section, we work either in the category Ans of complex analytic spaces or in the category of algebraic schemes over an algebraically closed field $\boldsymbol{k}$ of characteristic zero. We start with the following geometric data.

- Let $E \xrightarrow{p} T$ be a flat *projective* morphism of relative dimension one between reduced complex spaces and denote by $\breve{E}$ the smooth locus of p.
- We assume there exists a section $\imath : T \longrightarrow \breve{E}$ of p. Let $\Sigma := \imath(T) \subset E$ denote the corresponding Cartier divisor.
- Moreover, we assume that for all points $t \in T$ the fibre E_t is a *reduced* and *irreducible* projective curve of *arithmetic genus one*.
- The fibration $E \xrightarrow{p} T$ is embeddable into a smooth fibration of projective surfaces over T and $\omega_{E/T} \cong \mathcal{O}_E$.
- For our applications it is convenient to assume the fibration $E \xrightarrow{p} T$ is the *analytification* of an algebraic fibration.

6.1. **The construction.** For a pair of coprime integers $(n,d) \in \mathbb{N}\times\mathbb{Z}$ we denote by $\underline{\mathsf{M}}^{(n,d)}_{E/T} : \mathsf{Ans}_T \longrightarrow \mathsf{Sets}$ the moduli functor of relatively stable vector bundles of rank n and degree d. In particular, $\underline{\mathsf{M}}^{(1,d)}_{E/T} = \underline{\mathsf{Pic}}^d_{E/T}$ are the relative Picard functors and $\underline{\mathsf{M}}^{(1,0)}_{E/T} = \underline{\mathsf{Pic}}^0_{E/T}$ is the relative Jacobian. The assumption that $p : E \longrightarrow T$ has a section is only needed to ascertain that these functors have fine moduli spaces.

Theorem 6.1. *In the above notation we have:*

- *The relative Jacobian $\underline{\mathsf{Pic}}^0_{E/T}$ is representable by the fibration $\breve{E} \xrightarrow{p} T$ and the universal line bundle $\mathcal{L} = \mathcal{O}_{\breve{E}\times_T E}(-\Delta) \otimes \pi_2^*\mathcal{O}_E(\Sigma)$, where $\Delta \subset \breve{E} \times_T E$ denotes the diagonal and $\pi_2 : \breve{E} \times_T E \longrightarrow E$ is the natural projection.*
- *The functors $\underline{\mathsf{Pic}}^d_{E/T}$ for all $d \in \mathbb{Z}$ are isomorphic to each other and these isomorphisms are induced by tensoring with the line bundle $\mathcal{O}_E(\Sigma)$.*
- *The natural transformation of functors $\underline{\det} : \underline{\mathsf{M}}^{(n,d)}_{E/T} \longrightarrow \underline{\mathsf{Pic}}^d_{E/T}$ is an isomorphism. In particular, the moduli functor $\underline{\mathsf{M}}^{(n,d)}_{E/T}$ is representable for all pairs of coprime integers $(n, d) \in \mathbb{N} \times \mathbb{Z}$.*

Proof. The first part of this theorem can be found in [2] the second statement is trivial. The third part seems to be well-known, see for example [52] for the case of an elliptic curve and [21] for the proof of a more general statement and further details. □

From now on, by $M \xrightarrow{l} T$ we denote a fibration which, together with a universal family $\mathcal{P} = \mathcal{P}(n, d) \in \mathsf{VB}(M \times_T E)$, represents the functor $\underline{\mathsf{M}}^{(n,d)}_{E/T}$. For a closed point $t \in T$ we denote by $\mathcal{P}_t \in \mathsf{VB}(M_t \times E_t)$ the restriction of $\mathcal{P}$ to $M_t \times E_t$.
The morphisms p and l induce a morphism $C := M \times_T M \times_T \breve{E} \times_T \breve{E} \xrightarrow{g} T$, from which we obtain a Cartesian diagram of complex spaces:

$$\begin{tikzcd} M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \arrow[r, "f"] \arrow[d, "q"'] & E \arrow[d, "p"] \\ C \arrow[r, "g"] & T. \end{tikzcd}$$

Observe that $q : M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \longrightarrow C$ is again a genus one fibration satisfying all the conditions listed at the beginning of this section.

Definition 6.2. The diagonal map $\Delta : \breve{E} \longrightarrow \breve{E}\times_T E$ induces two *canonical* sections

$$h_1, h_2 : \quad M \times_T M \times_T \breve{E} \times_T \breve{E} \longrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E$$

of the morphism q, given by the rule $h_i(v_1, v_2; y_1, y_2) = (v_1, v_2; y_1, y_2, y_i)$ for $i = 1, 2$. Let D_i be the reduced image of h_i. Next, consider the two projection maps

$$\pi_i : M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \longrightarrow M \times_T E,$$

given by $\pi_i(v_1, v_2; y_1, y_2, y) = (v_i, y)$ for $i = 1, 2$. For any base point $x = (v_1, v_2; y_1, y_2) \in M \times_T M \times_T \breve{E} \times_T \breve{E}$ with $t = g(x)$ we denote:

$$\mathcal{P}^{v_i} := \pi_i^*\mathcal{P}|_{q^{-1}(x)} \cong \mathcal{P}_t|_{\{v_i\}\times E_t} \in \mathsf{VB}(E_t).$$

Consider the following closed subsets of the basis C:

$$\Delta_M = \bigl\{(v_1, v_2; y_1, y_2) \in C \mid v_1 = v_2\bigr\} \quad \text{and} \quad \Delta_E = \bigl\{(v_1, v_2; y_1, y_2) \in C \mid y_1 = y_2\bigr\}$$

and their complement $B = C \setminus \bigl(\Delta_M \cup \Delta_E\bigr)$. Then we have the induced genus one fibration:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, "q|_X"'] & M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T E \arrow[r, "f"] \arrow[d, "q"] & E \arrow[d, "p"] \\
B \arrow[r, hook] & C \arrow[r, "g"] & T.
\end{tikzcd}$$

Note that the images of the sections $h_1, h_2 : B \longrightarrow X$ are disjoint and for any point $x = (v_1, v_2; y_1, y_2) \in B$ we have: $\pi_1^*\mathcal{P}|_{q^{-1}(x)} = \mathcal{P}^{v_1} \ncong \mathcal{P}^{v_2} = \pi_2^*\mathcal{P}|_{q^{-1}(x)}$. Occasionally we shall use the abbreviation $\mathcal{V}_i = \pi_i^*\mathcal{P}|_X \in \mathsf{VB}(X)$ for $i = 1, 2$.

Lemma 6.3. *The set of points* $\bar{\Delta} = \bigl\{x \in B \mid \mathcal{V}_1(D_2)|_{q^{-1}(x)} \cong \mathcal{V}_2(D_1)|_{q^{-1}(x)}\bigr\}$ *is a closed analytic subset of* B.

Proof. Since the morphism q is projective, the sheaf $q_*\mathcal{H}om\bigl(\mathcal{V}_1(D_2), \mathcal{V}_2(D_1)\bigr)$ is coherent. Moreover, if $\mathcal{V}$ and $\mathcal{W}$ are two stable vector bundles on an irreducible projective curve E_t of arithmetic genus one having the same rank and degree, then $\mathsf{Hom}_{E_t}(\mathcal{V}, \mathcal{W}) \neq 0$ if and only if $\mathcal{V} \cong \mathcal{W}$. Since the sheaf $\mathcal{H}om\bigl(\mathcal{V}_1(D_2), \mathcal{V}_2(D_1)\bigr)$ is locally free, it is flat over B. Therefore, the base-change formula implies that for a point $x = (v_1, v_2; y_1, y_2) \in B$ with $t = g(x)$, after identifying $q^{-1}(x)$ with E_t, we have:

$$q_*\mathcal{H}om\bigl(\mathcal{V}_1(D_2), \mathcal{V}_2(D_1)\bigr)\Big|_x \cong \mathsf{Hom}_{E_t}\bigl(\mathcal{V}_1|_{E_t}(y_2), \mathcal{V}_2|_{E_t}(y_1)\bigr).$$

Therefore, the set $\bar{\Delta}$ coincides with the reduced support of $q_*\mathcal{H}om\bigl(\mathcal{V}_1(D_2), \mathcal{V}_2(D_1)\bigr)$, hence it is a closed analytic subset. □

Definition 6.4. Let $\omega \in H^0(\omega_{E/T})$ be a nowhere vanishing section of the dualising sheaf $\omega_{E/T}$ and $f^*(\omega) \in H^0(\omega_{X/B})$ its pull-back to X. Theorem 5.17 provides us with a canonical homomorphism of vector bundles on B (see also Remark 5.21):

$$\tilde{r} = \tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) := \tilde{r}^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}\bigl(f^*(\omega)\bigr) : \quad h_1^*\mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2) \longrightarrow h_2^*\mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2)$$

and a canonical holomorphic section

$$r = r^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) \in H^0\bigl(B, h_1^*\mathcal{H}om_X(\mathcal{V}_2, \mathcal{V}_1) \otimes h_2^*\mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2)\bigr).$$

We call $\tilde{r}$ and r the *geometric associative r-matrix* of the fibration $E \to T$.

Note that r and $\tilde{r}$ depend on the pair of coprime integers (n, d), the fibration $E \to T$ and the section $\omega \in H^0(\omega_{E/T})$ only. Two different choices of a universal bundle $\mathcal{P}$ lead to a canonical isomorphism between the corresponding section spaces $H^0\bigl(B, h_1^*\mathcal{H}om_X(\mathcal{V}_2, \mathcal{V}_1) \otimes h_2^*\mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2)\bigr)$ under which the constructed sections r

are identified. This isomorphism may involve an automorphism of the moduli space M and the tensor product with the pull back of a line bundle on M (see Prop. 5.24).

To formulate the base-change property (Proposition 6.5) for the geometric associative r-matrix r, let $\tilde{g} : T' \longrightarrow T$ be any morphism between reduced analytic spaces and

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ T' \arrow[r, "\tilde{g}"] & T \end{tikzcd}$$

the corresponding base-change diagram. From the representability of the functors $\underline{\mathsf{M}}^{(n,d)}_{E/T}$ and $\underline{\mathsf{M}}^{(n,d)}_{E'/T'}$ it follows easily that $M' := M \times_T T'$ with $\mathcal{P}' := (u \times \tilde{f})^* \mathcal{P}$ represents the functor $\underline{\mathsf{M}}^{(n,d)}_{E'/T'}$. Here we denoted by $u : M \times_T T' \longrightarrow M$ the first projection and $u \times \tilde{f} : M' \times_{T'} E' \longrightarrow M \times_T E$ coincides with $u \times \mathrm{id}_E : M' \times_T E \longrightarrow M \times_T E$ under the canonical identification $M' \times_{T'} E' = M' \times_T E$.

By X', B' we denote the spaces obtained from $E' \longrightarrow T'$ in the same way as X, B were obtained from $E \longrightarrow T$. Using $\hat{g} = u \times u \times \tilde{f} \times \tilde{f} \times \tilde{f} : X' \longrightarrow X$ and $\tilde{g}_B = u \times u \times \tilde{f} \times \tilde{f} : B' \longrightarrow B$, we obtain the Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\hat{g}"] \arrow[d, "q'"'] & X \arrow[d, "q"] \\ B' \arrow[r, "\tilde{g}_B"] & B. \end{tikzcd}$$

Note that there exist canonical isomorphisms $\phi_i : \hat{g}^* \mathcal{V}_i = \hat{g}^* \pi_i^* \mathcal{P} \longrightarrow \mathcal{V}'_i := \pi_i'^* \mathcal{P}'$, using the notation of Definition 6.2.

Proposition 6.5. *Let $\omega \in H^0(\omega_{E/T})$ and $\omega' = \tilde{f}^*(\omega) \in H^0(\omega_{E'/T'})$, then the image of the section $r = r^{\mathcal{V}_1,\mathcal{V}_2}_{h_1,h_2}(\omega) \in H^0\big(h_1^* \mathcal{H}om_X(\mathcal{V}_2, \mathcal{V}_1) \otimes h_2^* \mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2)\big)$ under the chain of canonical morphisms*

$$\begin{array}{c} H^0\big(h_1^* \mathcal{H}om_X(\mathcal{V}_2, \mathcal{V}_1) \otimes h_2^* \mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2)\big) \\ \Big\downarrow {\scriptstyle H^0(\tilde{g}_B^*)} \\ H^0\big(\tilde{g}_B^*(h_1^* \mathcal{H}om_X(\mathcal{V}_2, \mathcal{V}_1) \otimes h_2^* \mathcal{H}om_X(\mathcal{V}_1, \mathcal{V}_2))\big) \\ \Big\downarrow {\scriptstyle \eta_{\mathcal{V}_1,\mathcal{V}_2}} \\ H^0\big(h_1'^* \mathcal{H}om_{X'}(\hat{g}^*\mathcal{V}_2, \hat{g}^*\mathcal{V}_1) \otimes h_2'^* \mathcal{H}om_{X'}(\hat{g}^*\mathcal{V}_1, \hat{g}^*\mathcal{V}_2)\big) \\ \Big\downarrow {\scriptstyle H^0\left(h_1'^* \mathrm{cnj}(\phi_2,\phi_1) \otimes h_2'^* \mathrm{cnj}(\phi_1,\phi_2)\right)} \\ H^0\big(h_1'^* \mathcal{H}om_{X'}(\mathcal{V}'_2, \mathcal{V}'_1) \otimes h_2'^* \mathcal{H}om_{X'}(\mathcal{V}'_1, \mathcal{V}'_2)\big) \end{array}$$

is $r' = r^{\mathcal{V}'_1,\mathcal{V}'_2}_{h'_1,h'_2}(\omega')$, *where the first arrow is induced by the functor* $\tilde{g}_B^*$, *the second by the canonical isomorphisms of functors* $\tilde{g}_B^* h_i^* \cong h_i'^* \hat{g}^*$ *and the third by the isomorphisms of vector bundles* $\phi_i : \hat{g}^*\mathcal{V}_i \longrightarrow \mathcal{V}'_i$, $i = 1, 2$.

Proof. This proposition is an immediate consequence of Corollary 5.23. $\square$

Corollary 6.6. *In the notations as above, let* $x = (v_1, v_2; y_1, y_2) \in B$ *and* $t = g(x) \in T$. *Let* $\omega \in H^0(\omega_{E/T})$ *be a nowhere vanishing section and* ω_t *be its restriction to* E_t. *Then the image of the section* $r = r(\omega)$ *under the chain of canonical morphisms*

$$\begin{array}{c} H^0\big(B, h_1^*\mathcal{H}om_X(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om_X(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\big) \\ \Big\downarrow{\scriptstyle \mathsf{can}} \\ H^0\big(B, h_1^*\mathcal{H}om_X(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om_X(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P}) \otimes \mathbb{C}_x\big) \\ \Big\downarrow{\scriptstyle \cong} \\ \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_2}|_{y_1}, \mathcal{P}^{v_1}|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_1}|_{y_2}, \mathcal{P}^{v_2}|_{y_2}) \end{array}$$

is the tensor $r^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}_{y_1,y_2}(\omega_t)$ *obtained by the construction in Section 3 on the curve* E_t. *In particular, the section* r *is non-degenerate on* $B \setminus \bar{\Delta}$. *Equivalently, the morphism of vector bundles*

$$\tilde{r}(\omega) : h_1^*\mathcal{H}om_X(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P}) \longrightarrow h_2^*\mathcal{H}om_X(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})$$

is an isomorphism over $B \setminus \bar{\Delta}$.

Remark 6.7. Since we assume the fibration $E \xrightarrow{p} T$ is *algebraic*, the above construction yields a *meromorphic* section $r(\omega)$ of the vector bundle

$$h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})$$

over $M \times_T M \times_T \breve{E} \times_T \breve{E}$, which is holomorphic on $B = M \times_T M \times_T \breve{E} \times_T \breve{E} \setminus (\Delta_M \cup \Delta_E)$ and non-degenerate on $B \setminus \bar{\Delta}$, see Remark 5.21.

Our next goal is to show that the constructed canonical section $r = r(\omega)$ satisfies a version of the *associative Yang–Baxter equation.* For this purpose we need further notation. Let

$$p_{kl}^{ij} : M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E} \longrightarrow M \times_T M \times_T \breve{E} \times_T \breve{E}$$

be the projection $p_{kl}^{ij}(v_1, v_2, v_3; y_1, y_2, y_3) = (v_i, v_j; y_k, y_l)$, where $1 \le i \ne j \le 3$ and $1 \le k \ne l \le 3$. We also denote by

$$\hat{\pi}_j : M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E} \times_T E \longrightarrow M \times_T E$$

the projection given by the formula $\hat{\pi}_j(v_1, v_2, v_3; y_1, y_2, y_3, y) = (v_j; y)$, where $1 \le j \le 3$. Similarly, we have three canonical sections

$$\hat{h}_i : M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E} \longrightarrow M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E} \times_T E$$

given by the formulae $\hat{h}_i(v_1, v_2, v_3, y_1, y_2, y_3) = (v_1, v_2, v_3, y_1, y_2, y_3, y_i)$, $1 \le i \le 3$.

The set over which the Yang-Baxter relation will be defined is

$$\hat{B} := \bigcap_{i,j,k,l} \left(p^{ij}_{kl}\right)^{-1}(B) = \left(p^{12}_{12}\right)^{-1}(B) \cap \left(p^{13}_{13}\right)^{-1}(B) \cap \left(p^{23}_{23}\right)^{-1}(B).$$

Let $r \in H^0\big(B, h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\big)$ be the canonical holomorphic section constructed above. Observe that

$$\begin{aligned}(p^{ij}_{kl})^*\big(h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\big)& \\ \cong \hat{h}_k^*\mathcal{H}om(\hat{\pi}_j^*\mathcal{P}, \hat{\pi}_i^*\mathcal{P}) \otimes \hat{h}_l^*\mathcal{H}om(\hat{\pi}_i^*\mathcal{P}, \hat{\pi}_j^*\mathcal{P}).&\end{aligned}$$

Let $r^{ij}_{kl} = \left(p^{ij}_{kl}\right)^* r$ be the pull-back to $\hat{B} \subset M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E}$. Then we have:

- r^{32}_{12} is a meromorphic section of $\hat{h}_1^*\mathcal{H}om(\hat{\pi}_2^*\mathcal{P}, \hat{\pi}_3^*\mathcal{P}) \otimes \hat{h}_2^*\mathcal{H}om(\hat{\pi}_3^*\mathcal{P}, \hat{\pi}_2^*\mathcal{P})$.
- r^{13}_{13} is a meromorphic section of $\hat{h}_1^*\mathcal{H}om(\hat{\pi}_3^*\mathcal{P}, \hat{\pi}_1^*\mathcal{P}) \otimes \hat{h}_3^*\mathcal{H}om(\hat{\pi}_1^*\mathcal{P}, \hat{\pi}_3^*\mathcal{P})$.

Taking their composition, we obtain a meromorphic section $(r^{13}_{13})^{13}(r^{32}_{12})^{12}$ of the holomorphic vector bundle $\hat{h}_1^*\mathcal{H}om(\hat{\pi}_2^*\mathcal{P}, \hat{\pi}_1^*\mathcal{P}) \otimes \hat{h}_2^*\mathcal{H}om(\hat{\pi}_3^*\mathcal{P}, \hat{\pi}_2^*\mathcal{P}) \otimes \hat{h}_3^*\mathcal{H}om(\hat{\pi}_1^*\mathcal{P}, \hat{\pi}_3^*\mathcal{P})$. In a similar way, two other meromorphic sections $(r^{12}_{12})^{12}(r^{13}_{23})^{23}$ and $(r^{23}_{23})^{23}(r^{12}_{13})^{13}$ of this vector bundle can be defined. These sections are holomorphic over $\hat{B}$.

Let $x = (v_1, v_2, v_3; y_1, y_2, y_3)$ be a point in $\hat{B} \subset M \times_T M \times_T M \times_T \breve{E} \times_T \breve{E} \times_T \breve{E}$ lying over the point $t \in T$. Because $\hat{h}_k^*\mathcal{H}om(\hat{\pi}_i^*\mathcal{P}, \hat{\pi}_j^*\mathcal{P})\big|_x$ is canonically isomorphic to $\mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_i}|_{y_k}, \mathcal{P}^{v_j}|_{y_k})$, we may consider $r^{ij}_{kl}(x)$ as an element of the tensor product of vector spaces $\mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_j}|_{y_k}, \mathcal{P}^{v_i}|_{y_k}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_i}|_{y_l}, \mathcal{P}^{v_j}|_{y_l})$ and we have a canonical isomorphism of vector spaces

$$\begin{aligned}\big(\hat{h}_1^*\mathcal{H}om(\hat{\pi}_2^*\mathcal{P}, \hat{\pi}_1^*\mathcal{P}) \otimes \hat{h}_2^*\mathcal{H}om(\hat{\pi}_3^*\mathcal{P}, \hat{\pi}_2^*\mathcal{P}) \otimes \hat{h}_3^*\mathcal{H}om(\hat{\pi}_1^*\mathcal{P}, \hat{\pi}_3^*\mathcal{P})\big)\big|_x \cong& \\ \cong \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_2}|_{y_1}, \mathcal{P}^{v_1}|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_3}|_{y_2}, \mathcal{P}^{v_2}|_{y_2}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_1}|_{y_3}, \mathcal{P}^{v_3}|_{y_3}).&\end{aligned}$$

Definition 6.8. Assume $E \xrightarrow{p} T$ is a genus one fibration which satisfies the conditions set out at the beginning of Section 6 and fix (n, d) and ω as before. We call

$$(r^{13}_{13})^{13}(r^{32}_{12})^{12} - (r^{12}_{12})^{12}(r^{13}_{23})^{23} + (r^{23}_{23})^{23}(r^{12}_{13})^{13} = 0 \tag{23}$$

the *Yang-Baxter relation*. The left-hand side of this equation is a holomorphic section of the vector bundle $\hat{h}_1^*\mathcal{H}om(\hat{\pi}_2^*\mathcal{P}, \hat{\pi}_1^*\mathcal{P}) \otimes \hat{h}_2^*\mathcal{H}om(\hat{\pi}_3^*\mathcal{P}, \hat{\pi}_2^*\mathcal{P}) \otimes \hat{h}_3^*\mathcal{H}om(\hat{\pi}_1^*\mathcal{P}, \hat{\pi}_3^*\mathcal{P})$ over $\hat{B}$.

Let τ be the involution of $M \times_T M \times_T \breve{E} \times_T \breve{E}$ which is given by $\tau(v_1, v_2; y_1, y_2) = (v_2, v_1; y_2, y_1)$. We say that $r \in H^0\big(B, h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\big)$ is *unitary*, if

$$r_{h_1,h_2}^{\pi_1^*\mathcal{P},\pi_2^*\mathcal{P}}(\omega) = -\tau^*\left(r_{h_2,h_1}^{\pi_2^*\mathcal{P},\pi_1^*\mathcal{P}}(\omega)\right). \tag{24}$$

This means that the section $\tau^*(r)$ is mapped to $-r$ under the composition of the canonical isomorphisms

$$\tau^*\big(h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\big) \cong$$

$$h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P}) \otimes h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \cong h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P}).$$

The purpose of the following lemma is to use the relations (15) and (16), which were shown for tensors $r_{y_1,y_2}^{\mathcal{V}_1,\mathcal{V}_2}$ on smooth curves in Section 3, in order to prove that r satisfies the Yang-Baxter relation (23) and is unitary (24) in the case of elliptic fibrations with arbitrary fibers.

Lemma 6.9. *Assume $E \xrightarrow{p} T$ is a genus one fibration which satisfies the conditions set out at the beginning of Section 6 and fix (n, d) and ω as before. Denote r by r_T if constructed from the family $E \to T$. Let $U \subset T$ be a dense subset. Use the restriction of ω and the same pair of integers (n, d) to construct the section r_U and r_t from the induced families $E|_U \to U$ and $E_t \to \{t\}$. Then the following conditions are equivalent.*

1. *The Yang-Baxter relation (23) holds for r_U.*
2. *The Yang-Baxter relation (23) holds for r_T.*
3. *The Yang-Baxter relation (23) holds for r_t for all $t \in T$.*
4. *The Yang-Baxter relation (23) holds for r_t for all $t \in U$.*

If E_t is smooth, the Yang-Baxter relation (23) holds for r_t.

Similar statements hold for unitarity (24).

Proof. Let $B_U := B \times_T U, B_t := B \times_T \{t\}$ and define $\hat{B}_U, \hat{B}_t$ in a similar way. Note that $\hat{B}_U \subset \hat{B}$ is dense. If we apply Proposition 6.5 to the base-change $U \subset T$, we obtain that $r|_{B_U}$ corresponds (under a certain canonical isomorphism) to the section r_U obtained from the family $E_U \to U$. Similarly, $r|_{B_t}$ corresponds to the section r_t obtained from the family $E_t \to \{t\}$.

Let us denote the left hand side of the relation (23) by R_T, if it is a relation for r_T. Because the projections p_{kl}^{ij} are compatibel with restrictions to the subsets $B_U \subset B$ and $\hat{B}_U \subset \hat{B}$, we obtain from the above that $R_T|_{\hat{B}_U}$ corresponds to R_U under a certain canonical isomorphism. Similarly, $R_T|_{\hat{B}_t}$ corresponds to R_t for each $t \in T$. In particular, $R_T|_{\hat{B}_U}$ vanishes if and only if R_U does so and the vanishing of $R_T|_{\hat{B}_t}$ is equivalent to the vanishing of R_t.

As T is reduced, $R_T = 0$ is equivalent to $R_T(x) = 0$ for all $x \in \hat{B}$. Similar statements hold for R_U and R_t. Because the zero locus of R_T, which is a section of a coherent sheaf on $\hat{B}$, is a closed subset of $\hat{B}$, the equivalence of the statements (1)–(4) is now obvious.

Corollary 6.6 says that the restriction $r(\omega)|_{(v_1,v_2;y_1,y_2)}$ corresponds to

$$r^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}_{y_1,y_2}(\omega_t) \in \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_2}|_{y_1}, \mathcal{P}^{v_1}|_{y_1}) \otimes \mathsf{Hom}_{\mathbb{C}}(\mathcal{P}^{v_1}|_{y_2}, \mathcal{P}^{v_2}|_{y_2})$$

under a certain canonical isomorphism. For each $x = (v_1, v_2, v_3; y_1, y_2, y_3) \in \hat{B}_t$ this implies that $R_t(x) = 0$ is equivalent to (15) which was shown to be true in Section 3 for smooth curves.

The proofs for unitarity are similar. □

Theorem 6.10. (a) *For each Weierstraß curve E, smooth or singular, the section $r \in H^0\big(B, h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\big)$ from Definition 6.2 for the constant family $E \to \mathsf{Spec}(\mathbb{C})$ satisfies the Yang–Baxter relation (23) and the unitarity condition (24) for each choice of $(n,d), \mathcal{P}$ and ω.*

(b) *Let $E \xrightarrow{p} T$ be a genus one fibration satisfying the conditions from the beginning of this section. Let $\omega \in H^0(\omega_{E/T})$ be a nowhere vanishing differential form, (n,d) a pair of coprime integers and $\mathcal{P} = \mathcal{P}(n,d)$ a universal family. Then, the universal section*

$$r = r(\omega) \in H^0\big(B, h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\big)$$

satisfies the Yang–Baxter relation (23) and the unitarity condition (24). Moreover, r depends holomorphically (and in particular, continuously) on the parameter $t \in T$.

Proof. (a) Let $E_T \subset \mathbb{P}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2 =: T$ be the elliptic fibration given by the equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$ and let $\Delta(g_2, g_3) = g_2^3 - 27g_3^2$ be the discriminant of this family. This fibration has a section $(g_2, g_3) \mapsto \big((0:1:0), (g_2, g_3)\big)$ and satisfies the condition $\omega_{E/T} \cong \mathcal{O}_E$. Let $\omega \in H^0(\omega_{E/T})$ be a nowhere vanishing differential form.

There exists $t \in T$, such that the given curve E is isomorphic to the fibre E_t. The chosen differential form on E coincides with ω_t up to a constant factor. The restriction $\mathcal{P}|_{M_t \times E_t}$ of a universal family $\mathcal{P} \in \mathsf{VB}(M \times_T E)$ is a universal family of stable vector bundles of rank n and degree d on the curve E_t.

Using the open dense subset $U = T \setminus \Delta \subset T$ in the equivalence of the statements (3) and (4) in Lemma 6.9 and the fact that (23) and (24) are satisfied for smooth fibres by Lemma 6.9, we obtain the claim.

(b) Because each fibre of $E \to T$ is isomorphic to a Weierstraß curve, part (a) and Lemma 6.9 (2) and (3) imply the claim. □

6.2. **Passing to Matrices.** Our next goal is to pass from the categorical version of the associative Yang–Baxter equation (23) to the one which was studied in Section 2. Our construction is based on the choice of a trivialization of the universal bundle

$\mathcal{P}$ and on the choice of local coordinates on M and E. These two choices can be made independently.

Let $o = (m_0, e_0) \in M \times_T \breve{E} \subset M \times_T E$ be an arbitrary point, which lies over $t_0 \in T$. Consider a small open neighbourhood $V \subset M \times_T \breve{E}$ of the point o such that there exists an isomorphism of vector bundles $\xi : \mathcal{P}|_V \xrightarrow{\cong} V \times \mathbb{C}^n$.

Let $\varphi_{ij} := \pi_j \circ h_i : M \times_T M \times_T \breve{E} \times_T \breve{E} \longrightarrow M \times_T E$, $B_0 := \bigcap_{i,j=1}^{2} \varphi_{ij}^{-1}(V)$ and $O = \mathcal{O}_B(B_0 \cap B)$ be the ring of holomorphic functions on $B_0 \cap B$. The isomorphism ξ induces trivializations $\varphi_{ij}^*(\xi) : \varphi_{ij}^*\mathcal{P}|_{B_0} \longrightarrow B_0 \times \mathbb{C}^n$ from which we obtain isomorphisms $H^0\big(B_0 \cap B, h_k^*\mathcal{H}om(\pi_i^*\mathcal{P}, \pi_j^*\mathcal{P})\big) \xrightarrow{\cong} \mathsf{Mat}_{n\times n}(O)$. Under the induced isomorphism

$$H^0\big(B_0 \cap B, h_1^*\mathcal{H}om(\pi_2^*\mathcal{P}, \pi_1^*\mathcal{P}) \otimes h_2^*\mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\big) \xrightarrow{\cong} \mathsf{Mat}_{n\times n}(O) \otimes_O \mathsf{Mat}_{n\times n}(O),$$

the section r is mapped to a tensor

$$r^\xi = r^\xi(v_1, v_2; y_1, y_2) = \sum_\nu a_\nu(v_1, v_2; y_1, y_2) \otimes b_\nu(v_1, v_2; y_1, y_2)$$

in $\mathsf{Mat}_{n\times n}(O) \otimes_O \mathsf{Mat}_{n\times n}(O)$, where $a_\nu = a_\nu(v_1, v_2; y_1, y_2)$ and $b_\nu = b_\nu(v_1, v_2; y_1, y_2)$ are holomorphic functions on $B_0 \cap B$.

Because the fibration $p : E \to T$ is smooth at e_0 and so also $M \to T$ at m_0, there exist an open neighbourhood T_0 of $t_0 \in T$ and an open disc $\mathbb{D} \subset \mathbb{C}$ such that there are open neighbourhoods of $e_0 \subset E$ and of $m_0 \subset M$ which are isomorphic to $T_0 \times \mathbb{D}$ and such that the following diagrams of complex spaces are commutative:

$$\begin{tikzcd} E & T_0 \times \mathbb{D} \arrow[l, hook'] \arrow[d, "\mathsf{pr}"] \\ T \arrow[from=u] & T_0 \arrow[l, hook'] \end{tikzcd} \qquad \begin{tikzcd} M & T_0 \times \mathbb{D} \arrow[l, hook'] \arrow[d, "\mathsf{pr}"] \\ T \arrow[from=u] & T_0. \arrow[l, hook'] \end{tikzcd}$$

We assume that V is isomorphic to the fibred product of open neighbourhoods of the form $T_0 \times \mathbb{D}$, so that $V \cong T_0 \times \mathbb{D}^2$. With such V we obtain an isomorphism $B_0 \cong T_0 \times \mathbb{D}^4$ and the tensor $r^\xi = r^\xi(v_1, v_2; y_1, y_2)$ will be written in such coordinates as $r^\xi(t; v_1, v_2; y_1, y_2)$ with $t \in T_0$ and $v_1, v_2, y_1, y_2 \in \mathbb{D}$. We also define $\hat{B}_0 := \bigcap_{i,j,k,l} \big(p_{kl}^{ij}\big)^{-1}(B_0)$ and obtain an isomorphism $\hat{B}_0 \cong T_0 \times \mathbb{D}^6$. In these coordinates, we have $p_{kl}^{ij}(t; v_1, v_2, v_3; y_1, y_2, y_3) = (t; v_i, v_j; y_k, y_l)$. This equation implies that the Yang-Baxter relation (23) and unitarity (24) translate into (5) and (4) respectively. Therefore, Theorem 6.10 (b) implies the following corollary.

Corollary 6.11. *Let $E \xrightarrow{p} T$ be an elliptic fibration satisfying all the conditions from the beginning of this section. Let $\omega \in H^0(\omega_{E/T})$ be a nowhere vanishing section, (n, d) be coprime integers, $M = M_{E/T}^{(n,d)} \xrightarrow{l} T$ be the moduli space of relatively stable vector bundles of rank n and degree d, $\mathcal{P} = \mathcal{P}(n, d) \in \mathsf{VB}(M \times_T E)$ be a universal family. Let $o = (m_0, e_0) \in M \times_T \breve{E}$ be an arbitrary point lying over $t_0 \in T$ and choose* *coordinates around e_0 and m_0 as described above. Finally, let $\xi : \mathcal{P}|_V \longrightarrow \mathcal{O}_V^n$ be a trivialization of the universal family over a neighbourhood V of the point o.*

Then, the tensor $r^\xi(t; v_1, v_2; y_1, y_2)$ is unitary (i.e. it fulfils (5)) and satisfies the associative Yang-Baxter equation (4).

Our next goal is to explain how the tensor r^ξ depends on the choice of the trivialization ξ. We do not need to choose coordinates here. If $\mathcal{P}|_V \xrightarrow{\zeta} V \times \mathbb{C}^n$ is another trivialization of $\mathcal{P}$, we obtain a commutative diagram

$$\begin{tikzcd} & \mathcal{P}|_V \arrow[dl, "\xi"'] \arrow[dr, "\zeta"] & \\ V \times \mathbb{C}^n \arrow[rr, "\mathrm{id}\times\phi"] & & V \times \mathbb{C}^n, \end{tikzcd}$$

where $\phi = \phi(v, y) : V \longrightarrow \mathsf{GL}_n(\mathbb{C})$ is a holomorphic function.

Proposition 6.12. *The solutions r^ξ and r^ζ are gauge equivalent and such an equivalence is given by the function ϕ. In other words,* gauge transformations *of the solutions of the associative Yang–Baxter equation, which are obtained from a geometric associative r-matrix, correspond exactly to a* change of trivialization *of the universal family $\mathcal{P}$.*

Proof. With respect to the second trivialization ζ, the section r can be written as a tensor

$$r^\zeta = r^\zeta(v_1, v_2; y_1, y_2) = \sum_\nu a'_\nu(v_1, v_2; y_1, y_2) \otimes b'_\nu(v_1, v_2; y_1, y_2).$$

The functions a_ν and a'_ν are related by the following commutative diagram:

$$\begin{tikzcd} \mathbb{C}^n \arrow[rr, "a_\nu"] \arrow[dd, bend right=60, "\phi(v_2,y_1)"'] & & \mathbb{C}^n \arrow[dd, bend left=60, "\phi(v_1,y_1)"] \\ \mathcal{P}^{v_2}|_{y_1} \arrow[u, "\xi(v_2,y_1)"] \arrow[rr] \arrow[d, "\zeta(v_2,y_1)"] & & \mathcal{P}^{v_1}|_{y_1} \arrow[u, "\xi(v_1,y_1)"'] \arrow[d, "\zeta(v_1,y_1)"'] \\ \mathbb{C}^n \arrow[rr, "a'_\nu"] & & \mathbb{C}^n. \end{tikzcd}$$

Similarly for b_ν and b'_ν we have:

$$\begin{tikzcd} \mathbb{C}^n \arrow[rr, "b_\nu"] \arrow[dd, bend right=60, "\phi(v_1,y_2)"'] & & \mathbb{C}^n \arrow[dd, bend left=60, "\phi(v_2,y_2)"] \\ \mathcal{P}^{v_1}|_{y_2} \arrow[u, "\xi(v_1,y_2)"] \arrow[rr] \arrow[d, "\zeta(v_1,y_2)"] & & \mathcal{P}^{v_2}|_{y_2} \arrow[u, "\xi(v_2,y_2)"'] \arrow[d, "\zeta(v_2,y_2)"'] \\ \mathbb{C}^n \arrow[rr, "b'_\nu"] & & \mathbb{C}^n. \end{tikzcd}$$

These diagrams imply the following transformation rules:

$$\begin{array}{rcl} a'_\nu & = & \phi(v_1, y_1)\, a_\nu\, \phi^{-1}(v_2, y_1), \\ b'_\nu & = & \phi(v_2, y_2)\, b_\nu\, \phi^{-1}(v_1, y_2). \end{array}$$

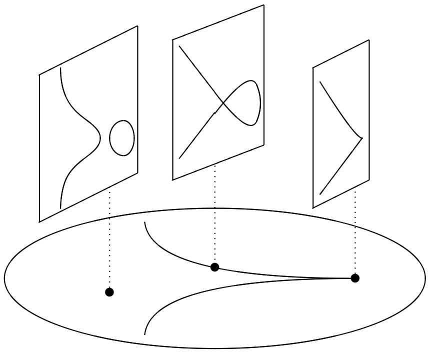
This means that the choice of a different trivialization ζ of the universal bundle $\mathcal{P}$ leads to the transformation rule

$$r^\xi \mapsto \big(\phi(v_1, y_1) \otimes \phi(v_2, y_2)\big)\, r^\xi\, \big(\phi(v_2, y_1)^{-1} \otimes \phi(v_1, y_2)^{-1}\big) = r^\zeta,$$

which means that r^ξ and r^ζ are gauge equivalent. Conversely, if we start with r^ξ and apply a gauge transformation ϕ, the same calculation shows that the result is r^ζ where $\zeta := (\mathrm{id} \times \phi) \circ \xi$. $\square$

Summing up all results of this section, we get the following theorem, which is one of the main results of this article.

Theorem 6.13. *Let $E \xrightarrow{p} T$ be a genus one fibration satisfying the conditions from the beginning of this section.*



Let $M = M_{E/T}^{(n,d)} \xrightarrow{l} T$ be the moduli space of relatively stable vector bundles of rank n and degree d and $\mathcal{P} = \mathcal{P}(n, d) \in \mathsf{VB}(M \times_T E)$ be a universal family on M. We fix a nowhere vanishing differential form $\omega \in H^0(\omega_{E/T})$. Let $e_0 \in \breve{E}$ and $m_0 \in M$ be arbitrary points and choose coordinate neighbourhoods of the form $T_0 \times \mathbb{D}$ around these two points. Let ξ be a trivialization of $\mathcal{P}$ in the corresponding neighbourhood of $o := (m_0, e_0)$. Let $\hat{o} = (m_0, m_0, e_0, e_0) \in M \times_T M \times_T \breve{E} \times_T \breve{E} \cong T_0 \times \mathbb{D}^4$.
Then, we get the germ of a meromorphic function

$$r^\xi = \big(r_{E/T}^{(n,d)}(\omega)\big)^\xi : \big(M \times_T M \times_T \breve{E} \times_T \breve{E}, \hat{o}\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

which satisfies the associative Yang-Baxter equation

$$r^\xi(t; v_1, v_2; y_1, y_2)^{12} r^\xi(t; v_1, v_3; y_2, y_3)^{23} =$$
$$r^\xi(t; v_1, v_3; y_1, y_3)^{13} r^\xi(t; v_3, v_2; y_1, y_2)^{12} + r^\xi(t; v_2, v_3; y_2, y_3)^{23} r^\xi(t; v_1, v_2; y_1, y_3)^{13}$$

and its “dual”

$$r^{\xi}(t; v_2, v_3; y_1, y_2)^{23} r^{\xi}(t; v_1, v_3; y_1, y_2)^{12} =$$
$$r^{\xi}(t; v_1, v_2; y_1, y_2)^{12} r^{\xi}(t; v_2, v_3; y_1, y_3)^{13} + r^{\xi}(t; v_1, v_3; y_1, y_3)^{13} r^{\xi}(t; v_2, v_1; y_2, y_3)^{23}.$$

Moreover, it fulfills the unitarity condition

$$r^{\xi}(t; v_1, v_2; y_1, y_2) = -\tau\big(r^{\xi}(t; v_2, v_1; y_2, y_1)\big),$$

where $\tau(a \otimes b) = b \otimes a$. *The function* $r^{\xi}(t; v_1, v_2; y_1, y_2)$ *depends analytically on the parameter* $t \in T$ *and its poles lie on the hypersurfaces* $v_1 = v_2$ *and* $y_1 = y_2$.
Different choices of trivializations of the universal family $\mathcal{P}$ *lead to equivalent solutions: if* ζ *is another trivialization of* $\mathcal{P}$ *and* $\phi = \zeta \circ \xi^{-1} : (M \times_T E, o) \longrightarrow \mathsf{GL}_n(\mathbb{C})$ *is the corresponding holomorphic function, then we have:*

$$r^{\zeta} = \big(\phi(t, v_1, y_1) \otimes \phi(t, v_2, y_2)\big) r^{\xi} \big(\phi(t, v_2, y_1)^{-1} \otimes \phi(t, v_1, y_2)^{-1}\big).$$

Finally, let $T' \xrightarrow{g} T$ *be an arbitrary base change. Let* $E' = E \times_T T' \xrightarrow{p'} T'$ *be the induced genus one fibration. Then the corresponding moduli space* $M' = M_{E'/T'}^{(n,d)}$ *of relatively stable vector bundles of rank* n *and degree* d *is isomorphic to* $M \times_T T'$. *Let* $\omega' \in H^0(\omega_{E'/T'})$ *be the pull-back of* ω *and* ξ' *be the induced trivialization of the pull-back* $\mathcal{P}' \in \mathsf{VB}(M' \times_{T'} E')$ *of the universal family* $\mathcal{P}$. *Let* $e_0' \in \breve{E}'$ *and* $m_0' \in M'$ *be points which are mapped to* e_0 *and* m_0 *respectively. After choosing coordinate neighbourhoods* $T_0 \times \mathbb{D}$ *of* $e_0 \in E$ *and* $m_0 \in M$, *we have induced neighbourhoods* $T_0' \times \mathbb{D}$ *of* $e_0' \in E'$ *and* $m_0' \in M'$ *so that the morphism between* E' *and* E *(and between* M' *and* M*) is given by* $g \times \mathrm{id}_{\mathbb{D}}$. *Then we have*

$$r^{\xi}(g(t); v_1, v_2; y_1, y_2) = r^{\xi'}(t; v_1, v_2; y_1, y_2)$$

for all $t \in T_0'$ *and all* $v_1, v_2, y_1, y_2 \in \mathbb{D}$. *In other words, the tensor* $r^{\xi}(t; v_1, v_2; y_1, y_2)$ *is compatible with base change in the variable* t.

6.3. **Comment on reducible curves.** The developed theory of the geometric r-matrices can be generalized literally to the case of reduced but reducible curves of arithmetic genus one with trivial dualizing sheaf. In this subsection we discuss some necessary technical results which are not yet available.

Throughout this section, E is a reduced projective curve with trivial dualizing sheaf.

- If E is smooth then it is isomorphic to an elliptic curve.
- Assume E is singular with singularities of embedding dimension *equal to two*. Then Kodaira's classification of degenerations of elliptic curves implies that E is either a cycle of m projective lines (type I_m), a cuspidal cubic curve (type II), a tachnode curve (type III) or a configuration of three concurrent lines in a plane (type IV).
- However, the class of reduced genus one curve with trivial dualizing sheaf is larger. For example, a generic configuration of m concurrent lines in $\mathbb{P}^{m-1}$ for $m \geq 4$ is such a curve.

Let $\pi : \widetilde{E} \to E$ be the normalization of E and $\widetilde{E} = \widetilde{E}_1 \cup \cdots \cup \widetilde{E}_m$ be the decomposition into irreducible components. If E was not smooth, then $\widetilde{E}_i \cong \mathbb{P}^1$ for all i. For a vector bundle $\mathcal{F}$ on E we define its *multi-degree* to be

$$\underline{\deg}(\mathcal{F}) = \big(d_1(\mathcal{F}), \dots, d_m(\mathcal{F})\big) \in \mathbb{Z}^m$$

where $d_i(\mathcal{F}) = \deg\big(\pi^*(\mathcal{F})\big|_{\widetilde{E}_i}\big)$. The following lemma can be shown in the same way as Lemma 9.6.

Lemma 6.14. *Let $\mathcal{F}$ be a vector bundle on E. Then we have:*

$$\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) = \deg(\mathcal{F}) := d_1(\mathcal{F}) + \cdots + d_m(\mathcal{F}).$$

For fixed $n \in \mathbb{Z}_+$ and $\mathbb{d} \in \mathbb{Z}^m$ denote by $\mathsf{Spl}^{(n,\mathbb{d})}(E)$ the category of simple vector bundles of rank n and multi-degree $\mathbb{d}$. Then we have the following result.

