SS 2022

Prof. Hansjörg Geiges
Tilman Becker, M.Sc.
Dr. Rima Chatterjee

Geometrische Topologie

Übungsblatt 5

Aufgabe 1. Let $L(p,q)$ be a lens space as defined on Übungsblatt 4.

(a) Use the theorem of Seifert–van-Kampen to show that $\pi_1(L(p,q)) \cong \mathbb{Z}_p$.

(b) Give an alternative proof of this fact, using the covering $S^3 \to L(p,q)$. To this end, convince yourself that $\pi_1(L(p,q))$ is generated by the homotopy class of the loop $t \mapsto (\mathrm{e}^{2\pi \mathrm{i} t/p}, 0)$, $t \in [0,1]$.

Aufgabe 2. (a) Let σ be a generator of the cyclic group $\mathbb{Z}_p$. Show that for any $q' \in \mathbb{N}$ coprime with p, the element $\sigma^{q'}$ is likewise a generator of $\mathbb{Z}_p$.

(b) Let $q, q' \in \mathbb{N}$ with $qq' \equiv \pm 1 \mod p$. Show that $L(p,q)$ and $L(p,q')$ are homeomorphic

(i) with the help of part (a) and Aufgabe 3, Übungsblatt 4;

(ii) with the description of $L(p,q)$ via a Heegaard splitting. (Hint: exchange the roles of the two solid tori.)

Remark. More generally, the following is true:

1. $L(p,q)$ is homeomorphic to $L(p,q')$ if and only if

$$\pm q' \equiv q^{\pm 1} \mod p.$$

   (The direction "$\Leftarrow$" follows from the considerations above.)

2. $L(p,q)$ is homotopy equivalent to $L(p,q')$ if and only if $\pm qq'$ is a quadratic residue modulo p, that is, $\pm qq' \equiv m^2 \mod p$ for some integer m.

For instance, $L(7,1)$ and $L(7,2)$ have the same homotopy type, but they are not homeomorphic.

Aufgabe 3. Let D^3 be the closed unit ball in $\mathbb{R}^3$, and let p, q be as in Aufgabe 1. Identify each point on the lower hemisphere ($z \leq 0$) of ∂D^3 with the point on the upper hemisphere obtained by a rotation through $2\pi q/p$ about the z-axis and reflection in the xy-plane. Let $M(p, q)$ be the resulting 3-manifold.

(a) Show that the cylinder

$$Z = D^3 \cap \left\{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 \leq 1/2\right\}$$

becomes a solid torus under this identification.

(b) Cut the complement $D^3 \setminus Z$ suitably into p pieces to recognize this complement, after the boundary identification, as another solid torus.

(c) Deduce that $M(p, q) = L(p, q)$.

Aufgabe 4. Think of the 3-torus T^3 as the manifold obtained by identifying opposite sides of a cube in $\mathbb{R}^3$. Show that T^3 has a Heegaard decomposition of genus 3.

Abgabe: Mittwoch 11.5.22
bis spätestens 17:00 Uhr im Raum 206/7 des MI