Prof. Hansjörg Geiges
Tilman Becker, M.Sc.
Dr. Rima Chatterjee

SS 2022

Geometrische Topologie

Übungsblatt 7

Aufgabe 1. (a) The **Klein bottle** is the closed, non-orientable surface obtained by gluing two Möbius bands along their boundary circle. Show that the Klein bottle is the boundary of a compact 3-manifold.

(b) Show that, alternatively, the Klein bottle can be obtained as the connected sum of two real projective planes $\mathbb{R}\mathrm{P}^2$. Here $\mathbb{R}\mathrm{P}^2$ is the quotient space $S^2/x \sim -x$; the **connected sum** is obtained by removing an open disc from each copy of $\mathbb{R}\mathrm{P}^2$, and gluing the remaining pieces along their boundary circle.

(c) Use (b) to give an alternative proof of (a).

(d) (optional; requires Algebraic Topology) Show that $\mathbb{R}\mathrm{P}^2$ is *not* the boundary of a compact 3-manifold.

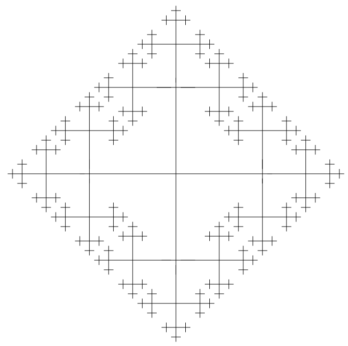
Hint: Suppose $\mathbb{R}\mathrm{P}^2$ were the boundary of the compact 3-manifold M. Then one could form the closed 3-manifold $M \cup_{\mathbb{R}\mathrm{P}^2} M$. The corresponding Mayer–Vietoris sequence (use $\mathbb{Z}_2$-coefficients!) leads to a contradiction. Alternatively, using a triangulation and its dual triangulation, one can see that every odd-dimensional manifold has Euler characteristic $\chi = 0$. On the other hand, we would have $\chi(\mathbb{R}\mathrm{P}^2) \equiv \chi(M \cup_{\mathbb{R}\mathrm{P}^2} M) \bmod 2$.

Aufgabe 2. In the proof of Satz 5.1' we had tacitly assumed that the surface F has genus $g \geq 1$ or $k \geq 1$ boundary components, so that after suitable cutting we obtain D^2_{k+2g-1} with $k+2g-1 \geq 0$.

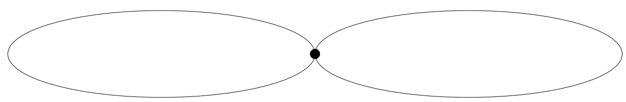
In the case $g = 0 = k$, i.e. the 2-sphere, one may argue with the theorem of Baer. This says that two homotopic simple closed curves on a surface (or its interior, in case there is boundary) are actually ambient isotopic. In other words, there is a homeomorphism of the surface, isotopic to the identity, that maps one curve to the other.

Use this to prove Satz 5.1' for S^2.

Aufgabe 3. Let $\widetilde{X}$ be the infinite tree as indicated in the figure.



Let $X = S^1 \vee S^1$ be the one-point union of two copies of S^1



Define $p\colon \widetilde{X} \to X$ as follows. All vertices of $\widetilde{X}$ are mapped to the common point of the two circles. The horizontal branches are mapped (for increasing x) in the mathematical positive sense to the first circle, the vertical branches (for increasing y), to the second circle. Show that this defines a covering. Is it regular? If yes, what group acts on $\widetilde{X}$ such that the orbit space equals X?

Aufgabe 4. Verify the Riemann–Hurwitz formula for the branched coverings $\Sigma_g \to S^2$ with three branch points constructed in the lectures, that is, both for the explicitly constructed one (using symmetries of Σ_g) and for the one coming from an arbitrary triangulation of Σ_g.

Abgabe: Mittwoch 25.5.22

bis spätestens 17:00 Uhr im Raum 206/7 des MI