A boundary layer solution to a semilinear elliptic system of FitzHugh-Nagumo type

Carolus Reinecke and Guido Sweers

Abstract – In this note we study the semilinear elliptic system (P_λ) on a bounded domain, where $\lambda, \delta, \gamma > 0$ and f has two consecutive zeroes, for example $f(u) = u(u-a)(1-u)$ with $0 < a < 1/2$. The existence, for λ large, of a positive solution with boundary layer behavior as well as uniqueness of this solution in an order interval are proven.

Une solution avec couche limite d'un système elliptique semilinéaire du type FitzHugh-Nagumo

Résumé – *Dans cette note nous considérons le système elliptique semilinéaire* (P_λ) *sur un domaine borné, où* $\lambda, \delta, \gamma > 0$ *et* f *possède deux zéros consécutifs, par exemple* $f(u) = u(1-u)(u-a)$ *avec* $0 < a < \frac{1}{2}$. *Nous établissons l'existence et l'unicité dans un intervalle d'ordre d'une solution avec couche limite pour* λ *suffisamment grand.*

Version française abrégée – Considérons le problème aux valeurs propres nonlinéaire

$$\begin{cases} -\Delta u = \lambda(f(u) - v) & \text{dans } \Omega, \\ -\Delta v = \lambda(\delta u - \gamma v) & \text{dans } \Omega, \\ u = v = 0 & \text{sur } \Gamma = \partial\Omega, \end{cases} \qquad (\mathrm{P}_\lambda)$$

où $\lambda, \gamma, \delta > 0$ et Ω est un domaine régulier, borné de $\mathbb{R}^n$. Nous supposons que la fonction f satisfait les hypothèses suivantes:

(A) $f \in C^{1,1}(\mathbb{R})$ *avec* $f(0) \geq 0$. *De plus il existe* $\sigma_1 > 0$ *tel que pour tout* $0 \leq \sigma < \sigma_1$ *il existe deux nombres* $\rho_\sigma^- < \rho_\sigma^+$ *avec* $\rho_\sigma^+ > 0$ *tels que*

- (i) $f(\rho_\sigma^\pm) = \sigma\rho_\sigma^\pm$ *et* $f(\rho) > \sigma\rho$ *pour* $\rho_\sigma^- < \rho < \rho_\sigma^+$.
- (ii) $f'(\rho) < 0$ *pour* $\rho \in (\rho_{\sigma_1}^+, \rho_0^+)$.
- (iii) $J_\sigma(\rho) := \int_\rho^{\rho_\sigma^+} (f(s) - \sigma s)\, ds > 0$ *sur* $[0, \rho_\sigma^+)$ *pour tout* $\sigma \in [0, \sigma_1)$.

La fonction $f(u) = u(a-u)(u-1)$ avec $0 < a < 1/2$ et $\sigma_1 = (2a^2 - 5a + 2)/9$ satisfait la condition (A). Dans ce cas, le système (P_λ) est une extension de l'equation de FitzHugh-Nagumo (cf. [4], [6]).

Klaasen et Mitidieri ([6]) ont démontré par un argument variationnel qu'il existe des solutions nontriviales pour $\lambda = 1$ si δ/γ est suffisamment petit et Ω contient une boule suffisamment grande, ce qui par changement d'échelle correspond à λ grand. Dans cette note, nous établissons non seulement l'existence d'une solution (u_λ, v_λ) pour λ grand comme dans [6], mais aussi, ce qui est beaucoup plus difficile, l'existence d'une courbe régulière de solutions voisines de $(\rho_{\delta/\gamma}^+, (\delta/\gamma)\,\rho_{\delta/\gamma}^+)$ en dehors d'une couche limite d'ordre $\sqrt{\lambda^{-1}}$, paramétrisée par λ en démontrant que le théorème des fonctions implicites est applicable dans un espace fonctionnel approprié. Dans le cas où $\delta = 0$, le système (P_λ) correspond à l'equation $-\Delta u = \lambda f(u)$ dans Ω et $u = 0$

sur Γ, qui a été traité par exemple dans [1], [2] et [3]. Pour une discussion générale de cette équation voir [8].