Theorem 6.15. *Let E be a reduced plane cubic curve.*

- *If $\mathcal{F}$ is a simple vector bundle on E then* $\gcd\big(\mathrm{rk}(\mathcal{F}), \chi(\mathcal{F})\big) = 1$*;*
- *If $\mathbb{d} = (d_1, d_2, \dots, d_m)$ satisfies* $\gcd(n, d_1 + \cdots + d_m) = 1$*, then the functor*

$$\det : \mathsf{Spl}^{(n,\mathbb{d})}(E) \longrightarrow \mathsf{Pic}^{\mathbb{d}}(E)$$

is an equivalence of categories. In particular, for any pair $\mathcal{F}', \mathcal{F}'' \in \mathsf{Spl}^{(n,\mathbb{d})}(E)$ such that $\mathcal{F}' \not\cong \mathcal{F}''$ we have $\mathsf{Hom}_E(\mathcal{F}', \mathcal{F}'') = 0$*.*

The case of a smooth curve E is due to Atiyah [5]. A proof for the irreducible singular Weierstraß cubic curves can be found in [17] (nodal case) and [13] (cuspidal case), see also Section 9. The remaining cases, i.e. the Kodaira fibers of type I_2, I_3, III and IV, were considered in a recent paper of Bodnarchuk, Drozd and Greuel [12].

Conjecture 6.16. Let E be an arbitrary reduced projective curve with trivial dualizing sheaf and m irreducible components. Then we have:

(1) The description of simple vector bundles on E given in Theorem 6.15 remains true: the rank and degree of a simple vector bundle are coprime; a simple vector bundle is determined by its rank and determinant; for given n and $\mathbb{d} = (d_1, \dots, d_m)$ satisfying $\gcd(n, d_1 + \cdots + d_m) = 1$, the category $\mathsf{Spl}^{(n,\mathbb{d})}(E)$ is equivalent to $\mathsf{Pic}^{\mathbb{d}}(E)$, in particular, it is non-empty.

(2) For any pair $(n, \mathbb{d}) \in \mathbb{Z}_+ \times \mathbb{Z}^m$ as above, there exists an auto-equivalence $\mathbb{F} \in \big\langle T_{\mathcal{O}}, \mathsf{Pic}(E), [1]\big\rangle$ of the derived category $D^b\big(\mathsf{Coh}(E)\big)$ inducing an equivalence between $\mathsf{Spl}^{(n,\mathbb{d})}(E)$ and the category of torsion sheaves of length one supported at the regular part of a single irreducible component of E.

(3) Consider the functor $\underline{\mathsf{M}}_E^{(n,\mathbb{d})} : \mathsf{Ans} \longrightarrow \mathsf{Sets}$ given by

$$\underline{\mathsf{M}}_E^{(n,\mathbb{d})}(T) = \Big\{\mathcal{F} \in \mathsf{VB}(E \times T) \,\Big|\, \mathcal{F}|_{E \times \{t\}} \in \mathsf{Spl}^{(n,\mathbb{d})}(E) \quad \text{for all} \quad t \in T\Big\} \Big/ \sim$$

where $\mathcal{F}_1 \sim \mathcal{F}_2$ if and only if there exists $\mathcal{L} \in \mathsf{Pic}(T)$ such that $\mathcal{F}_1 \cong \mathcal{F}_2 \otimes \mathrm{pr}_T^*(\mathcal{L})$. Then $\underline{\mathsf{M}}_E^{n,\mathbb{d}}$ is isomorphic to $\mathsf{Hom}_{\mathsf{Ans}}(- , G)$, where $G = \mathbb{C}^*$ for a cycle of projective lines and $G = \mathbb{C}$ in the other cases.

A proof of the first part of this conjecture in the case of cycles of projective lines was recently announced by Bodnarchuk and Drozd. Note that in this case, a description of simple vector bundles in terms of étale coverings is also known [18].

Let E_* be an irreducible component of the curve E. Provided a universal family $\mathcal{P}(n, \mathbb{d}) \in \mathsf{VB}(E \times G)$ of simple vector bundles of rank n and multi-degree $\mathbb{d}$ does exist, one can proceed in the same way as in the present section to end up with a solution $r_E^{(n,\mathbb{d},*)}$ of the associative Yang–Baxter equation with all the properties of Theorem 6.13.

Conjecture 6.16 is closely related to the study of moduli spaces of Simpson stable sheaves on degenerations of elliptic curves, see recent papers by Hernández Ruipérez, López Martín, Sánchez Gómez and Tejero Prieto [35], and Lowrey [43].

Conjecture 6.17. Let E be as in Conjecture 6.16 and $(n, \mathbb{d}) \in \mathbb{Z}_+ \times \mathbb{Z}^m$ be such that $\gcd(n, d_1 + \cdots + d_m) = 1$. Then there exists a polarization H of the curve E and a Hilbert polynomial $p(t) \in \mathbb{Q}[t]$ such that all simple vector bundles of rank n and multi-degree $\mathbb{d}$ are Simpson-stable with Hilbert polynomial equal to p. Moreover, the moduli space $M_E(n, \mathbb{d})$ is an open and dense subset of an irreducible component of the moduli space $M(p)$ of Simpson-stable sheaves with Hilbert polynomial p.

Since the exact combinatorics of simple vector bundles on reduced projective curves with trivial dualizing sheaf is still to be clarified and their relationship to Simpson-stable sheaves is not completely clear, we postpone a discussion of possible generalizations of the results of Section 6 to a future publication.

7. Action of the Jacobian and geometric associative r–matrices

Let $E = V(zy^2 - 4x^3 + g_2xz^2 + g_3z^3) \subseteq \mathbb{P}^2$ be a Weierstraß cubic curve and denote by $\breve{E}$ the regular part of E. Let $e \in \breve{E}$ be any point. In this section we are dealing with a single curve, not with a family of curves.

Fix a pair of coprime integers $(n, d) \in \mathbb{N} \times \mathbb{Z}$ and let $(M, \mathcal{P})$ be a pair which represents the moduli functor $\underline{\mathsf{M}}_E^{(n,d)}$. In the previous section, we have shown how to construct a tensor $r^\xi(v_1, v_2; y_1, y_2)$ satisfying the associative Yang-Baxter equation. In order to do so, we had to choose a point $m \in M$, a trivialization ξ of $\mathcal{P}$ over a neighbourhood of (m, e) and coordinates around $e \in E$ and around $m \in M$.

The main goal of this section is to show that it is possible to choose coordinates on M such that the associative r-matrix r^ξ is gauge equivalent to a solution r^ζ depending only on the difference $v = v_2 - v_1$ of the "vector bundle" spectral parameters. More precisely, we are going to prove that there are coordinates on M and there exists a gauge transformation $\phi : \big(\mathbb{C}^2_{(v,y)}, 0\big) \to \mathsf{GL}_n(\mathbb{C})$ such that the function

$$r^\zeta(v_1, v_2; y_1, y_2) = \big(\phi(v_1, y_1) \otimes \phi(v_2, y_2)\big) r^\xi(v_1, v_2; y_1, y_2) \big(\phi(v_2, y_1)^{-1} \otimes \phi(v_1, y_2)^{-1}\big)$$

is invariant under transformations $(v_1, v_2) \mapsto (v_1 + v, v_2 + v)$. In other words, there exists a trivialization ζ of the universal family $\mathcal{P}$ in a neighbourhood of the point $(m, e) \in M \times E$ such that we have

$$r^\zeta(v_1 + v, v_2 + v; y_1, y_2) = r^\zeta(v_1, v_2; y_1, y_2).$$

For simplicity of notation, we shall write $r^\zeta(v_1, v_2; y_1, y_2) = r^\zeta(v_1 - v_2; y_1, y_2) = r^\zeta(v; y_1, y_2)$, where $v = v_1 - v_2$. As it was explained in Section 2, the tensor $r^\zeta(v; y_1, y_2)$ satisfies the quantum Yang–Baxter equation and defines interesting first order differential operators.

The key idea to find such a distinguished trivialization ζ is to study the behaviour of the geometric r-matrix under the *action of the Jacobian* J on M. The coordinates on M are obtained from an isomorphism $M \longrightarrow J$ and a surjective homomorphism of groups $\mathbb{C} \longrightarrow J$. Using the isomorphism $M \longrightarrow J$ only, allows to give a coordinate free description of the "dependence on the difference of the v_i", because J has the structure of a group. The coordinates are used because they link the general discussion of this section with the explicit calculations of the subsequent sections.

The functors $\underline{\mathsf{M}}_E^{(n,d)}$, $\underline{\mathsf{Pic}}_E^0$ and $\underline{\mathsf{Pic}}_E^d$ are representable. Let $(M, \mathcal{P})$, $(J, \mathcal{L})$ and $(J^d, \mathcal{L}^{(d)})$ be spaces with universal families which represent these functors, so that we have isomorphisms of functors as follows

$$\alpha : \underline{\mathsf{M}}_E^{(n,d)} \longrightarrow \mathsf{Mor}(\,-\,, M),$$
$$\beta : \underline{\mathsf{Pic}}_E^0 \longrightarrow \mathsf{Mor}(\,-\,, J) \qquad \text{and} \qquad \beta^d : \underline{\mathsf{Pic}}_E^d \longrightarrow \mathsf{Mor}(\,-\,, J^d).$$

Recall that the product $F \times G : \mathsf{Ans} \longrightarrow \mathsf{Sets}$ of two functors $F, G : \mathsf{Ans} \longrightarrow \mathsf{Sets}$ is defined by $(F \times G)(S) = F(S) \times G(S)$. Because the tensor product of a stable vector bundle of rank n and degree d with a line bundle of degree zero is again a stable vector bundle of rank n and degree d, the Jacobian acts on the moduli space. On the functorial level, this action is described as a natural transformation of functors

$$\underline{\tau} : \underline{\mathsf{Pic}}_E^0 \times \underline{\mathsf{M}}_E^{(n,d)} \longrightarrow \underline{\mathsf{M}}_E^{(n,d)},$$

which is defined as $\underline{\tau}_S(\mathcal{N}, \mathcal{F}) = \mathcal{N} \otimes \mathcal{F}$ for any complex space S, any line bundle $\mathcal{N} \in \underline{\mathsf{Pic}}_E^0(S)$ and any vector bundle $\mathcal{F} \in \underline{\mathsf{M}}_E^{(n,d)}(S)$. The natural transformation $\underline{\tau}$ induces a morphism of complex spaces $\tau : J \times M \longrightarrow M$ making the following diagram commutative

$$\begin{tikzcd}
\underline{\mathsf{Pic}}_E^0 \times \underline{\mathsf{M}}_E^{(r,d)} \arrow[r, "\beta\times\alpha"] \arrow[d, "\underline{\tau}"'] & \mathsf{Mor}(\,-\,, J) \times \mathsf{Mor}(\,-\,, M) \arrow[r] & \mathsf{Mor}(\,-\,, J \times M) \arrow[d, "\tau_*"] \\
\underline{\mathsf{M}}_E^{(r,d)} \arrow[rr, "\alpha"] & & \mathsf{Mor}(\,-\,, M).
\end{tikzcd}$$

The unlabelled horizontal arrow is the isomorphism which results from the universal property of the product of two complex spaces. The morphism $J \times M \xrightarrow{\tau} M$ corresponds to the natural transformation τ_* using Yoneda's Lemma

$$\mathsf{Hom}\big(\mathsf{Mor}(\,-\,, J \times M), \mathsf{Mor}(\,-\,, M)\big) \cong \mathsf{Mor}(J \times M, M).$$

The following lemma describes the equivalence class of vector bundles on $J \times M$ which corresponds to the morphism $\tau \in \mathsf{Mor}(J \times M, M)$ under $\alpha_{J\times M}$.

Lemma 7.1. *Denote $\tilde{\tau} := \tau \times \mathrm{id}_E : J \times M \times E \longrightarrow M \times E$ and let p, q, t be the natural projections*

$$\begin{tikzcd} & J \times M \times E \arrow[dl, "p"'] \arrow[d, "q"] \arrow[dr, "t"] & \\ M \times E & J \times E & J \times M \end{tikzcd}$$

Then $p^*\mathcal{P} \otimes q^*\mathcal{L} \sim \tilde{\tau}^*\mathcal{P}$, i.e. there exists a line bundle $\mathcal{N} \in \mathsf{Pic}(J \times M)$ such that

$$p^*\mathcal{P} \otimes q^*\mathcal{L} \otimes t^*\mathcal{N} \cong \tilde{\tau}^*\mathcal{P}.$$

Proof. Note that we have the commutative diagram

$$\begin{tikzcd} & \mathsf{Mor}(J \times M, J) \times \mathsf{Mor}(J \times M, M) \arrow[dr, "\mathrm{can}"] & \\ \underline{\mathsf{Pic}}^0_E(J \times M) \times \underline{\mathsf{M}}_E^{(n,d)}(J \times M) \arrow[ur, "\beta\times\alpha"] \arrow[d, "\tau_{J\times M}"] & & \mathsf{Mor}(J \times M, J \times M) \arrow[d, "\tau_*"] \\ \underline{\mathsf{M}}_E^{(n,d)}(J \times M) \arrow[rr, "\alpha_{J\times M}"] & & \mathsf{Mor}(J \times M, M) \\ \underline{\mathsf{M}}_E^{(n,d)}(M) \arrow[u, "\tau^*"] \arrow[rr, "\alpha_M"] & & \mathsf{Mor}(M, M). \arrow[u, "\tau^*"'] \end{tikzcd}$$

Since $\tau_*(\mathrm{id}_{J\times M}) = \tau^*(\mathrm{id}_M) = \tau$, we obtain: $p^*\mathcal{P} \otimes q^*\mathcal{L} \sim (\tau \times \mathrm{id}_E)^*\mathcal{P}$. $\square$

Let o denote the neutral element of J and recall that we have chosen $e \in E$ and $m \in M$. Lemma 7.1 implies that there exist open neighbourhoods $o \in J' \subseteq J$ and $m \in M' \subseteq M$ such that $q^*\mathcal{L} \otimes p^*\mathcal{P}|_{J'\times M'\times E} \cong \tilde{\tau}^*\mathcal{P}|_{J'\times M'\times E}$. The following proposition is crucial.

Proposition 7.2. *Let E be a Weierstraß cubic curve, $M = M_E^{(n,d)}$ be the moduli space of stable vector bundles of rank n and degree d and J the Jacobian of E. Then there exist open neighbourhoods $o \in J' \subseteq J$ and $m \in M' \subseteq M$, trivializations $\xi : \mathcal{P}|_{M'\times E'} \longrightarrow \mathcal{O}^n_{M'\times E'}$, $\eta : \mathcal{L}|_{J'\times E'} \longrightarrow \mathcal{O}^n_{J'\times E'}$ and an isomorphism*

$$\varphi : q^*\mathcal{L} \otimes p^*\mathcal{P}|_{J'\times M'\times E} \longrightarrow \tilde{\tau}^*\mathcal{P}|_{J'\times M'\times E},$$

such that the following diagram in the category of vector bundles $\mathsf{VB}(J' \times M' \times E')$ *is commutative:*

$$\begin{tikzcd}
\big(q^*\mathcal{L} \otimes p^*\mathcal{P}\big)|_{J'\times M'\times E'} \arrow[r, "\varphi|_{J'\times M'\times E'}"] \arrow[d] & \tilde{\tau}^*\mathcal{P}|_{J'\times M'\times E'} \arrow[d] \\
q^*\big(\mathcal{L}|_{J'\times E'}\big) \otimes p^*\big(\mathcal{P}|_{M'\times E'}\big) \arrow[d, "q^*(\eta)\otimes p^*(\xi)"'] & \tilde{\tau}^*\big(\mathcal{P}|_{M'\times E'}\big) \arrow[dd, "\tilde{\tau}^*(\xi)"] \\
\mathcal{O}_{J'\times M'\times E'} \otimes \mathcal{O}^n_{J'\times M'\times E'} \arrow[d, "\mathsf{mult}"'] & \\
\mathcal{O}^n_{J'\times M'\times E'} \arrow[r, "I_n"] & \mathcal{O}^n_{J'\times M'\times E'},
\end{tikzcd}$$

where I_n *is the identity morphism.*

Proof. This proposition follows from a case-by-case analysis made below for each of the three types of Weierstraß cubic curves, see Proposition 8.11 and Theorem 9.44. $\square$

Remark 7.3. Note that we require the existence of a *global* isomorphism φ on the whole space $J' \times M' \times E$, although the condition on φ is *local*. The necessity of these assumptions is explained in the course of the proof of Theorem 7.5, which is one of the main results of this article.

To describe the correct coordinates on the moduli space M, we use a canonical isomorphism $M \longrightarrow J^d$ and a non-canonical one $J \longrightarrow J^d$, which depends on the chosen point $e \in E$. To define the canonical one, which is given by taking the determinant bundle, recall from Theorem 6.1 that we have an isomorphism of functors $\underline{\det} : \underline{\mathsf{M}}_E^{(r,d)} \longrightarrow \underline{\mathsf{Pic}}_E^d$. Using α and β^d, it defines an isomorphism of complex spaces $\det : M \longrightarrow J^d$, such that $\det^* \mathcal{L}^{(d)} \sim \det(\mathcal{P})$.

Lemma 7.4. *The following diagram is commutative*

$$\begin{tikzcd}
J \times M \arrow[r, "\tau"] \arrow[d, "\mathrm{id}_J\times\det"'] & M \arrow[d, "\det"] \\
J \times J^d \arrow[r, "\sigma"] & J^d,
\end{tikzcd}$$

where $\sigma : J \times J^d \longrightarrow J^d$ *corresponds (with the aid of* β *and* β^d*) to the natural transformation of functors* $\underline{\sigma} : \underline{\mathsf{Pic}}_E^0 \times \underline{\mathsf{Pic}}_E^d \longrightarrow \underline{\mathsf{Pic}}_E^d$ *which sends* $(\mathcal{L}', \mathcal{L}'')$ *to* $\mathcal{L}'^{\otimes n} \otimes \mathcal{L}''$.

Proof. This result follows from the isomorphism $\det(\mathcal{L} \otimes \mathcal{F}) \cong \mathcal{L}^{\otimes n} \otimes \det(\mathcal{F})$, where $\mathcal{F}$ is a vector bundle of rank n and $\mathcal{L}$ a line bundle. $\square$

The second isomorphism, $t^e : J \longrightarrow J^d$, is defined by taking the tensor product with $\mathcal{O}_E(de)$. More precisely, on functorial level, it is given by

$$\underline{t}^e_S(\mathcal{M}) = \mathcal{M} \otimes \mathcal{O}_{S\times E}\big(d(S \times \{e\})\big).$$

This gives us a commutative diagram

$$\begin{tikzcd} J \times J \arrow[r, "\sigma'"] \arrow[d, "\mathrm{id}_J \times t^e"'] & J \arrow[d, "t^e"] \\ J \times J^d \arrow[r, "\sigma"] & J^d, \end{tikzcd} \tag{25}$$

in which σ' is defined on functorial level by the same formula as σ in Lemma 7.4.

Finally, recall that the Jacobian $J = \mathsf{Pic}^0(E)$ has the following description:

$$J \cong \begin{cases} \mathbb{C}/\Gamma & \text{if E is elliptic,} \\ \mathbb{C}^* \cong \mathbb{C}/\mathbb{Z} & \text{if E is nodal,} \\ \mathbb{C} & \text{if E is cuspidal.} \end{cases}$$

In particular, in all three cases we have a surjective homomorphism of Lie groups $\pi_J : \mathbb{C} \longrightarrow J$. We combine π_J with the two isomorphisms $\det : M \longrightarrow J^d$ and $t^e : J \longrightarrow J^d$ to obtain a local isomorphism $\pi_M : \mathbb{C} \longrightarrow M$, which gives us local coordinates on M. This local isomorphism sits in the commutative diagram

$$\begin{tikzcd} \mathbb{C} \times \mathbb{C} \arrow[r, "\sigma"] \arrow[d, "\pi_J \times \pi_M"'] & \mathbb{C} \arrow[d, "\pi_M"] \\ J \times M \arrow[r, "\tau"] & M, \end{tikzcd} \qquad \text{where } \sigma(a,b) = na + b.$$

It is obtained by combining diagram (25) with the one in Lemma 7.4 and the local isomorphism $\pi_J : \mathbb{C} \longrightarrow J$, which is a homomorphism of groups.

Theorem 7.5. *Let E be a Weierstraß cubic curve, $\breve{E}$ its smooth part, $(n,d) \in \mathbb{N} \times \mathbb{Z}$ a pair of coprime integers, $M = M_E^{(n,d)}$ denote the moduli space of stable vector bundles of rank n and degree d and $J = \mathsf{Pic}^0(E)$ be the Jacobian of E. Let $\mathcal{P}|_{M'\times E'} \xrightarrow{\xi} \mathcal{O}^n_{M'\times E'}$ be a trivialization satisfying the conditions of Proposition 7.2. Then there exist coordinates on M' and on E', such that the corresponding associative r-matrix $r^\xi(v_1, v_2; y_1, y_2)$ satisfies*

$$r^\xi(v_1 + v, v_2 + v; y_1, y_2) = r^\xi(v_1, v_2; y_1, y_2).$$

Proof. Introduce the following notation. As in the previous section, for $i = 1, 2$ let

$$\pi_i : M \times M \times \breve{E} \times \breve{E} \times E \longrightarrow M \times E$$

be the canonical projections $\pi_i(v_1, v_2; y_1, y_2; y) = (v_i, y)$ and let

$$h_i : M \times M \times \breve{E} \times \breve{E} \longrightarrow M \times M \times \breve{E} \times \breve{E} \times E$$

be the canonical sections, given by $h_i(v_1, v_2; y_1, y_2) = (v_1, v_2; y_1, y_2; y_i)$. Let

$$\bar{\pi} : J \times M \times M \times \breve{E} \times \breve{E} \longrightarrow M \times M \times \breve{E} \times \breve{E}$$

be the canonical projection and

$$\bar{\tau} : J \times M \times M \times \breve{E} \times \breve{E} \longrightarrow M \times M \times \breve{E} \times \breve{E}$$

be the "diagonal" action of the Jacobian J, which is given by $\bar{\tau}(g; v_1, v_2; y_1, y_2) := \bigl(\tau(g, v_1), \tau(g, v_2); y_1, y_2\bigr)$. Let $\hat{\tau} = \bar{\tau} \times \mathrm{id}_E$ and $\hat{\pi} = \bar{\pi} \times \mathrm{id}_E$ be the morphisms

$$\hat{\tau}, \hat{\pi} :\ J \times M \times M \times \breve{E} \times \breve{E} \times E \longrightarrow M \times M \times \breve{E} \times \breve{E} \times E,$$

$\hat{q} : J \times M \times M \times \breve{E} \times \breve{E} \times E \longrightarrow J \times E$ the canonical projection. Finally, for $i = 1, 2$, define the projections

$$\hat{\pi}_i := \pi_i \circ \hat{\pi} : J \times M \times M \times \breve{E} \times \breve{E} \times E \longrightarrow M \times E$$

and the sections

$$\hat{h}_i := \mathrm{id}_J \times h_i : J \times M \times M \times \breve{E} \times \breve{E} \longrightarrow J \times M \times M \times \breve{E} \times \breve{E} \times E.$$

Using Theorem 5.17 and Proposition 5.24, we obtain the following commutative diagram of vector bundles on the complex manifold $U := J' \times M' \times M' \times \breve{E} \times \breve{E}$

$$\begin{tikzcd}
\bar{\pi}^* h_1^* \mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P}) \arrow[r, "\bar{\pi}^*\left(\tilde{r}_{h_1,h_2}^{\pi_1^*\mathcal{P},\pi_2^*\mathcal{P}}\right)"] \arrow[d, "\mathrm{can}"'] & \bar{\pi}^* h_2^* \mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P}) \arrow[d, "\mathrm{can}"] \\
\hat{h}_1^* \mathcal{H}om(\hat{\pi}_1^*\mathcal{P}, \hat{\pi}_2^*\mathcal{P}) \arrow[r, "\tilde{r}_{\hat{h}_1,\hat{h}_2}^{\hat{\pi}_1^*\mathcal{P},\hat{\pi}_2^*\mathcal{P}}"] \arrow[d, "\mathrm{can}"'] & \hat{h}_2^* \mathcal{H}om(\hat{\pi}_1^*\mathcal{P}, \hat{\pi}_2^*\mathcal{P}) \arrow[d, "\mathrm{can}"] \\
\hat{h}_1^* \mathcal{H}om(\hat{\pi}_1^*\mathcal{P} \otimes \hat{q}^*\mathcal{L}, \hat{\pi}_2^*\mathcal{P} \otimes \hat{q}^*\mathcal{L}) \arrow[r, "\tilde{r}_{\hat{h}_1,\hat{h}_2}^{\hat{\pi}_1^*\mathcal{P}\otimes\hat{q}^*\mathcal{L},\hat{\pi}_2^*\mathcal{P}\otimes\hat{q}^*\mathcal{L}}"] \arrow[d, "\hat{h}_1^*\left(\mathrm{cnj}(\hat{\varphi}_1,\hat{\varphi}_2)\right)"'] & \hat{h}_2^* \mathcal{H}om(\hat{\pi}_1^*\mathcal{P} \otimes \hat{q}^*\mathcal{L}, \hat{\pi}_2^*\mathcal{P} \otimes \hat{q}^*\mathcal{L}) \arrow[d, "\hat{h}_2^*\left(\mathrm{cnj}(\hat{\varphi}_1,\hat{\varphi}_2)\right)"] \\
\hat{h}_1^* \mathcal{H}om(\hat{\tau}^*\pi_1^*\mathcal{P}, \hat{\tau}^*\pi_2^*\mathcal{P}) \arrow[r, "\tilde{r}_{\hat{h}_1,\hat{h}_2}^{\hat{\tau}^*\pi_1^*\mathcal{P},\hat{\tau}^*\pi_2^*\mathcal{P}}"] \arrow[d, "\mathrm{can}"'] & \hat{h}_2^* \mathcal{H}om(\hat{\tau}^*\pi_1^*\mathcal{P}, \hat{\tau}^*\pi_2^*\mathcal{P}) \arrow[d, "\mathrm{can}"] \\
\bar{\tau}^* h_1^* \mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P}) \arrow[r, "\bar{\tau}^*\left(\tilde{r}_{h_1,h_2}^{\pi_1^*\mathcal{P},\pi_2^*\mathcal{P}}\right)"] & \bar{\tau}^* h_2^* \mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P}).
\end{tikzcd}$$

In this diagram, the isomorphisms of vector bundles $\hat{\varphi}_i : \hat{q}^*\mathcal{L} \otimes \hat{\pi}_i^*\mathcal{P} \longrightarrow \hat{\tau}^*\pi_i^*\mathcal{P}$ are defined as follows. For $i = 1, 2$ let

$$p_i : J' \times M' \times M' \times \breve{E} \times \breve{E} \times E \longrightarrow J' \times M' \times E$$

be the natural projection $p_i(g; v_1, v_2; y_1, y_2; y) = (g, v_i, y)$. Let $\xi : \mathcal{P}|_{M' \times E'} \to \mathcal{O}^n_{M' \times E'}$ be a trivialization and $\varphi : q^*\mathcal{L} \otimes p^*\mathcal{P}|_{J' \times M' \times E} \longrightarrow \tilde{\tau}^*\mathcal{P}_{J' \times M' \times E}$ an isomorphism, both satisfying the conditions of Proposition 7.2. Then we set $\hat{\varphi}_i$ to be the composition of morphisms of vector bundles on $U \times E$

$$\hat{q}^*\mathcal{L} \otimes \hat{\pi}_i^*\mathcal{P} \xrightarrow{\text{can}} p_i^*(q^*\mathcal{L} \otimes p^*\mathcal{P}) \xrightarrow{p_i^*(\varphi)} p_i^*\tilde{\tau}^*\mathcal{P} \xrightarrow{\text{can}} \hat{\tau}^*\pi_i^*\mathcal{P}.$$

Note that the commutative square which involves the $\hat{\varphi}_i$ is only available if $\hat{\varphi}_i$ is defined on $U \times E$ and not only on $U \times E'$. The reason is that $\underline{\mathrm{res}}$ in the definition of $\tilde{r}$ would not be an isomorphism on E' (see the proof of Theorem 5.17).

Let O denote the ring of holomorphic functions on $U' := J' \times M' \times M' \times E' \times E' \subset U$. With the aid of the trivialization ξ, for $i = 1, 2$ we get isomorphisms

$$H^0\left(U', \bar{\pi}^* h_i^* \mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\right) \cong \mathsf{Mat}_{n\times n}(O),$$
$$H^0\left(U', \bar{\tau}^* h_i^* \mathcal{H}om(\pi_1^*\mathcal{P}, \pi_2^*\mathcal{P})\right) \cong \mathsf{Mat}_{n\times n}(O).$$

Under these identifications, we can write the morphisms $H^0\left(\bar{\pi}^*\left(\tilde{r}_{h_1,h_2}^{\pi_1^*\mathcal{P},\pi_2^*\mathcal{P}}\right)\right)$ and $H^0\left(\bar{\tau}^*\left(\tilde{r}_{h_1,h_2}^{\pi_1^*\mathcal{P},\pi_2^*\mathcal{P}}\right)\right)$ as matrices $\tilde{\mathfrak{r}}$ and $\tilde{\mathfrak{r}}'$, such that the large diagram we set up earlier in this proof boils down to

$$\begin{tikzcd}
\mathsf{Mat}_{n\times n}(O) \arrow[r, "\tilde{\mathfrak{r}}"] \arrow[d, "\mathsf{Id}"'] & \mathsf{Mat}_{n\times n}(O) \arrow[d, "\mathsf{Id}"] \\
\mathsf{Mat}_{n\times n}(O) \arrow[r, "\tilde{\mathfrak{r}}'"] & \mathsf{Mat}_{n\times n}(O),
\end{tikzcd} \qquad \text{hence } \tilde{\mathfrak{r}} = \tilde{\mathfrak{r}}'.$$

That the vertical arrows are identities is a consequence of Proposition 7.2.

If we choose arbitrary coordinates on J', M' and E', we have $\tilde{\mathfrak{r}}(g; v_1, v_2; y_1, y_2) = \tilde{r}^\xi(v_1, v_2; y_1, y_2)$ and $\tilde{\mathfrak{r}}'(g; v_1, v_2; y_1, y_2) = \tilde{r}^\xi\big(\tau(g, v_1), \tau(g, v_2); y_1, y_2\big)$. If we use the special coordinates on J' and M' introduce above with the aid of $\pi_J : \mathbb{C} \longrightarrow J$, we obtain $\tilde{r}^\xi\big(\tau(g, v_1), \tau(g, v_2); y_1, y_2\big) = \tilde{r}^\xi(v_1 + ng, v_2 + ng; y_1, y_2)$. This implies

$$\tilde{r}^\xi(v_1 + ng, v_2 + ng; y_1, y_2) = \tilde{\mathfrak{r}}'(g; v_1, v_2; y_1, y_2) = \tilde{\mathfrak{r}}(g; v_1, v_2; y_1, y_2) = \tilde{r}^\xi(v_1, v_2; y_1, y_2),$$

which gives the desired property of the tensor $\tilde{r}^\xi(v_1, v_2; y_1, y_2)$. $\square$

Remark 7.6. Proposition 7.2 and Theorem 7.5 remain valid if the open neighbourhoods J' and M' are replaced by the maps $\pi_J : \mathbb{C} \longrightarrow J$ and $\pi_M : \mathbb{C} \longrightarrow M$. Similarly, by identifying $\breve{E}$ with J^1 and then proceeding as in the case of M, we may define a map $\pi_E : \mathbb{C} \longrightarrow \breve{E}$, which can be used instead of E'. The advantage of this point of view is that v_i, v, y_i can be arbitrary complex numbers in the statement of Theorem 7.5, whereas, if small neighbourhoods J', M' and E' are used, we have to make sure that v_i and $v_i + v$ are in M' and $y_i \in E'$.

Remark 7.7. Unfortunately, we have not found a "conceptual way" to prove Proposition 7.2, without going to a case-by-case analysis. As a consequence, we do not know whether this result generalizes to the relative case, when we replace a Weierstraß curve E by the Weierstraß fibration $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$.

Motivated by the corresponding result for the classical r–matrices [9], it is natural to conjecture, that the statement of Theorem 7.5 holds for the other pair of spectral variables, the "skyscraper" variables (y_1, y_2). Namely, there should exist coordinates on E and a trivialization ξ of the universal family $\mathcal{P}$ such that we have

$$r^{\xi}(v_1+v, v_2+v; y_1+y, y_2+y) = r^{\xi}(v_1, v_2; y_1, y_2).$$

Definition 7.8. Let $r(v; y_1, y_2) \in \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ be a non-degenerate unitary solution of the associative Yang–Baxter equation such that there exists the limit $\bar{r}(y_1, y_2) = \lim\limits_{v\to e}\bigl(\mathrm{pr}\otimes\mathrm{pr}\bigr)r(v; y_1, y_2)$. We say that r is of *elliptic type* if $\bar{r}$ is an elliptic classical r-matrix, of *trigonometric type* if $\bar{r}$ is trigonometric and of *rational type* if $\bar{r}$ is rational.

It was shown by Polishchuk [56, 57] that in the case of elliptic curves one always gets an associative r-matrix of elliptic type and in the case of Kodaira cycles a solution of trigonometric type.

Remark 7.9. It is natural to conjecture that for any pair of coprime integers (n, d) the geometric r-matrix corresponding to a cuspidal cubic curve always is of rational type.

The goal of the following three sections is to get an explicit form of the geometric r-matrix attached to the Weierstraß fibration $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$ and the pair $(n, d) = (2, 1)$ at any given point $(g_2, g_3) \in \mathbb{C}^2$.

8. Elliptic solutions of the associative Yang–Baxter equation

In this section we are going to compute the solution of the associative Yang–Baxter equation and the corresponding classical r–matrix, obtained from the universal family of stable vector bundles of rank two and degree one on a smooth elliptic curve. In [56, Section 2], Polishchuk computed the corresponding triple Massey products using homological mirror symmetry and formulae for higher products in the Fukaya category of an elliptic curve.

It is very instructive, however, to carry out a direct computation of the geometric triple Massey products for an elliptic curve, independent of homological mirror symmetry. This approach allows us to express the resulting associative r–matrix in terms of Jacobi's theta-functions and the corresponding classical r–matrix in terms of the elliptic functions $\mathrm{cn}(z)$, $\mathrm{sn}(z)$ and $\mathrm{dn}(z)$.

In order to proceed with the necessary calculations we recall some standard results about holomorphic vector bundles on one-dimensional complex tori, a description of morphisms between them in terms of theta-functions etc.

8.1. **Vector bundles on a one-dimensional complex torus.**
We start with some classical results about vector bundles on smooth elliptic curves.

Theorem 8.1 (Atiyah, Theorem 7 in [5])**.** *Let E be a smooth elliptic curve over $\mathbb{C}$ and $\mathcal{V}$ a vector bundle on E.*

- *If* $\mathsf{End}_E(\mathcal{V}) = \mathbb{C}$ *then* $\gcd\big(\mathrm{rk}(\mathcal{V}), \deg(\mathcal{V})\big) = 1$*,* $\mathcal{V}$ *is* stable *and is determined by* $\big(\mathrm{rk}(\mathcal{V}), \deg(\mathcal{V}), \det(\mathcal{V})\big) \in \mathbb{N}\times\mathbb{Z}\times E$*, where we use an isomorphism* $\mathsf{Pic}^d(E) \cong E$*.*
- *If* $\mathcal{V}$ *is indecomposable and* $m = \gcd\big(\mathrm{rk}(V), \deg(V)\big)$ *then there exists a unique stable vector bundle* $\mathcal{V}'$ *such that* $\mathcal{V} = \mathcal{V}' \otimes \mathcal{A}_m$*, where* $\mathcal{A}_m$ *is the indecomposable vector bundle of rank* m *and degree* 0 *recursively defined by non-split the exact sequences*
$$0 \longrightarrow \mathcal{A}_{m-1} \longrightarrow \mathcal{A}_m \longrightarrow \mathcal{O} \longrightarrow 0, \qquad\qquad \mathcal{A}_1 = \mathcal{O}.$$

In the complex-analytic case, one can give an explicit description of the stable holomorphic vector bundles on a one-dimensional complex torus.

Theorem 8.2 (Oda, Theorem 1.2 in [52])**.** *Let E be an elliptic curve and $\pi_n : E' \to E$ be an étale covering of degree n.*

- *If* $\mathcal{V}$ *is a stable vector bundle on* E *of rank* n *and degree* d*, then there exists a line bundle* $\mathcal{L} \in \mathsf{Pic}^d(E')$ *such that* $\mathcal{V} \cong \pi_{n*}(\mathcal{L})$*. Conversely, if* $\gcd(n, d) = 1$*, then for any* $\mathcal{L} \in \mathsf{Pic}^d(E')$ *the vector bundle* $\mathcal{V} \cong \pi_{n*}(\mathcal{L})$ *is stable of rank* n *and degree* d*.*
- *If* $\mathcal{L}, \mathcal{N} \in \mathsf{Pic}^d(E')$ *satisfy* $\pi_{n*}(\mathcal{L}) \cong \pi_{n*}(\mathcal{N})$*, then* $(\mathcal{L} \otimes \mathcal{N}^{-1})^{\otimes n} \cong \mathcal{O}_{E'}$*.*

A very convenient way to carry out calculations with vector bundles on complex tori is provided by the theory of automorphy factors, see [47] or [50, Section I.2].

Definition 8.3. Let $\tau \in \mathbb{C}$ be a complex number such that $\mathrm{Im}(\tau) > 0$. The category of *automorphy factors* AF_τ is defined as follows. Its objects are pairs (Φ, n) where $n \geq 0$ is an integer and $\Phi : \mathbb{C} \to \mathsf{GL}_n(\mathbb{C})$ is a holomorphic function such that for all $z \in \mathbb{C}$ we have: $\Phi(z+1) = \Phi(z)$. Given two automorphy factors (Φ, n) and (Ψ, m), we define
$$\mathsf{Hom}_{\mathsf{AF}_\tau}\big((\Phi, n), (\Psi, m)\big) = \left\{ A : \mathbb{C} \to \mathsf{Mat}_{m\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} A \quad \text{is holomorphic} \\ A(z+1) = A(z) \\ A(z+\tau)\Phi(z) = \Psi(z)A(z) \end{array} \right\}$$
and the composition of morphisms in AF_τ is given by the multiplication of matrices. In what follows, we shall frequently denote the object (Φ, n) of AF_τ by Φ. Note that one can define an interior tensor product in the category AF_τ induced by the tensor product of matrices.

Let $\Lambda = \Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}^2$ be the lattice defined by τ, $E = E_\tau = \mathbb{C}/\Lambda_\tau$ the corresponding complex torus and $\pi : \mathbb{C} \to E$ the universal covering of E. For an object (Φ, n) of the category AF_τ we define the sheaf $\mathcal{E}(\Phi)$ of $\mathcal{O}_E$–modules and an embedding of sheaves $\mathsf{m}_\Phi : \mathcal{E}(\Phi) \to \pi_*\mathcal{O}_{\mathbb{C}}^n$ by the following rule.

The open subsets $U \subset E$ for which all connected components of $\pi^{-1}(U)$ map isomorphically to U, form a basis of the topology of E. For such U, we let U_0 be a connected component of $\pi^{-1}(U)$ and denote $U_\gamma = \gamma + U_0$ for all $\gamma \in \Lambda_\tau$. Then $\pi_*\mathcal{O}_{\mathbb{C}}^n\bigl(U\bigr) = \prod_{\gamma\in\Lambda} \mathcal{O}_{\mathbb{C}}^n(U_\gamma)$ and we define

$$\mathcal{E}(\Phi)\bigl(U\bigr) := \left\{ (F_\gamma(z))_{\gamma\in\Lambda} \in \pi_*(\mathcal{O}_{\mathbb{C}}^n)\bigl(U\bigr) \;\middle|\; \begin{array}{l} F_{\gamma+1}(z+1) = F_\gamma(z) \\ F_{\gamma+\tau}(z+\tau) = \Phi(z)F_\gamma(z) \end{array} \right\}.$$

By m_Φ we denote the canonical embedding $\mathcal{E}(\Phi) \subset \pi_*\mathcal{O}_{\mathbb{C}}^n$. The next theorem plays a key role in our computation of elliptic r-matrices.

Theorem 8.4. *In the notations as above the following properties are true.*

- *Let (Φ, n) be an object of AF_τ then the corresponding sheaf $\mathcal{E}(\Phi)$ is locally free of rank n.*
- *The map $(\Phi, n) \mapsto \mathcal{E}(\Phi)$ extends to a functor $\mathbb{F} : \mathsf{AF}_\tau \longrightarrow \mathsf{VB}(E_\tau)$ which is an equivalence of categories*
- *The functor $\mathbb{F}$ commutes with tensor products: $\mathcal{E}(\Phi \otimes \Psi) \cong \mathcal{E}(\Phi) \otimes \mathcal{E}(\Psi)$.*
- *Let $\mathsf{For} : \mathsf{AF}_\tau \longrightarrow \mathsf{VB}(\mathbb{C})$ be the forgetful functor, i.e. $\mathsf{For}(\Phi, n) = \mathcal{O}_{\mathbb{C}}^n$ and $\mathsf{For}(A) = A$ for any object (Φ, n) and any morphism A. Then there is an isomorphism of functors $\gamma : \pi^* \circ \mathbb{F} \longrightarrow \mathsf{For}$, where for an automorphy factor Φ we set γ_Φ to be the composition $\pi^*\mathcal{E}(\Phi) \xrightarrow{\pi^*(\mathsf{m}_\Phi)} \pi^*\pi_*\mathcal{O}_{\mathbb{C}}^n \xrightarrow{\text{can}} \mathcal{O}_{\mathbb{C}}^n$.*
- *The natural transformation γ is compatible with tensor products, i.e. for any pair $(\Phi, n), (\Psi, m)$ of automorphy factors we have a commutative diagram:*

$$\begin{tikzcd} \pi^*\bigl(\mathcal{E}(\Phi \otimes \Psi)\bigr) \arrow[r, "\gamma_{\Phi\otimes\Psi}"] \arrow[d, "\cong"'] & \mathcal{O}_{\mathbb{C}}^{mn} \\ \pi^*\mathcal{E}(\Phi) \otimes \pi^*\mathcal{E}(\Psi) \arrow[r, "\gamma_\Phi \otimes \gamma_\Psi"] & \mathcal{O}_{\mathbb{C}}^m \otimes \mathcal{O}_{\mathbb{C}}^n \arrow[u, "\mathsf{mult}"'] \end{tikzcd}$$

- *If $\pi_n : E_{n\tau} \to E_\tau$ is the étale covering given by the inclusion of lattices $\Lambda_{n\tau} \subset \Lambda_\tau$, then $\pi_n^*\bigl(\mathcal{E}(\Phi)\bigr) \cong \mathcal{E}(\widetilde{\Phi})$, where $\widetilde{\Phi}(z) := \Phi(z+(n-1)\tau) \cdot \ldots \cdot \Phi(z+\tau)\Phi(z)$.*
- *The direct image $\pi_{n*}\bigl(\mathcal{E}(\Phi, m)\bigr) \cong \mathcal{E}(\widetilde{\Phi}, mn)$ of a vector bundle $\mathcal{E}(\Phi, m)$ is given by the automorphy factor*[3]

$$\widetilde{\Phi} = \begin{pmatrix} 0 & I_m & 0 & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & I_m \\ \Phi & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

In particular, if $\Phi(z)$ is an automorphy factor and $A : \mathbb{C} \to \mathsf{GL}_n(\mathbb{C})$ is a holomorphic function such that $A(z+1) = A(z)$, then $\Psi(z) = A(z+\tau)^{-1}\Phi(z)A(z)$ defines an isomorphic locally free sheaf $\mathcal{E}(\Phi) \cong \mathcal{E}(\Psi)$.

[3] we thank Oleksandr Iena for helping us at this point

Remark 8.5. There is another way to describe the functor $\mathbb{F}$. Let (Φ, n) be an automorphy factor then the corresponding holomorphic vector bundle $\mathcal{E}(\Phi)$ can be defined as the quotient $\mathbb{C} \times \mathbb{C}^n/\sim$, where the equivalence relation is generated by $(z, v) \sim (z+1, v) \sim \bigl(z+\tau, \Phi(z)v\bigr)$. Using this description, we have the following commutative diagram of complex manifolds

$$\begin{tikzcd}
\mathbb{C} \times \mathbb{C}^n \arrow[r] \arrow[d, "\mathrm{pr}_1"'] & \mathcal{E}(\Phi) = \mathbb{C} \times \mathbb{C}^n/\sim \arrow[d] \\
\mathbb{C} \arrow[r, "\pi"] & E.
\end{tikzcd}$$

The natural transformation γ can be constructed using the fact that this diagram is Cartesian.

Our next goal is to give a description of those automorphy factors which correspond to indecomposable vector bundles on E. To do this, recall that the holomorphic morphism $\mathbb{C} \longrightarrow \mathbb{C}^*$ sending z to $\exp(2\pi i z)$ identifies $\mathbb{C}^*$ with $\mathbb{C}/\mathbb{Z}$ and maps τ to q^2, where $q = \exp(\pi i \tau)$. Hence, it induces an isomorphism $E \cong \mathbb{C}^*/q^2$, where the quotient is formed modulo the multiplicative subgroup generated by q^2.

