MATHEMATICAL INSTITUTE
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Winter 2009/10
February 9

Mathematical Methods for Economists
5th Assignment

Exercise 31 (Definiteness)

Examine the definiteness of the following quadratic forms:

a) $Q = 3x_1^2 + 6x_1x_3 + x_2^2 - 4x_2x_3 + 8x_3^2$ b) $Q = -x_1^2 + 6x_1x_2 - 9x_2^2 - 2x_3^2$

Exercise 32 (Concave Function)

Show that $f(x, y) = 1 - x^2$ is concave by definition. Is f strictly concave?

Exercise 33 (Strictly Concave Function)

Consider $f(x, y, z) = -x^2 - 2y^2 - z - xy - e^{x+2y+2z} + 1$. Show that f is strictly concave.

Exercise 34 (Quasi-Concave Function)

Consider the function $f(x, y) = e^{-x^2-y^2}$; that is, f is proportional to the bivariate normal distribution function. Show that f is quasi-concave.

Exercise 35 (Cobb Douglas Function)

Show that the Cobb-Douglas function $z = cx_1^{a_1} \cdots x_n^{a_n}$, defined for $x_i > 0$, with $c, a_i > 0$, is homogenous of degree $\sum_{i=1}^n a_i$. Moreover, prove that the following statements are true:

- z is quasi-concave for all a_i
- z is concave for $\sum_{i=1}^n a_i \leq 1$
- z is strictly concave for $\sum_{i=1}^n a_i < 1$

Exercise 36 (CES Function)

The generalized CES function is defined for $x_i > 0,\ i = 1, ..., n$ by

$$z = c(d_1x_1^{-\rho} + ... + d_nx_n^{-\rho})^{-\mu/\rho}$$

$c, \mu, d_i > 0,\ i = 1, ..., n,\ \rho \neq 0$. Show that z is homogenous of degree μ, quasi-convex for $\rho \leq -1$, quasi-concave for $\rho \geq -1$, and concave for $\mu \leq 1,\ \rho \geq -1$.

Exercise 37 (Concave Functions)

Use the properties of the generalized CES function and the Cobb-Douglas function to examine the concavity of the following functions:

a) $F(x,y) = 20x^{2/3}y^{1/4}$ b) $F(x,y) = 10x^{1/3}y^{2/3}$ c) $F(x,y) = x^2y^4$

d) $F(x,y) = (x^{-1/4} + y^{-1/4})^{-3/4}$ e) $F(x,y) = (x^{1/3} + y^{1/3})^3$

Exercise 38 (Minimization)

Consider the function $f(x,y,z) = x^2 + 2y^2 + 3z^2 + xy + 2xz$. Find a global minimum of f.

Exercise 39 (Maximization)

Consider a competitive firm with the following profit function:

$$\pi = PQ - wL - rC$$

with price P, output Q, labor L, capital C; w and r are the input prices for L and C, respectively. Suppose that the firm operates in a competitive market; the exogenous variables are P,w,r, and the endogenous variables are C,L,Q. Moreover, the production function is given by

$$Q = Q(C,L) = L^{\alpha}C^{\beta},\ \alpha,\beta > 0$$

with $\alpha + \beta < 1$ (decreasing returns to scale), $\alpha = \beta < 0.5$. Determine the maximum point for the profit function.

Exercise 40 (Maximization)

Consider a firm which produces two goods, X and Y. The cost per day is given by

$$C(x,y) = 0.04x^2 - 0.02xy + 0.01y^2 + 5x + 3y + 600$$

when x units of X and y units of Y are produced ($x > 0,\ y > 0$). The firm sells its products for 14$ per unit of X and 9$ per unit of Y. Determine the profit function and sketch it. What are the values of x and y at which the profit function attains its maximum?

Exercise 41

Prove Corollary 2.4.16 stated in the lecture.