Une étape importante dans notre analyse consiste à transformer (P_λ) et à modifier la fonction f de façon à obtenir un système quasimonotone. Une telle transformation a été utilisée dans [9]. Dans le cas quasimonotone il est possible d'utiliser un principe de comparaison ([13], [11]) de balayage analogue à celui de MacNabb ([10], [2]). Pour obtenir cette transformation nous imposons la condition suivante pour les paramètres δ et γ :

(B1) $\gamma - 2\sqrt{\delta} > m$ *où* $m := \max\left\{-f'(s)\ ;\ 0 \le s \le \rho_0^+\right\}$.

Pour δ et γ fixés tels que (B1) soit satisfaite nous définissons

$$\beta := \frac{1}{2}(\gamma - m) - \frac{1}{2}\sqrt{(\gamma - m)^2 - 4\delta}. \tag{1}$$

Soit $\alpha := \gamma - \beta$. Il s'en suit que $-\beta(\beta + m) = \delta - \gamma\beta$ et que $\alpha, \beta > 0$. Soit (u, w) est une solution de

$$\left\{\begin{array}{rcll} -\Delta u &=& \lambda(f(u) - \beta u + \beta w) & \text{dans } \Omega, \\ -\Delta w &=& \lambda(f(u) + mu - \alpha w) & \text{dans } \Omega, \\ u &=& w = 0 & \text{sur } \Gamma, \end{array}\right. \tag{Q_λ}$$

alors (u, v) est une solution de (P_λ) avec $v := \beta u - \beta w$ et réciproquement. Puisque nous sommes intéressés par des solutions (u, v) de (P_λ) où u est positive et $\max u \le \rho_{\delta/\gamma}^+$ nous pouvons, sans perte de généralité, modifier la fonction f en dehors de l'intervalle $[0, \rho_0^+]$ de telle sorte que la condition suivante soit satisfaite:

(A*) *f satisfait* (A) *avec f, f' bornées et $f'(s) + m \ge 0$ pour tout $s \in \mathbb{R}$.*

Si f satisfait (A*) alors (Q_λ) est quasimonotone et afin d'obtenir une solution nontriviale nous imposons la condition suivante aux paramètres δ, γ.

(B2) $\beta < \sigma_1$ *avec* β *defini en* (1).

Il est important d'observer que les conditions (B1) et (B2) sont toujours satisfaites en choisissant γ suffisamment grand et δ suffisamment petit. En effet, pour $\delta > 0$ fixé, (B1) et (B2) sont satisfaites si

$$\gamma > \left\{\begin{array}{lll} m + 2\sqrt{\delta} & \text{si} & \delta < \sigma_1^2, \\ m + \sigma_1 + \delta/\sigma_1 & \text{si} & \delta \ge \sigma_1^2. \end{array}\right. \tag{2}$$

Nous avons alors le résultat suivant:

Théorème 1 *Supposons que $\Gamma \in C^3$. Si la fonction f satisfait* (A) *et si les paramètres δ, γ satisfont* (B1) *et* (B2) *alors:*

1. *il existe $\lambda^* > 0$ et une fonction $\Lambda \in C^1\left([\lambda^*, +\infty), C^2(\bar\Omega) \times C^2(\bar\Omega)\right)$ telle que $(u_\lambda, v_\lambda) := \Lambda(\lambda)$ est une solution positive de (P_λ) pour tout $\lambda \ge \lambda^*$;*
2. *$\max u_\lambda \in (\rho_{\delta/\gamma}^-, \rho_{\delta/\gamma}^+)$, $\max v_\lambda \in (\delta/\gamma)(\rho_{\delta/\gamma}^-, \rho_{\delta/\gamma}^+)$ et $(u_\lambda, v_\lambda) \to (\rho_{\delta/\gamma}^+, (\delta/\gamma)\rho_{\delta/\gamma}^+)$ uniformément sur les sous-ensembles compacts de Ω lorsque $\lambda \to \infty$.*
3. *pour toute fonction $z \in C_0^\infty(\Omega)$ avec $z \ge 0$ et $\max z \in (\rho_\beta^-, \rho_{\delta/\gamma}^+)$, il existe $\lambda_z > \lambda^*$ tel que si (u, v) est une solution de (P_λ) avec $\lambda > \lambda_z$ et $z \le u < \rho_{\delta/\gamma}^+$, alors $(u, v) = \Lambda(\lambda)$.*

