SS 2022 Prof. Hansjörg Geiges
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Geometrische Topologie

Übungsblatt 4

Aufgabe 1. (a) Write the torus knot $T(p,q)$ (see Übungsblatt 3) as the closure of a suitable braid. Use this description to show that $T(2,3)$ and $T(3,2)$ are isotopic to the (left- or right-handed?) trefoil knot (Kleeblattknoten). How does one obtain the other trefoil as the closure of a braid?

(b) Give an example of a braid $b \in B_n$ with the property that $\sigma(b) \in S_n$ is a permutation of order n, but the closure $\beta(b)$ is not a knot. (There is an example with $n = 6$.)

(c) Show that $b := b_1 b_2^{-1} b_3^2 b_4^{-2} b_2^{-1} b_1 \in B_5$ is a pure braid, i.e. $b \in K_5$. Write b as a product of the generators b_{ij} (and their inverses) of K_5.

Aufgabe 2. Let $f_i, g_i \colon X \to X$, $i = 0, 1$, be homeomorphisms of a topological space X, with f_0 isotopic to f_1, and g_0 isotopic to g_1. Show that $f_0 \circ g_0$ is isotopic to $f_1 \circ g_1$. Use this to explain how one obtains a group structure on the set of isotopy classes of homeomorphisms $X \to X$. (In the lectures we used this, for example, to define the group H_n of isotopy classes of homeomorphisms of D^2 with n open discs removed.)

Aufgabe 3. Let p, q be coprime integers with $p \geq 2$. Regard S^3 as the unit sphere in $\mathbb{C}^2$. Show the following:

(a) The map

$$\sigma(z,w) = \left(\mathrm{e}^{2\pi \mathrm{i}/p} z, \mathrm{e}^{2\pi \mathrm{i} q/p} w\right)$$

defines an action of the cyclic group $\mathbb{Z}_p$ on S^3 with generator σ.

(b) This action is fixed point free.

(c) The quotient $L(p,q) := S^3/\mathbb{Z}_p$ is a 3-manifold. This manifold is called a **lens space** (Linsenraum). Notice that $L(p,q)$ depends only on p and q mod p.

(d) The lens space $L(p,q)$ admits a Heegaard decomposition of genus 1. Describe the gluing map of this Heegaard decomposition.

(e) There is an orientation-reversing homeomorphism $L(p,q) \to L(p,-q)$.

Aufgabe 4. On the 2-torus $T^2 = S^1 \times S^1$ we denote the curve $S^1 \times \{*\}$ by μ, and the curve $\{*\} \times S^1$ by λ. Here $*$ is a chosen point of S^1. The fundamental group $\pi_1(T^2)$ with base point $(*, *)$ can be identified with $\mathbb{Z} \oplus \mathbb{Z}$, generated by $[\mu]$ and $[\lambda]$.

(a) Describe the effect on the fundamental group of a Dehn twist along $\mu' := S^1 \times \{-*\}$ and $\lambda' := \{-*\} \times S^1$, respectively.

(b) Let $h \mapsto h_*$ be the map that associates with every homeomorphism of T^2, fixing the base point $(*, *)$, the induced homomorphism h_* on the fundamental group. Show that this defines a surjective homomorphism

$$\mathrm{Homeo}(T^2) \longrightarrow \mathrm{GL}(2, \mathbb{Z}).$$

(In fact, it can be shown that this is an isomorphism.)

Abgabe: Mittwoch 4.5.22
bis spätestens 17:00 Uhr im Raum 206/7 des MI