Note that in the case of line bundles, automorphy factors are holomorphic functions $\varphi : \mathbb{C} \longrightarrow \mathbb{C}^*$ which satisfy $\varphi(z+1) = \varphi(z)$. In what follows we shall use the notation $\mathcal{L}(\varphi) := \mathcal{E}(\varphi)$. Line bundles of degree zero can be given by constant automorphy factors, for example $\mathcal{L}(1) = \mathcal{O}_E$. Because the function $a(z) = \exp(2\pi i z)$ satisfies $a(z+1) = a(z)$ and $a(z+\tau) = q^2 a(z)$ with $q = \exp(\pi i \tau)$ as above, the constants $\varphi \in \mathbb{C}^*$ and $q^2\varphi$ define isomorphic line bundles on E. In fact, the map $E := \mathbb{C}^*/q^2 \longrightarrow \mathsf{Pic}^0(E)$ sending $\varphi \in \mathbb{C}^*$ to $\mathcal{L}(\varphi) \in \mathsf{Pic}^0(E)$, is an isomorphism.

To describe line bundles of non-zero degree, we denote $p_0 = \dfrac{1+\tau}{2} \in E = \mathbb{C}/\Lambda$. The automorphy factor

$$\varphi_0(z) = \exp(-\pi i \tau - 2\pi i z)$$

satisfies $\mathcal{L}(\varphi_0) = \mathcal{O}_E(p_0)$. To see this, recall that, by definition,

$$H^0\bigl(\mathcal{L}(\varphi_0)\bigr) \cong \mathsf{Hom}\bigl(\mathcal{L}(1), \mathcal{L}(\varphi_0)\bigr) = \left\{ f : \mathbb{C} \to \mathbb{C} \;\middle|\; \begin{array}{l} f \quad \text{is holomorphic} \\ f(z+1) = f(z) \\ f(z+\tau) = \varphi_0(z) f(z) \end{array} \right\}$$

and that this vector space is *one-dimensional* and is generated by the *basic theta function*

$$\theta(z|\tau) = \theta_3(z|\tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z),$$

see for example [49]. It is well-known that $\theta(z|\tau)$ vanishes at $p_0 = \dfrac{1+\tau}{2}$. Moreover, this is the only zero in the fundamental parallelogram of Λ_τ. Hence, $H^0\bigl(\mathcal{L}(\varphi_0)\bigr) \cong \mathbb{C}$ and by the Riemann-Roch theorem the line bundle $\mathcal{L}(\varphi_0)$ has degree one. Moreover, if $\underline{\theta}(z|\tau)$ it its non-zero global section then $\mathsf{div}\big(\underline{\theta}(z|\tau)\big) = [p_0]$ and hence $\mathcal{L}(\varphi_0) \cong \mathcal{O}_E(p_0)$. Because $\theta\left(z + \frac{1+\tau}{2} - x \middle| \tau\right)$ has its unique zero at $x \in E$, we obtain

$$\mathcal{O}_E(x) \cong \mathcal{L}\left(t^*_{\frac{1+\tau}{2}-x}\varphi_0\right). \tag{26}$$

where $t_x^*\varphi_0(z) := \varphi_0(z+x)$. This gives a complete description of $\mathsf{Pic}^1(E)$.

Finally, any line bundle of degree d can be written as $\mathcal{O}_E\big([(d-1)p_0] + [p_0 - x]\big)$ for some point $x \in E$. To complete the description of $\mathsf{Pic}(E)$, it remains to observe that

$$\mathcal{O}_E\big([p_0 - x] + (d-1)[p_0]\big) \cong \mathcal{L}\big(t_x^*\varphi_0 \cdot \varphi_0^{d-1}\big).$$

Our next aim is to find an explicit family of automorphy factors describing the set of stable vector bundles of rank n and degree d on E, where $\gcd(n,d) = 1$. Interpreting Oda's description from Theorem 8.2 in terms of automorphy factors we immediately obtain the following proposition.

Proposition 8.6. *Let $(n,d) \in \mathbb{N} \times \mathbb{Z}$ be coprime. For $x \in \mathbb{C}/\langle 1, \tau\rangle$ let $\widetilde{\varphi}_{n,d}(z,x) := \exp(-\pi i n d\tau - 2\pi i d z - 2\pi i x)$. Then the family of automorphy factors*

$$\widetilde{\Phi}_{n,d}(z,x) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \widetilde{\varphi}_{n,d} & 0 & 0 & \dots & 0 \end{pmatrix}$$

describes the set of stable vector bundles of rank n and degree d on E.

However, these automorphy factors are not compatible with the action of the Jacobian $\mathsf{Pic}^0(E)$. In order to overcome this problem, denote $q_{\frac{x}{n}} = \exp\left(-\frac{2\pi i x}{n}\right)$ and let

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q_{\frac{x}{n}} & 0 & \dots & 0 \\ 0 & 0 & q_{\frac{x}{n}}^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q_{\frac{x}{n}}^{n-1} \end{pmatrix}.$$

Then $A^{-1}\widetilde{\Phi}_{n,d}A =: \Phi_{n,d}$ is the following matrix-valued function:

$$\Phi_{n,d}(z,x) = \begin{pmatrix} 0 & q_{\frac{x}{n}} & 0 & \dots & 0 \\ 0 & 0 & q_{\frac{x}{n}} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & q_{\frac{x}{n}} \\ q_{\frac{x}{n}}\varphi_{n,d} & 0 & 0 & \dots & 0 \end{pmatrix} = q_{\frac{x}{n}} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \varphi_n^d & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{27}$$

where $\varphi_n(z) = \exp(-\pi i n\tau - 2\pi i z)$ and $\varphi_{n,d} = \varphi_n^d$. Note that we have the equality

$$\exp(-2\pi i y) \cdot \Phi_{n,d}(z,x) = \Phi_{n,d}(z, x+ny).$$

Lemma 8.7. *In the notations as above, for two points $x, x' \in \mathbb{C}/\langle 1, n\tau\rangle$ we have: $\mathcal{E}\bigl(\widetilde{\Phi}_{n,d}(z,x)\bigr) \cong \mathcal{E}\bigl(\widetilde{\Phi}_{n,d}(z,x')\bigr)$ if and only if $x - x' \in \Lambda_\tau$.*

Proof. By Theorem 8.2, the vector bundle $\mathcal{E}^x := \mathcal{E}\bigl(\widetilde{\Phi}_{n,d}(z,x)\bigr)$ is stable of rank n and degree d for any point $x \in E_{n\tau}$. By [5, Theorem 7] the Jacobian $\mathsf{Pic}^0(E)$ acts transitively on the moduli space $M_E^{(n,d)}$. Moreover, $\mathcal{E}^x \cong \mathcal{E}^x \otimes \mathcal{L}$ for a line bundle $\mathcal{L} \in \mathsf{Pic}^0(E)$ if and only if $\mathcal{L}^{\otimes n} \cong \mathcal{O}$. For $a \in \mathbb{C}$, the line bundle $\mathcal{L}^a := \mathcal{L}\bigl(\exp(2\pi i a)\bigr)$ fulfills the property $\mathcal{L}^{\otimes n} \cong \mathcal{O}$ precisely if $na \in \Lambda_\tau$. Observe that $\mathcal{L}^a \otimes \mathcal{E}^x \cong \mathcal{E}^{x+na}$. In particular, it shows that $\mathcal{E}^x \cong \mathcal{E}^{x+a+b\tau}$ for any $x \in E_{n\tau}$ and $a, b \in \mathbb{Z}$. □

Our next goal is to explain how the language of automorphy factors can be used to describe a universal family of stable vector bundles of rank n and degree d on E as well as to construct a trivialization of it. In order to do this, we need the following generalization of Theorem 8.4.

As usual, let $\tau \in \mathbb{C}$ be such that $\mathrm{Im}(\tau) > 0$ and M be a complex manifold. Then we define a category $\mathsf{AF}_\tau(M)$, whose objects are pairs (Φ, n), where $n \geq 1$ is an integer and Φ is a holomorphic function $\Phi : \mathbb{C} \times M \longrightarrow \mathsf{GL}_n(\mathbb{C})$ such that $\Phi(z+1, y) = \Phi(z, y)$ for all $(z, y) \in \mathbb{C} \times M$. For a pair of automorphy factors (Φ, n) and (Ψ, m) we set

$$\mathsf{Hom}_{\mathsf{AF}_\tau(M)}\bigl(\Phi, \Psi\bigr) = \left\{ \mathbb{C} \times M \xrightarrow{A} \mathsf{Mat}_{m\times n}(\mathbb{C}) \;\middle|\; \begin{array}{l} A \quad \text{is holomorphic} \\ A(z+1, y) = A(z, y) \\ A(z+\tau, y)\Phi(z, y) = \Psi(z, y)A(z, y) \end{array} \right\}$$

and the composition of morphisms in $\mathsf{AF}_\tau(M)$ is given by the multiplication of matrices. As before, we have a fully faithful functor

$$\mathbb{F}_M : \mathsf{AF}_\tau(M) \longrightarrow \mathsf{VB}(E \times M)$$

mapping an automorphy factor (Φ, n) to the subsheaf $\mathcal{E}(\Phi)$ of the sheaf $\pi_{M*}\mathcal{O}^n_{\mathbb{C}\times M}$, where $\pi_M = \pi \times \mathrm{id} : \mathbb{C} \times M \longrightarrow E \times M$. The sheaf $\mathcal{E}(\Phi)$ is defined exactly as in the absolute case. Moreover, $\mathbb{F}_M$ is dense (hence an equivalence of categories) if, for example, $M \cong \Delta_1 \times \cdots \times \Delta_m \subseteq \mathbb{C}^m$, where each $\Delta_i \subseteq \mathbb{C}, 1 \leq i \leq m$ is either an open disc or $\mathbb{C}$ itself. This functor maps the tensor product of automorphy factors into the tensor product of the corresponding vector bundles. Next, there is an isomorphism of functors $\gamma : \pi_M^* \circ \mathbb{F} \longrightarrow \mathsf{For}$, where $\mathsf{For} : \mathsf{AF}_\tau(M) \longrightarrow \mathsf{VB}(\mathbb{C} \times M)$ is the forgetful functor. For an automorphy factor Φ the morphism γ_Φ is the composition

$$\pi_M^*\mathcal{E}(\Phi) \xrightarrow{\pi_M^*(\mathsf{m}_\Phi)} \pi_M^*\pi_{M*}\mathcal{O}^n_{\mathbb{C}\times M} \xrightarrow{\mathrm{can}} \mathcal{O}^n_{\mathbb{C}\times M}.$$

Let U be an open subset of E such that there exists a connected component $\widetilde{U}$ of $\pi^{-1}(U)$ which maps isomorphically to U. Hence, $\pi : \widetilde{U} \longrightarrow U$ is an isomorphism of

Riemann surfaces and the morphism γ_Φ induces a trivialization of the vector bundle $\mathcal{E}(\Phi)|_{U\times M}$.

It is important to note that the natural transformation γ is compatible with tensor products, i.e. for any pair (Φ, n) and (Ψ, m) of automorphy factors we have a commutative diagram:

$$\begin{tikzcd}
\pi^*\big(\mathcal{E}(\Phi\otimes\Psi)\big) \arrow[rr, "\gamma_{\Phi\otimes\Psi}"] \arrow[d, "\cong"'] & & \mathcal{O}^{mn}_{\mathbb{C}\times M} \\
\pi^*\mathcal{E}(\Phi)\otimes\pi^*\mathcal{E}(\Psi) \arrow[rr, "\gamma_\Phi\otimes\gamma_\Psi"] & & \mathcal{O}^{m}_{\mathbb{C}\times M}\otimes\mathcal{O}^{n}_{\mathbb{C}\times M}. \arrow[u, "\mathsf{mult}"']
\end{tikzcd}$$

Moreover, any holomorphic map $f: M \longrightarrow N$ of open domains induces a functor $f^*: \mathsf{AF}_\tau(N) \longrightarrow \mathsf{AF}_\tau(M)$ mapping an automorphy factor $\Phi: \mathbb{C}\times N \to \mathsf{GL}_n(\mathbb{C})$ to the automorphy factor $f^*(\Phi): \mathbb{C}\times M \xrightarrow{\mathrm{id}\times f} \mathbb{C}\times N \xrightarrow{\Phi} \mathsf{GL}_n(\mathbb{C})$. In these terms, we have the following diagram of functors

$$\begin{tikzcd}
\mathsf{AF}_\tau(N) \arrow[r, "f^*"] \arrow[d, "\mathbb{F}_N"'] & \mathsf{AF}_\tau(M) \arrow[d, "\mathbb{F}_M"] \\
\mathsf{VB}(E\times N) \arrow[r, "(1\times f)^*"] & \mathsf{VB}(E\times M),
\end{tikzcd}$$

where both compositions $(\mathrm{id}\times f)^*\circ\mathbb{F}_N$ and $\mathbb{F}_M\circ f^*$ are canonically isomorphic.

Our next goal is to give an explicit description of a universal family of stable vector bundles of rank n and degree d on the complex torus $E = E_\tau$. In what follows, $M = E_\tau$ stands for the moduli space of such bundles. Consider a pair of matrix-valued functions $\Phi, \Psi: \mathbb{C}\times\mathbb{C} \to \mathsf{GL}_n(\mathbb{C})$ given by the formulae:

$$\Phi(z,x) = \exp(-2\pi i x)\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \varphi_n^d & 0 & 0 & \dots & 0 \end{pmatrix} \qquad \Psi(z,x) = \exp(-2\pi i z) I_n, \tag{28}$$

where $\varphi_n(z)$ is the same as in (27). As in Remark 8.5, we define the vector bundle $\mathcal{E}(\Phi,\Psi) \in \mathsf{VB}(E\times M)$ via the following commutative diagram of complex manifolds:

$$\begin{tikzcd}
\mathbb{C}\times\mathbb{C}\times\mathbb{C}^n \arrow[r] \arrow[d, "\mathrm{pr}_{1,2}"'] & \mathcal{E}(\Phi,\Psi) = \mathbb{C}\times\mathbb{C}\times\mathbb{C}^n/\sim \arrow[d] \\
\mathbb{C}\times\mathbb{C} \arrow[r, "\pi\times\pi"] & E\times M,
\end{tikzcd}$$

where the equivalence relation is given by the formulae:

$$\begin{aligned}
&(z,x;v) \sim (z+1,x;v) \sim (z,x+1;v),\\
&(z,x;v) \sim \bigl(z+\tau,x;\Phi(z,x)v\bigr),\\
&(z,x;v) \sim \bigl(z,x+\tau;\Psi(z,x)v\bigr).
\end{aligned}$$

Note that the following equalities are fulfilled:

$$\Psi(z+\tau,x)\Phi(z,x) = \Phi(z,x+\tau)\Psi(z,x)$$

as well as

$$\begin{aligned}
\Phi(z+1,x) &= \Phi(z,x) & \qquad \Phi(z,x+1) &= \Phi(z,x)\\
\Psi(z+1,x) &= \Psi(z,x) & \qquad \Psi(z,x+1) &= \Psi(z,x).
\end{aligned}$$

Hence the equivalence relation $\sim$ is well-defined and $\mathcal{E}(\Phi,\Psi)$ is a holomorphic vector bundle on $E \times M$. Note that for any point $x_0 \in M$ we have:

$$\mathcal{E}(\Phi,\Psi)\big|_{E\times\{x_0\}} \cong \mathcal{E}\bigl(\Phi(\,-\,,x_0)\bigr).$$

In particular, $\mathcal{E}(\Phi,\Psi)$ is a family of stable vector bundles of rank n and degree d on E parameterized by the manifold M. By Lemma 8.7 we know that

$$\mathcal{E}(\Phi,\Psi)\big|_{E\times\{x_0\}} \cong \mathcal{E}(\Phi,\Psi)\big|_{E\times\{x_0'\}} \iff n(x_0 - x_0') \in \Lambda_\tau. \tag{29}$$

Lemma 8.8. *Let $\mu_n : E_\tau \to E_\tau$ be an étale covering of degree n^2 given by the rule $\mu_n(x) = n \cdot x$. Then there exists a universal family $\mathcal{P} = \mathcal{P}(n,d)$ of stable vector bundles on $E \times M$ such that $\mathcal{E}(\Phi,\Psi) \sim (1 \times \mu_n)^*\mathcal{P}$.*

Proof. We know that $M = E_\tau$ is the moduli space of stable vector bundles of rank n and degree d. Let $\mathcal{Q} \in \mathsf{VB}(E \times M)$ be any universal family, then by the universal property there exists a unique morphism $\nu : M \to M$ such that $\mathcal{E}(\Phi,\Psi) \sim (1\times\nu)^*\mathcal{Q}$. Since a morphism between two compact Riemann surfaces is either surjective or constant, the morphism ν is surjective. From the equality (29) we obtain that ν factorizes as

$$\begin{tikzcd}
M \arrow[rr, "\nu"] \arrow[dr, "\mu_n"'] & & M \\
& M \arrow[ur, "\hat{\nu}"'] &
\end{tikzcd}$$

and the induced map $\hat{\nu}$ is both injective and surjective, hence biholomorphic. Then the universal family $\mathcal{P} = (1 \times \hat{\nu})^*\mathcal{Q} \in \mathsf{VB}(E \times M)$ is the one we are looking for. $\square$

Remark 8.9. In a similar way, the functions $\varphi, \psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^*$ given by $\varphi(z,x) = \exp(-2\pi i x)$ and $\psi(z,x) = \exp(-2\pi i z)$ define a universal family

$$\mathcal{L} = \mathcal{L}(\varphi,\psi) \in \mathsf{Pic}(E \times J),$$

of degree zero line bundles on $E = E_\tau$ where $J = E_\tau$ is the Jacobian of E.

Lemma 8.10. *Let $\mathcal{P} \in \mathsf{VB}(E \times M)$ and $\mathcal{L} \in \mathsf{Pic}(E \times J)$ be as in Lemma 8.8 and Remark 8.9. Then the following diagram is commutative:*

$$\begin{tikzcd} \mathbb{C} \times \mathbb{C} \arrow[r, "\sigma"] \arrow[d, "\pi\times\bar{\pi}"'] & \mathbb{C} \arrow[d, "\bar{\pi}"] \\ J \times M \arrow[r, "\tau"] & M, \end{tikzcd} \tag{30}$$

where $\sigma(a,b) = a+b$, $\pi : \mathbb{C} \to \mathbb{C}/\langle 1, \tau\rangle$ is the canonical projection and $\bar{\pi}(z) = \pi(n\cdot z)$.

Proof. Recall that the morphism $\tau : J \times M \to M$ is uniquely determined by a choice of universal families $\mathcal{P}$ and $\mathcal{L}$ by the following property: the isomorphism

$$\mathcal{P}\big|_{E\times\{\tau(a,b)\}} \cong \mathcal{L}\big|_{E\times\{a\}} \otimes \mathcal{P}|_{E\times\{b\}}$$

holds for all points $(a,b) \in J \times E$. Consider the morphisms

$$\tilde{p}: \quad E \times \mathbb{C} \times \mathbb{C} \xrightarrow{\mathrm{pr}_{1,3}} E \times \mathbb{C} \xrightarrow{1\times\bar{\pi}} E \times M,$$

$$\tilde{q}: \quad E \times \mathbb{C} \times \mathbb{C} \xrightarrow{\mathrm{pr}_{1,2}} E \times \mathbb{C} \xrightarrow{1\times\pi} E \times J$$

and

$$\tilde{\sigma}: \quad E \times \mathbb{C} \times \mathbb{C} \xrightarrow{1\times\sigma} E \times \mathbb{C} \xrightarrow{1\times\bar{\pi}} E \times M.$$

The commutativity of diagram (30) is equivalent to the fact that $\tilde{q}^*\mathcal{L} \otimes \tilde{p}^*\mathcal{P} \sim \tilde{\sigma}^*\mathcal{P}$. This is furthermore equivalent that for all $(a,b) \in \mathbb{C} \times \mathbb{C}$ we have:

$$\tilde{q}^*\mathcal{L} \otimes \tilde{p}^*\mathcal{P}\big|_{E\times\{a\}\times\{b\}} \cong \tilde{\sigma}^*\mathcal{P}\big|_{E\times\{a\}\times\{b\}}.$$

The last isomorphism can be rewritten as

$$\mathcal{L}\big(\varphi(a)\big) \otimes \mathcal{E}\big(\Phi(\,-\,,b)\big) \cong \mathcal{E}\big(\Phi(\,-\,,a+b)\big).$$

Since for all $(a,b) \in \mathbb{C}\times\mathbb{C}$ we have $\varphi(z,a)\cdot\Phi(z,b) = \Phi(z,a+b)$, the result follows. $\square$

Consider the morphisms $\hat{p}$, $\hat{q}$, $\hat{\tau} : E \times \mathbb{C} \times \mathbb{C} \to E \times \mathbb{C}$, where $\hat{p} = \mathrm{pr}_{1,3}$, $\hat{q} = \mathrm{pr}_{1,2}$ and $\hat{\sigma} = 1 \times \sigma$. We define the morphism $\hat{q}^*\mathcal{L}(\varphi) \otimes \hat{p}^*\mathcal{E}(\Phi) \xrightarrow{\phi} \hat{\sigma}^*\mathcal{E}(\Phi)$ by the following commutative diagram:

$$\begin{tikzcd} \hat{q}^*\mathcal{L}(\varphi) \otimes \hat{p}^*\mathcal{E}(\Phi) \arrow[r, "\phi"] \arrow[d, "\cong"'] & \hat{\sigma}^*\mathcal{E}(\Phi) \arrow[d, "\cong"] \\ \mathcal{L}(\hat{q}^*\varphi) \otimes \mathcal{E}(\hat{p}^*\Phi) \arrow[r] \arrow[d, "\cong"'] & \mathcal{E}(\hat{\sigma}^*\Phi) \arrow[d, "="] \\ \mathcal{E}(\hat{q}^*\varphi \cdot \hat{p}^*\Phi) \arrow[r, "\bar{\phi}"] & \mathcal{E}(\hat{\sigma}^*\Phi), \end{tikzcd}$$

where all vertical isomorphisms are canonical and $\bar{\phi}$ corresponds to the morphism in the category $\mathsf{AF}_\tau(\mathbb{C} \times \mathbb{C})$ given by the identity matrix I_n. In particular, the morphism ϕ is identity in the trivializations of $\hat{p}^*\mathcal{E}(\Phi)$, $\hat{\sigma}^*\mathcal{E}(\Phi)$ and $\hat{q}^*\mathcal{L}(\varphi)$ induced by γ_Φ and γ_φ. Summing up, we get the following proposition

Proposition 8.11. *Let $\Phi : \mathbb{C} \times \mathbb{C} \longrightarrow \mathsf{GL}_n(\mathbb{C})$ be as in (28), $\varphi(y) = \exp(-2\pi y)$ and $\hat{q}^*\mathcal{L}(\varphi) \otimes \hat{p}^*\mathcal{E}(\Phi) \xrightarrow{\phi} \hat{\sigma}^*\mathcal{E}(\Phi)$ be the morphism of vector bundles on $E \times \mathbb{C} \times \mathbb{C}$ constructed above. Take small local neighbourhoods in the moduli space M and Jacobian J corresponding to small neighbourhoods of $0 \in \mathbb{C}$ with respect to the diagram (30). Then the induced trivializations $\xi^{\mathcal{P}}$ and $\xi^{\mathcal{L}}$ of the universal families $\mathcal{P} = \mathcal{P}(n,d)$ and $\mathcal{L} = \mathcal{P}(1,0)$ and the morphism ϕ are the ones we are looking for in Proposition 7.2. In particular, Theorem 7.5 is true in the elliptic case.*

Corollary 8.12. *Let $\tilde{r}^{(n,d)}_{\mathsf{ell}} = \tilde{r}^{\xi}_{\tau}(v_1, v_2; y_1, y_2)$ be the associative r–matrix obtained from the universal family $\mathcal{P}(n,d)$ of stable vector bundles of rank n and degree d on the elliptic curve E_τ using the trivialization $\xi = \gamma_\Phi$ described above. Then we have:*

$$\tilde{r}^{(n,d)}_{\mathsf{ell}}(v_1, v_2; y_1, y_2) = \tilde{r}^{(n,d)}_{\mathsf{ell}}(v_1 + v, v_2 + v; y_1, y_2)$$

for all values v_1, v_2, v and y_1, y_2 from a small neighbourhood of 0.

8.2. **Rules to calculate the evaluation and the residue maps.** In this subsection, we consistently denote the vector space of complex linear maps between two complex vector spaces V, W by $\mathsf{Lin}(V, W)$. We reserve here the notation $\mathsf{Hom}_{\mathbb{C}}(\ ,\ )$ for the vector space of morphisms of sheaves on the complex manifold $\mathbb{C}$.

Let $E = E_\tau$ be a complex torus, Ω_E the sheaf of holomorphic differential one-forms and $\omega = dz \in H^0(\Omega_E)$ a global section. We fix a pair of coprime integers $(n,d) \in \mathbb{Z}^+ \times \mathbb{Z}$ and let $M = M_E^{(n,d)}$ be the moduli space of stable holomorphic vector bundles of rank n and degree d on E. By $\mathcal{P} = \mathcal{P}(n,d) \in \mathsf{VB}(E \times M)$ we denote the universal family and by $\xi^{\mathcal{P}} : \mathcal{P}|_{U \times M'} \longrightarrow \mathcal{O}|^n_{U \times M'}$ a trivialization, as constructed in the previous subsection. Recall that these data define the germ of a meromorphic function

$$\tilde{r} = \tilde{r}^{\xi} : \big(M \times M \times E \times E, o\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}),$$

whose value at the point $(v_1, v_2; y_1, y_2)$, where $v_1 \neq v_1$ and $y_1 \neq y_2$, is defined via the commutative diagram

$$\begin{tikzcd}[column sep=small]
& \mathsf{Hom}_E\big(\mathcal{P}^{v_1}, \mathcal{P}^{v_2}(y_1)\big) \arrow[dl, "\mathrm{res}_{y_1}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}(\omega)"'] \arrow[dr, "\mathrm{ev}_{y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}(y_1)}"] & \\
\mathsf{Lin}(\mathcal{P}^{v_1}|_{y_1}, \mathcal{P}^{v_2}|_{y_1}) \arrow[rr, "\tilde{r}_{y_1,y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_1}}(\omega)"] \arrow[d, "\mathrm{cnj}(\bar{\xi}^{v_1}, \bar{\xi}^{v_2})"'] & & \mathsf{Lin}(\mathcal{P}^{v_1}|_{y_2}, \mathcal{P}^{v_1}|_{y_2}) \arrow[d, "\mathrm{cnj}(\bar{\xi}^{v_1}, \bar{\xi}^{v_2})"] \\
\mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[rr, "\tilde{r}^{\xi}(v_1,v_2;y_1,y_2)"] & & \mathsf{Mat}_{n\times n}(\mathbb{C}),
\end{tikzcd} \tag{31}$$

where $\mathrm{res}_{y_1}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}(\omega)$ and $\mathrm{ev}_{y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}(y_1)}$ are the maps defined in Section 4 and the vertical isomorphisms are induced by the trivialization ξ of the universal bundle.

Let $\pi : \mathbb{C} \longrightarrow \mathbb{C}/\Lambda_\tau = E$ be the universal covering, $o = \pi(0) \in E$. Take an open neighbourhood U of the point o in E such that there exists a connected component $\widetilde{U}$ of $\pi^{-1}(U)$ which maps isomorphically to U. For the sake of convenience, we denote the preimage $\pi^{-1}(y) \in \widetilde{U}$ of a point $y \in U$ by the same letter y. By Theorem 4.23 we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Lin}\big(\mathcal{P}^{v_1}|_{y_1}, \mathcal{P}^{v_2}|_{y_1}\big) \arrow[r, "\pi^*"] & \mathsf{Lin}\big(\pi^*\mathcal{P}^{v_1}|_{y_1}, \pi^*\mathcal{P}^{v_2}|_{y_1}\big) \\
\mathsf{Hom}_E\big(\mathcal{P}^{v_1}, \mathcal{P}^{v_2}(y_1)\big) \arrow[r, "\pi^*"] \arrow[u, "\mathrm{res}_{y_1}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}(\omega)"] \arrow[d, "\mathrm{ev}_{y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}(y_1)}"'] & \mathsf{Hom}_{\mathbb{C}}\big(\pi^*\mathcal{P}^{v_1}, \pi^*\mathcal{P}^{v_2}(D_1)\big) \arrow[u, "\mathrm{res}_{y_1}^{\pi^*\mathcal{P}^{v_1},\pi^*\mathcal{P}^{v_2}}(\tilde{\omega})"'] \arrow[d, "\mathrm{ev}_{y_2}^{\pi^*\mathcal{P}^{v_1},\pi^*\mathcal{P}^{v_2}(D_1)}"] \\
\mathsf{Lin}\big(\mathcal{P}^{v_1}|_{y_2}, \mathcal{P}^{v_2}|_{y_2}\big) \arrow[r, "\pi^*"] & \mathsf{Lin}\big(\pi^*\mathcal{P}^{v_1}|_{y_2}, \pi^*\mathcal{P}^{v_2}|_{y_2}\big)
\end{tikzcd}$$

where $\mathcal{O}_{\mathbb{C}}(D_1) = \pi^*\mathcal{O}_E(y_1)$ is the subsheaf of the sheaf $\mathcal{M}_{\mathbb{C}}$ of meromorphic functions on $\mathbb{C}$, whose local sections are meromorphic functions having at most simple poles along the infinite, but locally finite set $D_1 = \big\{y_1 + \gamma \,\big|\, \gamma \in \Lambda_\tau\big\} = \pi^{-1}(y_1)$.

Recall that the description of a universal family $\mathcal{P}$ in terms of automorphy factors yields an isomorphism of vector bundles $\gamma : \pi_M^*\mathcal{P} \longrightarrow \mathcal{O}^n_{\mathbb{C}\times M}$. For any point $v \in M$ it induces an isomorphism $\gamma^v : \pi^*\mathcal{P}^v \longrightarrow \mathcal{O}^n_{\mathbb{C}}$. If we apply Lemma 4.10 and Proposition 4.12 to these isomorphisms and use the morphisms $\mathrm{res}_{y_1}(\tilde{\omega})$ and ev_{y_2} from Lemma 4.5 and Lemma 4.13, which are given by $\mathrm{res}_{y_1}(\tilde{\omega})\big(F\big) = \mathrm{res}_{y_1}(F dz)$ and $\mathrm{ev}_{y_2}(F) = F(y_2)$, we obtain the following commutative diagram

$$\begin{tikzcd}
\mathsf{Lin}\big(\pi^*\mathcal{P}^{v_1}|_{y_1}, \pi^*\mathcal{P}^{v_2}|_{y_1}\big) \arrow[r, "\mathrm{cnj}(\bar{\gamma}^{v_1},\bar{\gamma}^{v_2})"] & \mathsf{Mat}_{n\times n}(\mathbb{C}) \\
\mathsf{Hom}_{\mathbb{C}}\big(\pi^*\mathcal{P}^{v_1}, \pi^*\mathcal{P}^{v_2}(D_1)\big) \arrow[r, "\mathrm{cnj}\left(\gamma^{v_1},\gamma^{v_2}(D_1)\right)"] \arrow[u, "\mathrm{res}_{y_1}^{\pi^*\mathcal{P}^{v_1},\pi^*\mathcal{P}^{v_2}}(\tilde{\omega})"] \arrow[d, "\mathrm{ev}_{y_2}^{\pi^*\mathcal{P}^{v_1},\pi^*\mathcal{P}^{v_2}(D_1)}"'] & \mathsf{Mat}_{n\times n}\big(\mathcal{O}_{\mathbb{C}}(D_1)\big) \arrow[u, "\mathrm{res}_{y_1}(\tilde{\omega})"'] \arrow[d, "\mathrm{ev}_{y_2}"] \\
\mathsf{Lin}\big(\pi^*\mathcal{P}^{v_1}|_{y_2}, \pi^*\mathcal{P}^{v_2}|_{y_2}\big) \arrow[r, "\mathrm{cnj}(\bar{\gamma}^{v_1},\bar{\gamma}^{v_2})"] & \mathsf{Mat}_{n\times n}(\mathbb{C}).
\end{tikzcd}$$

The three previous diagrams in this subsection give us another one:

$$
\begin{tikzcd}
\mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[rr, "\tilde{r}^{\xi}(v_1,v_2;y_1,y_2)"] & & \mathsf{Mat}_{n\times n}(\mathbb{C}) \\
& \mathsf{Hom}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}}^n, \mathcal{O}_{\mathbb{C}}^n(D_1)\big) \arrow[ul, "\mathrm{res}_{y_1}(\tilde{\omega})"'] \arrow[ur, "\mathrm{ev}_{y_2}"] & \\
& \mathsf{Hom}_E\big(\mathcal{P}^{v_1}, \mathcal{P}^{v_2}(y_1)\big) \arrow[u, "\mathrm{cnj}\left(\gamma^{v_1}, \gamma^{v_2}(D_1)\right)\circ \pi^*"] \arrow[dl, "\mathrm{res}_{y_1}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}(\omega)"'] \arrow[dr, "\mathrm{ev}_{y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}(y_1)}"] & \\
\mathsf{Lin}(\mathcal{P}^{v_1}|_{y_1}, \mathcal{P}^{v_2}|_{y_1}) \arrow[uuu, "\mathrm{cnj}(\bar{\xi}^{v_1},\bar{\xi}^{v_2})"] \arrow[rr, "\tilde{r}_{y_1,y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_1}}(\omega)"] & & \mathsf{Lin}(\mathcal{P}^{v_1}|_{y_2}, \mathcal{P}^{v_1}|_{y_2}). \arrow[uuu, "\mathrm{cnj}(\bar{\xi}^{v_1},\bar{\xi}^{v_2})"']
\end{tikzcd}
$$

Our next goal is to describe the image of the morphism $\mathrm{cnj}\big(\gamma^{v_1}, \gamma^{v_2}(D_1)\big) \circ \pi^*$. Let $\psi_y(z) = -\exp(-2\pi i z + 2\pi i y - 2\pi i\tau)$. It is easy to see that $\mathsf{Hom}_E\big(\mathcal{O}, \mathcal{L}(\psi_y)\big)$ is one-dimensional and generated by the section $\underline{\theta}_y$ corresponding to the theta-function $\theta_y(z) = \theta\left(z + \frac{1+\tau}{2} - y\,\middle|\,\tau\right)$. Note that θ_y is a holomorphic function on $\mathbb{C}$ having only one simple zero at y inside a fundamental parallelogram of Λ_τ. Hence, we have an isomorphism $\alpha : \mathcal{O}_E(y) \longrightarrow \mathcal{L}(\psi_y)$.

In order to be more precise, recall that $\mathcal{L}(\psi_y)$ is a subsheaf of the sheaf $\pi_*\mathcal{O}_{\mathbb{C}}$. In particular, we have: $H^0\big(\mathcal{L}(\psi_y)\big) = \mathbb{C}\cdot\theta_y \subset H^0(\mathcal{O}_{\mathbb{C}})$. On the other hand, the sheaf $\mathcal{O}_E(y)$ is a subsheaf of the sheaf of meromorphic functions $\mathcal{M}_E$ and $H^0\big(\mathcal{O}_E(y)\big) = \mathbb{C}\cdot 1$. Without loss of generality we may assume that $H^0(\alpha)(1) = \theta_y$. This choice fixes the isomorphism α.

Recall that for a point $x \in M$ we have: $\mathcal{P}^v = \mathcal{E}(\Phi_v)$, where $\Phi_v = \Phi(-, v)$ is the function defined by the equality (27). Then we have an isomorphism

$$\mathcal{P}^{v_2}(y_1) \xrightarrow{\mathsf{id}\otimes\alpha} \mathcal{P}^{v_2}\otimes\mathcal{L}(\psi_{y_1}) = \mathcal{E}\big(\psi_{y_1}\cdot\Phi_{v_2}\big)$$

and the following diagram is commutative:

$$
\begin{tikzcd}
\pi^*\mathcal{E}(\psi_{y_1}\Phi_{v_2}) \arrow[rr, "\gamma^{y_1,v_2}"] & & \mathcal{O}_{\mathbb{C}}^n \\
\pi^*\mathcal{E}(\psi_{y_1})\otimes\pi^*\mathcal{E}(\Phi_{v_2}) \arrow[u, "\mathsf{can}"] \arrow[rr, "\gamma^{y_1}\otimes\gamma^{v_2}"] & & \mathcal{O}_{\mathbb{C}}\otimes\mathcal{O}_{\mathbb{C}}^n \arrow[u, "\mathsf{mult}"'] \\
& \pi^*\mathcal{O}_E(y_1)\otimes\pi^*\mathcal{E}(\Phi_{v_2}) \arrow[ul, "\pi^*(\alpha)\otimes\mathsf{id}"] \arrow[ur, "\tilde{\alpha}\otimes\gamma^{v_2}"'] &
\end{tikzcd}
$$

where $\tilde{\alpha} : \pi^*\big(\mathcal{O}_E(y_1)\big) = \mathcal{O}_{\mathbb{C}}(D_1) \longrightarrow \mathcal{O}_{\mathbb{C}}$ is defined to be $\tilde{\alpha}(f) = f\theta_{y_1}$, γ^{y_1,v_2} corresponds to the automorphy factor $\psi_{y_1}\cdot\Phi_{v_2}$, γ^{y_1} to ψ_{y_1} and γ^{v_2} to Φ_{v_2}. As a result, we get the following commutative diagram:

$$\begin{tikzcd}
\mathsf{Hom}_{\mathbb{C}}\Big(\pi^*\mathcal{P}^{v_1}, \pi^*\big(\mathcal{P}^{v_2} \otimes \mathcal{O}_E(y_1)\big)\Big) \arrow[rr, "\mathrm{cnj}\left(\gamma^{v_1}, \gamma^{v_2}(D_1)\right)"] & & \mathsf{Hom}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}}^n, \mathcal{O}_{\mathbb{C}}^n(D_1)\big) \\
\mathsf{Hom}_{\mathbb{C}}\Big(\pi^*\mathcal{P}^{v_1}, \pi^*\big(\mathcal{P}^{v_2} \otimes \mathcal{L}(\psi_{y_1})\big)\Big) \arrow[u, "(\mathsf{id}\otimes\alpha)_*"] \arrow[rr, "\mathrm{cnj}\left(\gamma^{v_1}, \gamma^{v_2}(D_1)\right)"] & & \mathsf{Hom}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}}^n, \mathcal{O}_{\mathbb{C}}^n\big) \arrow[u, "\tilde{\alpha}_*"'] \\
& \mathsf{Hom}_{\mathsf{AF}_\tau}(\Phi_{v_1}, \psi_{y_1}\Phi_{v_2}) \arrow[ul, "\pi^*"] \arrow[ur, "\mathsf{For}"'] &
\end{tikzcd}$$

and $\tilde{\alpha}_*$ is given by the formula $\tilde{\alpha}_*(F) = \dfrac{F}{\theta_{y_1}}$.

Corollary 8.13. *Let* $O = \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}})$, $O(D_1) = \Gamma\big(\mathbb{C}, \mathcal{O}_{\mathbb{C}}(D_1)\big)$ *and*

$$\Pi_{y_1}^{v_1,v_2} = \mathsf{Im}\big(\mathsf{Hom}_{\mathsf{AF}_\tau}(\Phi_{v_1}, \psi_{y_1}\Phi_{v_2}) \longrightarrow \mathsf{Mat}_{n\times n}(O)\big).$$

Then the following diagram is commutative:

$$\begin{tikzcd}
& \Pi_{y_1}^{v_1,v_2} \arrow[dl, bend right, "\overline{\mathrm{res}}_{y_1}"'] \arrow[d, "\tilde{\alpha}_*"] \arrow[dr, bend left, "\overline{\mathrm{ev}}_{y_2}"] & \\
& \mathsf{Mat}_{n\times n}\big(O(D_1)\big) \arrow[dl, "\mathrm{res}_{y_1}(\tilde{\omega})"'] \arrow[dr, "\mathrm{ev}_{y_2}"] & \\
\mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[rr, "\tilde{r}^{\xi}(v_1,v_2;y_1,y_2)"] & & \mathsf{Mat}_{n\times n}(\mathbb{C}).
\end{tikzcd}$$

In particular, this gives the following algorithm to compute the value of an associative geometric r–matrix of elliptic type at a point $(v_1, v_2; y_1, y_2) \in M \times M \times E \times E$ *with respect to the trivialization* ξ*:*

(1) *First describe the vector space*

$$\Pi_{y_1}^{v_1,v_2} = \mathsf{Im}\Big(\mathsf{Hom}_{\mathsf{AF}_\tau}\big(\Phi_{v_1}, \psi_{y_1} \cdot \Phi_{v_2}\big) \longrightarrow \mathsf{Mat}_{n\times n}(O)\Big)$$

(2) *The morphism* $\overline{\mathrm{res}}_{y_1} : \mathsf{Mat}_{n\times n}(O) \to \mathsf{Mat}_{n\times n}(\mathbb{C})$ *is given by the formula*

$$F(z) \mapsto \overline{\mathrm{res}}_{y_1}\left(\frac{F(z)}{\theta_{y_1}(z)}dz\right) = \frac{F(y_1)}{\theta'_{y_1}(y_1)} = \frac{F(y_1)}{\theta'\left(\frac{1+\tau}{2}\,\middle|\,\tau\right)}$$

and the morphism $\overline{\mathrm{ev}}_{y_2} : \mathsf{Mat}_{n\times n}(O) \to \mathsf{Mat}_{n\times n}(\mathbb{C})$ *is given by the formula*

$$F(z) \mapsto \frac{F(y_2)}{\theta_{y_1}(y_2)} = \frac{F(y_2)}{\theta\left(y + \frac{1+\tau}{2}\,\middle|\,\tau\right)}.$$

(3) *Compute* $\tilde{r}^{\xi}(v_1, v_2; y_1, y_2)$ *as the composition*

$$\mathsf{Mat}_{n\times n}(\mathbb{C}) \xrightarrow{\left(\overline{\mathrm{res}}_{y_1}(\tilde{\omega})\right)^{-1}} \Pi_{y_1}^{v_1,v_2} \xrightarrow{\overline{\mathrm{ev}}_{y_2}} \mathsf{Mat}_{n\times n}(\mathbb{C}).$$

Note that there is an ambiguity in choosing the morphism α. *Another choice of* α *corresponds to rescaling the section* $\underline{\theta}_{y_1}$ *by* $\lambda \in \mathbb{C}^*$ *to* $\lambda\underline{\theta}_{y_1}$. *However, it is easy to see from the algorithm above, that the resulting linear map* $\tilde{r}^{\xi}(v_1, v_2; y_1, y_2)$ *does not depend on this choice.*

8.3. **Calculation of the elliptic r–matrix corresponding to $M_E^{(2,1)}$.** Let $\mathcal{P}^{x_1}$ and $\mathcal{P}^{x_i}$ be a pair of non-isomorphic simple vector bundles of rank two and degree one on the elliptic curve $E = E_\tau$, y_1 and y_2 two distinct points. In what follows we denote $q = \exp(\pi i\tau)$, $q_x = \exp(-\pi i x)$, $e(z) = \exp(-2\pi i z)$, $\varphi(z) = e(z+\tau)$, $x = x_2 - x_1$ and $y = y_2 - y_1$.
As we have seen in the previous subsection, one can write $\mathcal{P}^{x_i} = \mathcal{E}\big(\mathbb{C}^2, \Phi_{2,1,x_i}(z)\big)$, where

$$\Phi_{2,1,x_i}(z) = q_{x_i}\begin{pmatrix} 0 & 1 \\ \varphi(z) & 0 \end{pmatrix} =: q_{x_i}\Phi(z),$$

and the line bundle $\mathcal{O}_E(y_1)$ corresponds to the automorphy factor

$$\psi_{y_1}(z) = -e(z+\tau-y_1).$$

Recall that $\Pi_{y_2}^{x_1,x_2} =$

$$\left\{ A(z) = \begin{pmatrix} u(z) & v(z) \\ w(z) & t(z) \end{pmatrix} \middle| \; A(z+1) = A(z),\; A(z+\tau)\Phi(z) = q_x\psi_{y_1}(z)\Phi(z)A(z) \right\}$$

This leads to two systems of functional equations

$$\begin{cases} u(z+\tau) &= q_x\psi_{y_1}(z)t(z) \\ t(z+\tau) &= q_x\psi_{y_1}(z)u(z) \end{cases} \quad \text{and} \quad \begin{cases} \varphi(z)v(z+\tau) &= q_x\psi_{y_1}(z)w(z) \\ w(z+\tau) &= q_x\varphi(z)\psi_{y_1}(z)v(z) \end{cases}$$

which are equivalent to

$$\begin{cases} u(z+2\tau) &= a(z)u(z) \\ u(z+1) &= u(z) \\ t(z) &= \dfrac{u(z+\tau)}{q_x\psi_{y_1}(z)} \end{cases} \quad \text{and} \quad \begin{cases} v(z+2\tau) &= b(z)v(z) \\ v(z+1) &= v(z) \\ w(z) &= \dfrac{\varphi(z)}{q_x\psi_{y_1}(z)}v(z+\tau) \end{cases}$$

where

$$a(z) = \exp\left(-2\pi i\tau - 2\pi i\left(z + \frac{x+\tau}{2} - y_1\right)\right)^2 \quad \text{and}$$

$$b(z) = \exp\left(-2\pi i\tau - 2\pi i\left(z + \frac{x}{2} - y_1\right)\right)^2.$$

Lemma 8.14. *Let $E = E_\tau$ be an elliptic curve, $\varphi_0(z) = \exp(-\pi i\tau - 2\pi i z)$, $l \in \mathbb{N}$. Then $H^0\bigl(\mathcal{L}(\varphi_0^l)\bigr)$ has a basis $\left\{\theta\left[\frac{a}{l},0\right](lz|l\tau)\,\middle|\, 0 \le a < l, a \in \mathbb{Z}\right\}$, where we use Mumford's notation*

$$\theta[a,b](z|\tau) = \sum_{n\in\mathbb{Z}} \exp\bigl(\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)\bigr).$$

In the particular case of bundles of rank two and degree one it is convenient to use instead the four classical theta-functions of Jacobi:

$$\begin{aligned}
\theta_1(z|\tau) &= 2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin\bigl((2n+1)\pi z\bigr),\\
\theta_2(z|\tau) &= 2q^{\frac{1}{4}}\sum_{n=0}^{\infty} q^{n(n+1)}\cos\bigl((2n+1)\pi z\bigr),\\
\theta_3(z|\tau) &= 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2\pi n z),\\
\theta_4(z|\tau) &= 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}\cos(2\pi n z).
\end{aligned}$$

Remark 8.15. In Mumford's notation it holds:

$$\begin{aligned}
\theta_1(z|\tau) &= -\theta\left[\tfrac{1}{2},\tfrac{1}{2}\right](z|\tau) &\qquad \theta_2(z|\tau) &= \theta\left[\tfrac{1}{2},0\right](z|\tau)\\
\theta_3(z|\tau) &= \quad\theta\left[0,0\right](z|\tau) & \theta_4(z|\tau) &= \theta\left[0,\tfrac{1}{2}\right](z|\tau).
\end{aligned}$$

In what follows we shall express all our computations in terms of Jacobi's theta-functions. From Lemma 8.14 and Remark 8.15 we immediately obtain:

Corollary 8.16. *If we let*

$$\begin{aligned}
u_1(z) &= \theta_3\left(2\left(z - y_1 + \frac{x+\tau}{2}\right)\middle| 4\tau\right) &\qquad v_1(z) &= \theta_3\left(2\left(z - y_1 + \frac{x}{2}\right)\middle| 4\tau\right)\\
u_2(z) &= \theta_2\left(2\left(z - y_1 + \frac{x+\tau}{2}\right)\middle| 4\tau\right) & v_2(z) &= \theta_2\left(2\left(z - y_1 + \frac{x}{2}\right)\middle| 4\tau\right)
\end{aligned}$$

and

$$F_k(z) = \begin{pmatrix} u_k(z) & 0 \\ 0 & \dfrac{u_k(z+\tau)}{q_x\psi_{y_1}(z)} \end{pmatrix}, G_k(z) = \begin{pmatrix} 0 & v_k(z) \\ \dfrac{\varphi(z)}{q_x\psi_{y_1}(z)}v_k(z) & 0 \end{pmatrix}, \quad k = 1,2,$$

then $F_1(z), F_2(z), G_1(z), G_2(z)$ is a basis of $\Pi_{y_1}^{x_1,x_2}$.