1. ***Introduction and statement of main results*** – We consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta u &= \lambda(f(u) - v) & \text{in } \Omega, \\ -\Delta v &= \lambda(\delta u - \gamma v) & \text{in } \Omega, \\ u &= v = 0 & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{P_λ}$$

where $\lambda, \gamma, \delta > 0$ and $\Omega \subset \mathbb{R}^n$ is bounded and smooth. The nonlinearity f is assumed to be smooth and cubic-like. In [6] and [4] variational arguments are used to obtain nontrivial solutions to (P_λ) with $\lambda = 1$ if δ/γ is small and Ω satisfies certain conditions. Their results imply existence of nontrivial solutions to (P_λ) for δ/γ small and λ large. We not only prove, via appropriate sub- and supersolutions for a transformed system, the existence of a solution (u_λ, v_λ) to (P_λ) in an order interval, but also establish the existence of a curve of nondegenerate solutions parameterized by λ for λ large. These solutions are, except for a boundary layer of width $\sqrt{\lambda^{-1}}$, close to $(\rho, (\delta/\gamma)\rho)$ with ρ a positive zero of $f(s) - (\delta/\gamma)s$. Moreover, the estimates obtained enable us to prove uniqueness of such a boundary layer type solution.

We observe that if δ is taken to be zero, then problem (P_λ) corresponds to the scalar problem $-\Delta u = \lambda f(u)$ in Ω, $u = 0$ on Γ. There is an extensive literature on this kind of problems, see for example [1], [2] and [8] more recently [3].

We make the following assumption on f:

(A) $f \in C^{1,1}(\mathbb{R})$ *with* $f(0) \geq 0$. *Furthermore there exists* $\sigma_1 > 0$ *such that for every* $0 \leq \sigma < \sigma_1$ *there exist* $\rho_\sigma^- < \rho_\sigma^+$ *with* $\rho_\sigma^+ > 0$ *such that*

(i) $f(\rho_\sigma^\pm) = \sigma\rho_\sigma^\pm$ *and* $f(\rho) > \sigma\rho$ *for* $\rho_\sigma^- < \rho < \rho_\sigma^+$.

(ii) $f'(\rho) < 0$ *for all* $\rho \in (\rho_{\sigma_1}^+, \rho_0^+)$.

(iii) $J_\sigma(\rho) := \int_\rho^{\rho_\sigma^+} (f(s) - \sigma s)\, ds > 0$ *on* $[0, \rho_\sigma^+)$ *for all* $0 \leq \sigma < \sigma_1$.

A typical example is $f(u) = u(a-u)(u-1)$, with $0 < a < 1/2$. Condition (A) is satisfied for $\sigma_1 = \left(2a^2 - 5a + 2\right)/9$. With this nonlinearity problem (P_λ) is an extension of the FitzHugh-Nagumo equation, see [4] and [6]. For $f(u) = au - u^3$ with $a > 0$ (see [7] and [4]) condition (A) holds for $\sigma_1 = 2a/3$.

An important step in our analysis is to transform (P_λ) and to modify f to obtain a quasimonotone system. A similar transformation has also been used in [9] for a different system. A quasimonotone system shares many of the properties of a scalar equation. One has for example a comparison principle and the existence of a solution between an ordered pair of sub- and supersolutions, see [13] and [11]. There also exists a direct analogue of MacNabb's sweeping principle ([10], [2]) which will be used to determine the behavior of solutions.

