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Spring School on Nonlocal Problems and Related PDEs

Positivity Preserving Results and Maximum Principles

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Abstract: The general setting that we will discuss, can be written as follows:

$$Mu = f.$$

Here $M : X \to Y$ is some given operator for ordered spaces X, Y. The question for this generic equation is:

Can one classify M for which it holds that $f \geq 0 \implies u \geq 0$?

In other words, when do we a have a positivity preserving property in the sense that a positive source gives a positive solution.

- Our main focus is on partial differential operators defined on ordered function spaces but since the structure also appears for $u, f \in \mathbb{R}^n$, we will start with those problems. For matrices the crucial condition is that M is a so called M-matrix.
- Next we recall some maximum principles for second order elliptic pde, which are the basic tool for invers-positivity of second order elliptic boundary value problems. In is no coincidence that most discretisations of second order elliptic differential operators lead to M-matrices.
- For the problem above we do need a solution operator and we will reflect on the different possibilities. The existence of the principal eigenvalue follows for example from a Krein-Rutman theorem. We also discuss the almost similar situation for weakly coupled systems of cooperative type. Assuming that the equation cannot be split in independent subsystems, one obtains a generic answer. Considering $(\lambda I + M)u = f$ one will that the positivity preserving property will hold if and only if $\lambda > \lambda_1$, with λ_1 the principal eigenvalue.
- In the last part we consider some special cases where positivity is preserved but without the M-matrix type condition. This is the case for example in some 'real' fourth order problems.

Contents

Session 1

Positivity and matrices

1.1 Positivity preserving for matrix problems

Let $M \in M^{n\times n}(\mathbb{R})$ and consider for $u, f \in \mathbb{R}^n$ the system:

$$Mu = f \tag{1.1}$$

Specific names for a matrix having the properties mentioned above are as follows:

Definition 1.1.1 *Let $M \in M^{n\times n}(\mathbb{R})$.*

- *M is called **inverse-positive**, when for all $u \in \mathbb{R}^n$ with $Mu \geq 0$, it holds that $u \geq 0$.*
- *M is called **strongly inverse-positive**, when for all $u \in \mathbb{R}^n$ with $Mu \gneq 0$, it holds that $u > 0$.*

Remark 1.1.2
- *$u \geq 0$ means $\forall i \in \{1, \ldots, n\}: \ u_i \geq 0$,*
- *$u \gneq 0$ means $\forall i \in \{1, \ldots, n\}: \ u_i \geq 0$ and $\exists i \in \{1, \ldots, n\}: \ u_i > 0$,*
- *$u > 0$ means $\forall i \in \{1, \ldots, n\}: \ u_i > 0$.*

We will see that inverse-positive matrices are related with the following class of matrices.

Definition 1.1.3 *$M \in M^{n\times n}(\mathbb{R})$ is called a **nonsingular M-matrix**, if $M = sI - B$ with*

1. *B is nonnegative, i.e. $B = (b_{ij})$ satisfies $b_{ij} \geq 0$,*
2. *$s > \rho(B)$, the spectral radius of B.*

The spectral radius is defined by

$$\rho(B) = \limsup_{n\to\infty} \sqrt[n]{\|B^n\|}.$$

For a diagonal matrix B one finds that $\rho(B) = \max_{1\leq i\leq n} |b_{ii}|$ and if B has nonnegative coefficients $\rho(B)$ is an eigenvalue. We will see that $\rho(B)$ is an eigenvalue for arbitrary nonnegative matrices, that is, the spectral radius is an eigenvalue with a nonnegative eigenvector φ_1, i.e. $B\varphi_1 = \rho(B)\varphi_1$. This eigenvector might not be unique as we can see by taking for example the identity matrix. To have uniqueness, meaning a one-dimensional eigenspace, one needs an additional condition.

Definition 1.1.4 *A nonnegative matrix $B = (b_{ij})$ is called*

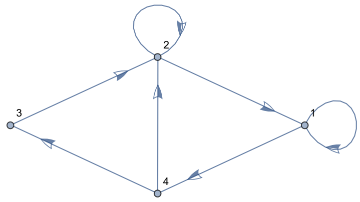
3. *irreducible, if $(sign\,(b_{ij}))$ is the adjacency matrix of a strongly connected directed graph.*

Remark 1.1.5 *An equivalent condition is: there exists $k \in \mathbb{N}$ such that $(I+B)^k$ has only strictly positive entries.*

Remark 1.1.6 *An M-matrix is called irreducible, whenever B is irreducible.*

Example 1.1.7 *Here is a example of a nonnegative matrix and the corresponding adjacency matrix.*

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$



With these three conditions in Definitions 1.1.3 and 1.1.4 satisfied, the following result holds true for the eigenvalue problem

$$Bu = \mu u \tag{1.2}$$

with $(\mu, u) \in \mathbb{C} \times \mathbb{C}^n$.