The following proposition summarises the main properties of Jacobi's theta-functions which we need in our calculation of the associative r–matrix corresponding to the universal family of stable vector bundles of rank two and degree one.

Proposition 8.17 (see [26] and Section I.4 in [42]). *The transformation rules for shifts of theta-functions are given by the table*

$\theta(z)$	$\theta(-z)$	$\theta(z+1)$	$\theta(z+\tau)$	$\theta(z+1+\tau)$	$\theta(z+\frac{1}{2})$	$\theta(z+\frac{\tau}{2})$
$\theta_1(z)$	$-\theta_1(z)$	$-\theta_1(z)$	$-p(z)\theta_1(z)$	$p(z)\theta_1(z)$	$\theta_2(z)$	$iq(z)\theta_4(z)$
$\theta_2(z)$	$\theta_2(z)$	$-\theta_2(z)$	$p(z)\theta_2(z)$	$-p(z)\theta_2(z)$	$-\theta_1(z)$	$q(z)\theta_3(z)$
$\theta_3(z)$	$\theta_3(z)$	$\theta_3(z)$	$p(z)\theta_3(z)$	$p(z)\theta_3(z)$	$\theta_4(z)$	$q(z)\theta_2(z)$
$\theta_4(z)$	$\theta_4(z)$	$\theta_4(z)$	$-p(z)\theta_4(z)$	$-p(z)\theta_4(z)$	$\theta_3(z)$	$iq(z)\theta_1(z)$

where $p(z) = \exp\bigl(-\pi i(2z+\tau)\bigr)$ *and* $q(z) = \exp\left(-\pi i\left(z+\dfrac{\tau}{4}\right)\right)$. *Moreover, Jacobi's theta-functions satisfy the so-called Watson's determinantal identities:*

$$\begin{aligned}
\theta_3(2x|2\tau)\theta_2(2y|2\tau) - \theta_3(2y|2\tau)\theta_2(2x|2\tau) &= \theta_1(x+y|\tau)\theta_1(x-y|\tau),\\
\theta_1(2x|2\tau)\theta_4(2y|2\tau) - \theta_1(2y|2\tau)\theta_4(2x|2\tau) &= \theta_2(x+y|\tau)\theta_1(x-y|\tau),\\
\theta_1(2x|2\tau)\theta_4(2y|2\tau) + \theta_1(2y|2\tau)\theta_4(2x|2\tau) &= \theta_1(x+y|\tau)\theta_2(x-y|\tau),\\
\theta_4(2x|2\tau)\theta_4(2y|2\tau) - \theta_1(2y|2\tau)\theta_1(2x|2\tau) &= \theta_3(x+y|\tau)\theta_4(x-y|\tau),\\
\theta_4(2x|2\tau)\theta_4(2y|2\tau) + \theta_1(2y|2\tau)\theta_1(2x|2\tau) &= \theta_4(x+y|\tau)\theta_3(x-y|\tau).
\end{aligned}$$

By Corollary 8.16, any element of $\Pi_{y_1}^{x_1,x_2}$ can be written as a sum

$$A(z) = \alpha F_1(z) + \beta F_2(z) + \gamma G_1(z) + \delta G_2(z)$$

for some $\alpha,\beta,\gamma,\delta \in \mathbb{C}$. In order to calculate the geometric associative r–matrix $r^\xi(x_1,x_2;y_1,y_2)$ we have to solve the system of linear equations

$$\overline{\mathrm{res}}_{y_1}\bigl(A(z)\bigr) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the linear map $\tilde{r}^\xi(x_1,x_2;y_1,y_2) : \mathsf{Mat}_{2\times 2}(\mathbb{C}) \longrightarrow \mathsf{Mat}_{2\times 2}(\mathbb{C})$ is given by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\mathrm{res}_{y_1}^{-1}}{\longmapsto} A(z) \overset{\mathrm{ev}_{y_2}}{\longmapsto} \frac{1}{\theta_3(y+\frac{1+\tau}{2}|\tau)} A(y_2).$$

It is easy to see that the system of linear equations

$$\overline{\mathrm{res}}_{y_1}\bigl(\alpha F_1(z) + \beta F_2(z) + \gamma G_1(z) + \delta G_2(z)\bigr) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

splits into two independent systems

$$\overline{\mathrm{res}}_{y_1}\bigl(F(z)\bigr) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \quad \text{and} \quad \overline{\mathrm{res}}_{y_1}\bigl(G(z)\bigr) = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix},$$

where $F(z) = \alpha F_1(z) + \beta F_2(z)$ and $G(z) = \gamma G_1(z) + \delta G_2(z)$.

Computation of the "diagonal terms". The system of linear equations

$$\overline{\mathrm{res}}_{y_1}\bigl(F(z)\bigr) := \frac{1}{\theta_3'(\frac{1+\tau}{2}|\tau)} F(y_1) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

reads as

$$\left\{\begin{array}{lcl} \theta_3(x+\tau|4\tau)\alpha + \theta_2(x+\tau|4\tau)\beta & = & \theta_3'(\frac{1+\tau}{2}|\tau)a \\ \theta_3(x+3\tau|4\tau)\alpha + \theta_2(x+3\tau|4\tau)\beta & = & -e(\tau+\frac{x}{2})\theta_3'(\frac{1+\tau}{2}|\tau)d. \end{array}\right.$$

By Watson's identity, the determinant of this system is

$$\Delta_1 = \begin{vmatrix} \theta_3(x+\tau|4\tau) & \theta_2(x+\tau|4\tau) \\ \theta_3(x+3\tau|4\tau) & \theta_2(x+3\tau|4\tau) \end{vmatrix} = \theta_1(x+2\tau|2\tau)\theta_1(-\tau|2\tau) =$$

$$= e(x+\tau)\theta_1(x|2\tau)\theta_1(\tau|2\tau)$$

and we obtain:

$$\left\{\begin{array}{lcl} \alpha & = & \frac{\theta_3'(\frac{1+\tau}{2}|\tau)}{\Delta_1}\big(\theta_2(x+3\tau|4\tau)a + e(\tau+\frac{x}{2})\theta_2(x+\tau|4\tau)d\big) \\ \beta & = & -\frac{\theta_3'(\frac{1+\tau}{2}|\tau)}{\Delta_1}\big(\theta_3(x+3\tau|4\tau)a + e(\tau+\frac{x}{2})\theta_3(x+\tau|4\tau)d\big). \end{array}\right.$$

This implies:

$$\tilde{r}^{\xi}(x_1,x_2;y_1,y_2)\left[\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right] =$$

$$\frac{\theta_3'(\frac{1+\tau}{2}|\tau)}{\theta_3(y+\frac{1+\tau}{2}|\tau)\Delta_1} \times \left[p_1(z)\begin{pmatrix} \theta_3(2y+x+\tau|4\tau) & 0 \\ 0 & -\dfrac{\theta_3(2y+x+3\tau|4\tau)}{e(x/2+y+\tau)} \end{pmatrix}\right.$$

$$\left. -p_2(z)\begin{pmatrix} \theta_2(2y+x+\tau|4\tau) & 0 \\ 0 & -\dfrac{\theta_2(2y+x+3\tau|4\tau)}{e(x/2+y+\tau)} \end{pmatrix}\right].$$

where

$$\begin{aligned} p_1(z) &= \theta_2(x+3\tau|4\tau)a + e(x/2+\tau)\theta_2(x+\tau|4\tau)d, \\ p_2(z) &= \theta_3(x+3\tau|4\tau)a + e(x/2+\tau)\theta_3(x+\tau|4\tau)d. \end{aligned}$$

In order to calculate the "diagonal part" of the corresponding tensor $r(x_1,x_2;y_1,y_2)$ we use the inverse of the canonical isomorphism

$$\mathsf{Mat}_{2\times 2}(\mathbb{C}) \otimes \mathsf{Mat}_{2\times 2}(\mathbb{C}) \longrightarrow \mathsf{Lin}\big(\mathsf{Mat}_{2\times 2}(\mathbb{C}), \mathsf{Mat}_{2\times 2}(\mathbb{C})\big)$$

given by the formula $X \otimes Y \mapsto \mathrm{tr}(X \circ -)Y$. It is easy to see that under the map

$$\mathsf{Lin}\big(\mathsf{Mat}_{2\times 2}(\mathbb{C}), \mathsf{Mat}_{2\times 2}(\mathbb{C})\big) \longrightarrow \mathsf{Mat}_{2\times 2}(\mathbb{C}) \otimes \mathsf{Mat}_{2\times 2}(\mathbb{C})$$

a linear function $e_{ij} \mapsto \alpha_{ij}^{kl} e_{kl}, \alpha_{ij}^{kl} \in \mathbb{C}^*$ corresponds to the tensor $\alpha_{ij}^{kl} e_{ji} \otimes e_{kl}$.

Again, Watson's identities imply:

- The coefficient at $e_{11} \otimes e_{11}$ is

$$\theta_3(2y+x+\tau|4\tau)\theta_2(x+3\tau|4\tau) - \theta_2(2y+x+\tau|4\tau)\theta_3(x+3\tau|4\tau) =$$

$$\theta_1(x+y+2\tau|2\tau)\theta_1(y-\tau|2\tau).$$

- The coefficient at $e_{22} \otimes e_{22}$ is

$$e(-y)\big(\theta_3(x+\tau|4\tau)\theta_2(2y+x+3\tau|4\tau) - \theta_2(x+\tau|4\tau)\theta_3(2y+x+3\tau|4\tau)\big) =$$

$$= \theta_1(x+y+2\tau|2\tau)\theta_1(y-\tau|2\tau).$$

• The coefficient at $e_{22} \otimes e_{11}$ is

$$e(x/2+\tau)\big(\theta_2(x+\tau|4\tau)\theta_3(2y+x+\tau)-\theta_3(x+\tau|4\tau)\theta_2(2y+x+\tau)\big)=$$
$$=e(x/2+\tau)\theta_1(y+x+\tau|2\tau)\theta_1(y|2\tau).$$

• The coefficient at $e_{11} \otimes e_{22}$ is

$$e(-y-x/2-\tau)\big(\theta_2(2y+x+3\tau|4\tau)\theta_3(x+3\tau|4\tau)-\theta_3(2y+x+3\tau|4\tau)\theta_3(x+3\tau|4\tau)\big)$$
$$=e(x/2+\tau)\theta_1(y+x+\tau|2\tau)\theta_1(y|2\tau).$$

Now observe that

$$\theta_1(x+y+2\tau|2\tau)\theta_1(y-\tau|2\tau)=ie(x+y/2+5\tau/4)\theta_1(x+y|2\tau)\theta_4(y|2\tau)$$

and

$$e(x/2+\tau)\theta_1(y+x+\tau|2\tau)\theta_1(y|2\tau)=ie(x+y/2+5\tau/4)\theta_4(x+y|2\tau)\theta_1(y|2\tau).$$

Hence, the “diagonal part” of $r(x_1,x_2;y_1,y_2)$ is

$$C\big[\theta_1(x+y|2\tau)\theta_4(y|2\tau)(e_{11}\otimes e_{11}+e_{22}\otimes e_{22})+\theta_4(x+y|2\tau)\theta_1(y|2\tau)(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})\big],$$

where

$$C=\frac{\theta_3'(\frac{1+\tau}{2}|\tau)e(x+\frac{y}{2}+\frac{5\tau}{4})}{\theta_3(y+\frac{1+\tau}{2}|\tau)\Delta_1}.$$

From the identities $\theta_3(y+\frac{1+\tau}{2}|\tau)=i\exp(-\pi i(y+\tau/4))\theta_1(y|\tau)$ and $\theta_1(0|\tau)=0$ it follows: $\theta_3'(\frac{1+\tau}{2}|\tau)=ie(\frac{\tau}{8})\theta'(0|\tau)$. Using the transformation rules from Proposition 8.17 we get:

$$C=\frac{\theta_1'(0|\tau)}{\theta_4(0|2\tau)\theta_1(x|2\tau)\theta_1(y|\tau)}=\frac{\theta_1'(0|\tau)}{\theta_1\left(\frac{x}{2}\Big|\,\tau\right)\theta_2\left(\frac{x}{2}\Big|\,\tau\right)\theta_1\left(y|\tau\right)},$$

where we have used Landen’s transform

$$\theta_4(0|2\tau)\theta_1(2x|2\tau)=\theta_1(x|\tau)\theta_2(x|\tau).$$

It remains to observe that $A(e_{11}\otimes e_{11}+e_{22}\otimes e_{22})+B(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})=$

$$\frac{1}{2}(A+B)(e_{11}+e_{22})\otimes(e_{11}+e_{22})-\frac{1}{2}(A-B)(e_{11}-e_{22})\otimes(e_{11}-e_{22}),$$

and that by Watson’s identities we have

$$\theta_1(x+y|2\tau)\theta_4(y|2\tau)+\theta_4(x+y|2\tau)\theta_1(y|2\tau)=\theta_1\left(y+\frac{x}{2}\Big|\,\tau\right)\theta_2\left(\frac{x}{2}\Big|\,\tau\right)$$

and

$$\theta_1(x+y|2\tau)\theta_4(y|2\tau)-\theta_4(x+y|2\tau)\theta_1(y|2\tau)=\theta_2\left(y+\frac{x}{2}\Big|\,\tau\right)\theta_1\left(\frac{x}{2}\Big|\,\tau\right),$$

so the contribution of the "diagonal terms" is

$$\frac{1}{2}\frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\frac{\theta_1(y+\frac{x}{2}|\tau)}{\theta_1(\frac{x}{2}|\tau)}\mathbb{1}\otimes\mathbb{1}+\frac{1}{2}\frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\frac{\theta_2(y+\frac{x}{2}|\tau)}{\theta_2(\frac{x}{2}|\tau)}h\otimes h.$$

Contribution of the "skew terms". We have to solve the system of linear equations

$$\overline{\mathrm{res}}_{y_1}\big(\gamma G_1(z)+\delta G_2(z)\big)=\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}.$$

In explicit form this system reads as

$$\left\{\begin{array}{ccc}\theta_3(x|4\tau)\gamma+\theta_2(x|4\tau)\delta & = & \theta_3'(\frac{1+\tau}{2}|\tau)b\\ \theta_3(x+2\tau|4\tau)\gamma+\theta_2(x+2\tau|4\tau)\delta & = & -e(x/2-y_1)\theta_3'(\frac{1+\tau}{2}|\tau)c.\end{array}\right.$$

By Watson's formulas the determinant of this system is

$$\Delta_2=\begin{vmatrix}\theta_3(x|4\tau) & \theta_2(x|4\tau)\\ \theta_3(x+2\tau|4\tau) & \theta_2(x+2\tau|4\tau)\end{vmatrix}=-\theta_1(x+\tau|2\tau)\theta_1(\tau|2\tau).$$

Hence, the solution of this system of equations is

$$\left\{\begin{array}{ccc}\gamma & = & \theta_3'(\frac{1+\tau}{2}|\tau)\big(\theta_2(x+2\tau|4\tau)b+e(x/2-y_1)\theta_2(x|4\tau)c\big)\\ \delta & = & \theta_3'(\frac{1+\tau}{2}|\tau)\big(\theta_3(x+2\tau|4\tau)b+e(x/2-y_1)\theta_3(x|4\tau)c.\big)\end{array}\right.$$

As a result, we obtain:

$$\tilde{r}^{\xi}(x_1,x_2;y_1,y_2)\left[\begin{pmatrix}0 & b\\ c & 0\end{pmatrix}\right]=$$

$$\frac{\theta_3'(\frac{1+\tau}{2}|\tau)}{\theta_3(y+\frac{1+\tau}{2}|\tau)\Delta_2}\times\left[q_1(z)\begin{pmatrix}0 & \theta_3(2y+x|4\tau)\\ -e(y_1-x/2)\theta_3(2y+x+2\tau|4\tau) & 0\end{pmatrix}\right.$$
$$\left.-q_2(z)\begin{pmatrix}0 & \theta_2(2y+x|4\tau)\\ -e(y_1-x/2)\theta_2(2y+x+2\tau|4\tau) & 0\end{pmatrix}\right],$$

where

$$q_1(z)=\theta_2(x+2\tau|4\tau)b+e(x/2-y_1)\theta_2(x|4\tau)c,$$
$$q_2(z)=\theta_3(x+2\tau|4\tau)b+e(x/2-y_1)\theta_3(x|4\tau)c.$$

Again, Watson's identities imply:

- The coefficient at $e_{21}\otimes e_{12}$ is

$$\theta_2(x+2\tau|4\tau)\theta_3(2y+x|4\tau)-\theta_3(x+2\tau|4\tau)\theta_2(2y+x|4\tau)=$$
$$=e\left(\frac{1}{2}(x+\tau)\right)\theta_4(x+y|2\tau)\theta_4(y|2\tau).$$

- The coefficient at $e_{12}\otimes e_{21}$ is

$$\theta_3(x|4\tau)\theta_2(2y+x+2\tau|4\tau)-\theta_2(x|4\tau)\theta_3(2y+x+2\tau|4\tau)=$$

$$= e\left(\frac{1}{2}(x+\tau)+y\right)\theta_4(x+y|2\tau)\theta_4(y|2\tau).$$

• The coefficient at $e_{12}\otimes e_{12}$ is

$$e(x/2-y_1)\big(\theta_2(x|4\tau)\theta_3(2y+x+2\tau|4\tau)-\theta_3(x|4\tau)\theta_2(2y+x+2\tau|4\tau)\big) =$$
$$= e(x/2-y_1)\theta_1(x+y|2\tau)\theta_1(y|2\tau).$$

• The coefficient at $e_{21}\otimes e_{21}$ is

$$e(y_1-x/2)\big(\theta_3(x+2\tau|4\tau)\theta_2(2y+x+2\tau|4\tau)-\theta_2(x+2\tau|4\tau)\theta_3(2y+x+2\tau|4\tau)\big) =$$
$$= e(y_2+x/2+\tau)\theta_1(y+x|2\tau)\theta_1(y|2\tau).$$

Note that the coefficients of the tensors $e_{12}\otimes e_{12}$ and $e_{21}\otimes e_{21}$ are not functions of $y = y_2 - y_1$. In order to overcome this problem we take $\phi(y) = \begin{pmatrix} e(y/2) & 0 \\ 0 & e(-\tau/4)\end{pmatrix}$ and consider the gauge transformation

$$r(x;y_1,y_2)\mapsto \big(\phi(y_1)\otimes\phi(y_2)\big)r(x;y_1,y_2)\big(\phi^{-1}(y_1)\otimes\phi^{-1}(y_2)\big).$$

It is easy to see that the "diagonal tensors" $e_{kk}\otimes e_{ll}(k,l=1,2)$ remain unchanged (and, in particular, this gauge transformation does not influence the final answer for the "diagonal terms" obtained before) and the transformation rule for the "skew tensors" is the following:

$$\begin{array}{lll} e_{12}\otimes e_{12} & \mapsto & e(\frac{\tau}{2})e(\frac{y_1+y_2}{2})e_{12}\otimes e_{12}, \\ e_{21}\otimes e_{21} & \mapsto & e(-\frac{\tau}{2})e(-\frac{y_1+y_2}{2})e_{21}\otimes e_{21}, \\ e_{12}\otimes e_{21} & \mapsto & e(-\frac{y}{2})e_{12}\otimes e_{21}, \\ e_{21}\otimes e_{12} & \mapsto & e(\frac{y}{2})e_{21}\otimes e_{12}. \end{array}$$

Hence, the new tensor of "skew terms" is

$$C\big[\theta_4(x+y|2\tau)\theta_4(y|2\tau)(e_{21}\otimes e_{12}+e_{12}\otimes e_{21})+\theta_1(x+y|2\tau)\theta_1(y|2\tau)(e_{12}\otimes e_{12}+e_{21}\otimes e_{21})\big],$$

where

$$C = \frac{\theta_3'(\frac{1+\tau}{2}|\tau)e(\frac{1}{2}(x+y+\tau))}{\theta_3(y+\frac{1+\tau}{2}|\tau)\Delta_2} = \frac{\theta_1'(0|\tau)}{\theta_4(0|2\tau)\theta_4(x|2\tau)\theta_1(y|\tau)}$$
$$= \frac{\theta_1'(0|\tau)}{\theta_3(x/2|\tau)\theta_4(x/2|\tau)\theta_1(y|\tau)}.$$

Using the equality

$$A\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\otimes\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} + B\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}\otimes\begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix} =$$
$$= (A+B)(e_{12}\otimes e_{21}+e_{21}\otimes e_{12}) + (A-B)(e_{21}\otimes e_{21}+e_{12}\otimes e_{12})$$

and Watson's identities

$$\theta_4(y+x|2\tau)\theta_4(y|2\tau)+\theta_1(y+x|2\tau)\theta_1(y|2\tau)=\theta_4\left(y+\frac{x}{2}\Big|\,\tau\right)\theta_3\left(\frac{x}{2}\Big|\,\tau\right)$$

$$\theta_4(y+x|2\tau)\theta_4(y|2\tau)-\theta_1(y+x|2\tau)\theta_1(y|2\tau)=\theta_3\left(y+\frac{x}{2}\Big|\,\tau\right)\theta_4\left(\frac{x}{2}\Big|\,\tau\right)$$

it follows that the contribution of the "skew terms" is

$$\frac{1}{2}\frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\left(\frac{\theta_3(y+\frac{x}{2}|\tau)}{\theta_3(\frac{x}{2}|\tau)}\sigma\otimes\sigma+\frac{\theta_4(y+\frac{x}{2}|\tau)}{\theta_4(\frac{x}{2}|\tau)}\gamma\otimes\gamma\right),$$

where

$$\sigma=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\qquad \gamma=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

In summary, we obtain the following theorem.

Theorem 8.18. *The universal family of stable vector bundles of rank two and degree one on an elliptic curve E_τ gives the following solution of the associative Yang–Baxter equation:*

$$\begin{aligned}r^{(2,1)}_{\mathsf{ell}}(x;y)=\frac{1}{2}\frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\Bigg(&\frac{\theta_1(y+\frac{x}{2}|\tau)}{\theta_1(\frac{x}{2}|\tau)}\mathbb{1}\otimes\mathbb{1}+\frac{\theta_2(y+\frac{x}{2}|\tau)}{\theta_2(\frac{x}{2}|\tau)}h\otimes h+\\ &+\frac{\theta_3(y+\frac{x}{2}|\tau)}{\theta_3(\frac{x}{2}|\tau)}\sigma\otimes\sigma+\frac{\theta_4(y+\frac{x}{2}|\tau)}{\theta_4(\frac{x}{2}|\tau)}\gamma\otimes\gamma\Bigg).\end{aligned}$$

Recall that

$$\mathrm{cn}(z)=\frac{\theta_4(0|\tau)\theta_2(z|\tau)}{\theta_2(0|\tau)\theta_4(z|\tau)},\quad \mathrm{sn}(z)=\frac{\theta_3(0|\tau)\theta_1(z|\tau)}{\theta_2(0|\tau)\theta_4(z|\tau)},\quad \mathrm{dn}(z)=\frac{\theta_4(0|\tau)\theta_3(z|\tau)}{\theta_3(0|\tau)\theta_4(z|\tau)}$$

and

$$\theta_1'(0|\tau)=\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau),$$

see [42, Sections I.5 and Section II.1]. Let $\bar{r}(y)=\lim\limits_{x\to 0}(\mathrm{pr}\otimes\mathrm{pr})r(x;y)$ then we have:

Theorem 8.19. *The solution of the classical Yang–Baxter equation obtained from the universal family of stable vector bundles of rank two and degree one on a complex torus E_τ is*

$$\bar{r}(y)=\frac{1}{2}\left(\frac{\mathrm{cn}(y)}{\mathrm{sn}(y)}h\otimes h+\frac{1}{\mathrm{sn}(y)}\gamma\otimes\gamma+\frac{\mathrm{dn}(y)}{\mathrm{sn}(y)}\sigma\otimes\sigma\right).$$

Remark 8.20. Note that $\mathrm{res}_x\big(r(x;y)\big)=\frac{1}{4}\mathbb{1}\otimes\mathbb{1}$, hence the tensor $r_x(y):=r(x;y)$ also satisfies the quantum Yang–Baxter equation for $x\neq 0$. In fact, it is the well-known solution of the QYBE which was found and studied by Baxter.

Remark 8.21 (see for example Section VII.3 in [22]). Let

$$\wp(z) = \frac{1}{z^2} + \sum_{(n,m)\in\mathbb{Z}^2\setminus\{0,0\}} \left(\frac{1}{(z-n\tau-m)^2} - \frac{1}{(n\tau+m)^2}\right)$$

be the Weierstraß $\wp$–function. Then $\wp'(\frac{1}{2}) = \wp'(\frac{\tau}{2}) = \wp'(\frac{1+\tau}{2}) = 0$ and $\frac{1}{2}, \frac{\tau}{2}$ and $\frac{1+\tau}{2}$ are the only branch points of $\wp(z)$ in the fundamental parallelogram of Λ_τ. Denote $e_1 = \wp(\frac{1}{2})$, $e_2 = \wp(\frac{\tau}{2})$ and $e_3 = \wp(\frac{1+\tau}{2})$. Then we have:

$$\wp(z) - e_1 = \left(\frac{\mathrm{cn}(z)}{\mathrm{sn}(z)}\right)^2, \quad \wp(z) - e_2 = \left(\frac{1}{\mathrm{sn}(z)}\right)^2, \quad \wp(z) - e_3 = \left(\frac{\mathrm{dn}(z)}{\mathrm{sn}(z)}\right)^2.$$

9. Vector bundles on singular cubic curves

To compute the associative r-matrices coming from the nodal and cuspidal Weierstraß cubic curves, we use a description of vector bundles on singular projective curves via the formalism of matrix problems [25], see also [16, 11]. The purpose of this section is to set up a clear language and provide the core technical tools necessary for our applications.

9.1. **Description of vector bundles on singular curves.** We start with recalling the general approach of Drozd and Greuel to study torsion free sheaves on singular projective curves [25]. In our applications, the normalization of the curve will always be rational.

Let X be a reduced singular (projective) curve, $\pi : \widetilde{X} \to X$ its normalization, $\mathcal{I} := \mathcal{H}om_{\mathcal{O}}\big(\pi_*(\mathcal{O}_{\widetilde{X}}), \mathcal{O}\big) = \mathcal{A}nn_{\mathcal{O}}\big(\pi_*(\mathcal{O}_{\widetilde{X}})/\mathcal{O}\big)$ the conductor ideal sheaf. Denote by $\eta : Z = V(\mathcal{I}) \longrightarrow X$ the closed artinian subspace defined by $\mathcal{I}$ (its topological support is precisely the singular locus of X) and by $\tilde{\eta} : \widetilde{Z} \longrightarrow \widetilde{X}$ its preimage in $\widetilde{X}$, defined by the Cartesian diagram

(32)
$$\begin{tikzcd} \widetilde{Z} \arrow[r, "\tilde{\eta}"] \arrow[d, "\tilde{\pi}"'] & \widetilde{X} \arrow[d, "\pi"] \\ Z \arrow[r, "\eta"] & X. \end{tikzcd}$$

The proof of the following lemma is straightforward.

Lemma 9.1. *The diagram (32) is also a* push-down *diagram. Moreover, denote $\nu = \eta\tilde{\pi} = \pi\tilde{\eta}$ and consider the following natural transformations of functors:*

$$\begin{cases} \mathsf{j}: & \mathbb{1}_X & \longrightarrow & \pi_*\pi^*, \\ \mathsf{q}: & \mathbb{1}_X & \longrightarrow & \eta_*\eta^*, \\ \mathsf{c}: & \pi_*\pi^* & \longrightarrow & \pi_*\tilde{\eta}_*\tilde{\eta}^*\pi^* \xrightarrow{\sim} \nu_*\nu^*, \\ \mathsf{m}: & \eta_*\eta^* & \longrightarrow & \eta_*\tilde{\pi}_*\tilde{\pi}^*\eta^* \xrightarrow{\sim} \nu_*\nu^*. \end{cases}$$

Then for any vector bundle $\mathcal{V}$ on X we have a short exact sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{\begin{pmatrix} -\mathsf{j}_{\mathcal{V}} \\ \mathsf{q}_{\mathcal{V}} \end{pmatrix}} \pi_*\pi^*\mathcal{V} \oplus \eta_*\eta^*\mathcal{V} \xrightarrow{\begin{pmatrix} \mathsf{c}_{\mathcal{V}} & \mathsf{m}_{\mathcal{V}} \end{pmatrix}} \nu_*\nu^*\mathcal{V} \longrightarrow 0.$$

In order to relate vector bundles on X and $\widetilde{X}$ we use the following definition.

Definition 9.2. The category $\mathsf{Tri}(X)$ is defined as follows.

- Its objects are triples $\big(\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}}\big)$, where $\widetilde{\mathcal{V}} \in \mathsf{VB}(\widetilde{X})$, $\mathcal{N} \in \mathsf{VB}(Z)$ and
$$\widetilde{\mathsf{m}} : \tilde{\pi}^*\mathcal{N} \longrightarrow \tilde{\eta}^*\widetilde{\mathcal{V}}$$
is an isomorphism of $\mathcal{O}_{\widetilde{Z}}$-modules, called the *gluing map*.
- The set of morphisms $\mathsf{Hom}_{\mathsf{Tri}(X)}\big((\widetilde{\mathcal{V}}_1, \mathcal{N}_1, \widetilde{\mathsf{m}}_1), (\widetilde{\mathcal{V}}_2, \mathcal{N}_2, \widetilde{\mathsf{m}}_2)\big)$ consists of all pairs (F, f), where $F : \widetilde{\mathcal{V}}_1 \to \widetilde{\mathcal{V}}_2$ and $f : \mathcal{N}_1 \to \mathcal{N}_2$ are morphisms of vector bundles such that the following diagram is commutative
$$\begin{tikzcd} \tilde{\pi}^*\mathcal{N}_1 \arrow[r, "\widetilde{\mathsf{m}}_1"] \arrow[d, "\tilde{\pi}^*(f)"'] & \tilde{\eta}^*\widetilde{\mathcal{V}}_1 \arrow[d, "\tilde{\eta}^*(F)"] \\ \tilde{\pi}^*\mathcal{N}_2 \arrow[r, "\widetilde{\mathsf{m}}_2"] & \tilde{\eta}^*\widetilde{\mathcal{V}}_2. \end{tikzcd}$$

Remark 9.3. The category $\mathsf{Tri}(X)$ is endowed with an interior tensor product:
$$(\widetilde{\mathcal{V}}_1, \mathcal{N}_1, \widetilde{\mathsf{m}}_1) \otimes (\widetilde{\mathcal{V}}_2, \mathcal{N}_2, \widetilde{\mathsf{m}}_2) = (\widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2, \mathcal{N}_1 \otimes \mathcal{N}_2, \widetilde{\mathsf{m}}),$$
where $\widetilde{\mathsf{m}}$ is defined to be the composition
$$\tilde{\pi}^*(\mathcal{N}_1 \otimes \mathcal{N}_2) \xrightarrow{\cong} \tilde{\pi}^*\mathcal{N}_1 \otimes \tilde{\pi}^*\mathcal{N}_2 \xrightarrow{\widetilde{\mathsf{m}}_1 \otimes \widetilde{\mathsf{m}}_2} \tilde{\eta}^*\widetilde{\mathcal{V}}_1 \otimes \tilde{\eta}^*\widetilde{\mathcal{V}}_2 \xrightarrow{\cong} \tilde{\eta}^*(\widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2).$$
Similarly, we define the functor $\det : \mathsf{Tri}_n(X) \longrightarrow \mathsf{Tri}_1(X)$, where $\mathsf{Tri}_n(X)$ denotes the full subcategory of $\mathsf{Tri}(X)$ whose objects $(\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}})$ satisfy $\mathrm{rk}(\widetilde{\mathcal{V}}) = \mathrm{rk}(\mathcal{N}) = n$.

The following theorem summarizes main results about the category $\mathsf{Tri}(X)$ and its relations with the category of vector bundles $\mathsf{VB}(X)$.

Theorem 9.4 (Lemma 2.4 in [25] and Theorem 1.3 in [16])**.** *Let X be a reduced curve.*

- *Let $\mathbb{F} : \mathsf{VB}(X) \longrightarrow \mathsf{Tri}(X)$ be the functor assigning to a vector bundle $\mathcal{V}$ the triple $(\pi^*\mathcal{V}, \eta^*\mathcal{V}, \widetilde{\mathsf{m}}_{\mathcal{V}})$, where $\widetilde{\mathsf{m}}_{\mathcal{V}} : \tilde{\pi}^*(\eta^*\mathcal{V}) \longrightarrow \tilde{\eta}^*(\pi^*\mathcal{V})$ is the canonical isomorphism. Then $\mathbb{F}$ is an equivalence of categories.*
- *The functor $\mathbb{F}$ commutes with tensor products: we have a bifunctorial isomorphism*
$$\mathbb{F}(\mathcal{V}_1 \otimes \mathcal{V}_2) \xrightarrow{\cong} \mathbb{F}(\mathcal{V}_1) \otimes \mathbb{F}(\mathcal{V}_2).$$

Moreover, we have an isomorphism $\mathbb{F} \circ \det \xrightarrow{\cong} \det \circ \mathbb{F}$ of functors $\mathsf{VB}_n(E) \to \mathsf{Tri}_1(E)$, where $\mathsf{VB}_n(E)$ denotes the category of vector bundles of fixed rank n.

• *Let* $\mathbb{G} : \mathsf{Tri}(X) \longrightarrow \mathsf{Coh}(X)$ *be the functor assigning to a triple* $(\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}})$ *the coherent sheaf*

$$\mathcal{V} := \ker\bigl(\pi_*\widetilde{\mathcal{V}} \oplus \eta_*\mathcal{N} \xrightarrow{(\mathsf{c}\ \mathsf{m})} \nu_*\tilde{\eta}^*\widetilde{\mathcal{V}}\bigr),$$

where $\mathsf{c} = \mathsf{c}^{\widetilde{\mathcal{V}}}$ *is the canonical morphism* $\pi_*\widetilde{\mathcal{V}} \longrightarrow \pi_*\tilde{\eta}_*\tilde{\eta}^*\widetilde{\mathcal{V}} = \nu_*\tilde{\eta}^*\widetilde{\mathcal{V}}$ *and* m *is the composition* $\eta_*\mathcal{N} \xrightarrow{\mathrm{can}} \eta_*\tilde{\pi}_*\tilde{\pi}^*\mathcal{N} \xrightarrow{=} \nu_*\tilde{\pi}^*\mathcal{N} \xrightarrow{\nu_*(\widetilde{\mathsf{m}})} \nu_*\tilde{\eta}^*\widetilde{\mathcal{V}}$. *Then the coherent sheaf* $\mathcal{V}$ *is locally free. Moreover, the functor* $\mathbb{G}$ *is quasi-inverse to* $\mathbb{F}$.

Being more precise, let $\mathcal{V} \xrightarrow{\left(\begin{smallmatrix} -\mathsf{p} \\ \mathsf{q} \end{smallmatrix}\right)} \pi_*\widetilde{\mathcal{V}} \oplus \eta_*\mathcal{N}$ *be the canonical inclusion. Then the morphisms* $\widetilde{\mathsf{p}} : \pi^*\mathcal{V} \xrightarrow{\pi^*(\mathsf{p})} \pi^*\pi_*\widetilde{\mathcal{V}} \xrightarrow{\mathrm{can}} \widetilde{\mathcal{V}}$ *and* $\widetilde{\mathsf{q}} : \eta^*\mathcal{V} \xrightarrow{\eta^*(\mathsf{q})} \eta^*\eta_*\mathcal{N} \xrightarrow{\mathrm{can}} \mathcal{N}$ *are isomorphisms and* $(\pi^*\mathcal{V}, \eta^*\mathcal{V}, \widetilde{\mathsf{m}}_{\mathcal{V}}) \xrightarrow{(\widetilde{\mathsf{p}}, \widetilde{\mathsf{q}})} (\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}})$ *is an isomorphism in the category* $\mathsf{Tri}(X)$.

• *Let* $\mathcal{T}_i = (\widetilde{\mathcal{V}}_i, \mathcal{N}_i, \widetilde{\mathsf{m}}_i)$, $i = 1, 2$ *be objects of* $\mathsf{Tri}(X)$ *and* $\mathcal{V}_i = \mathbb{G}(\mathcal{T}_i)$. *Consider the short exact sequences defining* $\mathcal{V}_i = \mathbb{G}(\mathcal{T}_i)$:

$$0 \longrightarrow \mathcal{V}_i \xrightarrow{\left(\begin{smallmatrix} -\mathsf{p}_i \\ \mathsf{q}_i \end{smallmatrix}\right)} \pi_*\widetilde{\mathcal{V}}_i \oplus \eta_*\mathcal{N}_i \xrightarrow{(\mathsf{c}_i\ \mathsf{m}_i)} \nu_*\tilde{\eta}^*\widetilde{\mathcal{V}}_i \longrightarrow 0.$$

Then the sequence

$$0 \longrightarrow \mathcal{V}_1 \otimes \mathcal{V}_2 \xrightarrow{\left(\begin{smallmatrix} -\mathsf{p} \\ \mathsf{q} \end{smallmatrix}\right)} \pi_*(\widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2) \oplus \eta_*(\mathcal{N}_1 \otimes \mathcal{N}_2) \xrightarrow{(\mathsf{c}\ \mathsf{m})} \nu_*\tilde{\eta}^*(\widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2) \longrightarrow 0$$

is exact, where

$$\begin{cases} \mathsf{p} : & \mathcal{V}_1 \otimes \mathcal{V}_2 \xrightarrow{\mathsf{p}_1 \otimes \mathsf{p}_2} \pi_*\widetilde{\mathcal{V}}_1 \otimes \pi_*\widetilde{\mathcal{V}}_2 \xrightarrow{\mathrm{can}} \pi_*(\widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2) \\ \mathsf{q} : & \mathcal{V}_1 \otimes \mathcal{V}_2 \xrightarrow{\mathsf{q}_1 \otimes \mathsf{q}_2} \eta_*\mathcal{N}_1 \otimes \eta_*\mathcal{N}_2 \xrightarrow{\mathrm{can}} \eta_*(\mathcal{N}_1 \otimes \mathcal{N}_2) \\ \mathsf{c} : & \pi_*(\widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2) \xrightarrow{\pi_*(\mathrm{can})} \pi_*\tilde{\eta}_*\tilde{\eta}^*(\widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2) = \nu_*\tilde{\eta}^*(\widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2) \\ \mathsf{m} : & \eta_*(\mathcal{N}_1 \otimes \mathcal{N}_2) \xrightarrow{\eta_*(\mathrm{can})} \eta_*\tilde{\pi}_*\tilde{\pi}^*(\mathcal{N}_1 \otimes \mathcal{N}_2) \xrightarrow{\nu_*(\widetilde{\mathsf{m}})} \nu_*\eta^*(\mathcal{V}_1 \otimes \mathcal{V}_2)\ , \end{cases}$$

using $\widetilde{\mathsf{m}}$ *from Remark 9.3. This means that* $\left(\begin{smallmatrix} -\mathsf{p} \\ \mathsf{q} \end{smallmatrix}\right)$ *gives us a bifunctorial isomorphism*

$$\alpha_{\mathcal{T}_1, \mathcal{T}_2} : \mathbb{G}(\mathcal{T}_1) \otimes \mathbb{G}(\mathcal{T}_2) \xrightarrow{\sim} \mathbb{G}(\mathcal{T}_1 \otimes \mathcal{T}_2).$$

• *Let* $\mathsf{For} : \mathsf{Tri}(X) \longrightarrow \mathsf{VB}(\widetilde{X})$ *be the forgetful functor mapping a triple* $\mathcal{T} = (\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}})$ *to* $\widetilde{\mathcal{V}}$. *Let* $\mathcal{V} = \mathbb{G}(\mathcal{T})$ *and* $\gamma_{\mathcal{T}} = \widetilde{\mathsf{p}} : \pi^*\mathcal{V} \longrightarrow \widetilde{\mathcal{V}}$ *be the isomorphism introduced above. Then we obtain an isomorphism of functors* $\gamma : \pi^* \circ \mathbb{G} \longrightarrow \mathsf{For}$, *In particular, we have a commutative diagram:*

$$\begin{tikzcd} \mathsf{Hom}_{\mathsf{Tri}(X)}(\mathcal{T}_1, \mathcal{T}_2) \arrow[r, "\mathbb{G}"] \arrow[d, "\mathsf{For}"'] & \mathsf{Hom}_X(\mathcal{V}_1, \mathcal{V}_2) \arrow[d, "\pi^*"] \\ \mathsf{Hom}_{\widetilde{X}}(\widetilde{\mathcal{V}}_1, \widetilde{\mathcal{V}}_2) & \mathsf{Hom}_{\widetilde{X}}(\pi^*\mathcal{V}_1, \pi^*\mathcal{V}_2). \arrow[l, "\mathrm{cnj}(\gamma_{\mathcal{T}_1}, \gamma_{\mathcal{T}_2})"'] \end{tikzcd}$$

Moreover, γ is compatible with tensor products: for $\mathcal{T}_i = (\tilde{\mathcal{V}}_i, \mathcal{N}_i, \widetilde{\mathsf{m}}_i)$ and $\mathcal{V}_i = \mathbb{G}(\mathcal{T}_i)$ $(i = 1, 2)$ the diagram

$$\begin{tikzcd}
\pi^*\mathbb{G}(\mathcal{T}_1) \otimes \pi^*\mathbb{G}(\mathcal{T}_2) \arrow[rr, "\pi^*(\alpha_{\mathcal{T}_1,\mathcal{T}_2})"] \arrow[dr, "\gamma_{\mathcal{T}_1} \otimes \gamma_{\mathcal{T}_2}"'] & & \pi^*\mathbb{G}(\mathcal{T}_1 \otimes \mathcal{T}_2) \arrow[dl, "\gamma_{\mathcal{T}_1 \otimes \mathcal{T}_2}"] \\
& \widetilde{\mathcal{V}}_1 \otimes \widetilde{\mathcal{V}}_2 &
\end{tikzcd}$$

is commutative.

Our next goal is to obtain an explicit description of stable vector bundles on a singular Weierstraß curve E. In this context, we replace X by E and note that the normalisation is $\widetilde{E} \cong \mathbb{P}^1$. In order to obtain a clearer description of objects of $\mathsf{Tri}(E)$, recall the following well-known theorem.

Theorem 9.5 (Birkhoff-Grothendieck)**.** *On the projective line $\mathbb{P}^1$, taking the degree gives an isomorphism $\mathsf{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$. Any vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ splits into a direct sum of line bundles: $\mathcal{E} \cong \oplus_{n\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(n)^{m_n}$.*

This implies that if $(\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}})$ is an object of $\mathsf{Tri}(E)$ with $\mathrm{rk}(\widetilde{\mathcal{V}}) = n$, we have

$$\widetilde{\mathcal{V}} = \bigoplus_{l\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{k_l} \quad \text{and} \quad \mathcal{N} \cong \mathcal{O}_Z^n, \quad \text{where } \sum_{l\in\mathbb{Z}} k_l = n.$$

Note that $\mathcal{N}$ is in fact free, because Z is artinian. From now on we shall always fix a decomposition of $\widetilde{\mathcal{V}}$ as above.