To enable us to transform the system we impose the following condition on the parameters δ and γ:

(B1) $\gamma - 2\sqrt{\delta} > m$ *where* $m := \max\left\{-f'(s)\ ;\ 0 \leq s \leq \rho_0^+\right\}$.

Fix δ and γ such that (B1) holds and define β as in (1) by $\beta := \frac{1}{2}(\gamma - m) - \frac{1}{2}\sqrt{(\gamma-m)^2 - 4\delta}$. Let $\alpha := \gamma - \beta$. Observe that $-\beta(\beta + m) = \delta - \gamma\beta$ and that $\alpha, \beta > 0$. Suppose that (u, w) is a solution to

$$\begin{cases} -\Delta u &= \lambda(f(u) - \beta u + \beta w) & \text{in } \Omega, \\ -\Delta w &= \lambda(f(u) + mu - \alpha w) & \text{in } \Omega, \\ u &= w = 0 & \text{on } \Gamma. \end{cases} \tag{Q_λ}$$

Setting $v := \beta u - \beta w$ one finds that $-\Delta v = \lambda\left(-\beta\left(\beta+m\right)u + \beta\left(\beta+\alpha\right)w\right) = \lambda(\delta u - \gamma v)$. Hence (u,v) is a solution to (P_λ) if and only if (u,w) is a solution of (Q_λ). Since we are interested in solutions (u,v) with u positive and $\max u \leq \rho^+_{\delta/\gamma}$ we can modify f outside $[0,\rho^+_0]$ and assume without loss of generality that f is as follows:

(A*) *f satisfies* (A) *with f and f' bounded and* $f'(s)+m \geq 0$ for all $s \in \mathbb{R}$.

If f satisfies (A*) then (Q_λ) is a quasimonotone system. In order to have nontrivial solutions to (Q_λ), for λ large, we impose the following condition on the parameters.

(B2) $\beta < \sigma_1$ *with β as defined in* (1).

We observe that (B1) and (B2) can always be satisfied by choosing γ sufficiently large and δ sufficiently small. Indeed, for fixed $\delta > 0$ conditions (B1) and (B2) hold true if (2) is satisfied.

Our main result is the following.

Theorem 1 *Let f satisfy* (A), *let δ, γ be such that* (B1) *and* (B2) *hold and assume that Γ is C^3. The following results hold:*

1. *there exist $\lambda^* > 0$ and a function $\Lambda \in C^1\left([\lambda^*, +\infty), C^2(\bar{\Omega}) \times C^2(\bar{\Omega})\right)$ such that $(u_\lambda, v_\lambda) := \Lambda(\lambda)$ is a positive solution to (P_λ) for all $\lambda \geq \lambda^*$;*
2. $\max u_\lambda \in (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$, $\max v_\lambda \in (\delta/\gamma)(\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$ *and* $(u_\lambda, v_\lambda) \to (\rho^+_{\delta/\gamma}, (\delta/\gamma)\rho^+_{\delta/\gamma})$ *uniformly on compact subsets of Ω for $\lambda \to \infty$;*
3. *for every function $z \in C_0^\infty(\Omega)$ with $z \geq 0$ and $\max z \in (\rho^-_\beta, \rho^+_{\delta/\gamma})$ there exists $\lambda_z > \lambda^*$ such that if (u,v) is a solution to (P_λ) with $\lambda > \lambda_z$ and $z \leq u < \rho^+_{\delta/\gamma}$ then $(u,v) = \Lambda(\lambda)$.*

A detailed version of the proof will appear in [12].

2. *Sketch of the proof* – *Existence of a solution in an order interval.* Let f satisfy (A*) and suppose that δ, γ is such that (B1) and (B2) hold. Let B be the unit ball in $\mathbb{R}^n$. There exist $\lambda_B > 0$ and positive functions U and W such that

$$\left\{\begin{array}{rcll} -\Delta U &=& \lambda_B(f(U) - \beta U + \beta W) & \text{in } B, \\ -\Delta W &=& \lambda_B(f(U) + mU - \alpha W) & \text{in } B, \\ U &=& W = 0 & \text{on } \partial B, \end{array}\right. \tag{3}$$

with $\max U \in (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$, $\max W \in (\vartheta\rho^-_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$ and $\vartheta = 1 - \frac{\delta}{\gamma\beta} > 0$. Indeed, since $\beta < \sigma_1$ we have that $J_\beta(0) > 0$ and hence by [2] there exists for $\lambda = \lambda_B$ sufficiently large a nontrivial positive solution, say $\bar{u}$, to $-\Delta u = \lambda(f(u) - \beta u)$ in B, $u = 0$ on ∂B with $\max \bar{u} \in (\rho^-_\beta, \rho^+_\beta) \subset (\rho^-_{\delta/\gamma}, \rho^+_{\delta/\gamma})$. Since $(\bar{u}, 0)$ is a subsolution and $(\rho^+_\beta, \vartheta\rho^+_\beta)$ is a supersolution to (3), we have a solution (U,W) to (3) with $(\bar{u}, 0) \leq (U,W) \leq (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$. Some extra steps show $\max W > \vartheta\rho^-_{\delta/\gamma}$. Using an extension due to Troy [13] of a result of Gidas, Ni and Nirenberg [5] to quasimonotone systems, the functions U and W are seen to be radially symmetric and decreasing. In particular $(\rho^-_{\delta/\gamma}, \vartheta\rho^-_{\delta/\gamma}) < (\tau, \kappa) := (U(0), W(0)) < (\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$ and it also holds that $U'(1) < W'(1) < 0$. On Ω we then define

$$Z_{\lambda,y}(x) := \left\{\begin{array}{ll} (U,W)\left((\lambda/\lambda_B)^{1/2}(y-x)\right) & \text{if } |y-x| < (\lambda/\lambda_B)^{-1/2}; \\ 0 & \text{if } |y-x| \geq (\lambda/\lambda_B)^{-1/2}. \end{array}\right.$$

Fix $y^\circ \in \Omega$ let $\lambda^\circ := \lambda_B \operatorname{dist}(y^\circ, \Gamma)^2$ and $Z^\circ_\lambda := Z_{\lambda, y^\circ}$. Since $Y{:=}(\rho^+_{\delta/\gamma}, \vartheta\rho^+_{\delta/\gamma})$ is a supersolution to (Q_λ) above the subsolution Z°_λ, there exists for all $\lambda > \lambda^\circ$ at least one solution (u, w) in the order interval $\mathcal{I}_\lambda = [Z^\circ_\lambda, Y]$.

– *Boundary layer estimates for solutions in $\mathcal{I}_\lambda$.* Set $\Omega(\varepsilon) = \{y \in \Omega; \operatorname{dist}(y, \Gamma) > \varepsilon\}$ and $B_\varepsilon(y) = \{x \in \mathbb{R}^n; |x - y| < \varepsilon\}$. Since $\partial\Omega \in C^3$ there is $\varepsilon_0 > 0$ such that $\Omega = \cup\{B_\varepsilon(y)\ ;\ y \in \Omega(\varepsilon)\}$ holds and such that $\Omega(\varepsilon)$ is arcwise connected for $\varepsilon \in (0, \varepsilon_0)$. Let $\lambda > \lambda^\dagger = \max\{\lambda_B \varepsilon_0^{-2}, \lambda^\circ\}$ and (u, w) be a solution of (Q_λ) in $[Z^\circ_\lambda, Y]$. A sweeping principle argument shows that $(u, w) \geq Z^\circ_\lambda$ implies $(u, w) \geq Z_{\lambda, y}$ for all $y \in \Omega(\sqrt{\lambda_B/\lambda})$. Indeed, let $y : [0, 1] \to \Omega(\sqrt{\lambda_B/\lambda})$ be a continuous curve connecting y° with y and consider $S = \{t \in [0, 1]\,;(u, w) \geq Z_{\lambda, y(t)} \text{ in } \Omega\}$. By continuity S is closed, by the strong maximum principle and the fact that $Z_{\lambda, y(t)}$ is a strict subsolution, S is open. Since S is nonempty S equals $[0, 1]$. Hence for all solutions $(u, w) \in [Z^\circ_\lambda, Y]$ with $\lambda > \lambda^\dagger$, we have the estimate