An explicit description of morphisms between objects in $\mathsf{Tri}(E)$ requires to choose coordinates on $\mathbb{P}^1$. Let (z_0, z_1) be coordinates on $V = \mathbb{C}^2$. They induce homogeneous coordinates $(z_0 : z_1)$ on the projective line $\mathbb{P}^1(V) = (V \setminus \{0\})/\sim$, where $v \sim \lambda v$ for all $\lambda \in \mathbb{C}^*$.

We set $U_0 = \{(z_0 : z_1) | z_0 \neq 0\}$ and $U_\infty = \{(z_0 : z_1) | z_1 \neq 0\}$ and put $0 := (1 : 0)$, $\infty := (0 : 1)$, $z = z_1/z_0$ and $w = z_0/z_1$. So, z is a coordinate in a neighbourhood of 0. If $U = U_0 \cap U_\infty$ and $w = 1/z$ is used as a coordinate on U_∞, then the transition function of the line bundle $\mathcal{O}_{\mathbb{P}^1}(n)$ is

$$U_0 \times \mathbb{C} \supset U \times \mathbb{C} \xrightarrow{(z,v)\mapsto\left(\frac{1}{z}, \frac{v}{z^n}\right)} U \times \mathbb{C} \subset U_\infty \times \mathbb{C}.$$

The vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ is isomorphic to the sheaf of sections of the so-called tautological line bundle

$$\big\{(l, v) |\, v \in l\big\} \subset \mathbb{P}^1(V) \times V = \mathcal{O}_{\mathbb{P}^1}^2.$$

The choice of coordinates on $\mathbb{P}^1$ fixes two distinguished elements, z_0 and z_1, in the space $\mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}\bigr)$:

$$\begin{tikzcd}
\mathbb{P}^1 \times \mathbb{C}^2 \arrow[dr] & \mathcal{O}_{\mathbb{P}^1}(-1) \arrow[l, hook'] \arrow[r, "z_i"] \arrow[d] & \mathbb{P}^1 \times \mathbb{C} \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

where z_i maps $\bigl(l, (v_0, v_1)\bigr)$ to (l, v_i) for $i = 0, 1$. It is clear that the section z_0 vanishes at ∞ and z_1 vanishes at 0. After having made this choice, we may write

$$\mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m)\bigr) = \mathbb{C}[z_0, z_1]_{m-n} := \bigl\langle z_0^{m-n}, z_0^{m-n-1} z_1, \dots, z_1^{m-n} \bigr\rangle_{\mathbb{C}}.$$

Lemma 9.6. *Let E be a singular Weierstraß curve, $\pi : \mathbb{P}^1 \to E$ its normalization and $\mathcal{V}$ a vector bundle on E. Then $\deg_E(\mathcal{V}) = \deg_{\mathbb{P}^1}(\pi^*\mathcal{V})$.*

Proof. If $n = \mathrm{rk}(\mathcal{V})$ then $\pi^*\mathcal{V}$ is a vector bundle of rank n on $\mathbb{P}^1$. The canonical morphism $g : \mathcal{V} \to \pi_*\pi^*\mathcal{V}$ is generically injective and $\mathcal{V}$ is torsion free, hence $\ker(g) = 0$ and we have an exact sequence

$$0 \longrightarrow \mathcal{V} \xrightarrow{\;g\;} \pi_*\pi^*\mathcal{V} \longrightarrow \mathcal{S} \longrightarrow 0,$$

where $\mathcal{S}$ is a torsion sheaf supported at the singular point s of the curve E. Since g commutes with restrictions to an open set, we have

$$\mathcal{S} \cong \bigl(\mathrm{coker}(\mathcal{O}_E \to \pi_*(\mathcal{O}_{\mathbb{P}^1}))\bigr)^n.$$

Because s is either a node or a cusp, we obtain $h^0(\mathcal{S}) = n$. Using the Riemann-Roch formula, this implies

$$\deg_E(\mathcal{V}) = \chi(\mathcal{V}) = \chi(\pi_*\pi^*\mathcal{V}) - \chi(\mathcal{S}) = \chi(\pi^*\mathcal{V}) - n = \deg_{\mathbb{P}^1}(\pi^*\mathcal{V}).$$

□

Lemma 9.7. *Let E be a singular Weierstraß curve, $\pi : \mathbb{P}^1 \to E$ its normalization and $\mathcal{V}$ a simple vector bundle of rank n on E. Then*

- *$\mathcal{V}$ is stable.*
- *$\pi^*\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}(c)^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(c+1)^{n_2}$ for some integer $c \in \mathbb{Z}$ and some non-negative integers n_1, n_2 which satisfy $n = n_1 + n_2$.*

Proof. For the first statement see for example [20, Corollary 4.5]. To prove the second part, let $\mathbb{F}(\mathcal{V}) = (\widetilde{\mathcal{V}}, \mathcal{O}_Z^n, \widetilde{\mathsf{m}})$ and assume

$$\pi^*\mathcal{V} \cong \widetilde{\mathcal{V}} = \mathcal{O}_{\mathbb{P}^1}(c) \oplus \mathcal{O}_{\mathbb{P}^1}(d) \oplus \widetilde{\mathcal{V}}'',$$

where $d - c \geq 2$. Because the length of $\widetilde{Z}$ is two, we can find a non-zero homogeneous form $p = p(z_0, z_1) \in \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}(c), \mathcal{O}_{\mathbb{P}^1}(d)\bigr)$ such that $\tilde{\eta}^*(p) = 0$. This gives us a non-scalar endomorphism of $\mathcal{V}$ corresponding to the endomorphism (F, f) of the triple $\mathbb{F}(\mathcal{V})$ given by $f = \mathrm{id}$ and

$$F = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This contradicts our assumption that $\mathcal{V}$ was simple. $\square$

An explicit description of the morphism $\widetilde{\mathsf{m}}$ by a matrix requires to fix isomorphisms $\zeta_l : \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(l) \to \mathcal{O}_{\widetilde{Z}}$. In order to keep compatibility with tensor products in our description of vector bundles, we have to ensure that for all $k, l \in \mathbb{Z}$ the following diagram is commutative:

$$\begin{tikzcd} \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(l) \arrow[r, "\mathrm{can}"] \arrow[d, "\zeta_k \otimes \zeta_l"'] & \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(k+l) \arrow[d, "\zeta_{k+l}"] \\ \mathcal{O}_{\widetilde{Z}} \otimes \mathcal{O}_{\widetilde{Z}} \arrow[r, "\mathrm{mult}"] & \mathcal{O}_{\widetilde{Z}}. \end{tikzcd} \tag{33}$$

In the case of a nodal or cuspidal Weierstraß cubic curve we shall explicitly give our choice of these isomorphisms.

Remark 9.8. It is natural to assume $\zeta_0 = \mathsf{id}$. Then such a family of isomorphisms $\{\zeta_l\}_{l \in \mathbb{Z}}$ is uniquely determined by $\zeta = \zeta_1 : \tilde{\eta}^*\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow \mathcal{O}_{\widetilde{Z}}$. Moreover, the choice of a global section $p = p_\zeta = az_0 + bz_1 \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$, which does not vanish on $\widetilde{Z}$, determines ζ as follows: $\zeta(s) = \frac{s}{p}\Big|_{\widetilde{Z}}$. Modulo automorphisms of $\mathbb{P}^1$ such a section p is determined by its unique zero, which should belong to $\mathbb{P}^1 \setminus \widetilde{Z} \cong \breve{E}$. In other words, our choice of a set of trivializations $\{\zeta_l\}_{l \in \mathbb{Z}}$ corresponds to the choice of a smooth point in Atiyah's classification of vector bundles on an elliptic curve [5].

Note that, because we have fixed a decomposition $\widetilde{\mathcal{V}} = \bigoplus_{l \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{k_l}$, a family $\{\zeta_l\}_{l \in \mathbb{Z}}$ induces an isomorphism $\zeta^{\widetilde{\mathcal{V}}} : \tilde{\eta}^*\widetilde{\mathcal{V}} \to \mathcal{O}_{\widetilde{Z}}^n$. Because $\mathcal{N} \cong \mathcal{O}_Z^n$, we also get an isomorphism $\tilde{\pi}^*\mathcal{N} \cong \mathcal{O}_{\widetilde{Z}}^n$. This allows us to describe the map $\widetilde{\mathsf{m}} : \tilde{\pi}^*\mathcal{N} \longrightarrow \tilde{\eta}^*\widetilde{\mathcal{V}}$ as a matrix in $\mathsf{GL}_n(\mathcal{O}_{\widetilde{Z}})$.

Corollary 9.9. *Let $\mathsf{Mat}_{\widetilde{Z}}$ be the category of square matrices over the ring $\mathcal{O}_{\widetilde{Z}}$. The choice of isomorphisms $\{\zeta_l\}_{l \in \mathbb{Z}}$ yields a functor $\mathbb{P}^\zeta : \mathsf{Tri}(E) \longrightarrow \mathsf{Mat}_{\widetilde{Z}}$, assigning to a triple $(\widetilde{\mathcal{V}}, \mathcal{O}_Z^n, \widetilde{\mathsf{m}})$ the matrix of the $\mathcal{O}_{\widetilde{Z}}$-linear map*

$$\mathcal{O}_{\widetilde{Z}}^n \xrightarrow{\widetilde{\mathsf{m}}} \tilde{\eta}^*\widetilde{\mathcal{V}} \xrightarrow{\zeta^{\widetilde{\mathcal{V}}}} \mathcal{O}_{\widetilde{Z}}^n.$$

Moreover, let $\mathbb{H}^\zeta = \mathbb{P}^\zeta \circ \mathbb{F} : \mathsf{VB}(E) \longrightarrow \mathsf{Mat}_{\widetilde{Z}}$. Using (33)*, for any $\mathcal{L} \in \mathsf{Pic}(E)$ and $\mathcal{V} \in \mathsf{VB}(E)$ we obtain:*

$$\mathbb{H}^\zeta(\mathcal{L} \otimes \mathcal{V}) = \mathbb{H}^\zeta(\mathcal{L}) \cdot \mathbb{H}^\zeta(\mathcal{V}) \quad \textit{and} \quad \mathbb{H}^\zeta\big(\det(\mathcal{V})\big) = \det\big(\mathbb{H}^\zeta(\mathcal{V})\big).$$

Let $(\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}})$ be an object of $\mathsf{Tri}(E)$. We have a natural action of the group $\mathsf{Aut}_{\mathbb{P}^1}(\widetilde{\mathcal{V}}) \times \mathsf{Aut}_Z(\mathcal{N})$ on the vector space $\mathsf{Hom}_{\widetilde{Z}}(\tilde{\pi}^*\mathcal{N}, \tilde{\eta}^*\widetilde{\mathcal{V}})$. The orbits of this action correspond precisely to the points in the fibre of the functor $\pi^* : \mathsf{VB}(E) \to \mathsf{VB}(\mathbb{P}^1)$ over $\widetilde{\mathcal{V}}$. In what follows, we shall use this action to find a normal form for $\widetilde{\mathsf{m}}$. A description of the matrix problem describing *all* vector bundles on an irreducible Weierstraß cubic curve, can be found in [25] and [11]. In this article, we are mainly interested in a description of *simple* vector bundles. Having in mind Lemma 9.7, we introduce the following notation.

In order to recover a vector bundle $\mathcal{V}$ from the matrix $\mathbb{H}^\zeta(\mathcal{V})$, we need to specify $\widetilde{\mathcal{V}}$. For a singular Weierstraß cubic curve E, let $\mathsf{VB}^{(0,1)}(E)$ be the full subcategory of $\mathsf{VB}(E)$ consisting of vector bundles $\mathcal{V}$ such that $\pi^*\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$ for some non-negative integers $n_1, n_2 \in \mathbb{Z}$. In a similar way, let $\mathsf{Tri}^{(0,1)}(E)$ be the corresponding subcategory of the category $\mathsf{Tri}(E)$.

Definition 9.10. Let E be a Weierstraß cubic curve E. Consider the following category $\mathsf{BM}(E)$ of "block matrices":

- Its objects are invertible matrices over the ring $\mathcal{O}_{\widetilde{Z}}$ with a block structure:
$$M = \left(\begin{array}{c|c} M_{00} & M_{01} \\ \hline M_{10} & M_{11} \end{array}\right),$$
where M_{00} and M_{11} are square matrices, possibly of size zero.
- Let M and N be two objects of $\mathsf{BM}(E)$ of sizes $m = m_0 + m_1$ and $n = n_0 + n_1$ respectively, where the block M_{ij} has size $m_i \times m_j$ etc. Then a morphism from M to N in the category $\mathsf{BM}(E)$ is given by a pair of matrices (F, f), where $f \in \mathsf{Mat}_{n\times m}(\mathcal{O}_Z)$ and
$$F = \left(\begin{array}{c|c} F_{00} & 0 \\ \hline F_{10} & F_{11} \end{array}\right)$$
has blocks $F_{00} \in \mathsf{Mat}_{n_0\times m_0}(\mathbb{C})$, $F_{11} \in \mathsf{Mat}_{n_1\times m_1}(\mathbb{C})$ and $F_{10} \in \mathsf{Mat}_{n_1\times m_0}(\mathcal{O}_{\widetilde{Z}})$, such that $FM = N\tilde{f}$. Here $\tilde{f}$ is the image of the matrix f under the morphism $\mathsf{Mat}_{n\times m}(\mathcal{O}_Z) \longrightarrow \mathsf{Mat}_{n\times m}(\mathcal{O}_{\widetilde{Z}})$ induced by the ring homomorphism $\mathcal{O}_Z \longrightarrow \mathcal{O}_{\widetilde{Z}}$.
- The composition of morphisms in $\mathsf{BM}(E)$ is given by the matrix product.

Proposition 9.11. *Take some isomorphism $\zeta : \mathcal{O}_{\mathbb{P}^1}(1)|_{\widetilde{Z}} \longrightarrow \mathcal{O}_{\widetilde{Z}}$. Then in the notation of Remark 9.8 and Corollary 9.9, we have equivalences of categories:*
$$\mathsf{VB}^{(0,1)}(E) \xrightarrow{\mathbb{F}} \mathsf{Tri}^{(0,1)}(E) \xrightarrow{\mathbb{P}^\zeta} \mathsf{BM}(E)\,,$$
with block structure on $\mathbb{P}^\zeta\big(\widetilde{\mathcal{V}}, \mathcal{O}_Z^n, \widetilde{\mathsf{m}}\big)$ coming from the decomposition $\widetilde{\mathcal{V}} = \mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$. Moreover, the functor $\mathbb{P}^\zeta \circ \mathbb{F}$ sends $\det(\mathcal{V}) \in \mathsf{VB}(E)$ to the determinant of the corresponding matrix $\mathbb{P}^\zeta\big(\mathbb{F}(\mathcal{V})\big)$.

Proof. This result follows from Theorem 9.4 and the observation that the map

$$\tilde{\eta}^* : \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\big) \longrightarrow \mathsf{Hom}_{\widetilde{Z}}(\mathcal{O}_{\widetilde{Z}}, \mathcal{O}_{\widetilde{Z}})$$

is an isomorphism both for a nodal and a cuspidal cubic curve. □

9.2. **Simple vector bundles on a nodal Weierstraß curve.** The main aim of this subsection is an explicit description of those objects in $\mathsf{Tri}(E)$ which correspond to simple vector bundles on a nodal Weierstraß curve E. We give an algorithm which produces some kind of normal form of such triples for each given rank and degree. Crucial for our application to the Yang–Baxter equation is a description of the family of all simple vector bundles with fixed rank and degree in a way which is compatible with the action of the Jacobian. We shall also see that rank and degree of a simple vector bundle on a nodal Weierstraß curve are always coprime.

Let E be a nodal Weierstraß curve, e.g. given by $zy^2 = x^3 + x^2 z$, $s = (0:0:1)$ the singular point and $\pi : \mathbb{P}^1 \longrightarrow E$ its normalization. Choose homogeneous coordinates $(z_0 : z_1)$ on $\mathbb{P}^1$ in such a way that $\pi^{-1}(s) = \{0, \infty\}$. Then, in notations of the previous subsection, Z and $\widetilde{Z}$ are reduced complex spaces as follows

$$Z = \{s\} \quad \text{and} \quad \widetilde{Z} = \{0\} \cup \{\infty\}.$$

Hence, for $\big(\tilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}}\big) \in \mathsf{Tri}(E)$ the map $\widetilde{\mathsf{m}}$ is just an isomorphism of $\mathbb{C}\times\mathbb{C}$-modules, i.e. it is given by a pair of invertible matrices $\mathsf{m}(0)$ and $\mathsf{m}(\infty)$.

The linear form $p = p_\zeta(z_0, z_1) = z_1 - z_0 \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$ does not vanish on $\widetilde{Z}$. Following the recipe from Remark 9.8, we consider the collection of isomorphisms $\zeta_l : \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(l) \longrightarrow \mathcal{O}_{\widetilde{Z}}$ given by the formula $\zeta_l(s) = \left.\dfrac{s}{p^l}\right|_{\widetilde{Z}}$ for each open subset $V \subset \mathbb{P}^1$ not containing $(1:1)$, $l \in \mathbb{Z}$ and any $s \in \Gamma\big(V, \mathcal{O}_{\mathbb{P}^1}(l)\big)$.

This implies the following evaluation rule for morphisms of vector bundles on $\mathbb{P}^1$: if $q = a_0 z_0^{m-n} + a_1 z_0^{m-n-1} z_1 + \dots + a_{m-n} z_1^{m-n} \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m)\big)$ then we have a commutative diagram

$$\begin{tikzcd}
\tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(n) \arrow[rr, "\tilde{\eta}^*(q)"] \arrow[d, "\zeta_n"'] & & \tilde{\eta}^*\mathcal{O}_{\mathbb{P}^1}(m) \arrow[d, "\zeta_m"] \\
\mathbb{C}_0 \oplus \mathbb{C}_\infty \arrow[rr, "\begin{pmatrix} (-1)^{m-n}a_0 & 0 \\ 0 & a_{m-n} \end{pmatrix}"] & & \mathbb{C}_0 \oplus \mathbb{C}_\infty.
\end{tikzcd}$$

If the family $\{\zeta_l\}$ is understood, we shall often write $\widetilde{\eta}^*(q) = \big(\big((-1)^{m-n}a_0\big), \big(a_{m-n}\big)\big)$.

Our next goal is to describe the category $\mathsf{BM}(E)$ from Definition 9.10. An object of $\mathsf{BM}(E)$ is a pair of matrices $\mathsf{m}(0)$ and $\mathsf{m}(\infty)$ simultaneously divided into blocks

$$\mathsf{m}(0) = \left(\begin{array}{c|c} M_{00}(0) & M_{01}(0) \\ \hline M_{10}(0) & M_{11}(0) \end{array}\right), \quad \mathsf{m}(\infty) = \left(\begin{array}{c|c} M_{00}(\infty) & M_{01}(\infty) \\ \hline M_{10}(\infty) & M_{11}(\infty) \end{array}\right).$$

Two objects $\bigl(\mathsf{m}(0), \mathsf{m}(\infty)\bigr)$ and $\bigl(\mathsf{m}'(0), \mathsf{m}'(\infty)\bigr)$ of $\mathsf{BM}(E)$ are isomorphic if and only if the corresponding blocks have the same sizes and there exist matrices

$$F(0) = \left(\begin{array}{c|c} F_{11} & 0 \\ \hline F_{21}(0) & F_{22} \end{array}\right), \quad F(\infty) = \left(\begin{array}{c|c} F_{11} & 0 \\ \hline F_{21}(\infty) & F_{22} \end{array}\right)$$

and f such that

$$F(0)\mathsf{m}(0) = \mathsf{m}'(0)f, \quad F(\infty)\mathsf{m}(\infty) = \mathsf{m}'(\infty)f.$$

In particular, we have the following isomorphism in the category $\mathsf{BM}(E)$:

$$\bigl(\mathsf{m}(0), \mathsf{m}(\infty)\bigr) \cong \bigl(\mathsf{m}(0)\mathsf{m}(\infty)^{-1}, \mathsf{id}\bigr)$$

i.e. without loss of generality we may assume that the second matrix $\mathsf{m}(\infty)$ is the identity matrix.

To illustrate how the explicit identification of vector bundles on E and objects of $\mathsf{BM}(E)$ works in practice, we shall now consider the simplest interesting case: the description of $\mathsf{Pic}^1(E)$. We explicitly determine for each $y \in \breve{E}$ the object in $\mathsf{Tri}(E)$ which corresponds to the line bundle $\mathcal{O}_E(y)$.

The chosen coordinates provide us with an isomorphism $\mathbb{C}^* \cong U := \mathbb{P}^1 \setminus \{0, \infty\}$ mapping $y \in \mathbb{C}^*$ to $(1 : y) \in U$. As E is nodal, the normalization restricts to an isomorphism $\pi : \mathbb{P}^1 \setminus \{0, \infty\} \to \breve{E}$. Together, this gives us an identification $\breve{E} \cong \mathbb{C}^*$, under which $y \in \mathbb{C}^*$ corresponds to $\tilde{y} := \pi^{-1}(y) = (1 : y) \in \mathbb{P}^1$. Obviously, $\pi^*\bigl(\mathcal{O}_E(y)\bigr) = \mathcal{O}_{\mathbb{P}^1}(\tilde{y}) \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and the following lemma is true.

Lemma 9.12. *For the given choice of homogeneous coordinates on $\mathbb{P}^1$ and the set of trivializations $\{\zeta_l\}_{l\in\mathbb{Z}}$ described above, we obtain for all $y \in \breve{E} \cong \mathbb{C}^*$*

$$\mathbb{F}\bigl(\mathcal{O}_E(y)\bigr) = \Bigl(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, \bigl((y), (1)\bigr)\Bigr).$$

Proof. Assume $\mathcal{T}_{\mathcal{O}_E(y)} := \mathbb{F}\bigl(\mathcal{O}_E(y)\bigr) = \Bigl(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, \bigl((\lambda), (1)\bigr)\Bigr)$. It is clear that $\mathcal{T}_{\mathcal{O}_E} := \mathbb{F}(\mathcal{O}_E) = \Bigl(\mathcal{O}_{\mathbb{P}^1}, \mathbb{C}_s, \bigl((1), (1)\bigr)\Bigr)$. Moreover, by Theorem 9.4 we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Tri}(E)}(\mathcal{T}_{\mathcal{O}_E}, \mathcal{T}_{\mathcal{O}_E(y)}) \arrow[r, "\mathbb{G}"] \arrow[d, "\mathsf{For}"'] & \mathsf{Hom}_E\bigl(\mathcal{O}_E, \mathcal{O}_E(y)\bigr) \arrow[d, "\pi^*"] \\
\mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\bigr) & \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(\tilde{y})\bigr). \arrow[l, "\mathrm{cnj}\left(\gamma_{\mathcal{T}_{\mathcal{O}}}, \gamma_{\mathcal{T}_{\mathcal{O}(y)}}\right)"']
\end{tikzcd}$$

The section $z_1 - yz_0 \in \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\bigr)$ generates the image of π^*, hence belongs to the image of For. Using the description of morphisms in the category $\mathsf{Tri}(E)$ and the evaluation rule $\tilde{\eta}^*(z_1 - yz_0) = \bigl((y), (1)\bigr)$, this is equivalent to the existence of a constant $c \in \mathbb{C}^*$ making the following diagram commutative:

$$\begin{tikzcd}
\mathbb{C}_0 \oplus \mathbb{C}_\infty \arrow[r, "\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"'] & \mathbb{C}_0 \oplus \mathbb{C}_\infty \arrow[d, "\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}"] \\
\mathbb{C}_0 \oplus \mathbb{C}_\infty \arrow[r, "\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}"] & \mathbb{C}_0 \oplus \mathbb{C}_\infty.
\end{tikzcd}$$

This implies that $\lambda = y$ and $\mathbb{F}\big(\mathcal{O}_E(y)\big) \cong \big(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, \big((y), (1)\big)\big)$. □

Our next goal is to describe the so-called *Atiyah bundles.*

Lemma 9.13. *Let E be a nodal Weierstraß curve. Then there exists a unique indecomposable semi-stable vector bundle $\mathcal{A}_n$ of rank n and degree 0 such that all its Jordan-Hölder factors are isomorphic to $\mathcal{O}_E$. This vector bundle is called the Atiyah bundle of rank n and is given by the triple $\big(\mathcal{O}^n_{\mathbb{P}^1}, \mathbb{C}^n_s, \widetilde{\mathsf{m}}\big)$, where*

$$\mathsf{m}(0) = J_m(1) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad \mathsf{m}(\infty) = I_m = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Proof. The category of semi-stable vector bundles with the Jordan-Hölder factor $\mathcal{O}_E$ is equivalent to the category of finite-dimensional modules over $\mathbb{C}[[t]]$, see for example [29, Theorem 1.1 and Lemma 1.7]. Therefore, there exists a unique indecomposable vector bundle $\mathcal{A}_n$ of rank n recursively defined by the non-split exact sequences

$$0 \longrightarrow \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1} \longrightarrow \mathcal{O}_E \longrightarrow 0 \quad \text{and} \quad \mathcal{A}_1 = \mathcal{O}_E.$$

In order to get a description of $\mathcal{A}_n$ in terms of triples, first observe that $\pi^*\mathcal{A}_n \cong \mathcal{O}^n_{\mathbb{P}^1}$, hence $\mathbb{F}(\mathcal{A}_n) = \big(\mathcal{O}^n_{\mathbb{P}^1}, \mathbb{C}^n_s, \widetilde{\mathsf{m}}\big)$. The morphism $\widetilde{\mathsf{m}}$ is given by two invertible matrices $\mathsf{m}(0), \mathsf{m}(\infty) \in \mathsf{GL}_n(\mathbb{C})$. If $\widetilde{\mathsf{m}}' = \big(\mathsf{m}'(0), \mathsf{m}'(\infty)\big)$ is another pair such that

$$\mathsf{m}'(0) = S^{-1}\mathsf{m}(0)T, \quad \mathsf{m}'(\infty) = S^{-1}\mathsf{m}(\infty)T$$

with $S, T \in \mathsf{GL}_n(\mathbb{C})$, then $\big(\mathcal{O}^n_{\mathbb{P}^1}, \mathbb{C}^n_s, \widetilde{\mathsf{m}}'\big)$ and $\big(\mathcal{O}^n_{\mathbb{P}^1}, \mathbb{C}^n_s, \widetilde{\mathsf{m}}\big)$ define isomorphic vector bundles on E. We may, therefore, assume $\mathsf{m}(\infty) = I_n$. Keeping $\mathsf{m}(\infty) = I_n$ unchanged, the matrix $\mathsf{m}(0)$ can still be transformed to $S^{-1}\mathsf{m}(0)S$. Hence, $\mathsf{m}(0)$ splits into a direct sum of Jordan blocks. Since the vector bundle $\mathcal{A}_n$ is indecomposable, $\mathsf{m}(0) \sim J_n(\lambda)$ for some $\lambda \in \mathbb{C}^*$. From the condition $\mathsf{Hom}_E(\mathcal{A}_n, \mathcal{O}) = \mathbb{C}$ one can easily deduce $\lambda = 1$. □

Now we start to focus on simple vector bundles on a nodal Weierstraß curve E. We aim at giving a canonical form for those elements $(\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}})$ in $\mathsf{Tri}(E)$ which correspond to simple vector bundles on E under the functor $\mathbb{F}$ from Theorem 9.4.

Definition 9.14. Let E be a nodal cubic curve and $n_1 > 0$, $n_2 \geq 0$ integers. The category $\mathsf{MP}_{\mathrm{nd}}(n_1, n_2)$ is defined as follows.

- Its objects are invertible matrices with blocks $M_{ij} \in \mathsf{Mat}_{n_i \times n_j}(\mathbb{C})$
$$M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right).$$
- Morphisms are pairs of block matrices
$$\mathsf{Hom}_{\mathsf{MP}_{\mathrm{nd}}(n_1,n_2)}(M, M') = \{(S, T) \mid SM = M'T\},$$
with obvious composition and such that
$$S = \begin{pmatrix} A & 0 \\ C' & B \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} A & 0 \\ C'' & B \end{pmatrix}$$
have blocks of the same size as the blocks of M and M'.

By $\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$ we denote the full subcategory of *simple*[4] objects of $\mathsf{MP}_{\mathrm{nd}}(n_1, n_2)$.

The proof of the following lemma is straightforward.

Lemma 9.15. *Let $\mathsf{VB}^{(0,1)}_{n_1,n_2}(E)$ be the category of vector bundles $\mathcal{V} \in \mathsf{VB}(E)$ such that $\pi^*\mathcal{V} \cong \mathcal{O}^{n_1}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$. Then $\mathsf{VB}^{(0,1)}_{n_1,n_2}(E)$ and $\mathsf{MP}_{\mathrm{nd}}(n_1, n_2)$ are equivalent.*

Proof. Let $\mathsf{BM}_{n_1,n_2}(E)$ be the full subcategory of $\mathsf{BM}(E)$ consisting of matrices, whose diagonal blocks have sizes $n_1 \times n_1$ and $n_2 \times n_2$. By Proposition 9.11 the categories $\mathsf{VB}^{(0,1)}_{n_1,n_2}(E)$ and $\mathsf{BM}_{n_1,n_2}(E)$ are equivalent. But it is easy to see that sending and $M \in \mathsf{MP}_{\mathrm{nd}}(n_1, n_2)$ to $(M, \mathrm{id}) \in \mathsf{GL}_{n_1+n_2}(\mathcal{O}_{\widetilde{Z}})$ as an object in $\mathsf{BM}_{n_1,n_2}(E)$ with same block structure, is an equivalence. □

Remark 9.16. If $n_2 = 0$ the block structure becomes invisible and we end up in a situation of elementary linear algebra. Indeed, we have $\mathsf{MP}_{\mathrm{nd}}(n, 0) = \mathsf{GL}_n(\mathbb{C})$ and $\mathsf{Hom}_{\mathsf{MP}_{\mathrm{nd}}(n,0)}(M, M') = \{S \mid SM = M'S\}$. The indecomposable objects in this category are precisely those which are isomorphic to a Jordan block $J_n(\lambda)$, $\lambda \in \mathbb{C}^*$. The endomorphism ring of $J_n(\lambda)$ is isomorphic to $\mathbb{C}[t]/t^n$. Hence, $\mathsf{MP}^s_{\mathrm{nd}}(n, 0) = \emptyset$ if $n > 1$ and $\mathsf{MP}^s_{\mathrm{nd}}(1, 0) = \mathsf{GL}_1(\mathbb{C}) = \mathbb{C}^*$.

We aim now at finding a canonical form for objects $M \in \mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$. This means that we wish to find in each isomorphism class of $\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$ a unique object with a particularly "simple" structure. We shall often say that we can "reduce" a matrix M to a matrix N if M and N are isomorphic in $\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$ and N has a "simpler"

[4]Simple objects of $\mathsf{MP}_{\mathrm{nd}}(n_1, n_2)$ are by definition the objects having only scalar endomorphisms. They are sometimes called *Schurian* objects or *bricks*.

form than M. The reduction procedure described below is based on the following easy lemma.

Lemma 9.17. *The block M_{12} has full rank, if $M \in \mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$ is simple.*

Proof. If the matrix M_{12} does not have full rank, M can be reduced to the form

$$M = \left(\begin{array}{cc|cc} M_1 & M_2 & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline M_3 & M_4 & 0 & M_5 \\ M_6 & M_7 & 0 & M_8 \end{array}\right),$$

where M_{11}, M_{21} and M_{22} are split into blocks such that M_1 and M_8 are square matrices. As an object of $\mathsf{MP}_{\mathrm{nd}}(n_1, n_2)$, such a matrix has an endomorphism (S, T) with

$$S = \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline M_5 & 0 & I & 0 \\ M_8 & 0 & 0 & I \end{array}\right) \quad T = \left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ M_1 & M_2 & 0 & I \end{array}\right).$$

Since M is invertible, at least one of the matrices M_1 and M_2 is not the zero matrix, hence (S, T) is not a scalar multiple of the identity. This implies that M was not simple. $\square$

Example 9.18. The triple $\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^2_s, \widetilde{\mathsf{m}}\big)$ with

$$\widetilde{\mathsf{m}}(0) = \left(\begin{array}{c|c} 0 & 1 \\ \hline \lambda & 0 \end{array}\right) \qquad \text{and} \qquad \widetilde{\mathsf{m}}(\infty) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right),$$

defines for any $\lambda \in \mathbb{C}^*$ a simple vector bundle of rank 2 and degree 1 on E. The corresponding matrix for this vector bundle is $M_{1,1}(\lambda) := \left(\begin{smallmatrix} 0 & 1 \\ \lambda & 0 \end{smallmatrix}\right) \in \mathsf{MP}^s_{\mathrm{nd}}(1, 1)$. $\square$

Theorem 9.19. *Let E be a nodal Weierstraß curve and denote by $\mathsf{Spl}^{(n,d)}(E)$ the set of all isomorphism classes of simple vector bundles of rank n and degree d on E. If* $\gcd(n, d) = 1$ *the map* $\det : \mathsf{Spl}^{(n,d)}(E) \to \mathsf{Pic}^d(E) \cong \mathbb{C}^*$ *is bijective. If* $\gcd(n, d) > 1$ *we have $\mathsf{Spl}^{(n,d)}(E) = \emptyset$.*

This result can be proven by various methods, see for example [17, Theorem 3.6] for a description of simple vector bundles on E in terms of étale coverings. For the reader's convenience we shall outline another proof[5], which is parallel to the case of a cuspidal cubic curve [13].

Proof. First note that, without loss of generality, we may assume $0 \le d < n$. If $\mathsf{Spl}^{(n,d)}(E) \neq \emptyset$ and $\mathcal{V}$ is a non-zero element of $\mathsf{Spl}^{(n,d)}(E)$ then, by Lemma 9.7,

$$\mathbb{F}(\mathcal{V}) \cong \Big(\mathcal{O}^{n_1}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}, \mathbb{C}^{n_1+n_2}_s, \big(M, \mathsf{id}\big)\Big),$$

[5]This proof is due to Lesya Bodnarchuk.

where $n_2 = d$ and $n_1 = n - d$. By Lemma 9.15 we have an equivalence $\mathsf{Spl}^{(n,d)}(E) \cong \mathsf{MP}^s_{\mathrm{nd}}(n-d,d)$.

Assume first $n_2 = 0$ and $n_1 > 1$. In this case, we have seen in Remark 9.16 that $\mathsf{MP}^s_{\mathrm{nd}}(n,0) = \emptyset$. This implies $\mathsf{Spl}^{(n,0)}(E) = \emptyset$ for $n > 1$. On the other hand, $\mathsf{Spl}^{(1,0)}(E) = \mathsf{Pic}^0(E) \cong \breve{E}$.

For the rest of this proof we assume $n_2 > 0$. By Lemma 9.17, the block M_{12} of $M \in \mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2)$ has maximal rank. If $n_1 = n_2$ this means that M_{12} is invertible and M can be reduced to the form

$$M = \left(\begin{array}{c|c} 0 & I \\ \hline X & 0 \end{array}\right)$$

where X splits into a direct sum of Jordan blocks with non-zero eigenvalues. It is easy to see that M is decomposable in $\mathsf{MP}_{\mathrm{nd}}(n_1,n_2)$ unless $n_1 = n_2 = 1$. Hence, $\mathsf{MP}^s_{\mathrm{nd}}(m,m) = \emptyset$ if $m > 1$.

On the other hand, if $n_1 \neq n_2$, we can reduce M to the form (because both $n_i > 0$)

$$\left(\begin{array}{c|cc} 0 & I & 0 \\ \hline M'_{11} & 0 & M'_{12} \\ M'_{21} & 0 & M'_{22} \end{array}\right) \text{ if } n_2 > n_1, \quad \text{or to} \quad \left(\begin{array}{cc|c} M'_{11} & M'_{12} & 0 \\ 0 & 0 & I \\ \hline M'_{21} & M'_{22} & 0 \end{array}\right) \text{ if } n_1 > n_2.$$

In both cases, the additional split of the blocks is made in such a way that M'_{11} and M'_{22} are square matrices. A straightforward calculation shows that

$$M' = \begin{pmatrix} M'_{11} & M'_{12} \\ M'_{21} & M'_{22} \end{pmatrix}$$

is an object of $\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2 - n_1)$ or $\mathsf{MP}^s_{\mathrm{nd}}(n_1 - n_2, n_2)$ respectively. This implies that, in case $n_2 > n_1$, the fully faithful functor

$$\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2 - n_1) \longrightarrow \mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$$

which is defined on objects by sending

$$N' = \begin{pmatrix} N'_{11} & N'_{12} \\ N'_{21} & N'_{22} \end{pmatrix} \quad \text{to} \quad N = \left(\begin{array}{c|cc} 0 & I & 0 \\ \hline N'_{11} & 0 & N'_{12} \\ N'_{21} & 0 & N'_{22} \end{array}\right) \in \mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$$

and on morphisms by sending

$$\left(\begin{pmatrix} A & 0 \\ D' & E \end{pmatrix}, \begin{pmatrix} A & 0 \\ D'' & E \end{pmatrix} \right) \in \mathsf{Hom}_{\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2 - n_1)}(M', N')$$

to

$$\left(\left(\begin{array}{c|cc} A & 0 & 0 \\ \hline N'_{12}D' & A & 0 \\ N'_{22}D' & D' & E \end{array}\right), \left(\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & A & 0 \\ D'' & D' & E \end{array}\right) \right) \in \mathsf{Hom}_{\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)}(M, N)$$

is in fact an equivalence of categories. Similarly, if $n_1 > n_2$, we obtain an equivalence

$$\mathsf{MP}^s_{\mathrm{nd}}(n_1 - n_2, n_2) \longrightarrow \mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2).$$

If we start with any pair of positive integers $n_1 \neq n_2$ and continue to reduce the size of the matrix in the way described above, we obtain an equivalence of categories $\mathsf{MP}^s_{\mathrm{nd}}(m,m) \to \mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2)$, where $m = \gcd(n_1,n_2)$. Our assumption $\mathsf{Spl}^{(n,d)}(E) \neq \emptyset$ implies now $\gcd(n,d) = \gcd(n_1,n_2) = 1$ and our construction gives us an equivalence

$$\mathsf{MP}^s_{\mathrm{nd}}(1,1) \to \mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2).$$

Using Lemma 9.17 we see that each object in $\mathsf{MP}^s_{\mathrm{nd}}(1,1)$ is isomorphic to

$$M_{1,1}(\lambda) = \left(\begin{array}{c|c} 0 & 1 \\ \hline \lambda & 0 \end{array}\right)$$

for some $\lambda \in \mathbb{C}^*$. We consider $M_{1,1}(\lambda)$ to be the canonical form for objects in $\mathsf{MP}^s_{\mathrm{nd}}(1,1)$. Its image under the equivalence $\mathsf{MP}^s_{\mathrm{nd}}(1,1) \to \mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2)$ constructed above will be denoted by $M_{n_1,n_2}(\lambda)$. This is a *canonical form* for objects of the category $\mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2)$. An explicit description of $M_{n_1,n_2}(\lambda)$ is given in Algorithm 9.20 below.

Observe now that isomorphic objects of $\mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2)$ have the same determinant and that the functor $\mathsf{MP}^s_{\mathrm{nd}}(1,1) \to \mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2)$ respects the determinant up to a sign which depends on (n_1,n_2) only. As a consequence, we see that $M_{n_1,n_2}(\lambda) \cong M_{n_1,n_2}(\lambda')$ if and only if $\det\bigl(M_{n_1,n_2}(\lambda)\bigr) = \det\bigl(M_{n_1,n_2}(\lambda')\bigr)$, which is equivalent to $\lambda = \lambda'$. Because we have

$$\mathbb{F}\bigl(\det(\mathcal{V})\bigr) = \bigl(\mathcal{O}_{\mathbb{P}^1}(n_2), \mathbb{C}_s, (\det(M),1)\bigr)$$

for any $\mathcal{V} \in \mathsf{Spl}^{(n,d)}(E)$, we see now that $\det : \mathsf{Spl}^{(n,d)}(E) \to \mathsf{Pic}^d(E)$ is bijective if $\gcd(n,d) = 1$. $\square$

Algorithm 9.20. For any pair of positive coprime integers (n_1,n_2), the simple objects $M_{n_1,n_2}(\lambda) \in \mathsf{MP}^s_{\mathrm{nd}}(n_1,n_2)$ are described in the following way.

(1) First, we produce a sequence of pairs of coprime integers by replacing at each step a pair (n_1,n_2) by (n_1-n_2,n_2) if $n_1 > n_2$ and by (n_1,n_2-n_1) if $n_2 > n_1$. We continue until we arrive at $(1,1)$.

(2) Starting with the matrix $M_{1,1}(\lambda) \in \mathsf{MP}^s_{\mathrm{nd}}$ from Example 9.18 we recursively construct the matrix $M_{n_1,n_2}(\lambda)$ as follows. We follow the sequence constructed in part (1) in reverse order and

- if we go from (m_1,m_2) to (m_1+m_2,m_2) we proceed as follows

$$M_{m_1,m_2}(\lambda) = \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) \Rightarrow M_{m_1+m_2,m_2}(\lambda) = \left(\begin{array}{cc||c} X & Y & 0 \\ \hline 0 & 0 & I_{m_2} \\ \hline\hline Z & W & 0 \end{array}\right).$$

- and similarly, if we go from (m_1,m_2) to (m_1,m_1+m_2) we set

$$M_{m_1,m_2}(\lambda) = \left(\begin{array}{c|c} X & Y \\ \hline Z & W \end{array}\right) \Rightarrow M_{m_1,m_1+m_2}(\lambda) = \left(\begin{array}{c||c|c} 0 & I_{m_1} & 0 \\ \hline\hline X & 0 & Y \\ Z & 0 & W \end{array}\right).$$

Remark 9.21. From the construction it is clear that $M_{n_1,n_2}(\lambda)$ is an $n \times n$-matrix having exactly one non-zero entry in each column and each row. Exactly one of these non-zero entries is equal to $\lambda \in \mathbb{C}^*$ and this entry is found in the last row. All the other non-zero entries are equal to 1.

Remark 9.22. Because simple vector bundles on Weierstraß curves are stable ([20, Cor. 4.5]), we have $\mathsf{Spl}^{(n,d)}(E) = M_E^{(n,d)}$ and Theorem 9.19 provides another proof of the part of Theorem 6.1 which says that two stable vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ of the same rank on a nodal Weierstraß curve are isomorphic if and only if $\det(\mathcal{V}_1) \cong \det(\mathcal{V}_2)$.

Example 9.23. Let us apply Algorithm 9.20 to describe in terms of triples all simple vector bundles on E of rank 5 and degree 12. From earlier calculations we see that the normalisation of such a bundle is $\widetilde{\mathcal{V}} = \mathcal{O}_{\mathbb{P}^1}(2)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^2$, in particular $(n_1, n_2) = (3, 2)$. The sequence of reductions for sizes of matrices from the category $\mathsf{MP}_{\mathrm{nd}}$ is:

$$(3,2) \longrightarrow (1,2) \longrightarrow (1,1).$$

This induces a reverse sequence of functors

$$\mathsf{MP}^s_{\mathrm{nd}}(1,1) \longrightarrow \mathsf{MP}^s_{\mathrm{nd}}(1,2) \longrightarrow \mathsf{MP}^s_{\mathrm{nd}}(3,2),$$

giving the following sequence of canonical forms:

$$\left(\begin{array}{c|c} 0 & 1 \\ \hline \lambda & 0 \end{array}\right) \mapsto \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \lambda & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, the set of stable vector bundles of rank 5 and degree 12 is described by the family of triples $\big(\mathcal{O}_{\mathbb{P}^1}(2)^3 \oplus \mathcal{O}_{\mathbb{P}^1}(3)^2, \mathbb{C}^5_s, \widetilde{\mathsf{m}}\big)$, where

$$\mathsf{m}(0) = \left(\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \end{array}\right), \quad \mathsf{m}(\infty) = I_5.$$

Example 9.24. The indecomposable semi-stable vector bundles $\mathcal{V}$ of rank two and degree zero, whose Jordan-Hölder factors are locally free, are of the form $\mathcal{L} \otimes \mathcal{A}_2$, where $\mathcal{L} \in \mathsf{Pic}^0(E)$. Using Lemma 9.13 and compatibility with tensor products, we see that they are described by the triples $\big(\mathcal{O}^2_{\mathbb{P}^1}, \mathbb{C}^2_s, \widetilde{\mathsf{m}}\big) \in \mathsf{Tri}(E)$, where

$$\mathsf{m}(0) = \lambda J_2(1) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{C}^* \quad \text{and} \quad \mathsf{m}(\infty) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Lemma 9.25. *Let $\lambda \in \mathbb{C}^*$, $X = \operatorname{diag}(\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{C}^*$, $i = 1, \dots, n$ and $n = n_1 + n_2$, then $XM_{n_1,n_2}(\lambda) \cong M_{n_1,n_2}(\lambda \cdot \alpha_1 \cdot \dots \cdot \alpha_n)$ as objects of $\mathsf{MP}_{\mathrm{nd}}$.*

Proof. It is not hard to see that $XM_{n_1,n_2}(\lambda) \in \mathsf{MP}_{\mathrm{nd}}$ is again simple, hence it is isomorphic to a canonical form $M_{n_1,n_2}(\lambda')$. This means that there exist invertible matrices $S = \left(\begin{smallmatrix} A & 0 \\ C' & B \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} A & 0 \\ C'' & B \end{smallmatrix}\right)$ such that $M_{n_1,n_2}(\lambda') = S^{-1} X M_{n_1,n_2}(\lambda) T$. Because $\det(S) = \det(A)\det(B) = \det(T)$, we obtain $\lambda' = \lambda \det(X)$. □

Remark 9.26. If n_1, n_2 are fixed and for each $\lambda \in \mathbb{C}$ there is given an object $M(\lambda) \in \mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$, we obtain simple vector bundles $\mathcal{V}_\lambda$ of rank $n = n_1 + n_2$ on E such that $\mathbb{F}(\mathcal{V}_\lambda) \cong \big(\mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}, \mathbb{C}_s^{n_1+n_2}, (M(\lambda), \mathrm{id})\big)$. Similarly, for each $\beta \in \mathbb{C}^*$ there exists a unique line bundle $\mathcal{L}_\beta$ on E such that $\mathbb{F}(\mathcal{L}_\beta) \cong \big(\mathcal{O}_{\mathbb{P}^1}, \mathbb{C}, (\beta, 1)\big)$. We say that *the family $M(\lambda)$ is compatible with the action of the Jacobian*, if for all $\lambda, \beta \in \mathbb{C}^*$ we have $M(\beta^n\lambda) = \beta M(\lambda)$. This implies, but is stronger than $\mathcal{V}_{\beta^n\lambda} \cong \mathcal{L}_\beta \otimes \mathcal{V}_\lambda$.