$$(u(x), w(x)) \geq \bar{Z}_\lambda(x) := \sup\left\{Z_{\lambda, y}(x)\,;\, y \in \Omega(\sqrt{\lambda_B/\lambda})\right\} \geq \min\left\{b\sqrt{\lambda}\operatorname{dist}(x, \Gamma), \tau\right\}(1, \vartheta),$$

where $b > 0$ is choosen such that $b(1 - |x|) \leq \min\{U(x), \vartheta W(x)\}$ for $|x| < 1$. A second sweeping argument in 'vertical' direction is used to obtain interior estimates. Let $\varepsilon > 0$ and let $\ell_\varepsilon > 0$ be such that $f(s) - \frac{\delta}{\gamma}s \geq \ell_\varepsilon(s - \tau)$ for $s \in (\tau, \rho^+_{\delta/\gamma} - \varepsilon)$. Let ϕ be the positive eigenfunction with eigenvalue μ of the Dirichlet Laplacian on B extended by 0 outside of B and set $c = \sqrt{\lambda\ell_\varepsilon/\mu}$. Sweeping with $t \longmapsto (1, \vartheta)\left(\bar{Z}_\lambda + t\,\phi(c(\cdot - y))\right)$ for $y \in \Omega(c^{-1} + \frac{\tau}{b\sqrt{\lambda}})$ it follows that for every $\varepsilon > 0$ there exists a constant $b(\varepsilon)$, independent of λ, such that for every solution $(u, w) \in [\bar{Z}_\lambda, Y]$ with $\lambda > \lambda^\dagger$, and a fortiori for every solution $(u, w) \in [Z^\circ_\lambda, Y]$ with $\lambda > \lambda^\dagger$, it holds that

$$(u(x), w(x)) > \min\left\{b(\varepsilon)\sqrt{\lambda}\operatorname{dist}(x, \Gamma), \rho^+_{\delta/\gamma} - \varepsilon\right\}(1, \vartheta). \tag{4}$$

A similar sweeping argument can be used in order to prove that solutions with u in $\left[z, \rho^+_{\delta/\gamma}\right]$ lie in $\mathcal{I}_\lambda$ if λ is sufficiently large. Here z is the function in Theorem 1.

– *Uniqueness for the solutions in $\mathcal{I}_\lambda$.* Since all solutions in the order interval are of boundary layer type, it is sufficient to establish uniqueness for solutions that satisfy (4). The argument uses the Leray-Schauder degree. Let $\omega \geq \alpha$ be such that $f'(s) - \beta + \omega > 0$ for all $s \in [0, \rho^+_{\delta/\gamma}]$ and set $X = C(\bar{\Omega}) \times C(\bar{\Omega})$. For $\lambda > 0$ and $u \in C(\bar{\Omega})$ let $K_{\lambda, u} : X \to X$ be defined by

$$K_{\lambda, u} := \begin{pmatrix} \left(-\lambda^{-1}\Delta + \omega\right)_0^{-1} & 0 \\ 0 & \left(-\lambda^{-1}\Delta + \omega\right)_0^{-1} \end{pmatrix} \begin{pmatrix} f'(u) - \beta + \omega & \beta \\ f'(u) + m & \omega - \alpha \end{pmatrix}.$$

Here $h = \left(-\lambda^{-1}\Delta + \omega\right)_0^{-1} g$, $g \in C(\bar{\Omega})$, is the unique $u \in C(\bar{\Omega})$ such that $u = 0$ on Γ and $\left(-\lambda^{-1}\Delta + \omega\right) h = g$ in the sense of distributions. The operator $K_{\lambda, u}$ is positive and compact with a positive spectral radius $r(K_{\lambda, u})$. Hence by the Krein-Rutman Theorem $r(K_{\lambda, u})$ is an eigenvalue to which a positive eigenvalue corresponds.

A key lemma is the following.

Lemma 2 *There exists $\lambda^* > \lambda^\dagger$ such that for every solution (u, w) to (Q_λ) with $(u, w) \in [Z^\circ_\lambda, Y]$ and $\lambda > \lambda^*$ it holds for the spectral radius of $K_{\lambda, u}$ that $0 < r(K_{\lambda, u}) < 1$.*

The result is proven by contradiction through a 'blow-up' argument. Suppose there exists a sequence $\lambda_k \to \infty$ and solutions $(u_k, w_k) \in [Z^\circ_{\lambda_k}, Y]$ to (Q_{λ_k}) such that $r(K_{\lambda_k, u_k}) \geq 1$. Denote by (φ_k, ψ_k) the positive eigenfunction pertaining to $r(K_{\lambda_k, u_k})$. By rescaling one finds subsequences of these functions converging to $\left(\bar{U},\bar{W}\right)$, $\left(\bar{\Phi},\bar{\Psi}\right)$ in $C^2_{\mathrm{loc}}\left(\mathbb{R}^+\times\mathbb{R}^{n-1}\right)^2$. Both pairs are solutions to appropriate elliptic systems on $\mathbb{R}^+\times\mathbb{R}^{n-1}$. A tedious argument, using the estimate in (4), shows that $\left(\bar{U},\bar{W}\right)$ depends on the first variable only. Comparing $\left(\bar{U}_{x_1},\bar{W}_{x_1}\right)$ with $\sup\left\{\left(\bar{\Phi},\bar{\Psi}\right)(x_1,x')\,;\ x'\in\mathbb{R}^{n-1}\right\}$ one finds a contradiction.

By this lemma and a degree argument it follows that the operator $H_\lambda : X\to X$, defined by

$$H_\lambda(u,w)=\begin{pmatrix}\left(-\lambda^{-1}\Delta+\omega\right)_0^{-1} & 0\\ 0 & \left(-\lambda^{-1}\Delta+\omega\right)_0^{-1}\end{pmatrix}\begin{pmatrix}f(u)-\beta u+\omega u+\beta w\\ f(u)+mu-\alpha w+\omega w\end{pmatrix},$$

has one and only one fixed point in $\mathcal{I}_\lambda$, say $\tilde{\Lambda}(\lambda)$, for $\lambda>\lambda^*$.

– *Curve of solutions.* It follows from the lemma and the Implicit Function Theorem that $\tilde{\Lambda}\in C^1\left([\lambda^*,+\infty),\left(C\left(\bar{\Omega}\right)\right)^2\right)$. A bootstrap argument shows that $\tilde{\Lambda}\in C^1\left([\lambda^*,+\infty),\left(C^2\left(\bar{\Omega}\right)\right)^2\right)$. The curve of solutions to (P_λ) is given by $\Lambda(\lambda)=(u_\lambda,v_\lambda):=(u_\lambda,\beta(u_\lambda-w_\lambda))$ for $\lambda\in[\lambda^*,+\infty)$ with $(u_\lambda,w_\lambda):=\tilde{\Lambda}(\lambda)$. The boundary layer behavior of (u_λ,v_λ) follows from (4).

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C.R. & G.S., Department of Mathematics, Delft University of Technology
PObox 5031, 2600 GA Delft, The Netherlands
email: `reinecke@twi.tudelft.nl` `sweers@twi.tudelft.nl`
fax: ++31 15 2787245