Lemma 9.25 implies that the vector bundles $\mathcal{V}_\lambda$ given by the family $M_{n_1,n_2}(\lambda)$ satisfy $\mathcal{V}_{\beta^n\lambda} \cong \mathcal{L}_\beta \otimes \mathcal{V}_\lambda$, but $M_{n_1,n_2}(\lambda)$ is not compatible with the action of the Jacobian. However, if we replace all non-zero entries of $M_{n_1,n_2}(\lambda)$ by $\sqrt[n]{\lambda}$, some fixed n-th root of λ, we obtain an object $N_{n_1,n_2}(\lambda) \in \mathsf{MP}_{\mathrm{nd}}(n_1, n_2)$ which is isomorphic to $M_{n_1,n_2}(\lambda)$ and which is compatible with the action of the Jacobian: $N_{n_1,n_2}(\beta^n\lambda) = \beta N_{n_1,n_2}(\lambda)$. Another choice of $\sqrt[n]{\lambda}$ gives an isomorphic vector bundle.

Because there is no global choice of an n-th root, we define $\widetilde{N}_{n_1,n_2}(t) := N_{n_1,n_2}(t^n)$ for all $t \in \mathbb{C}^*$ in order to globalize this construction. Compatibility for this family has now the form $\widetilde{N}_{n_1,n_2}(\beta t) = \beta \widetilde{N}_{n_1,n_2}(t)$. As we shall see in Subsection 9.4, this will give us a trivialization of a universal family of stable vector bundles compatible with the action of Jacobian, necessary to construct a gauge transformation of the geometric associative r-matrix depending on the difference of the vector bundle spectral parameters.

If we apply this construction to the family of triples from Example 9.18, which describe the simple vector bundles of rank 2 and degree 1, we obtain a family of vector bundles described by triples $(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s^2, \widetilde{\mathsf{m}})$ with

$$\mathsf{m}(0) = \widetilde{N}_{1,1}(\lambda) = \left(\begin{array}{c|c} 0 & \lambda \\ \hline \lambda & 0 \end{array}\right), \lambda \in \mathbb{C}^* \quad \text{and} \quad \mathsf{m}(\infty) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right).$$

This family of matrices is compatible with the action of $\mathsf{Pic}^0(E)$. But for λ and $-\lambda$ we always obtain isomorphic vector bundles.

9.3. **Vector bundles on a cuspidal Weierstraß curve.** We now recall an explicit description of those objects in $\mathsf{Tri}(E)$ which correspond to simple vector bundles on a cuspidal Weierstraß curve E, following the approach of [13]. The exposition is very similar to the nodal case and includes an algorithm producing a normal form for such triples for each given rank and degree. Rank and degree of a simple vector bundle on a cuspidal Weierstraß curve also turn out to be coprime.

Let E be the cuspidal cubic curve, given by the equation $zy^2 = x^3$. Its normalisation $\pi : \mathbb{P}^1 \longrightarrow E$ is given by $\pi(z_0 : z_1) = (z_0^2 z_1 : z_0^3 : z_1^3)$. With these coordinates on $\mathbb{P}^1$ the preimage of the singular point $s = (0 : 0 : 1) \in E$ is $\pi^{-1}(s) = (0 : 1) = \infty$. Then Z is the reduced point $s \in E$ with the structure sheaf $\mathbb{C}$. Moreover, $\widetilde{Z}$ is non-reduced with support at $\infty = (0 : 1) \in \mathbb{P}^1$ and structure sheaf $\mathsf{R} = \mathbb{C}[\varepsilon]/\varepsilon^2$. The morphism $\tilde{\pi} : \widetilde{Z} \to Z$ corresponds to the canonical ring homomorphism $\mathbb{C} \to \mathsf{R}$.

Recall that $w = z_0/z_1$ is a coordinate in the neighbourhood U_∞ of the point $(0 : 1)$. The morphism $\tilde{\eta} : \widetilde{Z} \to \mathbb{P}^1$ corresponds to the map $\mathrm{ev}_{U_\infty} : \mathcal{O}_{\mathbb{P}^1}(U_\infty) \to \mathcal{O}_{\widetilde{Z}}(U_\infty) = \mathsf{R}$, given by $\mathrm{ev}_{U_\infty}(w) = \varepsilon$. Next, following the recipe of Remark 9.8 we use the section $p_\zeta(z_0, z_1) = z_1 \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$ to define the collection of isomorphisms $\zeta_l : \tilde{\eta}^* \mathcal{O}_{\mathbb{P}^1}(l) \longrightarrow \mathcal{O}_{\widetilde{Z}}$. They are given by the formula $\zeta_l(s) = \mathrm{ev}_V\left(\dfrac{s}{z_1^l}\right)$ for each open set $V \subset U_\infty$, all $l \in \mathbb{Z}$ and any $s \in \Gamma\big(V, \mathcal{O}_{\mathbb{P}^1}(l)\big)$. A morphism

$$q = q(z_0, z_1) = a_0 z_0^{m-n} + a_1 z_0^{m-n-1} z_1 + \cdots + a_{m-n} z_1^{m-n} \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}(n), \mathcal{O}_{\mathbb{P}^1}(m)\big)$$

is therefore evaluated according to the rule

$$\begin{array}{ccc} \tilde{\eta}^* \mathcal{O}_{\mathbb{P}^1}(n) & \xrightarrow{\tilde{\eta}^*(q)} & \tilde{\eta}^* \mathcal{O}_{\mathbb{P}^1}(m) \\ {\scriptstyle \zeta_n}\big\downarrow & & \big\downarrow{\scriptstyle \zeta_m} \\ \mathsf{R} & \xrightarrow{a_{m-n} + a_{m-n-1}\varepsilon} & \mathsf{R}. \end{array}$$

The following lemma shows how the explicit identification of $\widetilde{\mathsf{m}}$ with a matrix is carried out for line bundles of degree one.

The chosen coordinates provide us with an isomorphism $\mathbb{C} \cong U_\infty = \mathbb{P}^1 \setminus \{\infty\}$ mapping $y \in \mathbb{C}$ to $(1 : y) \in U_\infty$. In the cuspidal case, the normalization restricts to an isomorphism $\pi : \mathbb{P}^1 \setminus \{\infty\} \longrightarrow \breve{E}$. Together, this gives us an identification $\breve{E} \cong \mathbb{C}$, under which $y \in \mathbb{C}$ corresponds to $\tilde{y} := \pi^{-1}(y) = (1 : y) \in \mathbb{P}^1$.

Lemma 9.27. *With respect to the given choice of homogeneous coordinates on* $\mathbb{P}^1$ *and the set of trivializations* $\{\zeta_l\}_{l \in \mathbb{Z}}$ *described above, we have for all* $y \in \breve{E} \cong \mathbb{C}$

$$\mathbb{F}\big(\mathcal{O}_E(y)\big) \cong \big(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, 1 - y\varepsilon\big).$$

Proof. As in the case of a nodal cubic curve, because $\mathcal{O}_E(y)$ is a line bundle of degree one, we know $\mathcal{T}_{\mathcal{O}_E(y)} := \mathbb{F}\big(\mathcal{O}_E(y)\big) = \big(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, (1 + \lambda\varepsilon)\big)$ for some $\lambda \in \mathbb{C}$ and $\mathcal{T}_{\mathcal{O}_E} := \mathbb{F}(\mathcal{O}_E) = \big(\mathcal{O}_{\mathbb{P}^1}, \mathbb{C}_s, (1)\big)$. By Theorem 9.4 we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Tri}(E)}\big(\mathcal{T}_{\mathcal{O}}, \mathcal{T}_{\mathcal{O}_E(y)}\big) \arrow[r, "\mathbb{G}"] \arrow[d, "\mathsf{For}"'] & \mathsf{Hom}_E\big(\mathcal{O}_E, \mathcal{O}_E(y)\big) \arrow[d, "\pi^*"] \\
\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\big) & \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(\tilde{y})\big). \arrow[l, "\mathrm{cnj}\left(\gamma_{\mathcal{T}_{\mathcal{O}}}, \gamma_{\mathcal{T}_{\mathcal{O}(y)}}\right)"']
\end{tikzcd}$$

The section $z_1 - yz_0 \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\big)$ generates the image of π^*, hence belongs to the image of For and there exists $c \in \mathbb{C}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathsf{R} \arrow[r, "c"] \arrow[d, "1"'] & \mathsf{R} \arrow[d, "1+\lambda\varepsilon"] \\
\mathsf{R} \arrow[r, "1-y\varepsilon"] & \mathsf{R}.
\end{tikzcd}$$

This implies $c = 1$, $\lambda = -y$ and $\mathbb{F}\big(\mathcal{O}(y)\big) = \big(\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s, (1 - y\varepsilon)\big)$. $\square$

We aim now at giving a canonical form for those elements in $\mathsf{Tri}(E)$ which correspond to simple vector bundles on E having rank n and degree d. Just as in the nodal case, without loss of generality, we may assume $0 \le d < n$. Recall that $\mathsf{VB}^{(0,1)}_{n_1,n_2}(E)$ is the full subcategory of $\mathsf{VB}^{(0,1)}(E)$, whose objects are vector bundles $\mathcal{V}$ with fixed normalization $\pi^*\mathcal{V} \cong \widetilde{\mathcal{V}} := \mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$. The category $\mathsf{VB}^{(0,1)}_{n_1,n_2}(E)$ is equivalent to the full subcategory of $\mathsf{Tri}(E)$ whose objects $(\widetilde{\mathcal{V}}, \mathcal{N}, \widetilde{\mathsf{m}})$ satisfy $\mathcal{N} \cong \mathbb{C}_s^{n_1+n_2}$ and $\widetilde{\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$. Hence, these objects are described by an invertible matrix $\widetilde{\mathsf{m}} = \mathsf{m}_0 + \varepsilon\mathsf{m}_\varepsilon \in \mathsf{GL}_{n_1+n_2}(\mathsf{R})$. Note the following easy lemma.

Lemma 9.28. *Let* $\widetilde{\mathsf{m}} = \mathsf{m}_0 + \varepsilon\mathsf{m}_\varepsilon$ *be an element of* $\mathsf{Mat}_{n\times n}(\mathsf{R})$*, where* $\mathsf{m}_0, \mathsf{m}_\varepsilon \in \mathsf{Mat}_{n\times n}(\mathbb{C})$*. Then the matrix* m *is invertible if and only if* m_0 *is. Moreover, in the latter case we have:* $\det(\widetilde{\mathsf{m}}) = \det(\mathsf{m}_0)\left(1 + \varepsilon\,\mathrm{tr}(\mathsf{m}_0^{-1}\mathsf{m}_\varepsilon)\right)$.

Next, two such matrices $\widetilde{\mathsf{m}}, \widetilde{\mathsf{m}}' \in \mathsf{GL}_{n_1+n_2}(\mathsf{R})$ correspond to isomorphic vector bundles if and only if there exist a matrix $f \in \mathsf{GL}_{n_1+n_2}(\mathbb{C})$ and an automorphism F of $\mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}$ such that $\widetilde{\mathsf{m}}' = \tilde{\eta}^*(F)^{-1} \circ \widetilde{\mathsf{m}} \circ \tilde{\pi}^*(f)$. For any $\widetilde{\mathsf{m}}$, using $f = \mathsf{m}_0^{-1}$ and $F = \mathrm{id}$, we find an equivalent matrix $\widetilde{\mathsf{m}}'$ with $\mathsf{m}_0' = \mathrm{id}$. In order to reduce $\widetilde{\mathsf{m}} = \mathrm{id} + \varepsilon\mathsf{m}_\varepsilon$ further, we split m_ε into blocks $(\mathsf{m}_\varepsilon)_{ij} \in \mathsf{Mat}_{n_i\times n_j}(\mathbb{C})$ and let

$$F = \left(\begin{array}{c|c} I_{n_1} & 0 \\ \hline F_{21} & I_{n_2} \end{array}\right)$$

be the automorphism of $\widetilde{\mathcal{V}}$ with $F_{21} = z_0(\mathsf{m}_\varepsilon)_{21} \in \mathsf{Mat}_{n_2\times n_1}(\mathbb{C}[z_0, z_1]_1)$. With this choice of F and $f = \mathrm{id}$, a straightforward calculation, which uses

$$\tilde{\eta}^*(F) = \left(\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & I_{n_2} \end{array}\right) + \varepsilon\left(\begin{array}{c|c} 0 & 0 \\ \hline (\mathsf{m}_\varepsilon)_{21} & 0 \end{array}\right),$$

shows that we can reduce $\widetilde{\mathsf{m}} = \mathrm{id} + \varepsilon \mathsf{m}_\varepsilon$ further to the form

$$\widetilde{\mathsf{m}} = \left(\begin{array}{c|c} I_{n_1} & 0 \\ \hline 0 & I_{n_2} \end{array}\right) + \varepsilon \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline 0 & M_{22} \end{array}\right).$$

Therefore, triples $\bigl(\mathcal{O}_{\mathbb{P}^1}^{n_1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n_2}, \mathbb{C}_s^{n_1+n_2}, \widetilde{\mathsf{m}}\bigr)$ with such $\widetilde{\mathsf{m}}$ form a category which is equivalent to $\mathsf{VB}^{(0,1)}_{n_1,n_2}(E)$. This motivates the following definition.

Definition 9.29. Let E be a cuspidal cubic curve and $n_1 > 0$, $n_2 \geq 0$ integers. The category $\mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$ is defined as follows.

- Its objects are "matrices" with three blocks $M_{ij} \in \mathsf{Mat}_{n_i \times n_j}(\mathbb{C})$

$$M = \left(\begin{array}{c|c} M_{11} & M_{12} \\ \hline \times & M_{22} \end{array}\right),$$

  where $\times$ is an "empty" or "non-existing" block.
- Morphisms are given by "matrices"

$$\mathsf{Hom}_{\mathsf{MP}_{\mathrm{cp}}}(M, M') = \{S \mid SM = M'S\}.$$

  with obvious composition and such that $S = \left(\begin{array}{c|c} S_{11} & \times \\ \hline S_{21} & S_{22} \end{array}\right)$ has blocks of the same size as the blocks of M and M'. The condition $SM = M'S$ means that

$$\begin{aligned} S_{11}M_{11} &= M'_{11}S_{11} + M'_{12}S_{21} \\ S_{11}M_{12} &= M'_{12}S_{22} \\ S_{21}M_{12} + S_{22}M_{22} &= M'_{22}S_{22}, \end{aligned}$$

  in other words: we ignore the lower left block in SM and $M'S$.

As in the nodal case, we denote by $\mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2)$ the full subcategory of *simple* objects of $\mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$.

Remark 9.30. If $n_2 = 0$ the block structure and the non-existing block disappear and we end up in a situation similar to the one described in Remark 9.16. Here we have $\mathsf{MP}_{\mathrm{cp}}(n, 0) = \mathsf{Mat}_{n\times n}(\mathbb{C})$ and $\mathsf{Hom}_{\mathsf{MP}_{\mathrm{cp}}(n,0)}(M, M') = \{S \mid SM = M'S\}$. The indecomposable objects in this category are precisely those which are isomorphic to a Jordan block $J_n(\lambda)$, $\lambda \in \mathbb{C}$. As before, this implies $\mathsf{MP}^s_{\mathrm{cp}}(n, 0) = \emptyset$ if $n > 1$ and $\mathsf{MP}^s_{\mathrm{cp}}(1, 0) = \mathbb{C}$.

Lemma 9.31. *For any pair of non-negative integers (n_1, n_2) with $n_1 > 0$, the categories $\mathsf{VB}^{(0,1)}_{n_1,n_2}(E)$ and $\mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$ are equivalent. Under this equivalence, simple vector bundles correspond to objects of $\mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2)$.*

Proof. Sending $M \in \mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2)$ to $I_{n_1+n_2} + \varepsilon M \in \mathsf{GL}_{n_1+n_2}(\mathcal{O}_{\widetilde{Z}})$, with inherited block structure from M, gives an equivalence between $\mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2)$ and $\mathsf{BM}_{n_1,n_2}(E)$. The proof of the lemma is now completely parallel to the case of a nodal cubic curve and is, therefore, left to the reader. $\square$

Next, we wish to find a canonical form for objects $M \in \mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2)$. Similar to the nodal case, in each isomorphism class of $\mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2)$, we are going to describe a unique object with a particularly "simple" structure. Again, the reduction procedure described below is based on an easy lemma.

Lemma 9.32. *The block M_{12} has full rank, if $M \in \mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2)$ is simple.*

Proof. Just as in the nodal case (Lemma 9.17), if M_{12} does not have full rank, the matrix M can be reduced to the form

$$M = \left(\begin{array}{c|c||c|c} M_1 & M_2 & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline\hline \times & \times & M_3 & M_4 \\ \hline \times & \times & 0 & M_5 \end{array}\right)$$

and we obtain a non-scalar endomorphism

$$S = \left(\begin{array}{c|c||c|c} I & 0 & \times & \times \\ \hline 0 & I & \times & \times \\ \hline\hline 0 & 0 & I & 0 \\ \hline W & 0 & 0 & I \end{array}\right),$$

where W is an arbitrary matrix of appropriate size. $\square$

Example 9.33. For any $\lambda \in \mathbb{C}$, the triple $\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^2_s, \widetilde{\mathsf{m}}\big)$ with

$$\widetilde{\mathsf{m}} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) + \varepsilon \left(\begin{array}{c|c} \lambda & 1 \\ \hline 0 & 0 \end{array}\right)$$

defines a simple vector bundle of rank 2 and degree 1 on a cuspidal cubic curve E. It corresponds to $M_{1,1}(\lambda) := \left(\begin{smallmatrix} \lambda & 1 \\ \times & 0 \end{smallmatrix}\right) \in \mathsf{MP}^s_{\mathrm{cp}}(1,1)$.

Theorem 9.34 (see [13])**.** *Let E be a cuspidal cubic curve and denote by $\mathsf{Spl}^{(n,d)}(E)$ the set of all isomorphism classes of simple vector bundles of rank n and degree d on E. If $\gcd(n,d) = 1$ the map $\det : \mathsf{Spl}^{(n,d)}(E) \to \mathsf{Pic}^d(E) \cong \mathbb{C}$ is bijective. If $\gcd(n,d) > 1$ we have $\mathsf{Spl}^{(n,d)}(E) = \emptyset$.*

Proof. The proof very similar to the proof of Theorem 9.19. A first difference is that if $n_1 = n_2$, we can transform the matrix $M \in \mathsf{MP}^s_{\mathrm{cp}}(n,n)$ to the form

$$M = \left(\begin{array}{c|c} M_{11} & I \\ \hline \times & 0 \end{array}\right),$$

because the block M_{12} is square and invertible. We can further reduce the block M_{11} to its Jordan canonical form keeping the block $M_{12} = I$ unchanged. This implies that M splits into a direct sum of objects of the form

$$\left(\begin{array}{c|c} J_m(\lambda) & I_m \\ \hline \times & 0 \end{array}\right)$$

which are simple in $\mathsf{MP}_{\mathrm{cp}}(m,m)$ if and only if $m=1$.

The other difference is that a simple object $M \in \mathsf{MP}_{\mathrm{cp}}(n_1,n_2)$ can be reduced to

$$M = \left(\begin{array}{c|cc} 0 & I & 0 \\ \hline \times & M'_{11} & M'_{12} \\ \times & 0 & M'_{22} \end{array}\right) \text{ if } n_2 > n_1, \quad \text{or to} \quad M = \left(\begin{array}{cc|c} M'_{11} & M'_{12} & 0 \\ 0 & M'_{22} & I \\ \hline \times & \times & 0 \end{array}\right) \text{ if } n_1 > n_2.$$

A straightforward calculation shows that the matrix

$$M' = \left(\begin{array}{c|c} M'_{11} & M'_{12} \\ \hline \times & M'_{22} \end{array}\right)$$

is an object of $\mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2-n_1)$ or $\mathsf{MP}^s_{\mathrm{cp}}(n_1-n_2, n_2)$ respectively and that

$$\det(\mathrm{id}+\varepsilon M) = \det(\mathrm{id}+\varepsilon M').$$

If $\gcd(n_1,n_2)=1$, we end up with an equivalence $\mathsf{MP}^s_{\mathrm{cp}}(1,1) \to \mathsf{MP}^s_{\mathrm{cp}}(n_1,n_2)$ just as in the nodal case. Using Lemma 9.32 we see that each object in $\mathsf{MP}^s_{\mathrm{cp}}(1,1)$ is isomorphic to

$$M_{1,1}(\lambda) = \left(\begin{array}{c|c} \lambda & 1 \\ \hline \times & 0 \end{array}\right), \quad \lambda \in \mathbb{C}.$$

By definition, $M_{n_1,n_2}(\lambda) \in \mathsf{MP}^s_{\mathrm{cp}}(n_1,n_2)$ denotes the image of $M_{1,1}(\lambda)$ under the equivalence described above. Again, $M_{n_1,n_2}(\lambda) \cong M_{n_1,n_2}(\lambda')$ in $\mathsf{MP}^s_{\mathrm{cp}}(n_1,n_2)$ if and only if $\lambda=\lambda'$. From the identity $\det\bigl(\mathrm{id}+\varepsilon M_{1,1}(\lambda)\bigr) = 1+\varepsilon\lambda$, the bijectivity of the determinant map follows. □

Remark 9.35. Reversing the reduction step in the proof of Theorem 9.34 gives us an algorithm similar to Algorithm 9.20 which produces the matrix $M_{n_1,n_2}(\lambda)$ starting with $M_{1,1}(\lambda)$. The only non-zero diagonal element of $M_{n_1,n_2}(\lambda)$ will be the moduli parameter $\lambda \in \mathbb{C}$, i.e. $\lambda = \mathrm{tr}\bigl(M_{n_1,n_2}(\lambda)\bigr)$.

Lemma 9.36. *Let $\lambda \in \mathbb{C}$, $A = \mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_n)$ with $\alpha_i \in \mathbb{C}$ and $n = n_1+n_2$, then $A + M_{n_1,n_2}(\lambda) \cong M_{n_1,n_2}(\lambda+\alpha_1+\alpha_2+\cdots+\alpha_n)$ as objects of* $\mathsf{MP}_{\mathrm{cp}}$.

Proof. We proceed by induction on n, the size of the matrix $M_{n_1,n_2}(\lambda)$. The case $n_1=n_2=1$ is an easy calculation. Assume the statement is true for all pairs (n_1,n_2) of positive integers such that $n_1+n_2<n$. We shall deal with the case $n_1>n_2$, the opposite case is similar and left to the reader.

From the proof of Theorem 9.34 we know that $M_{n_1,n_2}(\lambda)$ has the structure

$$M_{n_1,n_2}(\lambda) = \left(\begin{array}{cc|c} M'_{11} & M'_{12} & 0 \\ 0 & M'_{22} & I_{n_2} \\ \hline \times & \times & 0 \end{array}\right) \quad \text{so that} \quad M_{n_1-n_2,n_2}(\lambda) = \left(\begin{array}{c|c} M'_{11} & M'_{12} \\ \hline \times & M'_{22} \end{array}\right),$$

with M'_{11}, M'_{22} being square matrices of sizes n_1-n_2 and n_2 respectively. If we write

$$A = \mathrm{diag}(\alpha_1,\alpha_2,\ldots,\alpha_{n_1-n_2}|\alpha_{n_1-n_2+1},\ldots,\alpha_{n_1}|\alpha_{n_1+1},\ldots,\alpha_n) = \mathrm{diag}(A_1|A_2|A_3),$$

with A_i being diagonal blocks, we obtain

$$A + M_{n_1,n_2}(\lambda) = \left(\begin{array}{cc|c} M'_{11} + A_1 & M'_{12} & 0 \\ 0 & M'_{22} + A_2 & I_{n_2} \\ \hline \times & \times & A_3 \end{array}\right).$$

A straightforward calculation shows that the matrix

$$S = \left(\begin{array}{cc|c} I_{n_1-n_2} & 0 & \times \\ 0 & I_{n_2} & \times \\ \hline 0 & -A_3 & I_{n_2} \end{array}\right)$$

defines an isomorphism in the category $\mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$ between $A + M_{n_1,n_2}(\lambda)$ and

$$\left(\begin{array}{cc|c} M'_{11} + A_1 & M'_{12} & 0 \\ 0 & M'_{22} + A_2 + A_3 & I_{n_2} \\ \hline \times & \times & 0 \end{array}\right).$$

The inductive hypothesis implies that $M_{n_1-n_2,n_2}(\lambda) + \mathrm{diag}(A_1|A_2 + A_3)$ is a simple object of $\mathsf{MP}_{\mathrm{cp}}(n_1-n_2, n_2)$ which is isomorphic to $M_{n_1-n_2,n_2}(\lambda+\alpha_1+\ldots+\alpha_n)$. This implies that $A + M_{n_1,n_2}(\lambda) \cong M_{n_1,n_2}(\lambda + \alpha_1 + \ldots + \alpha_n)$ in $\mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$. It was not possible to give a direct proof like for Lemma 9.25 in the nodal case, because it is not obvious at the beginning that $A + M_{n_1,n_2}(\lambda) \in \mathsf{MP}_{\mathrm{cp}}$ is again simple. $\square$

Remark 9.37. This lemma implies that the object $N_{n_1,n_2}(\lambda) \in \mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$, obtained from $M_{n_1,n_2}(\lambda)$ by replacing each diagonal entry by $\frac{\lambda}{n}$, is isomorphic to $M_{n_1,n_2}(\lambda)$ in $\mathsf{MP}_{\mathrm{cp}}(n_1, n_2)$. Moreover, this matrix is *compatible with the action of the Jacobian* in the sense that for all $\lambda, \beta \in \mathbb{C}$ we have $\beta I_n + N_{n_1,n_2}(\lambda) = N_{n_1,n_2}(n\beta+\lambda)$. This is equivalent to $(1 + \varepsilon\beta)(I_n + \varepsilon N_{n_1,n_2}(\lambda)) = I_n + \varepsilon N_{n_1,n_2}(n\beta + \lambda)$. The precise meaning of this condition will be clarified in Subsection 9.4.

Remark 9.38. Because simple vector bundles on Weierstraß curves are stable ([20, Cor. 4.5]), we have $\mathsf{Spl}^{(n,d)}(E) = M_E^{(n,d)}$ and Theorem 9.34 provides another proof of the part of Theorem 6.1 which says that two stable vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ of the same rank on a cuspidal Weierstraß curve are isomorphic if and only if $\det(\mathcal{V}_1) \cong \det(\mathcal{V}_2)$.

Moreover, because the group $\mathsf{Pic}^0(E) \cong \mathbb{C}$ is torsion free and divisible, it follows from $\det(\mathcal{V} \otimes \mathcal{L}) \cong \det(\mathcal{V}) \otimes \mathcal{L}^{\otimes n}$ that the action of the Jacobian $\mathsf{Pic}^0(E)$ on the set $M_E^{(n,d)}$ of stable vector bundles of rank n and degree d is simply transitive.

Example 9.39. The family of vector bundles on a cuspidal cubic curve, described by the triples $\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^2_s, \widetilde{\mathsf{m}}\big)$ with

$$\widetilde{\mathsf{m}} = I_2 + \varepsilon N_{1,1}(\lambda) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right) + \varepsilon \left(\begin{array}{c|c} \frac{\lambda}{2} & 1 \\ \hline 0 & \frac{\lambda}{2} \end{array}\right),$$

defines a universal family of stable vector bundles of rank 2 and degree 1. The family of matrices

$$N_{n_1,n_2}(\lambda) = \left(\begin{array}{c|c} \frac{\lambda}{2} & 1 \\ \hline 0 & \frac{\lambda}{2} \end{array}\right)$$

is compatible with the action of $\mathsf{Pic}^0(E)$.

9.4. **Universal families and their trivializations.** The goal of this subsection is an explicit description of a universal family on the moduli space of stable vector bundles on a singular Weierstraß cubic curve E.

For a *reduced* complex space B consider the Cartesian diagram

$$\begin{tikzcd} \widetilde{Z} \times B \arrow[r, "\tilde{\eta}_B = \tilde{\eta}\times \mathrm{id}"] \arrow[d, "\tilde{\pi}_B = \tilde{\pi}\times\mathrm{id}"'] & \mathbb{P}^1 \times B \arrow[d, "\pi_B = \pi \times \mathrm{id}"] \\ Z \times B \arrow[r, "\eta_B = \eta\times\mathrm{id}"] & E \times B \end{tikzcd}$$

and abbreviate $\nu_B = \pi_B \circ \tilde{\eta}_B = \eta_B \circ \tilde{\pi}_B$. If B is a point we omit the subscript B in the notation introduced.

Let us fix homogeneous coordinates $(z_0 : z_1)$ on $\mathbb{P}^1$ and denote $\mathcal{O}_{\mathbb{P}^1\times B}(l) = \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}^1}(l)$, where $\mathrm{pr}_1 : \mathbb{P}^1 \times B \to \mathbb{P}^1$ is the projection map. Let $p = p(z_0, z_1) \in H^0\bigl(\mathcal{O}_{\mathbb{P}^1}(1)\bigr)$ be a section, which is non-vanishing on $\widetilde{Z}$. The recipe of Remark 9.8 gives a family of isomorphisms $\zeta_l : \tilde{\eta}^*\bigl(\mathcal{O}_{\mathbb{P}^1}(l)\bigr) \longrightarrow \mathcal{O}_{\widetilde{Z}}$. Pulling back to $\mathbb{P}^1 \times B$, we obtain isomorphisms $\zeta_l : \tilde{\eta}_B^*\bigl(\mathcal{O}_{\mathbb{P}^1\times B}(l)\bigr) \longrightarrow \mathcal{O}_{\widetilde{Z}\times B}$ denoted for the sake of simplicity by the same letters.

Let $\widetilde{\mathcal{A}} = \bigoplus_{l\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1\times B}(l)^{n_l}$ be a vector bundle of rank n on $\mathbb{P}^1 \times B$. Then there are induced isomorphisms $\zeta^{\widetilde{\mathcal{A}}} : \tilde{\eta}_B^*\widetilde{\mathcal{A}} \longrightarrow \mathcal{O}^n_{\widetilde{Z}\times B}$ and $\zeta^{\widetilde{\mathcal{A}}} : \nu_{B*}\tilde{\eta}_B^*\widetilde{\mathcal{A}} \longrightarrow \nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B}$, denoted again by the same letters. For any point $b \in B$ we have: $\widetilde{\mathcal{A}}_b := \widetilde{\mathcal{A}}|_{\mathbb{P}^1\times\{b\}} = \bigoplus_{l\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(l)^{n_l}$. In a similar way, we obtain isomorphisms $\zeta^{\widetilde{\mathcal{A}}_b} : \tilde{\eta}^*\widetilde{\mathcal{A}}_b \longrightarrow \mathcal{O}^n_{\widetilde{Z}}$ and $\zeta^{\widetilde{\mathcal{A}}_b} : \nu_*\tilde{\eta}^*\widetilde{\mathcal{A}}_b \longrightarrow \nu_*\mathcal{O}^n_{\widetilde{Z}}$. Note that the isomorphisms $\zeta^{\widetilde{\mathcal{A}}}$ and $\zeta^{\widetilde{\mathcal{A}}_b}$ are related by the canonical base change diagrams.

Now we proceed with our construction of vector bundles on $E \times B$. We start with an invertible matrix $M \in \mathsf{GL}_n(\mathcal{O}_{\widetilde{Z}\times B})$. If convenient, we may start instead with a holomorphic function $M : B \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})$, denoted by the same letter. The corresponding element in $\mathsf{GL}_n(\mathcal{O}_{\widetilde{Z}\times B})$ will then be (M, id) in the nodal case and $1_n + \varepsilon M$ in the cuspidal case (see Lemmas 9.15 and 9.31). For any point $b \in B$ we denote by $M(b)$ the corresponding matrix in $\mathsf{GL}_n(\mathcal{O}_{\widetilde{Z}})$ or $\mathsf{Mat}_{n\times n}(\mathbb{C})$ respectively. Following the notation of Subsection 9.1, we denote

$$\mathsf{m} : \eta_{B*}\mathcal{O}^n_{Z\times B} \xrightarrow{\mathrm{can}} \nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B} \xrightarrow{\nu_{B*}(M)} \nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B}$$

and let $\widetilde{\mathsf{m}}$ be the unique map which makes the following diagram of isomorphisms commutative:

$$\begin{tikzcd} \mathcal{O}^n_{\widetilde{Z}\times B} \arrow[r, "\widetilde{\mathsf{m}}"] \arrow[rr, bend right, "M"'] & \widetilde{\eta}^*_B\widetilde{\mathcal{A}} \arrow[r, "\zeta^{\widetilde{\mathcal{A}}}"] & \mathcal{O}^n_{\widetilde{Z}\times B} \end{tikzcd}$$

In a similar way, let m_b and $\widetilde{\mathsf{m}}_b$ be the morphisms determined by the matrix $M(b)$. The following theorem is a mild generalization of Theorem 9.4.

Theorem 9.40. *Let $\bar{\mathcal{A}} = \oplus_{l\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^1}(l)^{n_l}$ be a vector bundle of rank n on $\mathbb{P}^1$, $\zeta : \mathcal{O}(1)_{\mathbb{P}^1}\big|_{\widetilde{Z}} \longrightarrow \mathcal{O}_{\widetilde{Z}}$ be the isomorphism induced by a section $p = p_\zeta(z_0, z_1) \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$, and $\widetilde{\mathcal{A}} = \oplus_{l\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^1\times B}(l)^{n_l}$ be the pull-back of $\bar{\mathcal{A}}$ to $\mathbb{P}^1 \times B$.*

- *Consider the coherent sheaf $\mathcal{A}$ on $E \times B$ given by the exact sequence*

$$0 \to \mathcal{A} \xrightarrow{\binom{\mathsf{i}}{\mathsf{p}}} \pi_{B*}\widetilde{\mathcal{A}} \oplus \eta_{B*}\mathcal{O}^n_{Z\times B} \xrightarrow{(\zeta^{\widetilde{\mathcal{A}}}\ \mathsf{m})} \nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B} \to 0. \tag{34}$$

Then $\mathcal{A} \in \mathsf{Coh}(E \times B)$ is locally free and for each $b \in B$ we have: $\mathcal{A}_b = \mathcal{A}|_{E\times\{b\}} \cong \mathbb{G}\big(\widetilde{\mathcal{A}}\big|_{E\times\{b\}}, \mathcal{O}^n_Z, \widetilde{\mathsf{m}}(b)\big)$, where $\mathbb{G}$ is the functor described in Theorem 9.4. In what follows, we shall use the notation $\mathcal{A} = \mathbb{G}\big(\widetilde{\mathcal{A}}, \mathcal{O}^n_{Z\times B}, \widetilde{\mathsf{m}}\big)$.

- *Let $\widetilde{\mathcal{B}} = \oplus_{l\in\mathbb{Z}}\mathcal{O}_{\mathbb{P}^1\times B}(l)^{m_l}$ be a vector bundle of rank m on $\mathbb{P}^1\times B$, $N \in \mathsf{GL}_m(\mathcal{O}_{\widetilde{Z}\times B})$, $\mathsf{n} : \eta_{B*}\mathcal{O}^m_{Z\times B} \xrightarrow{\text{can}} \nu_{B*}\mathcal{O}^m_{\widetilde{Z}\times B} \xrightarrow{\nu_{B*}(N)} \nu_{B*}\mathcal{O}^m_{\widetilde{Z}\times B}$ and $\mathcal{B} = \mathbb{G}\big(\widetilde{\mathcal{A}}, \mathcal{O}^m_{Z\times B}, \widetilde{\mathsf{n}}\big)$. Then*

$$\mathsf{Hom}_{E\times B}(\mathcal{A},\ \mathcal{B}) \subseteq \mathsf{Hom}_{\mathbb{P}^1\times B}(\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}}) \times \mathsf{Mat}_{m\times n}(\mathcal{O}_{Z\times B})$$

consists of those pairs (F, f) for which the following diagram is commutative:

$$\begin{tikzcd} \mathcal{O}^n_{\widetilde{Z}\times B} \arrow[r, "M"'] \arrow[rr, bend left, "\widetilde{\mathsf{m}}"] \arrow[d, "\widetilde{\pi}^*_B(f)"'] & \mathcal{O}^n_{\widetilde{Z}\times B} & \widetilde{\eta}^*_B\widetilde{\mathcal{A}} \arrow[l, "\zeta^{\widetilde{\mathcal{A}}}"] \arrow[d, "\widetilde{\eta}^*_B(F)"] \\ \mathcal{O}^m_{\widetilde{Z}\times B} \arrow[r, "N"] \arrow[rr, bend right, "\widetilde{\mathsf{n}}"'] & \mathcal{O}^m_{\widetilde{Z}\times B} & \widetilde{\eta}^*_B\widetilde{\mathcal{B}}. \arrow[l, "\zeta^{\widetilde{\mathcal{B}}}"'] \end{tikzcd}$$

- *Let $f : B' \longrightarrow B$ be a holomorphic map between reduced analytic spaces. Let $M' \in \mathsf{GL}_n(\mathcal{O}_{\widetilde{Z}\times B'})$ be the image of the matrix M under the morphism induced by $H^0(\mathcal{O}_{\widetilde{Z}\times B}) \to H^0(\mathcal{O}_{\widetilde{Z}\times B'})$, which is given by pull-back under f. Let $\mathcal{A}' = (\mathrm{id}\times f)^*\mathcal{A}$, $\widetilde{\mathcal{A}}' = (\mathrm{id} \times f)^*\widetilde{\mathcal{A}}$ etc. Then we have: $\mathcal{A}' \cong \mathbb{G}\big(\widetilde{\mathcal{A}}', \mathcal{O}^n_{Z\times B'}, \widetilde{\mathsf{m}}'\big)$. In other words, this construction is compatible with base change.*

• *Let* $N \in H^0(\mathcal{O}^*_{\widetilde{Z}\times B})$, $\widetilde{\mathcal{L}} = \mathcal{O}_{\mathbb{P}^1\times B}(c)$ *for some* $c \in \mathbb{Z}$ *and* $\mathcal{L} = \mathbb{G}\bigl(\widetilde{\mathcal{L}}, \mathcal{O}_{Z\times B}, \widetilde{\mathsf{n}}\bigr)$ *be the corresponding line bundle, i.e. we have an exact sequence*

$$0 \to \mathcal{L} \xrightarrow{\binom{\mathsf{j}}{\mathsf{q}}} \pi_{B*}\widetilde{\mathcal{L}} \oplus \eta_{B*}\mathcal{O}_{Z\times B} \xrightarrow{\left(\zeta^{\widetilde{\mathcal{L}}}\ \mathsf{n}\right)} \nu_{B*}\mathcal{O}_{\widetilde{Z}\times B} \to 0.$$

Then the following sequence is exact:

$$0 \to \mathcal{A}\otimes\mathcal{L} \xrightarrow{\begin{pmatrix}\mathsf{i}\boxtimes\mathsf{j}\\ \mathsf{p}\boxtimes\mathsf{q}\end{pmatrix}} \pi_{B*}\bigl(\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{L}}\bigr) \oplus \eta_{B*}\mathcal{O}^n_{Z\times B} \xrightarrow{\left(\zeta^{\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{L}}}\ \mathsf{m}\boxtimes\mathsf{n}\right)} \nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B} \to 0,$$

where the morphism $\mathsf{m}\boxtimes\mathsf{n}$ *is induced by the matrix* $M\cdot N$, $\mathsf{i}\boxtimes\mathsf{j}$ *is the morphism* $\mathcal{A}\otimes\mathcal{L} \xrightarrow{\mathsf{i}\otimes\mathsf{j}} \pi_{B*}\widetilde{\mathcal{A}}\otimes\pi_{B*}\widetilde{\mathcal{L}} \xrightarrow{\mathrm{can}} \pi_{B*}\bigl(\widetilde{\mathcal{A}}\otimes\widetilde{\mathcal{L}}\bigr)$ *and* $\mathsf{p}\boxtimes\mathsf{q}$ *is defined is a similar way. This means that the tensor product of a vector bundle with a line bundle is given by the product of the corresponding matrices. More generally, the tensor product of two vector bundles corresponds to the tensor product of the defining matrices.*

• *Finally, we have the following short exact sequence:*

$$0 \to \det\mathcal{A} \longrightarrow \pi_{B*}\det\widetilde{\mathcal{A}} \oplus \eta_{B*}\mathcal{O}_{Z\times B} \xrightarrow{\left(\zeta^{\det\widetilde{\mathcal{A}}}\ \det(\mathsf{m})\right)} \nu_{B*}\mathcal{O}_{\widetilde{Z}\times B} \to 0,$$

where the morphism $\det(\mathsf{m})$ *corresponds to the determinant* $\det(M) \in H^0(\mathcal{O}_{\widetilde{Z}\times B})$.

Proof. To prove the first part of the statement, note that the sheaf $\nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B}$ is B-flat. Hence, for any point $b \in B$ the restriction of the sequence (34) to $E\times\{b\}$

$$0 \to \mathcal{A}_b \to \pi_*\widetilde{\mathcal{A}}_b \oplus \eta_*\mathcal{O}^n_Z \xrightarrow{\left(\zeta^{\widetilde{\mathcal{A}}_b}\ \mathsf{m}_b\right)} \nu_*\mathcal{O}^n_{\widetilde{Z}} \to 0,$$

is exact again. By Theorem 9.4 the coherent sheaf $\mathcal{A}_b$ is locally free. Since B is reduced, $\mathcal{A}$ is locally free, too.

The description of morphisms between $\mathcal{A}$ and $\mathcal{B}$ in terms of morphisms between $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ and matrices M and N follows from the universal property of the kernel.

In order to show the base change property for $\mathbb{G}$ we use again that $\nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B}$ is flat over B. Let $\tilde{f} = \mathrm{id}\times f : E\times B' \longrightarrow E\times B$ then the functor $\tilde{f}^*$ induces a short exact sequence

$$0 \to \tilde{f}^*\mathcal{A} \xrightarrow{\tilde{f}^*\binom{\mathsf{i}}{\mathsf{p}}} \tilde{f}^*\bigl(\pi_{B*}\widetilde{\mathcal{A}} \oplus \eta_{B*}\mathcal{O}^n_{Z\times B}\bigr) \xrightarrow{\tilde{f}^*\left(\zeta^{\widetilde{\mathcal{A}}}\ \mathsf{m}\right)} \tilde{f}^*\bigl(\nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B}\bigr) \to 0.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^1 \times B' \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\pi_{B'}"'] & \mathbb{P}^1 \times B \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\pi_B"] & \mathbb{P}^1 \arrow[d, "\pi"] \\
E \times B' \arrow[r, "\tilde{f}"] \arrow[d, "\mathrm{pr}_2"'] & E \times B \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & E \\
B' \arrow[r, "f"] & B &
\end{tikzcd}$$

It implies that the base-change morphism $\tilde{f}^*\pi_{B*}\widetilde{\mathcal{A}} \longrightarrow \pi_{B'*}(\mathrm{id}\times f)^*\widetilde{\mathcal{A}}$ is an isomorphism. Indeed, it can be identified with the composition of isomorphisms

$$\tilde{f}^*\pi_{B*}\widetilde{\mathcal{A}} \cong \tilde{f}^*\pi_{B*}\mathrm{pr}_1^*\bar{\mathcal{A}} \cong \tilde{f}^*\mathrm{pr}_1^*\pi_*\bar{\mathcal{A}} \cong \pi_{B'*}(\mathrm{id}\times f)^*\mathrm{pr}_1^*\bar{\mathcal{A}} \cong \pi_{B'*}(\mathrm{id}\times f)^*\widetilde{\mathcal{A}},$$

given by the flat base change. Denote $\widetilde{\mathcal{A}}' = (\mathrm{id}\times f)^*\widetilde{\mathcal{A}}$. Then it is not difficult to show that the following diagrams are commutative:

$$\begin{tikzcd}
\tilde{f}^*\pi_{B*}\widetilde{\mathcal{A}} \arrow[r, "\tilde{f}^*(\zeta^{\widetilde{\mathcal{A}}})"] \arrow[d, "\mathrm{can}"'] & \tilde{f}^*\nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B} \arrow[d, "\mathrm{can}"] \\
\pi_{B'*}\widetilde{\mathcal{A}}' \arrow[r, "\zeta^{\widetilde{\mathcal{A}}'}"] & \nu_{B'*}\mathcal{O}^n_{\widetilde{Z}\times B'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{f}^*\eta_{B*}\mathcal{O}^n_{Z\times B} \arrow[r, "\tilde{f}^*(\mathsf{m})"] \arrow[d, "\mathrm{can}"'] & \tilde{f}^*\nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B} \arrow[d, "\mathrm{can}"] \\
\eta_{B'*}\mathcal{O}^n_{Z\times B'} \arrow[r, "\mathsf{m}'"] & \nu_{B'*}\mathcal{O}^n_{\widetilde{Z}\times B'}
\end{tikzcd},$$

in which the vertical morphisms are induced by the base change. This implies that we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{f}^*\mathcal{A} \arrow[r, "\tilde{f}^*\left(\begin{smallmatrix}\mathsf{i}\\ \mathsf{p}\end{smallmatrix}\right)"] \arrow[d] & \tilde{f}^*\big(\pi_{B*}\widetilde{\mathcal{A}} \oplus \eta_{B*}\mathcal{O}^n_{Z\times B}\big) \arrow[r, "\tilde{f}^*\left(\zeta^{\widetilde{\mathcal{A}}}\ \mathsf{m}\right)"] \arrow[d, "\mathrm{can}"'] & \tilde{f}^*\big(\nu_{B*}\mathcal{O}^n_{\widetilde{Z}\times B}\big) \arrow[r] \arrow[d, "\mathrm{can}"] & 0 \\
0 \arrow[r] & \tilde{f}^*\mathcal{A} \arrow[r] & \pi_{B'*}\widetilde{\mathcal{A}}' \oplus \eta_{B'*}\mathcal{O}^n_{Z\times B'} \arrow[r, "\left(\zeta^{\widetilde{\mathcal{A}}'}\ \mathsf{m}'\right)"] & \nu_{B'*}\mathcal{O}^n_{\widetilde{Z}\times B'} \arrow[r] & 0.
\end{tikzcd}$$

Finally, the compatibility of $\mathbb{G}$ with tensor products can be proven along similar lines as in the absolute case, see [16, Kapitel 2] for more details. □

It turns out that the description of simple vector bundles in terms of objects of $\mathsf{MP}^s_{\mathrm{nd}}(n_1, n_2)$ and $\mathsf{MP}^s_{\mathrm{cp}}(n_1, n_2)$ respectively, allows us to give an explicit description of a universal family of stable vector bundles of rank n and degree d on nodal and cuspidal Weierstraß cubic curves.

Proposition 9.41. *Let E be either a nodal or a cuspidal Weierstraß cubic curve, $0 \le d < n$ be coprime integers and $G = \mathbb{C}^*$ if E is nodal and $G = \mathbb{C}$ if E is cuspidal. If $M : G \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})$ is a holomorphic function such that the image of*

M contains exactly one representative of each isomorphism class in $\mathsf{MP}^s_{\mathrm{nd}}(n-d,d)$ or $\mathsf{MP}^s_{\mathrm{cp}}(n-d,d)$ respectively, then

$$\mathcal{P} := \mathbb{G}\big(\mathcal{O}^{n-d}_{\mathbb{P}^1\times G} \oplus \mathcal{O}_{\mathbb{P}^1\times G}(1)^d, \mathcal{O}^n_{Z\times G}, \widetilde{\mathsf{m}}\big)$$

is a universal family of stable vector bundles of rank n and degree d on E.

Proof. By Lemma 9.15 and Lemma 9.31 and because each simple vector bundle is stable ([20, Cor. 4.5]), for each stable vector bundle $\mathcal{V}$ of rank n and degree d there exists a unique $b \in G$ such that $\mathcal{V} \cong \mathcal{P}_b$.

Let $\mathcal{Q} \in \mathsf{VB}\big(E \times M_E^{(n,d)}\big)$ be a universal family of stable vector bundles of rank n and degree d on E. The universal property implies that there exists a unique morphism $f : G \longrightarrow M_E^{(n,d)}$ such that $\mathcal{P} = (\mathrm{id} \times f)^*\mathcal{Q} \otimes \mathrm{pr}_2^*\mathcal{L}$. Restricting on $E \times \{b\}$ shows that $f(b) = [\mathcal{P}_b]$, the point in $M_E^{(n,d)}$ which corresponds to the isomorphism class of the vector bundle $\mathcal{P}_b$. Since $\mathcal{P}_{b_1} \not\cong \mathcal{P}_{b_2}$ for $b_1 \neq b_2$, the map f is bijective. Because $M_E^{(n,d)}$ is known to be smooth, bijectivity of f implies that f is an isomorphism. Hence, the pair $(G, \mathcal{P})$ represents the moduli functor. □

Corollary 9.42. *Let E be a singular Weierstraß cubic curve, $G = \mathbb{C}^*$ if E is nodal and $G = \mathbb{C}$ if E is cuspidal, $0 \le d < n$ be coprime integers, $\widetilde{\mathcal{P}} = \mathcal{O}^{n-d}_{\mathbb{P}^1\times G} \oplus \mathcal{O}_{\mathbb{P}^1\times G}(1)^d$, and M be the canonical form from the proofs of Theorems 9.19 and 9.34 respectively. Then the coherent sheaf*

$$\mathcal{P} = \ker\Big(\pi_{G*}\widetilde{\mathcal{P}} \oplus \eta_{G*}\mathcal{O}^n_{Z\times G} \xrightarrow{\left(\zeta^{\widetilde{\mathcal{P}}}\ \mathsf{m}\right)} \nu_{G*}\mathcal{O}^n_{\widetilde{Z}\times G}\Big)$$

is a universal family of stable vector bundles of rank n and degree d on the curve E.

To construct a trivialization of a vector bundle $\mathcal{A}$ given by a matrix $A \in \mathsf{GL}_n\big(\mathcal{O}_{\widetilde{Z}\times G}\big)$ via the sequence (34), we pick a holomorphic section $p = p_\xi \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$ (in our applications we shall have $p = z_1$). This section induces a family of trivializations $\{\xi_l : \mathcal{O}_{\mathbb{P}^1}(l)|_{\widetilde{U}} \longrightarrow \mathcal{O}_{\widetilde{U}}\}_{l\in\mathbb{Z}}$, compatible with tensor products, from which we obtain isomorphisms $\xi^{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}}|_{\widetilde{U}\times G} \longrightarrow \mathcal{O}^n_{\widetilde{U}\times G}$, because $\widetilde{\mathcal{A}} = \bigoplus \mathcal{O}_{\mathbb{P}^1\times G}(l)^{k_l}$. Here, $\widetilde{U} \subset \mathbb{P}^1$ could be any subset on which p_ξ does not vanish, but we shall assume $\widetilde{U} \cap \widetilde{Z} = \emptyset$, which implies that $\pi_G|_{\widetilde{U}\times G} : \widetilde{U} \times G \longrightarrow U \times G$ is an isomorphism. Restricting the sequence (34) to the open subset $U \times G \subset E \times G$, we obtain a trivialization $\xi^{\mathcal{A}}$ of the family $\mathcal{A}$

$$\xi^{\mathcal{A}} : \mathcal{A}|_{U\times G} \xrightarrow{\mathsf{i}} \pi_{G*}\widetilde{\mathcal{A}}|_{U\times G} \xrightarrow{\pi_{G*}\left(\xi^{\widetilde{\mathcal{A}}}\right)} \mathcal{O}^n_{U\times G}. \tag{35}$$

Remark 9.43. Note that in the construction of all our families of stable vector bundles on a singular Weierstraß cubic curve we have chosen *two* sections $p_\zeta, p_\xi \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$. These choices are independent of each other! The section p_ζ is used to *define* a family $\mathcal{A}$ associated to a matrix $A \in \mathsf{GL}_n\big(\mathcal{O}_{\widetilde{Z}\times G}\big)$, whereas p_ξ is used to *trivialize* it on $U \times G$.

Even though in our application of Theorem 9.44 we shall come back to the framework of Section 7, we consider here a more general setting: J and M are arbitrary reduced complex spaces. We denote the canonical projections as before by $p : E \times J \times M \longrightarrow E \times M$, $q : E \times J \times M \longrightarrow E \times J$, $p_{\widetilde{Z}} : \widetilde{Z} \times J \times M \longrightarrow \widetilde{Z} \times M$ and $q_{\widetilde{Z}} : \widetilde{Z} \times J \times M \longrightarrow \widetilde{Z} \times J$.

Theorem 9.44. *Let $\tau : J \times M \longrightarrow M$ be a flat morphism, $\tilde{\tau} = \mathrm{id}_E \times \tau$, $\tau_{\widetilde{Z}} = \mathrm{id}_{\widetilde{Z}} \times \tau$ and fix $P \in \mathsf{GL}_n\big(\mathcal{O}_{\widetilde{Z}\times M}\big)$ and $N \in \mathsf{GL}_1\big(\mathcal{O}_{\widetilde{Z}\times M}\big)$ such that*

$$\tau_{\widetilde{Z}}^* P = q_{\widetilde{Z}}^* N \cdot p_{\widetilde{Z}}^* P \; . \tag{36}$$

Denote by $\mathcal{P} \in M_E^{(n,d)}$ and $\mathcal{N} \in \mathsf{Pic}^c(E)$ the bundles defined by P and N respectively. Let $p_\xi \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$ be a section which gives trivializations $\xi^{\mathcal{P}}$ of $\mathcal{P}$ on $U \times M$ and $\xi^{\mathcal{N}}$ of $\mathcal{N}$ on $U \times J$.

Then there exists an isomorphism $\varphi : q^*\mathcal{N} \otimes p^*\mathcal{P} \longrightarrow \tilde{\tau}^*\mathcal{P}$ which is represented by the identity with respect to the trivializations $\xi^{\mathcal{P}}$ and $\xi^{\mathcal{N}}$ on $U \times J \times M$ (compare with Proposition 7.2).

Proof. The bundles $\mathcal{P}$ and $\mathcal{N}$ are defined by the short exact sequences

$$0 \longrightarrow \mathcal{N} \xrightarrow{\binom{\mathsf{j}}{\mathsf{q}}} \pi_{J*}\widetilde{\mathcal{N}} \oplus \eta_{J*}\mathcal{O}_{Z\times J} \xrightarrow{\left(\zeta^{\widetilde{\mathcal{N}}}\ \mathsf{n}\right)} \nu_{J*}\mathcal{O}_{\widetilde{Z}\times J} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{P} \xrightarrow{\binom{\mathsf{i}}{\mathsf{r}}} \pi_{M*}\widetilde{\mathcal{P}} \oplus \eta_{M*}\mathcal{O}^n_{Z\times M} \xrightarrow{\left(\zeta^{\widetilde{\mathcal{P}}}\ \mathsf{p}\right)} \nu_{M*}\mathcal{O}^n_{\widetilde{Z}\times M} \longrightarrow 0,$$

where $\widetilde{\mathcal{P}} = \bigoplus_{l\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1\times M}(l)^{k_l}$ and $\widetilde{\mathcal{N}} = \mathcal{O}_{\mathbb{P}^1\times J}(c)$. Let $\widehat{\mathcal{P}} = \bigoplus_{l\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1\times J\times M}(l)^{k_l}$ and $\widehat{\mathcal{N}} = \mathcal{O}_{\mathbb{P}^1\times J\times M}(c)$. By Theorem 9.40 we have short exact sequences

$$0 \longrightarrow \tilde{\tau}^*\mathcal{P} \longrightarrow (\pi_{J\times M})_*\widehat{\mathcal{P}} \oplus (\eta_{J\times M})_*\mathcal{O}^n_{Z\times J\times M} \xrightarrow{\left(\zeta^{\widehat{\mathcal{P}}}\ \tilde{\tau}^*(\mathsf{p})\right)} (\nu_{J\times M})_*\mathcal{O}^n_{\widetilde{Z}\times J\times M} \longrightarrow 0,$$

$$0 \longrightarrow p^*\mathcal{P} \longrightarrow (\pi_{J\times M})_*\widehat{\mathcal{P}} \oplus (\eta_{J\times M})_*\mathcal{O}^n_{Z\times J\times M} \xrightarrow{\left(\zeta^{\widehat{\mathcal{P}}}\ p^*(\mathsf{p})\right)} (\nu_{J\times M})_*\mathcal{O}^n_{\widetilde{Z}\times J\times M} \longrightarrow 0,$$

and

$$0 \longrightarrow q^*\mathcal{N} \longrightarrow (\pi_{J\times M})_*\widehat{\mathcal{N}} \oplus (\eta_{J\times M})_*\mathcal{O}_{Z\times J\times M} \xrightarrow{\left(\zeta^{\widehat{\mathcal{N}}}\ q^*(\mathsf{n})\right)} (\nu_{J\times M})_*\mathcal{O}_{\widetilde{Z}\times J\times M} \longrightarrow 0.$$

Moreover, we also know that the sequence

$$0 \longrightarrow q^*\mathcal{N} \otimes p^*\mathcal{P} \longrightarrow (\pi_{J\times M})_*\widehat{\mathcal{P}} \oplus (\eta_{J\times M})_*\mathcal{O}^n_{Z\times J\times M} \xrightarrow{\beta} (\nu_{J\times M})_*\mathcal{O}^n_{\widetilde{Z}\times J\times M} \longrightarrow 0$$

is exact, where we abbreviated $\beta = \left(\zeta^{\widehat{\mathcal{P}}}\ q^*(\mathsf{n})\boxtimes p^*(\mathsf{p})\right)$. Because τ is flat, the morphism $\tilde{\tau}^*(\mathsf{p})$ is induced by the matrix $\tau_{\widetilde{Z}}^*(P)$ and the morphism $q^*(\mathsf{n}) \boxtimes p^*(\mathsf{p})$ by the matrix

$q^*_{\widetilde{Z}}(N) \cdot p^*_{\widetilde{Z}}(P)$. Hence, using (36), we obtain a commutative diagram

$$\begin{tikzcd}
(\pi_{J\times M})_*\widehat{\mathcal{P}} \oplus (\eta_{J\times M})_*\mathcal{O}^n_{Z\times J\times M} \arrow[rr, "\left(\zeta^{\widehat{\mathcal{P}}}\ \tilde{\tau}^*(\mathsf{m})\right)"] & & (\nu_{J\times M})_*\mathcal{O}^n_{\widetilde{Z}\times J\times M} \\
(\pi_{J\times M})_*\widehat{\mathcal{P}} \oplus (\eta_{J\times M})_*\mathcal{O}^n_{Z\times J\times M} \arrow[rr, "\left(\zeta^{\widehat{\mathcal{P}}}\ q^*(\mathsf{n})\boxtimes p^*(\mathsf{p})\right)"] \arrow[u, "\left(\begin{smallmatrix}\mathrm{id} & 0\\ 0 & \mathrm{id}\end{smallmatrix}\right)"] & & (\nu_{J\times M})_*\mathcal{O}^n_{\widetilde{Z}\times J\times M} \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

But this implies that we have an induced morphism $\varphi : q^*\mathcal{N} \otimes p^*\mathcal{P} \longrightarrow \tilde{\tau}^*\mathcal{P}$.

It remains to verify that the isomorphism φ is the *identity* with respect to the trivializations induced by $\xi^{\mathcal{P}}$ and $\xi^{\mathcal{N}}$. By Theorem 9.40, we have a commutative diagram

$$\begin{tikzcd}
\tilde{\tau}^*\mathcal{P} \arrow[r, "\tilde{\tau}^*(\mathsf{i})"] \arrow[d, "\mathrm{id}"'] & \tilde{\tau}^*(\pi_M)_*\widetilde{\mathcal{P}} \arrow[d, "\mathrm{can}"] \\
\tilde{\tau}^*\mathcal{P} \arrow[r] & (\pi_{J\times M})_*\widehat{\mathcal{P}} \\
q^*\mathcal{N} \otimes p^*\mathcal{P} \arrow[u, "\varphi"] \arrow[r, "q^*(\mathsf{j})\boxtimes p^*(\mathsf{i})"] \arrow[dr, "q^*(\mathsf{j})\otimes p^*(\mathsf{i})"'] & (\pi_{J\times M})_*\widehat{\mathcal{P}} \arrow[u, "\mathrm{id}"'] \\
 & q^*\pi_{J*}\widetilde{\mathcal{N}} \otimes p^*\pi_{M*}\widetilde{\mathcal{P}} \arrow[u, "\mathrm{can}"']
\end{tikzcd}$$

The key point is now that the trivializations $\xi^{\widetilde{\mathcal{P}}}$ and $\xi^{\widetilde{\mathcal{N}}}$ are the pull backs of trivializations $\xi^{\bar{\mathcal{P}}}_{\widetilde{U}} : \bigoplus \mathcal{O}_{\mathbb{P}^1}(l)^{k_l}|_{\widetilde{U}} \longrightarrow \mathcal{O}^n_{\widetilde{U}}$ and $\xi^{\bar{\mathcal{N}}}_{\widetilde{U}} : \mathcal{O}_{\mathbb{P}^1}(c)|_{\widetilde{U}} \longrightarrow \mathcal{O}_{\widetilde{U}}$, on $\mathbb{P}^1$. This implies that the base-change morphism $\mathrm{can} : \tilde{\tau}^*(\pi_M)_*\widetilde{\mathcal{P}} \longrightarrow (\pi_{J\times M})_*\widehat{\mathcal{P}}$ is the *identity* with respect to the trivializations $\xi^{\widetilde{\mathcal{P}}}$ and $\xi^{\widehat{\mathcal{P}}}$. It follows that, in these trivializations, the isomorphism φ is the identity. □

After choosing representing pairs $(J^d, \mathcal{L}^{(d)})$, $(J, \mathcal{L})$ and $(M, \mathcal{P})$ for the functors $\underline{\mathsf{Pic}}^d$, $\underline{\mathsf{Pic}}^0$ and $\underline{\mathsf{M}}^{n,d}_E$, in Section 7 we have constructed a morphism $\tau : J \times M \longrightarrow M$. To make this explicit, let $G = \mathbb{C}^*$ if E is nodal and $G = \mathbb{C}$ if E is cuspidal. For simplicity, we assume again $0 \le d < n$.

We define $J = J^d = M = G$ with universal bundles $\mathcal{L}^{(d)}$ and $\mathcal{L}$ both given by (y, id) in the nodal case and by $1 + \varepsilon y$ in the cuspidal case. We define the universal bundle $\mathcal{P}$ to be given by $(M_{n-d,d}, \mathrm{id})$ in the nodal case (proof of Theorem 9.19) and $1_n + \varepsilon M_{n-d,d}$ in the cuspidal case (proof of Theorem 9.34). Universality was shown in Corollary 9.42.

Using the notation of diagram (25) in Section 7, we obtain now $t^e = \mathrm{id}_G$ and $\det = \pm\mathrm{id}_G$, the sign depending on (n,d) only. Moreover, because $\big((y_1),(1)\big)\big((y_2),(1)\big) = \big((y_1y_2),(1)\big)$, the group structure on $J = G$ is multiplication in the nodal case and because $(1+y_1\varepsilon)(1+y_2\varepsilon) = 1+(y_1+y_2\varepsilon)$, the group structure is addition in the cuspidal case. Therefore, $\tau = \sigma' : G \times G \longrightarrow G$ has the description $\tau(a,b) = a^n b$ in the nodal case and $\tau(a,b) = na+b$ in the cuspidal case.

We also consider the vector bundle $\mathcal{P}'$ of rank n and degree d on $G \times E$, which is given by $(\widetilde{N}_{n-d,d}, \mathrm{id})$ in the nodal case (Remark 9.26) and $1_n + \varepsilon\widetilde{N}_{n-d,d}$ in the cuspidal case (Remark 9.37). These were constructed in such a way that (36) holds with respect to the morphism $\tau' : G \times G \longrightarrow G$ given by $\tau'(a,b) = ab$ (respectively $\tau'(a,b) = a+b$).

If $f_n : G \longrightarrow G$ is given by $f_n(t) = t^n$ in the nodal case and by $f_n(t) = nt$ in the cuspidal case, we have $(\mathrm{id}_E \times f_n) \circ \tilde{\tau}' = \tilde{\tau} \circ (\mathrm{id}_{E\times J} \times f_n)$, because $J = G$ is abelian.

From Remark 9.26 and Remark 9.37 respectively, it is clear that $(\mathrm{id}_E \times f_n)^*\mathcal{P}$ and $\mathcal{P}'$ are isomorphic after restriction to a fibre. This implies that these two bundles are locally isomorphic (with respect to the basis G, which is reduced). Equivalently, up to a twist by the pull-back of a line bundle on G, $(\mathrm{id}_E \times f_n)^*\mathcal{P}$ and $\mathcal{P}'$ are isomorphic. As G is a non-compact Riemann surface, we even get $(\mathrm{id}_E \times f_n)^*\mathcal{P} \cong \mathcal{P}'$, but we do not need this in the sequel.

Corollary 9.45. *The morphism τ' and the bundles $\mathcal{P}'$ and $\mathcal{L}$ satisfy the properties of Theorem 9.44. Moreover, each point of G has an open neighbourhood $M' \subset G$ such that there exits an isomorphism $\varphi : q^*\mathcal{L} \otimes p^*\mathcal{P}|_{E\times J\times M'} \longrightarrow \tilde{\tau}^*\mathcal{P}|_{E\times J\times M'}$, which is represented by the identity with respect to the trivializations $\xi^{\mathcal{P}}$ and $\xi^{\mathcal{L}}$ on $U \times J \times M'$.*

Proof. Because, up to a local isomorphism, τ is isomorphic to addition of complex numbers, it is flat. Now, the first statement is clear from the above. The prove the second, chose a sufficiently small open neighbourhood $M' \subset G$ around a given point on G such that f_n restricted to M' is an isomorphism and $(\mathrm{id}_E \times f_n)^*\mathcal{P}$ is isomorphic to $\mathcal{P}'$ over $E \times M'$. Because $q \circ (\mathrm{id}_{E\times J} \times f_n) = q$, $p \circ (\mathrm{id}_{E\times J} \times f_n) = (\mathrm{id}_E \times f_n) \circ p$ and $(\mathrm{id}_E \times f_n) \circ \tilde{\tau}' = \tilde{\tau} \circ (\mathrm{id}_{E\times J} \times f_n)$, with the aid of the isomorphism $\mathrm{id}_{E\times J} \times f_n\big|_{M'}$ the claim follows from the first part of the corollary. □

In the cuspidal case, f_n is an isomorphism, hence we may choose $M' = G$. In the nodal case, however, the matrix $\widetilde{N}_{n_1,n_2}$ does not descent to $M = G$, as it involves taking n-th roots; f_n is an unramified n-fold cover. Therefore, the isomorphism φ exists only locally on M in this case.

Remark 9.46. The map $x \mapsto \mathcal{O}_E(x)$ gives a canonical isomorphism $\breve{E} \to \mathsf{Pic}^1_E$. Let $e \in E$ be the point which has coordinate $z = 1$ in the nodal case (Subsection 9.2) and $z = 0$ in the cuspidal case (Subsection 9.3). This point corresponds to the neutral element of J under the isomorphisms $\breve{E} \xrightarrow{\mathrm{can}} J^1 \xleftarrow{t^e} J$ (see Section 7).

This isomorphism and our choice of the representing pair $(J, \mathcal{L})$ with $J = G$ induce coordinates on $\breve{E}$.

Lemma 9.12 shows that these coordinates coincide with the ones induced by the coordinates on $\mathbb{P}^1$ and the normalization morphism $\mathbb{P}^1 \setminus \widetilde{Z} \xrightarrow{\pi} \breve{E}$. However, by Lemma 9.27 we see that in the case of a cuspidal curve these two choices are *different*; they are related by the involution of $\mathbb{C}$ mapping z to $-z$.

Remark 9.47. If $(n, d) \in \mathbb{Z}^+ \times \mathbb{Z}$ are coprime integers, we let c be the unique integer for which $n_1 = (1 + c)n - d > 0$ and $n_2 = d - cn \geq 0$. With the aid of the equivalence between $\mathsf{VB}^{(c,c+1)}_{n_1,n_2}(E)$ and $\mathsf{VB}^{(0,1)}_{n_1,n_2}(E)$, given by the tensor product with $\mathcal{O}_E(ce)$, it can be shown that all the results of this section are also valid for such pairs (n, d).

10. Computations of r-matrices for singular Weierstrass curves

Let E be a singular Weierstraß cubic curve, Ω_E the sheaf of regular holomorphic 1-forms, $\omega \in H^0(\Omega_E)$ a no-where vanishing global section. As usual, for a pair of coprime integers $(n, d) \in \mathbb{Z}^+ \times \mathbb{Z}$, let $M = M_E^{(n,d)}$ be the moduli space of stable holomorphic vector bundles of rank n and degree d on E, $\mathcal{P} = \mathcal{P}(n, d) \in \mathsf{VB}(E \times M)$ be a universal family and $\xi^{\mathcal{P}} : \mathcal{P}|_{U \times M'} \longrightarrow \mathcal{O}|^n_{U \times M'}$ its trivialization, as constructed in the previous section. Recall that these data define the germ of a meromorphic function

$$\tilde{r} = \tilde{r}^{\xi} : \big(M \times M \times E \times E, o\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C}),$$

whose value at the point $(v_1, v_2; y_1, y_2)$, where $v_1 \neq v_1$ and $y_1 \neq y_2$, is defined via the commutative diagram (31). Our next goal is to get explicit formulae to calculate the morphisms $\mathrm{res}_{y_1}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}(\omega)$ and $\mathrm{ev}_{y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}(y_1)}$ in the case of nodal and cuspidal Weierstraß cubic curves. To do this, we consider first the case of vector bundles on a projective line $\mathbb{P}^1$.

10.1. Residue and evaluation morphisms on $\mathbb{P}^1$.

Let $(z_0 : z_1)$ be homogeneous coordinates on $\mathbb{P}^1$, $0 = (1 : 0)$ and $U = \big\{(z_0 : z_1) | z_0 \neq 0\big\}$. Let $z = \frac{z_1}{z_0}$ be a local coordinate on U. In what follows, we shall use the identification $\mathbb{C} \xrightarrow{\cong} U$ mapping $x \in \mathbb{C}$ to $(1 : x) \in U$.

Let $\mathcal{V} = \bigoplus_{j\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(j)^{n_j}$ and $\mathcal{W} = \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(i)^{m_i}$ be a pair of vector bundles on $\mathbb{P}^1$ of ranks n and m respectively. Then a morphism $F \in \mathsf{Hom}_{\mathbb{P}^1}(\mathcal{V}, \mathcal{W})$ can be written in matrix form: $F = (F_{ij})$, where $F_{ij} \in \mathsf{Mat}_{m_i\times n_j}\big(\mathbb{C}[z_0, z_1]_{i-j}\big)$.

Consider the set of trivializations $\big\{\xi_l : \mathcal{O}_{\mathbb{P}^1}(l)|_U \longrightarrow \mathcal{O}_U\big\}_{l\in\mathbb{Z}}$ mapping a local section p of $\mathcal{O}_{\mathbb{P}^1}(l)$ to the holomorphic function $\dfrac{p}{z_0^l|_U}$. They induce trivializations $\xi^{\mathcal{V}} : \mathcal{V}|_U \longrightarrow \mathcal{O}_U^n$ and $\xi^{\mathcal{W}} : \mathcal{W}|_U \longrightarrow \mathcal{O}_U^m$. Let $x, y \in U$ and ω be a meromorphic differential form on $\mathbb{P}^1$ holomorphic at x. Let $\bar{\xi}^{\mathcal{V}}$ be the morphism

$$\mathcal{V}|_U \otimes \mathbb{C}_x \xrightarrow{\xi^{\mathcal{V}}\otimes \mathsf{id}} \mathcal{O}_U^n \otimes \mathbb{C}_x \xrightarrow{\mathrm{can}} \mathbb{C}_x^n;$$

the morphism $\bar{\xi}^{\mathcal{W}}$ is defined in a similar way. Our goal is to get explicit formulae to calculate the morphisms:

$$\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(x)\big) \xrightarrow{\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)} \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}\otimes \mathbb{C}_x, \mathcal{W}\otimes \mathbb{C}_x\big) \xrightarrow{\mathrm{cnj}(\bar{\xi}^{\mathcal{V}},\bar{\xi}^{\mathcal{W}})} \mathsf{Mat}_{m\times n}(\mathbb{C})$$

and

$$\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(x)\big) \xrightarrow{\mathrm{ev}_y^{\mathcal{V},\mathcal{W}(x)}} \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}\otimes \mathbb{C}_y, \mathcal{W}\otimes \mathbb{C}_y\big) \xrightarrow{\mathrm{cnj}(\bar{\xi}^{\mathcal{V}},\bar{\xi}^{\mathcal{W}})} \mathsf{Mat}_{m\times n}(\mathbb{C}).$$

Let $\sigma = z_1 - xz_0 \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$ be a section such that $\mathsf{div}(\sigma) = [x]$. Then σ defines an isomorphism $\mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(x)$ mapping a global section $p = p(z_0, z_1)$ to the meromorphic function $\frac{p}{\sigma}$. Moreover, it induces an isomorphism

$$t_\sigma : \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(1)\big) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(x)\big).$$

Let $\mathcal{V}' = \mathcal{V}|_U$ and $\mathcal{W}' = \mathcal{W}|_U$. By Proposition 4.8 we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(x)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V},\mathcal{W}}(\omega)"] \arrow[d, "\mathrm{can}"'] & \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}\otimes \mathbb{C}_x, \mathcal{W}\otimes \mathbb{C}_x\big) \arrow[d, "\mathrm{can}"] \\
\mathsf{Hom}_U\big(\mathcal{V}', \mathcal{W}'(x)\big) \arrow[r, "\mathrm{res}_x^{\mathcal{V}',\mathcal{W}'}(\omega)"] \arrow[d, "\mathrm{cnj}\left(\xi^{\mathcal{V}},\xi^{\mathcal{W}(x)}\right)"'] & \mathsf{Hom}_U\big(\mathcal{V}'\otimes \mathbb{C}_x, \mathcal{W}'\otimes \mathbb{C}_x\big) \arrow[d, "\mathrm{cnj}\left(\bar{\xi}^{\mathcal{V}},\bar{\xi}^{\mathcal{W}}\right)"] \\
\mathsf{Hom}_U\big(\mathcal{O}_U^n, \mathcal{O}_U^m(x)\big) \arrow[r, "\mathrm{res}_x(\omega)"] \arrow[d, "\mathrm{can}"'] & \mathsf{Hom}_U\big(\mathbb{C}_x^n, \mathbb{C}_x^m\big) \arrow[d, "\mathrm{can}"] \\
\mathsf{Mat}_{m\times n}\big(O(x)\big) \arrow[r, "\mathrm{res}_x(\omega)"] & \mathsf{Mat}_{m\times n}(\mathbb{C}),
\end{tikzcd}$$

where $O(x)$ denotes the vector space of meromorphic functions on U which have at most a pole of order one at x. Let T be the composition

$$\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(1)\big) \xrightarrow{t_\sigma} \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(x)\big) \xrightarrow{\mathrm{cnj}\left(\xi^{\mathcal{V}},\xi^{\mathcal{W}(x)}\right)} \mathsf{Mat}_{m\times n}\big(O(x)\big).$$

Then we have the following result: if $F \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(1)\big)$ then $T(F) = \dfrac{F(1,z)}{z-x}$. Let $\omega = g(z)dz$ then by Lemma 4.5 we have:

$$\mathrm{res}_x(\omega)\Big(\frac{F(1,z)}{z-x}\Big) = g(x)F(1,x).$$

Corollary 10.1. *In the above notation, the morphism*

$$\overline{\mathrm{res}}_x := \mathrm{res}_x(\omega) \circ T : \quad \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(1)\big) \longrightarrow \mathsf{Mat}_{m\times n}(\mathbb{C})$$

has the following form:

- *if* $\omega = \dfrac{dz}{z}$ *then* $F(z_0, z_1)$ *is mapped to* $\frac{1}{x}F(1, x)$*;*
- *if* $\omega = dz$ *then* $F(z_0, z_1)$ *is mapped to* $F(1, x)$*.*

In a similar way, we compute the morphism $\mathrm{ev}_y^{\mathcal{V},\mathcal{W}(x)}$. Indeed, by Proposition 4.12 we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(x)\big) \arrow[r, "\mathrm{ev}_y^{\mathcal{V},\mathcal{W}(x)}"] \arrow[d, "\mathrm{can}"'] & \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V} \otimes \mathbb{C}_y, \mathcal{W} \otimes \mathbb{C}_y\big) \arrow[d, "\mathrm{can}"] \\
\mathsf{Hom}_U\big(\mathcal{V}', \mathcal{W}'(x)\big) \arrow[r, "\mathrm{ev}_y^{\mathcal{V}',\mathcal{W}'(x)}"] \arrow[d, "\mathrm{cnj}\left(\zeta^{\mathcal{V}}, \zeta^{\mathcal{W}(x)}\right)"'] & \mathsf{Hom}_U\big(\mathcal{V}' \otimes \mathbb{C}_y, \mathcal{W}' \otimes \mathbb{C}_y\big) \arrow[d, "\mathrm{cnj}\left(\bar{\zeta}^{\mathcal{V}}, \bar{\zeta}^{\mathcal{W}}\right)"] \\
\mathsf{Hom}_U\big(\mathcal{O}_U^n, \mathcal{O}_U^m(x)\big) \arrow[r, "\mathrm{ev}_y"] \arrow[d, "\mathrm{can}"'] & \mathsf{Hom}_U\big(\mathbb{C}_y^n, \mathbb{C}_y^m\big) \arrow[d, "\mathrm{can}"] \\
\mathsf{Mat}_{m\times n}\big(O(x)\big) \arrow[r, "\mathrm{ev}_y"] & \mathsf{Mat}_{m\times n}(\mathbb{C})
\end{tikzcd}$$

and for $A(z) \in \mathsf{Mat}_{m\times n}\big(O(x)\big)$ we have $\mathrm{ev}_y\big(A(z)\big) = A(y)$.

Corollary 10.2. *In the above notations, the morphism*

$$\overline{\mathrm{ev}}_y := \mathrm{ev}_y \circ T : \quad \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(1)\big) \longrightarrow \mathsf{Mat}_{m\times n}(\mathbb{C})$$

maps a matrix $F(z_0, z_1) \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(1)\big)$ *to* $\dfrac{1}{y-x}F(1, y)$*.*

Remark 10.3. The above morphisms can be included into the following diagram:

$$\begin{tikzcd}
 & \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(1)\big) \arrow[d, "t_\sigma"'] \arrow[dl, bend right, "\overline{\mathrm{res}}_x"'] \arrow[dr, bend left, "\overline{\mathrm{ev}}_y"] & \\
 & \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{V}, \mathcal{W}(x)\big) \arrow[dl, "\mathrm{res}_x(\omega)"] \arrow[dr, "\mathrm{ev}_y"'] & \\
\mathsf{Mat}_{m\times n}(\mathbb{C}) & & \mathsf{Mat}_{m\times n}(\mathbb{C})
\end{tikzcd}$$

In particular, the linear maps $\overline{\mathrm{res}}_x$ and $\overline{\mathrm{ev}}_y$ depend on the choice of a section $\sigma \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$ vanishing at x: such a σ is determined uniquely only up to a non-zero constant. However, as we shall see below, this choice does not affect the final formula to compute the triple Massey product $\tilde{r}_{y_1,y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_1}}(\omega)$ on a cubic curve.

10.2. **Residue and evaluations maps on singular Weierstraß curves.** Let E be a singular Weierstraß cubic curve, $s \in E$ its unique singular point, $\pi : \mathbb{P}^1 \longrightarrow E$ normalization of E. We choose homogeneous coordinates on E in such a way that $\pi^{-1}(s) = \{(0:1),(1:0)\}$ if E is nodal and $\pi^{-1}(s) = \{(0:1)\}$ if E is cuspidal. Let $\breve{E}$ be the regular part of E then the isomorphism $\pi : \mathbb{P}^1 \setminus \pi^{-1}(s) \longrightarrow \breve{E}$ induces local coordinates on $\breve{E}$. In what follows, we shall identify a point $y \in \breve{E} = G$ with its preimage $\tilde{y} = (1:y) \in \mathbb{P}^1$. Let $0 < d < n$ be a pair of mutually prime integers and $\mathcal{P} \in \mathsf{VB}(E \times M)$ a universal family of stable vector bundles of rank n and degree d on E. Recall that for $v_1 \neq v_2 \in M$ and $y_1 \neq y_2 \in \breve{E}$ we have a commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_E\big(\mathcal{P}^{v_1}|_{y_1}, \mathcal{P}^{v_2}|_{y_1}\big) \arrow[r, "\pi^*"] & \mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{P}^{v_1}|_{\tilde{y}_1}, \pi^*\mathcal{P}^{v_2}|_{\tilde{y}_1}\big) \\
\mathsf{Hom}_E\big(\mathcal{P}^{v_1}, \mathcal{P}^{v_2}(y_1)\big) \arrow[r, "\pi^*"] \arrow[u, "\mathrm{res}_{y_1}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}(\omega)"] \arrow[d, "\mathrm{ev}_{y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}(y_1)}"'] & \mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{P}^{v_1}, \pi^*\mathcal{P}^{v_2}(\tilde{y}_1)\big) \arrow[u, "\mathrm{res}_{\tilde{y}_1}^{\pi^*\mathcal{P}^{v_1},\pi^*\mathcal{P}^{v_2}}(\tilde{\omega})"'] \arrow[d, "\mathrm{ev}_{\tilde{y}_2}^{\pi^*\mathcal{P}^{v_1},\pi^*\mathcal{P}^{v_2}(\tilde{y}_1)}"] \\
\mathsf{Hom}_E\big(\mathcal{P}^{v_1} \otimes \mathbb{C}_{y_2}, \mathcal{P}^{v_2} \otimes \mathbb{C}_{y_2}\big) \arrow[r, "\pi^*"] & \mathsf{Hom}_{\mathbb{P}^1}\big(\pi^*\mathcal{P}^{v_1}|_{\tilde{y}_2}, \pi^*\mathcal{P}^{v_2}|_{\tilde{y}_2}\big).
\end{tikzcd}$$

In particular, the computation of the morphisms $\mathrm{res}_{y_1}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}(\omega)$ and $\mathrm{ev}_{y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}(y_1)}$ can be reduced to an analogous computation on $\mathbb{P}^1$.

Let E' and M' be open neighbourhoods of $e \in E$ and $m \in M$, $\xi : \mathcal{P}|_{E' \times M'} \longrightarrow \mathcal{O}^n_{E' \times M'}$ be a trivialization of the universal family $\mathcal{P}$ which is compatible with the action of the Jacobian. For $v \in M'$ let $\mathcal{P}^v = \mathcal{P}|_{E \times v}$ and ξ^v be the induced trivialization $\xi^v : \mathcal{P}^v|_{E'} \longrightarrow \mathcal{O}^n_{E'}$. Next, for $y \in E'$ let $\bar{\zeta}^v$ be the corresponding isomorphism $\mathcal{P}^v \otimes \mathbb{C}_y \longrightarrow \mathbb{C}^n_y$. Our goal is to compute the value of the geometric r–matrix r^ξ at the point $(v_1, v_2; y_1, y_2)$. This linear morphism $\tilde{r}^\xi(v_1, v_2; y_1, y_2)$ is defined via the commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_E(\mathcal{P}^{v_1} \otimes \mathbb{C}_{y_1}, \mathcal{P}^{v_2} \otimes \mathbb{C}_{y_1}) \arrow[r, "\mathrm{cnj}\left(\bar{\xi}^{v_1},\bar{\xi}^{v_2}\right)"] \arrow[d, "\tilde{r}_{y_1,y_2}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}(\omega)"'] & \mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[d, "\tilde{r}^\xi(v_1,v_2;y_1,y_2)"] \\
\mathsf{Hom}_E(\mathcal{P}^{v_1} \otimes \mathbb{C}_{y_2}, \mathcal{P}^{v_2} \otimes \mathbb{C}_{y_2}) \arrow[r, "\mathrm{cnj}\left(\bar{\xi}^{v_1},\bar{\xi}^{v_2}\right)"] & \mathsf{Mat}_{n\times n}(\mathbb{C}).
\end{tikzcd}$$

Let $\widetilde{\mathcal{P}} = \mathcal{O}^{n-d}_{\mathbb{P}^1} \oplus \mathcal{O}^d_{\mathbb{P}^1}(1)$ and $\widehat{\mathcal{P}}$ be the pull-back of $\widetilde{\mathcal{P}}$ to $\mathbb{P}^1 \times M$. Let $\zeta_1 : \mathcal{O}_{\mathbb{P}^1}(1)|_{\widetilde{Z}} \longrightarrow \mathcal{O}_{\widetilde{Z}}$ be the isomorphism used in Section 9 in the construction of the category $\mathsf{Tri}(E)$.

Recall that the universal family $\mathcal{P} = \mathcal{P}(n,d)$ was defined using the following short exact sequence

$$0 \longrightarrow \mathcal{P} \xrightarrow{\binom{\mathsf{i}}{\mathsf{p}}} \pi_{M*}\widehat{\mathcal{P}} \oplus \eta_{M*}\mathcal{O}^n_{Z\times M} \xrightarrow{(\zeta^{\widehat{\mathcal{P}}}\ \mathsf{m})} \nu_{M*}\mathcal{O}^n_{\widetilde{Z}\times M} \longrightarrow 0,$$

In particular, the trivialization $\hat{\xi}^{\widehat{\mathcal{P}}} : \widehat{\mathcal{P}}|_{U\times M'} \longrightarrow \mathcal{O}^n_{U\times M'}$ induces the trivialization

$$\xi^{\mathcal{P}} : \mathcal{P}|_{E'\times M'} \xrightarrow{\mathsf{i}} \pi_{M*}\widehat{\mathcal{P}}|_{E'\times M'} \xrightarrow{\xi^{\widehat{\mathcal{P}}}} \mathcal{O}^n_{E'\times M'}.$$

Moreover, by Theorem 9.4 we know that the morphism

$$\tilde{\mathsf{i}} : \pi_M^*\mathcal{P} \xrightarrow{\pi^*(\mathsf{i})} \pi_M^*\pi_{M*}\widehat{\mathcal{P}} \xrightarrow{\text{can}} \widehat{\mathcal{P}}$$

is an isomorphism. Hence, we have the following commutative diagram

$$\begin{tikzcd}
\mathsf{Hom}_E\big(\mathcal{P}^{v_1}\otimes\mathbb{C}_{y_1}, \mathcal{P}^{v_2}\otimes\mathbb{C}_{y_1}\big) \arrow[rr, "\mathrm{cnj}\circ\pi^*"] \arrow[dr, "\mathrm{cnj}\left(\bar{\xi}^{v_1},\bar{\xi}^{v_2}\right)"'] \arrow[dddd, "\tilde{r}^{\mathcal{P}^{v_1},\mathcal{P}^{v_2}}_{y_1,y_2}(\omega)"'] & & \mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_1}, \widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_1}) \arrow[dl, "\mathrm{cnj}\left(\tilde{\xi}^{v_1},\tilde{\xi}^{v_2}\right)"] \\
& \mathsf{Mat}_{n\times n}(\mathbb{C}) \arrow[dd, "\tilde{r}^{\xi}(v_1,v_2;y_1,y_2)"'] & \\
& & \Pi^{v_1,v_2}_{y_1} \arrow[uu, "\mathrm{res}_{\tilde{y}_1}"'] \arrow[dd, "\mathrm{ev}_{\tilde{y}_2}"] \\
& \mathsf{Mat}_{n\times n}(\mathbb{C}) & \\
\mathsf{Hom}_E\big(\mathcal{P}^{v_1}\otimes\mathbb{C}_{y_2}, \mathcal{P}^{v_2}\otimes\mathbb{C}_{y_2}\big) \arrow[rr, "\mathrm{cnj}\circ\pi^*"] \arrow[ur, "\mathrm{cnj}\left(\bar{\xi}^{v_1},\bar{\xi}^{v_2}\right)"] & & \mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_2}, \widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_2}) \arrow[ul, "\mathrm{cnj}\left(\tilde{\xi}^{v_1},\tilde{\xi}^{v_2}\right)"']
\end{tikzcd}$$

where $\Pi^{v_1,v_2}_{y_1} = \mathsf{Im}\Big(\mathsf{Hom}_E\big(\mathcal{P}^{v_1}, \mathcal{P}^{v_2}(y_1)\big) \xrightarrow{\mathrm{cnj}\left(\tilde{\mathsf{i}}^{v_1},\tilde{\mathsf{i}}^{v_2}(y_1)\right)\circ\pi^*} \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(\tilde{y}_1)\big)\Big)$.

The morphisms $\mathrm{res}_{\tilde{y}_1} : \Pi^{v_1,v_2}_{y_1} \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_1}, \widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_1})$ and $\mathrm{ev}_{\tilde{y}_2} : \Pi^{v_1,v_2}_{y_1} \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_2}, \widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_2})$ are isomorphisms. Hence, in order to compute the linear map $r^{\xi}(v_1,v_2;y_1,y_2)$, it suffices to get explicit formulae for the morphisms

$$\mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_1}, \widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_1}) \xrightarrow{\mathrm{res}^{-1}_{\tilde{y}_1}} \Pi^{v_1,v_2}_{y_1} \xrightarrow{\mathrm{ev}_{\tilde{y}_2}} \mathsf{Hom}_{\mathbb{P}^1}(\widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_2}, \widetilde{\mathcal{P}}\otimes\mathbb{C}_{\tilde{y}_2}).$$

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_E(y_1) \xrightarrow{\binom{\mathsf{j}}{\mathsf{q}}} \pi_*\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) \oplus \eta_*\mathcal{O}_Z \xrightarrow{(\,\zeta_1\;\; \mathsf{n}_y\,)} \nu_*\mathcal{O}_{\widetilde{Z}} \longrightarrow 0.$$

By Theorem 9.4 we know that the morphism

$$\tilde{\mathsf{j}} : \mathcal{O}_{\mathbb{P}^1}(\tilde{y}_1) = \pi^*\big(\mathcal{O}_E(y_1)\big) \xrightarrow{\pi^*(\mathsf{j})} \pi^*\pi_*\mathcal{O}_{\mathbb{P}^1}(1) \xrightarrow{\text{can}} \mathcal{O}_{\mathbb{P}^1}(1)$$

is an isomorphism. Let $\sigma = \tilde{\mathsf{j}}(1) \in H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big)$. Then the morphism

$$t_\sigma : \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(y_1)\big)$$

introduced in Subsection 10.1 is the inverse of the morphism

$$\tilde{\mathsf{j}}_* : \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(y_1)\big) \longrightarrow \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big)$$

induced by $\tilde{\mathsf{j}}$. Consider the short exact sequence

$$0 \longrightarrow \mathcal{P}^{v_2} \xrightarrow{\begin{pmatrix}\mathsf{i}^{v_2}\\ \mathsf{q}^{v_2}\end{pmatrix}} \pi_*\widetilde{\mathcal{P}} \oplus \eta_*\mathcal{O}_Z^n \xrightarrow{\begin{pmatrix}\zeta^{\widetilde{\mathcal{P}}} & \mathsf{m}(v_2)\end{pmatrix}} \nu_*\mathcal{O}_{\widetilde{Z}}^n \longrightarrow 0.$$

We know that the sequence

$$0 \longrightarrow \mathcal{P}^{v_2}(y_1) \xrightarrow{\begin{pmatrix}\mathsf{k}\\ \mathsf{l}\end{pmatrix}} \pi_*\widetilde{\mathcal{P}}(1) \oplus \eta_*\mathcal{O}_Z^n \xrightarrow{\begin{pmatrix}\zeta^{\widetilde{\mathcal{P}}(1)} & \mathsf{m}\end{pmatrix}} \nu_*\mathcal{O}_{\widetilde{Z}}^n \longrightarrow 0$$

is exact, where k is the morphism $\mathcal{P}^{v_2} \otimes \mathcal{O}_E(y_1) \xrightarrow{\mathsf{i}^{v_1}\otimes \mathsf{j}} \pi_*\widetilde{\mathcal{P}} \otimes \pi_*\mathcal{O}_{\mathbb{P}^1}(1) \xrightarrow{\text{can}} \pi_*\big(\widetilde{\mathcal{P}}(1)\big)$ and m corresponds to the tensor product of matrices $\mathsf{m}(v_2)$ and n.

Lemma 10.4. *In the above notation, the following diagram is commutative:*

$$\begin{tikzcd}
\widetilde{\mathcal{P}}(\tilde{y}_1) \arrow[rr, "\tilde{\mathsf{j}}"] & & \widetilde{\mathcal{P}}(1) \\
& \pi^*\big(\mathcal{P}^{v_2}(y_1)\big) \arrow[ul, "\tilde{\mathsf{j}}(\tilde{y}_1)"] \arrow[ur, "\tilde{\mathsf{k}}"'] &
\end{tikzcd}$$

Proof. This follows from the definition of the morphism k and the fact that the diagram

$$\begin{tikzcd}
\pi^*\big(\mathcal{P}(y_1)\big) \arrow[r] \arrow[d, "\text{can}"'] & \pi^*\pi_*\widetilde{\mathcal{P}} \otimes \pi^*\pi_*\mathcal{O}_{\mathbb{P}^1}(1) \arrow[r, "\pi^*(\text{can})"] \arrow[d, "="'] & \pi^*\pi_*\widetilde{\mathcal{P}}(1) \arrow[d, "\text{can}"] \\
\pi^*\mathcal{P} \otimes \mathcal{O}_{\mathbb{P}^1}(\tilde{y}_1) \arrow[r] & \pi^*\pi_*\widetilde{\mathcal{P}} \otimes \pi^*\pi_*\mathcal{O}_{\mathbb{P}^1}(1) \arrow[r, "\pi^*(\text{can})\otimes\pi^*(\text{can})"] & \widetilde{\mathcal{P}} \otimes \mathcal{O}_{\mathbb{P}^1}(1)
\end{tikzcd}$$

is commutative. □

Using Lemma 10.2, we obtain the following result.

Proposition 10.5. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathsf{Hom}_{\mathsf{Tri}(E)}\big(T_{\mathcal{P}^{v_1}}, T_{\mathcal{P}^{v_2}(y_1)}\big) \arrow[r, "\mathsf{For}"] \arrow[d, "\mathbb{G}"'] & \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big) \arrow[d, "t_\sigma"] \\
\mathsf{Hom}_E\big(\mathcal{P}^{v_1}, \mathcal{P}^{v_2}(y_1)\big) \arrow[r, "\text{cnj}\left(\tilde{\mathsf{i}}^{v_1}, \tilde{\mathsf{i}}^{v_2}(\tilde{y}_1)\right)\circ\pi^*"] & \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(\tilde{y}_1)\big),
\end{tikzcd}$$

where $T_{\mathcal{P}^v} = \mathbb{F}(\mathcal{P}^v)$ for all $v \in M$.

Proof. First note that by Theorem 9.4 the diagram

$$\begin{tikzcd}
 & \mathsf{Hom}_{\mathsf{Tri}(E)}\big(T_{\mathcal{P}^{v_1}}, T_{\mathcal{P}^{v_2}(y_1)}\big) \arrow[dl, "\mathbb{G}"'] \arrow[dr, "\mathsf{For}"] & \\
\mathsf{Hom}_E\big(\mathcal{P}^{v_1}, \mathcal{P}^{v_2}(y_1)\big) \arrow[rr, "\mathrm{cnj}\left(\tilde{\mathsf{i}}^{v_1}, \tilde{\mathsf{k}}\right) \circ \pi^*"] & & \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big)
\end{tikzcd}$$

is commutative. By Lemma 10.2 the diagram

$$\begin{tikzcd}
 & \mathsf{Hom}_E\big(\mathcal{P}^{v_1}, \mathcal{P}^{v_2}(y_1)\big) \arrow[dl, "\mathrm{cnj}\left(\tilde{\mathsf{i}}^{v_1}, \tilde{\mathsf{i}}^{v_2}(\tilde{y}_1)\right) \circ \pi^*"'] \arrow[dr, "\mathrm{cnj}\left(\tilde{\mathsf{i}}^{v_1}, \tilde{\mathsf{k}}\right) \circ \pi^*"] & \\
\mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(\tilde{y}_1)\big) \arrow[rr, "\tilde{\mathsf{j}}_*"] & & \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big)
\end{tikzcd}$$

is commutative, too. Patching both diagrams together and using that $\tilde{\mathsf{j}}_* = t_\sigma^{-1}$, we get the claim. □

Corollary 10.6. *Let* $\widetilde{\Pi}^{v_1,v_2}_{y_1} = \mathsf{Im}\Big(\mathsf{Hom}_{\mathsf{Tri}(E)}\big(T_{\mathcal{P}^{v_1}}, T_{\mathcal{P}^{v_2}(y_1)}\big) \xrightarrow{\mathsf{For}} \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big)\Big)$. *Then the following diagram is commutative:*

$$\begin{tikzcd}
\widetilde{\Pi}^{v_1,v_2}_{y_1} \arrow[r] \arrow[d, "t_\sigma"'] & \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big) \arrow[d, "t_\sigma"] \\
\Pi^{v_1,v_2}_{y_1} \arrow[r] & \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(\tilde{y}_1)\big)
\end{tikzcd}$$

Collecting everything together, we get the following algorithm for computing associative r–matrices coming from a singular Weierestraß cubic curve E.

Algorithm 10.7. Let $\omega \in H^0(\Omega^{1,R}_E) \subset H^0(\Omega^{1,M}_{\mathbb{P}^1})$ be the global regular differential one form on $\mathbb{P}^1$ equal to $\dfrac{dz}{z}$ if E is nodal and dz if E is cuspidal. The linear morphism $\tilde{r}^{\xi}_{\omega}(v_1, v_2; y_1, y_2)$ can be computed in the following way.

- Compute the vector space $\widetilde{\Pi}^{v_1,v_2}_{y_1} \subseteq \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big)$.
- Consider the morphism $\overline{\mathrm{res}}_{y_1} : \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})$ given by

$$\overline{\mathrm{res}}_{y_1}(F) = \begin{cases} \dfrac{1}{y_1}F(1, y_1) & \text{if E is nodal} \\ F(1, y_1) & \text{if E is cuspidal.} \end{cases}$$

- If E is either nodal or cuspidal, we set $\mathrm{ev}_{y_2} : \mathsf{Hom}_{\mathbb{P}^1}\big(\widetilde{\mathcal{P}}, \widetilde{\mathcal{P}}(1)\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})$ to be given by the formula

$$\overline{\mathrm{ev}}_{y_2}(F) = \frac{1}{y_2 - y_1} F(1, y_2).$$

- The linear morphism $\tilde{r}^{\xi}(v_1, v_2; y_1, y_2) : \mathsf{Mat}_{n\times n}(\mathbb{C}) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})$ can be computed as the composition

$$\mathsf{Mat}_{n\times n}(\mathbb{C}) \xrightarrow{\overline{\mathrm{res}}_{y_1}^{-1}} \widetilde{\Pi}_{y_1}^{v_1,v_2} \xrightarrow{\overline{\mathrm{ev}}_{y_2}} \mathsf{Mat}_{n\times n}(\mathbb{C}).$$

10.3. **A trigonometric solution obtained from a nodal cubic curve.** Let E be a nodal Weierstraß cubic curve. In this subsection we calculate the associative r-matrices corresponding to the moduli spaces of rank two (semi-)stable vector bundles on a nodal Weierstraß curve. We use the notation from Subsection 9.2.

We start with the case of the moduli space of stable vector bundles of rank 2 and degree 1, $M = M_E^{(2,1)} = \mathbb{C}^*$. It is convenient to use the local homeomorphism $\sigma : \mathbb{C}^* \to \mathbb{C}^*$ given by $\sigma(z) = z^2$, because, according to Example 9.18 and Remark 9.26, the family of stable vector bundles $(1 \times \sigma)^*\mathcal{P}(2, 1)$ is then given by the triple $\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}_s^2, \mathsf{m}\big)$, where

$$\mathsf{m}(0) = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \ \lambda \in \mathbb{C}^* \quad \text{and} \quad \mathsf{m}(\infty) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Our goal is to compute the map

$$r_{y_1,y_2}^{\lambda_1,\lambda_2} : \mathsf{Mat}_{2\times 2}(\mathbb{C}) \longrightarrow \mathsf{Mat}_{2\times 2}(\mathbb{C}), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \varphi & \psi \\ \eta & \xi \end{pmatrix}.$$

Step 1. In order to calculate the entries φ, ψ, η, ξ we first need to describe the subspace $\Pi_{y_1}^{\lambda_1,\lambda_2} \subset \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)\big)$. Following the recipe of Subsection 9.2, we take the section $p_\zeta = z_1 - z_0$ to evaluate a morphism

$$F = \begin{pmatrix} a'z_0 + a''z_1 & t \\ b'z_0^2 + b''z_0z_1 + b'''z_1^2 & d'z_0 + d''z_1 \end{pmatrix} \in \mathsf{Hom}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\big).$$

This gives the following evaluation rule:

$$F(0) = \begin{pmatrix} -a' & t \\ b' & -d' \end{pmatrix}, \qquad F(\infty) = \begin{pmatrix} a'' & t \\ b''' & d'' \end{pmatrix}.$$

From the definition of the category of triples we see that F belongs to $\Pi_{y_1}^{\lambda_1,\lambda_2}$ if and only if there exists a matrix $\varphi \in \mathsf{Mat}_{2\times 2}(\mathbb{C})$ making the following diagram commutative:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{C}_0^2 \oplus \mathbb{C}_\infty^2 \arrow[r, "\begin{pmatrix} F(0) & 0 \\ 0 & F(\infty) \end{pmatrix}"] & \mathbb{C}_0^2 \oplus \mathbb{C}_\infty^2 \\
\mathbb{C}_0^2 \oplus \mathbb{C}_\infty^2 \arrow[u, "\begin{pmatrix} \left(\begin{smallmatrix} 0 & \lambda_1 \\ \lambda_1 & 0 \end{smallmatrix}\right) & 0 \\ 0 & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{pmatrix}"] \arrow[r, "\begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix}"] & \mathbb{C}_0^2 \oplus \mathbb{C}_\infty^2 \arrow[u, "\begin{pmatrix} \left(\begin{smallmatrix} 0 & \lambda_2 y_1 \\ \lambda_2 y_1 & 0 \end{smallmatrix}\right) & 0 \\ 0 & \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \end{pmatrix}"']
\end{tikzcd}$$

This is equivalent to the equations:

$$F(\infty) = \varphi \quad \text{and} \quad F(0)\begin{pmatrix} 0 & \lambda_1 \\ \lambda_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda_2 y_1 \\ \lambda_2 y_1 & 0 \end{pmatrix}\varphi.$$

Taking a'', b'', b''', d'' as free variables and solving the above system we get

$$\begin{cases} a' &= & -\lambda y_1 d'' \\ d' &= & -\lambda y_1 a'' \\ t &= & \lambda y_1 b''' \\ b' &= & (\lambda y_1)^2 b''', \qquad \text{where } \lambda = \dfrac{\lambda_2}{\lambda_1}. \end{cases}$$

Step 2. Next, the equation $\overline{\mathrm{res}}_{y_1}(F) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ reads as

$$\begin{pmatrix} a' + a'' y_1 & t \\ b' + b'' y_1 + b''' y_1^2 & d' + d'' y_1 \end{pmatrix} = y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From this we obtain

$$\begin{cases} a' &= & -\dfrac{\lambda y_1}{1-\lambda^2}(d + \lambda a) \\ a'' &= & \dfrac{1}{1-\lambda^2}(a + \lambda d) \\ b' &= & \lambda y_1^2 b \\ b'' &= & c - \dfrac{\lambda^2+1}{\lambda} y_1 b \\ b''' &= & \dfrac{1}{\lambda} b \\ d' &= & -\dfrac{\lambda y_1}{1-\lambda^2}(a + \lambda d) \\ d'' &= & \dfrac{1}{1-\lambda^2}(d + \lambda a) \\ t &= & y_1 b. \end{cases}$$

Step 3. By the formula for the evaluation map we get:

$$\overline{\mathrm{ev}}_{y_2}(F) = \frac{1}{y_2 - y_1}\begin{pmatrix} a' + a''y_2 & t \\ b' + b''y_2 + b'''y_2^2 & d' + d''y_2 \end{pmatrix} = \frac{1}{y}\begin{pmatrix} \varphi & \psi \\ \eta & \xi \end{pmatrix},$$

where we denote

$$\begin{cases} \varphi &= \dfrac{y_2 - \lambda^2 y_1}{1-\lambda^2}a + \dfrac{\lambda(y_2 - y_1)}{1-\lambda^2}d \\ \psi &= y_1 b \\ \xi &= \dfrac{\lambda(y_2 - y_1)}{1-\lambda^2}a + \dfrac{y_2 - \lambda^2 y_1}{1-\lambda^2}d \\ \eta &= \dfrac{(y_2 - y_1)(y_2 - \lambda^2 y_1)}{\lambda}b + y_2 c. \end{cases}$$

In order to calculate the corresponding solution of the associative Yang–Baxter equation we use the inverse of the canonical isomorphism

$$\mathsf{Mat}_{2\times 2}(\mathbb{C}) \otimes \mathsf{Mat}_{2\times 2}(\mathbb{C}) \longrightarrow \mathsf{Lin}\big(\mathsf{Mat}_{2\times 2}(\mathbb{C}), \mathsf{Mat}_{2\times 2}(\mathbb{C})\big)$$

given by $X \otimes Y \mapsto \mathrm{tr}(X \circ -)Y$. It is easy to see that under this inverse

$$\mathsf{Lin}\big(\mathsf{Mat}_{2\times 2}(\mathbb{C}), \mathsf{Mat}_{2\times 2}(\mathbb{C})\big) \longrightarrow \mathsf{Mat}_{2\times 2}(\mathbb{C}) \otimes \mathsf{Mat}_{2\times 2}(\mathbb{C})$$

a linear function $e_{ij} \mapsto \alpha_{ij}^{kl}e_{kl}, \alpha_{ij}^{kl} \in \mathbb{C}$ corresponds to the tensor $\alpha_{ij}^{kl}e_{ji} \otimes e_{kl}$. Having this rule in mind we obtain the desired associative r–matrix:

$$r(\lambda; y_1, y_2) = \frac{y_2 - \lambda^2 y_1}{(y_2 - y_1)(1-\lambda^2)}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) + \frac{\lambda}{1-\lambda^2}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11})$$
$$+ \frac{y_1}{y_2 - y_1}e_{21} \otimes e_{12} + \frac{y_2}{y_2 - y_1}e_{12} \otimes e_{21} + \frac{y_2 - \lambda^2 y_1}{\lambda}e_{21} \otimes e_{21}.$$

The gauge transformation $\varphi(z) = \varphi(\mu; z) : (\mathbb{C}^2, 0) \longrightarrow \mathsf{Aut}\big(\mathsf{Mat}_n(\mathbb{C})\big)$ (see Definition 2.5) given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{z} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{z}} & 0 \\ 0 & 1 \end{pmatrix}$$

yields the transformation

$$\begin{cases} e_{ii} \otimes e_{jj} \mapsto e_{ii} \otimes e_{jj}, \quad i, j \in \{1, 2\} \\ e_{21} \otimes e_{12} \mapsto \sqrt{\frac{y_2}{y_1}}e_{21} \otimes e_{12} \\ e_{12} \otimes e_{21} \mapsto \sqrt{\frac{y_1}{y_2}}e_{12} \otimes e_{21} \\ e_{21} \otimes e_{21} \mapsto \frac{1}{\sqrt{y_1 y_2}}e_{21} \otimes e_{21}. \end{cases}$$

Thus, we end up with the solution

$$r(\lambda, y) = \frac{y - \lambda^2}{(y-1)(1-\lambda^2)}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) + \frac{\lambda}{1-\lambda^2}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) +$$
$$+ \frac{\sqrt{y}}{y-1}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \left(\frac{\sqrt{y}}{\lambda} - \frac{\lambda}{\sqrt{y}}\right)e_{21} \otimes e_{21},$$

where $y = \dfrac{y_2}{y_1}$. Using the notation $\mathbb{1} = e_{11} + e_{22}$, this can be rewritten as

$$r(\lambda, y) = \frac{\mathbb{1} \otimes \mathbb{1}}{1-\lambda^2} + \frac{1}{y-1}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) - \frac{1}{\lambda+1}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11})$$
$$+ \frac{\sqrt{y}}{y-1}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \left(\frac{\sqrt{y}}{\lambda} - \frac{\lambda}{\sqrt{y}}\right) e_{21} \otimes e_{21}.$$

This is a solution of the associative Yang–Baxter equation of type (10), and by Theorem 2.15 this tensor also satisfies the quantum Yang–Baxter equation.

In order to rewrite $r(\lambda; y)$ in the additive form, we make the change of variables $y = \exp(2iz), \lambda = \exp(iv)$. Making a gauge transformation we can multiply the tensor $e_{21} \otimes e_{12}$ with an arbitrary scalar without changing the coefficients of the other tensors. Therefore, we obtain

$$2r_{\mathrm{trg}}(v, z) = \frac{\sin(z+v)}{\sin(z)\sin(v)}(e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) + \frac{1}{\sin(v)}(e_{11} \otimes e_{22} + e_{22} \otimes e_{11}) +$$
$$+ \frac{1}{\sin(z)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \sin(z+v) e_{21} \otimes e_{21}.$$

Up to a scalar, the corresponding solution $\bar{r}(z) := \lim_{v \to 0}(\mathrm{pr} \otimes \mathrm{pr}) r(v; z)$ of the classical Yang–Baxter equation is the trigonometric solution of Cherednik:

$$\bar{r}_{\mathrm{trg}}(z) = \frac{1}{2}\cot(z) h \otimes h + \frac{1}{\sin(z)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \sin(z) e_{21} \otimes e_{21}.$$

10.4. **Trigonometric solutions coming from semi-stable vector bundles.** Our next goal is to construct a solution $r(v; y)$ of the associative Yang–Baxter equation (8) having a higher-order pole with respect to v. The triple $\left(\mathcal{O}^2_{\mathbb{P}^1}, \mathbb{C}^2_s, \mathsf{m}\right)$ with

$$\mathsf{m}(0) = \begin{pmatrix} \lambda & \lambda \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{C}^* \quad \text{and} \quad \mathsf{m}(\infty) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

describes a universal family of semi-stable indecomposable vector bundles of rank two and degree one, having locally free Jordan-Hölder factors.

Step 1. First we compute the subspace $\Pi^{\lambda_1, \lambda_2}_{y_1} \subset \mathsf{Hom}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right)$. Recall that for a morphism

$$F = \begin{pmatrix} a'z_0 + a''z_1 & b'z_0 + b''z_1 \\ c'z_0 + c''z_1 & d'z_0 + d''z_1 \end{pmatrix} \in \mathsf{Hom}_{\mathbb{P}^1}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right)$$

we take the evaluation rule induced by the section $p_\zeta = z_1 - z_0$:

$$F(0) = -F' := -\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \quad F(\infty) = F'' := \begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}.$$

Thus, F belongs to $\Pi_{y_1}^{\lambda_1,\lambda_2}$ if and only if we have

$$F(0)\begin{pmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_2 y_1 & \lambda_2 y_1 \\ 0 & \lambda_2 y_1 \end{pmatrix} F(\infty).$$

This implies that

$$F' = -\lambda y_1 F'' + \lambda y_1 \begin{pmatrix} -c'' & a'' + c'' - d'' \\ 0 & c'' \end{pmatrix}.$$

<u>Step 2</u>. The equation $\overline{\operatorname{res}}_{y_1}(F) = y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ reads $F' + y_1 F'' = y_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Solving this equation we obtain

$$\begin{cases} a'' &= \dfrac{1}{1-\lambda} a + \dfrac{\lambda}{(1-\lambda)^2} c \\ b'' &= \dfrac{\lambda}{1-\lambda} a + \dfrac{1}{1-\lambda} b - \dfrac{\lambda(\lambda+1)}{(1-\lambda)^3} c + \dfrac{\lambda}{(1-\lambda)^2} d \\ c'' &= \dfrac{1}{1-\lambda} c \\ d'' &= -\dfrac{\lambda}{(1-\lambda)^2} c + \dfrac{1}{1-\lambda} d. \end{cases}$$

<u>Step 3</u>. From the formula $\overline{\operatorname{ev}}_{y_2}(F) = \dfrac{1}{y_2 - y_1}(F' + y_2 F'')$ we obtain:

$$\tilde{r}_{y_1,y_2}^{\lambda_1,\lambda_2} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varphi & \psi \\ \eta & \zeta \end{pmatrix},$$

where

$$\begin{cases} \varphi = \dfrac{y-\lambda}{(y-1)(1-\lambda)} a + \dfrac{\lambda}{(1-\lambda)^2} c \\ \eta = \dfrac{y-\lambda}{(y-1)(1-\lambda)} c \\ \xi = -\dfrac{\lambda}{(1-\lambda)^2} c + \dfrac{y-\lambda}{(y-1)(1-\lambda)} d \\ \psi = -\dfrac{\lambda}{(1-\lambda)^2} a + \dfrac{y-\lambda}{(y-1)(1-\lambda)} b - \dfrac{\lambda(1+\lambda)}{(1-\lambda)^3} c + \dfrac{\lambda}{(1-\lambda)^2} d \end{cases}$$

and $y = \dfrac{y_2}{y_1}$, $\lambda = \dfrac{\lambda_2}{\lambda_1}$. Hence, we obtain the associative r–matrix

$$r(\lambda; y) = \frac{y-\lambda}{(y-1)(1-\lambda)}\bigl(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{21}\otimes e_{12} + e_{12}\otimes e_{21}\bigr) + \\ + \frac{\lambda}{(1-\lambda)^2}\bigl(e_{12}\otimes h - h\otimes e_{12}\bigr) - \frac{\lambda(1+\lambda)}{(1-\lambda)^3} e_{12}\otimes e_{12}.$$

Denoting $y = \exp(2iz), \lambda = \exp(-2iv)$ and making a gauge transformation

$$e_{11} \mapsto e_{11}, e_{22} \mapsto e_{22}, e_{12} \mapsto 2e_{12} \text{ and } e_{21} \mapsto \frac{1}{2}e_{21}$$

we finally end up with an associative r–matrix

$$\begin{aligned} r(v;z) = &\frac{\sin(z+v)}{2\sin(z)\sin(v)}\bigl(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{21}\otimes e_{12} + e_{12}\otimes e_{21}\bigr) \\ &+ \frac{1}{2\sin^2(v)}\bigl(e_{12}\otimes h - h\otimes e_{12}\bigr) - \frac{\cos(v)}{\sin^3(v)}e_{12}\otimes e_{12}. \end{aligned}$$

Remark 10.8. Since $\lim\limits_{v\to 0}\bigl(\mathrm{pr}\otimes\mathrm{pr}(r(v;z))\bigr)$ does not exist, the family of indecomposable semi-stable vector bundles of rank two and degree zero on a nodal Weierstraß curve E, whose Jordan-Hölder factors are locally free, does not give a solution of the classical Yang–Baxter equation.

10.5. **A rational solution obtained from a cuspidal cubic curve.** In this subsection we shall calculate the rational solution of the classical Yang–Baxter equation, obtained from a universal family of stable vector bundles of rank 2 and degree 1 on a cuspidal cubic curve. In terms of Subsection 9.3 the universal family is described by the family of triples $\bigl(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1), \mathbb{C}^2_s, \mathsf{m}\bigr)$, where

$$\mathsf{m} = \mathsf{m}_0 + \varepsilon\mathsf{m}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \ \lambda\in\mathbb{C}.$$

As in the previous subsection, let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Mat}_{2\times 2}(\mathbb{C}) \quad\text{and}\quad \begin{pmatrix} \varphi & \psi \\ \eta & \xi \end{pmatrix} = \tilde{r}^{\lambda_1,\lambda_2}_{y_1,y_2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

<u>Step 1</u>. Again, we start by calculating the linear subspace $\Pi^{\lambda_1,\lambda_2}_{y_1} \subset \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\bigr)$. Recall that, in the case of a cuspidal curve, a morphism

$$F = \begin{pmatrix} a'z_0 + a''z_1 & t \\ b'z_0^2 + b''z_0z_1 + b'''z_1^2 & d'z_0 + d''z_1 \end{pmatrix} \in \mathsf{Hom}_{\mathbb{P}^1}\bigl(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(1), \mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\bigr)$$

is evaluated on the analytic subspace $\widetilde{Z}$ using the section $p_\zeta = z_1$. This gives the following evaluation rule:

$$F \mapsto \begin{pmatrix} a'' + a'\varepsilon & t \\ b''' + b''\varepsilon & d'' + d'\varepsilon \end{pmatrix}.$$

From the definition of the category $\mathsf{Tri}(E)$ we see that F belongs to $\Pi_{y_1}^{\lambda_1,\lambda_2}$ if and only if there exists a matrix $f \in \mathsf{Mat}_{2\times 2}(\mathbb{C})$ making the following diagram commutative

$$\begin{tikzcd}
\mathsf{R}^2 \arrow[rrr, "\begin{pmatrix} a'' & t \\ b''' & d'' \end{pmatrix} + \varepsilon \begin{pmatrix} a' & 0 \\ b'' & d' \end{pmatrix}"] & & & \mathsf{R}^2 \\
\mathsf{R}^2 \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}"] \arrow[rrr, "f"] & & & \mathsf{R}^2, \arrow[u, "\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} \lambda_2 - y_1 & 1 \\ 0 & \lambda_2 - y_1 \end{pmatrix}"']
\end{tikzcd}$$

where $\mathsf{R} = \mathbb{C}[\varepsilon]/\varepsilon^2$. This leads to the equality

$$\begin{pmatrix} a' & 0 \\ b'' & d' \end{pmatrix} + \begin{pmatrix} a'' & t \\ b''' & d'' \end{pmatrix} \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_2 - y_1 & 1 \\ 0 & \lambda_2 - y_1 \end{pmatrix} \begin{pmatrix} a'' & t \\ b''' & d'' \end{pmatrix}.$$

Taking a'', b', b''' and t as free variables we obtain

$$\begin{cases} a' = (\lambda - y_1)a'' + b''' \\ b'' = (\lambda - y_1)b''' \\ d' = (\lambda - y_1)a'' - b''' - (\lambda - y_1)^2 t \\ d'' = a'' - (\lambda - y_1)t. \end{cases}$$

<u>Step 2</u>. By the formula for the residue map $\overline{\mathrm{res}}_{y_1}$ we have:

$$\overline{\mathrm{res}}_{y_1}(F) = \begin{pmatrix} a' + a''y_1 & t \\ b' + b''y_1 + b'''y_1^2 & d' + d''y_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

from which we get:

$$\begin{cases} t &= b \\ a'' &= \dfrac{1}{2\lambda}a + \dfrac{\lambda - y_1}{2}b + \dfrac{1}{2\lambda}d \\ b' &= -\dfrac{\lambda y_1}{2}a + \dfrac{\lambda^2 y_1(\lambda - y_1)}{2}b + c + \dfrac{\lambda y_1}{2}d \\ b''' &= \dfrac{1}{2}a - \dfrac{\lambda(\lambda - y_1)}{2}b - \dfrac{1}{2}d \end{cases}$$

<u>Step 3</u>. Since the formula for the map $\overline{\mathrm{ev}}_{y_2}$ is given by:

$$\overline{\mathrm{ev}}_{y_2}(F) = \frac{1}{y_2 - y_1} \left(\begin{array}{cc} a' + a''y_2 & t \\ b' + b''y_2 + b'''y_2^2 & d' + d''y_2 \end{array} \right) = \frac{1}{y} \left(\begin{array}{cc} \varphi & \psi \\ \eta & \xi \end{array} \right)$$

we obtain:

$$\begin{cases} \varphi &= (1 + \frac{y_2 - y_1}{2\lambda})a + \frac{(\lambda - y_1)(y_2 - y_1)}{2}b + \frac{y_2 - y_1}{2\lambda}d \\ \psi &= t \\ \eta &= \frac{(y_2 - y_1)(y_2 - y_1 + \lambda)}{2}a - \frac{\lambda(\lambda - y_1)(\lambda + y_2)(y_2 - y_1)}{2}b + \\ & + c - \frac{(y_2 - y_1)(\lambda + y_2)}{2}d \\ \xi &= \frac{y_2 - y_1}{2\lambda}a - \frac{(y_2 - y_1)(y_1 - \lambda)}{2}b + \left(1 + \frac{y_2 - y_1}{2\lambda}\right) d. \end{cases}$$

From this we get the following associative r-matrix:

$$r(\lambda, y_1, y_2) = \frac{1}{2\lambda}\mathbb{1} \otimes \mathbb{1} + \frac{1}{y_2 - y_1}\Big(e_{11} \otimes e_{11} + e_{22} \otimes e_{22} + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\Big) + \\ + \frac{\lambda - y_1}{2}e_{21} \otimes h + \frac{\lambda + y_2}{2}h \otimes e_{21} - \frac{\lambda(\lambda - y_1)(\lambda + y_2)}{2}e_{21} \otimes e_{21}. \tag{37}$$

Projecting this matrix to $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathfrak{sl}_2(\mathbb{C})$ we obtain the rational solution of the classical Yang–Baxter equation

$$\bar{r}(y_1, y_2) = \frac{1}{y_2 - y_1}\left(\frac{1}{2}h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\right) + \frac{y_2}{2}h \otimes e_{21} - \frac{y_1}{2}e_{21} \otimes h, \tag{38}$$

found for the first time by Stolin in [62]. It is easy to check that $\bar{r}(y_1, y_2)$ does not have infinitesimal symmetries, hence by Theorem 2.15 the tensor $r(\lambda, y_1, y_2)$ satisfies the Quantum Yang–Baxter equation. This solution was recently found by Khoroshkin, Stolin and Tolstoy [39].

Remark 10.9. By a result of Belavin and Drinfeld [9] it is known that the r–matrix (38) is equivalent to a solution depending on the difference of spectral parameters. However, we were not able to find the corresponding gauge transformation in the literature and the following form of Stolin's solution seems to be new. Consider the gauge transformation $\phi : (\mathbb{C}, 0) \longrightarrow \mathsf{Aut}\big(\mathfrak{sl}_2(\mathbb{C})\big)$ given by the formula

$$\phi(y)h = h - 2y^2 e_{21}, \quad \phi(y)e_{12} = -\frac{y^2}{2}h + \frac{1}{4}e_{12} - \frac{y^4}{4}e_{21} \quad \text{and} \quad \phi(y)e_{21} = 4e_{21}.$$

Then we have: $\big(\phi(y_1) \otimes \phi(y_2)\big)r(y_1, y_2) = s(y_2 - y_1)$, where

$$s(y) = \frac{1}{y}\Big(\frac{1}{2}h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\Big) + y(e_{21} \otimes h + h \otimes e_{21}) - y^3 e_{21} \otimes e_{21}. \tag{39}$$

Remark 10.10. For any $t \in \mathbb{C}^*$ consider the constant gauge transformation

$$e_{11} \mapsto e_{11}, \quad e_{22} \mapsto e_{22}, \quad e_{21} \mapsto t\, e_{21}, \quad e_{12} \mapsto \frac{1}{t}\, e_{12}.$$

Then the associative r–matrix (37) transforms into the solution

$$r_t(\lambda, y_1, y_2) = \frac{1}{2\lambda}\mathbb{1}\otimes\mathbb{1} + \frac{1}{y_2 - y_1}\Bigl(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\Bigr) + \\ + t\Bigl(\frac{\lambda - y_1}{2} e_{21}\otimes h + \frac{\lambda + y_2}{2} h\otimes e_{21}\Bigr) + \frac{t^2\lambda(\lambda - y_1)(\lambda + y_2)}{2} e_{21}\otimes e_{21}.$$

Taking the limit $t \to 0$, we get the following solution of the associative Yang–Baxter equation (12):

$$r(\lambda, y) = \frac{1}{2\lambda}\mathbb{1}\otimes\mathbb{1} + \frac{1}{y}\bigl(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\bigr). \tag{40}$$

Note that the corresponding solution of the classical Yang–Baxter equation (3) is the rational solution of Yang.

11. Summary

Let us summarise the main analytical results obtained is this article. We have shown that for any pair of coprime integers $0 < d < n$ and an irreducible reduced projective curve E with trivial dualizing sheaf one can *canonically* attach the germ of a tensor-valued function

$$r_E^{(n,d)}(v; y_1, y_2) : (\mathbb{C}^3, 0) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

satisfying the associative Yang–Baxter equation

$$r(u; y_1, y_2)^{12} r(u+v; y_2, y_3)^{23} = r(u+v; y_1, y_3)^{13} r(-v; y_1, y_2)^{12} \\ + r(v; y_2, y_3)^{23} r(u; y_1, y_3)^{13}.$$

By Proposition 2.9, the tensor $r_E^{(n,d)}(v; y_1, y_2)$ defines a family of commuting first order differential operators. Moreover, under certain conditions (which are always fulfilled at least for elliptic curves and for nodal cubic curves) it also satisfies the quantum Yang–Baxter equation with respect to the spectral variables y_1 and y_2 and a fixed value of $v \neq 0$. For $(n, d) = (2, 1)$ these tensors $r_E^{(2,1)}$ have the following explicit form

- For an elliptic curve $E = E_\tau = \mathbb{C}/\langle 1, \tau\rangle$ we get

$$r_{\mathrm{ell}}^{(2,1)}(v; y) = \frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\left[\frac{\theta_1(y+v|\tau)}{\theta_1(v|\tau)}\mathbb{1}\otimes\mathbb{1} + \frac{\theta_2(y+v|\tau)}{\theta_2(v|\tau)} h\otimes h + \right. \\ \left. + \frac{\theta_3(y+v|\tau)}{\theta_3(v|\tau)}\sigma\otimes\sigma + \frac{\theta_4(y+v|\tau)}{\theta_4(v|\tau)}\gamma\otimes\gamma\right],$$

where $\mathbb{1} = e_{11} + e_{22}$, $h = e_{11} - e_{22}$, $\sigma = i(e_{21} - e_{12})$ and $\gamma = e_{21} + e_{12}$. This solution is a quantization of the elliptic solution of the classical Yang–Baxter equation

$$\bar{r}_{\mathrm{ell}}^{(2,1)}(y) = \frac{1}{2}\left(\frac{\mathrm{cn}(y)}{\mathrm{sn}(y)} h\otimes h + \frac{1}{\mathrm{sn}(y)}\gamma\otimes\gamma + \frac{\mathrm{dn}(y)}{\mathrm{sn}(y)}\sigma\otimes\sigma\right).$$

studied by Baxter, Belavin and Sklyanin.
• For the plane nodal cubic curve $E = V(zy^2 - x^3 - zx^2) \subset \mathbb{P}^2$ we get a trigonometric solution

$$r_{\mathrm{trg}}^{(2,1)}(v,y) = \frac{\sin(y+v)}{\sin(y)\sin(v)}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}) + \frac{1}{\sin(v)}(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}) + \\ + \frac{1}{\sin(y)}(e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + \sin(y+v)e_{21}\otimes e_{21}.$$

This solution is a quantization of the trigonometric solution of Cherednik:

$$\bar{r}_{\mathrm{trg}}^{(2,1)}(y) = \frac{1}{2}\cot(z) h\otimes h + \frac{1}{\sin(y)}(e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + \sin(y)e_{21}\otimes e_{21}.$$

• For the cuspidal nodal cubic curve $E = V(zy^2 - x^3) \subset \mathbb{P}^2$ we get a rational solution

$$r_{\mathrm{rat}}^{(2,1)}(v,y_1,y_2) = \frac{1}{v}\mathbb{1}\otimes\mathbb{1} + \frac{2}{y_2 - y_1}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + \\ + (v - y_1)e_{21}\otimes h + (v + y_2)h\otimes e_{21} - v(v-y_1)(v+y_2)e_{21}\otimes e_{21}.$$

This solution of the quantum Yang–Baxter equation is a quantization of the rational solution of Stolin

$$\bar{r}(y_1,y_2) = \frac{1}{y_2 - y_1}\left(\frac{1}{2}h\otimes h + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\right) + \frac{y_2}{2}h\otimes e_{21} - \frac{y_1}{2}e_{21}\otimes h.$$

Note that this solution is gauge equivalent to the solution

$$s(y) = \frac{1}{y}\Big(\frac{1}{2}h\otimes h + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\Big) + y(e_{21}\otimes h + h\otimes e_{21}) - y^3 e_{21}\otimes e_{21}$$

depending on the difference of spectral parameters.
• Next, the following solution $r_{\mathrm{rat-deg}}^{(2,1)}(v;y)$ of the associative Yang–Baxter equation is a degeneration of the rational solution $r_{\mathrm{rat}}^{(2,1)}(v,y_1,y_2)$:

$$r_{\mathrm{rat-deg}}^{(2,1)}(v,y) = \frac{1}{2v}\mathbb{1}\otimes\mathbb{1} + \frac{1}{y}\big(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\big).$$

The corresponding solution of the classical Yang–Baxter equation is the rational solution of Yang:

$$r_{\mathrm{rat-deg}}^{(2,1)}(y) = \frac{1}{y}\Big(\frac{1}{2}h\otimes h + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\Big).$$

• In the case of a nodal cubic curve E, the universal family of *indecomposable semi-stable* vector bundles of rank 2 and degree 0 having *locally free* Jordan-Hölder factors gives the following solution of the associative Yang–Baxter equation:

$$r_{\mathrm{trg}}^{(2,0)}(v;y) = \frac{\sin(y+v)}{2\sin(y)\sin(v)}\bigl(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{21}\otimes e_{12} + e_{12}\otimes e_{21}\bigr) + \frac{1}{2\sin^2(v)}\bigl(e_{12}\otimes h - h\otimes e_{12}\bigr) - \frac{\cos(v)}{\sin^3(v)} e_{12}\otimes e_{12}.$$

This solution has higher order poles with respect to the spectral variable v and does not project to a solution of the classical Yang–Baxter equation. However, it still yields a family of commuting Dunkl operators.

The second analytic application of methods developed in this article, is the following. Consider the Weierstraß family of plane cubic curves $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$, where $g_2, g_3 \in \mathbb{C}$. Recall the following classical result.

Proposition 11.1 (see Section II.4 in [36])**.** *Let $\tau \in \mathbb{C}\setminus\mathbb{R}$ and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}^2$ be the corresponding lattice. Then the complex torus $\mathbb{C}/\Lambda_\tau$ is isomorphic to the projective cubic curve $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$, where*

(41)
$$g_2 = 60 \sum_{(m',m'')\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m'+m''\tau)^4}, \quad g_3 = 140 \sum_{(m',m'')\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m'+m''\tau)^6}.$$

Conversely, for any pair $(g_2, g_3) \in \mathbb{C}^2$ such that $\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \neq 0$ there exists a unique τ from the domain D given below, such that $(g_2, g_3) = \bigl(g_2(\tau), g_3(\tau)\bigr)$.

$$D = \left\{ \tau \in \mathbb{C} \;\middle|\; |\mathrm{Re}(\tau)| \le \frac{1}{2}, \quad |\tau| \ge 1 \;\; \textit{if} \;\; \mathrm{Re}(\tau) \le 0, \quad |\tau| > 1 \;\; \textit{if} \;\; \mathrm{Re}(\tau) > 0 \right\}$$

Let $(n,d) \in \mathbb{N}\times\mathbb{Z}$ be a pair of coprime integers and $M = M_{E/T}^{(n,d)} \cong \breve{E}$ be the moduli space of relatively stable vector bundles on E of rank n and degree d with universal family $\mathcal{P}(n,d)$. Let $t = (g_2, g_3)$ and $o = \bigl((0:1:0), t\bigr) \in E$, $m \in M$ be the point corresponding to o and ξ be some trivialization of $\mathcal{P}(n,d)$ in a neighbourhood of $(o,m) \in E\times_T M$ and $\omega \in H^0(\omega_{E/T})$ be a nowhere vanishing regular one-form. Then we get the germ of a meromorphic function

$$r^\xi := \bigl(r_{E/T}^{(n,d)}(\omega)\bigr)^\xi : \bigl(M\times_T M\times_T \breve{E}\times_T\breve{E}, \hat{o}\bigr) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})\times\mathsf{Mat}_{n\times n}(\mathbb{C})$$

which satisfies the associative Yang–Baxter equation

$$r^\xi(t;v_1,v_2;y_1,y_2)^{12} r^\xi(t;v_1,v_3;y_2,y_3)^{23} = r^\xi(t;v_1,v_3;y_1,y_3)^{13} r^\xi(t;v_3,v_2;y_1,y_2)^{12} + r^\xi(t;v_2,v_3;y_2,y_3)^{23} r^\xi(t;v_1,v_2;y_1,y_3)^{13}$$

and its "dual"

$$r^\xi(t;v_2,v_3;y_1,y_2)^{23} r^\xi(t;v_1,v_3;y_1,y_2)^{12} = r^\xi(t;v_1,v_2;y_1,y_2)^{12} r^\xi(t;v_2,v_3;y_1,y_3)^{13} + r^\xi(t;v_1,v_3;y_1,y_3)^{13} r^\xi(t;v_2,v_1;y_2,y_3)^{23}.$$

Moreover, it fulfills the unitarity condition

$$r^{\xi}(t; v_1, v_2; y_1, y_2) = -\tau\bigl(r^{\xi}(t; v_2, v_1; y_2, y_1)\bigr),$$

where $\tau(a \otimes b) = b \otimes a$. The function $r^{\xi}(t; v_1, v_2; y_1, y_2)$ depends analytically on the parameter $t \in T$ and its poles lie on the hypersurfaces $v_1 = v_2$ and $y_1 = y_2$.

Next, different choices of trivializations of the universal family $\mathcal{P}$ lead to equivalent solutions: if ζ is another trivialization of $\mathcal{P}$ and $\phi = \zeta \circ \xi^{-1} : (M \times_T E, o) \longrightarrow \mathsf{GL}_n(\mathbb{C})$ is the corresponding holomorphic function, then we have:

$$r^{\zeta} = \bigl(\phi(t, v_1, y_1) \otimes \phi(t, v_2, y_2)\bigr) r^{\xi}\bigl(\phi(t, v_2, y_1)^{-1} \otimes \phi(t, v_1, y_2)^{-1}\bigr).$$

We have shown that $r_{\mathrm{ell}}^{(2,1)}(v; y)$ is equivalent to the solution $r_{E/T}^{(2,1)}(t; v; y)$ for all $t = (g_2, g_3)$ such that $\Delta(t) \neq 0$. This equivalence relation is generated by the gauge transformations and coordinate changes.

The trigonometric solution $r_{\mathrm{trg}}^{(2,1)}(v; y)$ is equivalent to the solution $r_{E/T}^{(2,1)}(t; v; y)$ for $t = (g_2, g_3) \neq (0, 0)$ but such that $\Delta(t) = 0$. Finally, the rational solution $r_{\mathrm{rat}}^{(2,1)}(v; y_1, y_2)$ is equivalent to the solution $r_{E/T}^{(2,1)}\bigl((0, 0); v; y\bigr)$.

In other words, we get the following result, which seems to be difficult to show by direct computations: the rational solution $r_{\mathrm{rat}}^{(2,1)}(v; y_1, y_2)$ is *equivalent* to a solution, which is a *degeneration* of the trigonometric solution $r_{\mathrm{trg}}^{(2,1)}(v; y)$ and of the elliptic solution $r_{\mathrm{ell}}^{(2,1)}(v; y)$.

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