Theorem 1.1.8 (Part of Perron-Frobenius) *Suppose $B \in M^{n\times n}(\mathbb{R})$ is nonnegative and irreducible. Then the following holds:*

i) The in absolute sense largest eigenvalue μ_1 is unique and satisfies $\mu_1 = \rho(B)$.

ii) The algebraic multiplicity of μ_1 is one and the corresponding eigenspace is spanned by a strictly positive eigenvector φ_1.

iii) Except for positive multiples of φ_1 there are no other nonnegative eigenvectors.

For a proof see for example [3].

Remark 1.1.9 *When B is symmetric, things become much easier and one shows almost directly that*

$$\rho(B) = \mu_1 = \max_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v \cdot Bv}{v \cdot v}.$$

Theorem 1.1.10 (M-matrix and invers-positivity) *Let $M \in M^{n\times n}(\mathbb{R})$. Then the following are equivalent.*

- *$M = sI - B$ satisfies 1., 2. and 3. of Definitions 1.1.3 and 1.1.4.*
- *$\lambda I + M$ is strongly inverse-positive for all $\lambda \geq 0$.*

Proof. $(\Rightarrow)$ Since $s > \rho(B)$ the inverse of $\lambda I + M$ exists and can be written by a convergent Neumann series

$$\begin{aligned}(\lambda I + M)^{-1} &= ((s+\lambda) I - B)^{-1} = (s+\lambda)^{-1} \left(I - (s+\lambda)^{-1} B\right)^{-1} \\ &= (s+\lambda)^{-1} \sum_{k=0}^{\infty} \left((s+\lambda)^{-1} B\right)^k . \end{aligned} \tag{1.3}$$

Since B has only nonnegative entries, irreducibility implies that

$$\sum_{k=0}^{n}\left((s+\lambda)^{-1}B\right)^k$$

has only positive entries.

($\Leftarrow$) If $\lambda I+M$ for some $\lambda\geq 0$ has a zero eigenvalue with eigenvector φ_0, then $(\lambda I+M)\,\varphi_0=0$. But φ_0 or $-\varphi_0$ is nonpositive and that is a contradiction. So $\lambda I+M$ is invertible.

The matrices $(\lambda I+M)^{-1}$ have only strictly positive entries. If not, say a coefficient a_{ij} satisfies $a_{ij}\leq 0$, then $(\lambda I+M)^{-1}\vec{e}_j$ has a nonpositive entry and $(\lambda I+M)^{-1}$ is not strictly positive, a contradiction.

Set $\sigma=\max(m_{ii})$ and write $B=\sigma I-M$. So the diagonal entries of B are nonnegative. For all λ large enough, that is for all λ such that $\sigma+\lambda>\rho(B)$, we have from (1.3) that

$$\begin{aligned}(\lambda I+M)^{-1}&=((\sigma+\lambda)\,I-B)^{-1}=((\sigma+\lambda)\,I-B)^{-1}\\&=(\sigma+\lambda)^{-1}\left(I+(\sigma+\lambda)^{-1}B+\mathcal{O}\left((\sigma+\lambda)^{-2}\right)\right).\end{aligned}$$

Since $(\lambda I+M)^{-1}$ has only strictly positive entries for all $\lambda\geq 0$ the signs of the off-diagonal terms for large λ are determined by B. Hence all off-diagonal entries of B are also nonnegative.

So $M=\sigma I-B$ with $\sigma>\rho(B)$. Indeed, since B is nonnegative the spectral radius is an eigenvalue and hence $\sigma+\lambda\neq\rho(B)$ for all $\lambda\geq 0$, implying that $\sigma>\rho(B)$.

For λ large the formula in (1.3) holds. With B being nonnegative, the formula shows that strong positivity implies irreducibility of B. ∎

Example 1.1.11 *Consider*

$$M=\begin{pmatrix}3&-1&\frac{1}{5}&-1\\-1&4&-1&-1\\-1&-1&4&-1\\-1&-1&-1&3\end{pmatrix}. \tag{1.4}$$

Due to the entry $m_{1,3}=\frac{1}{5}>0$ this is not an M-matrix. The change is however small enough to keep a positive first eigenfunction and even such that the inverse of $M+tI$ is still positive for small t. However, for $t>t_0\approx 7.5$ entry $a_{1,3}$ in $(M+tI)^{-1}$ becomes and remains negative. See Figure 1.1, where scaled values of the individual off-diagonal entries $a_{ij}(t)$ of $(M+tI)^{-1}$ are plotted.

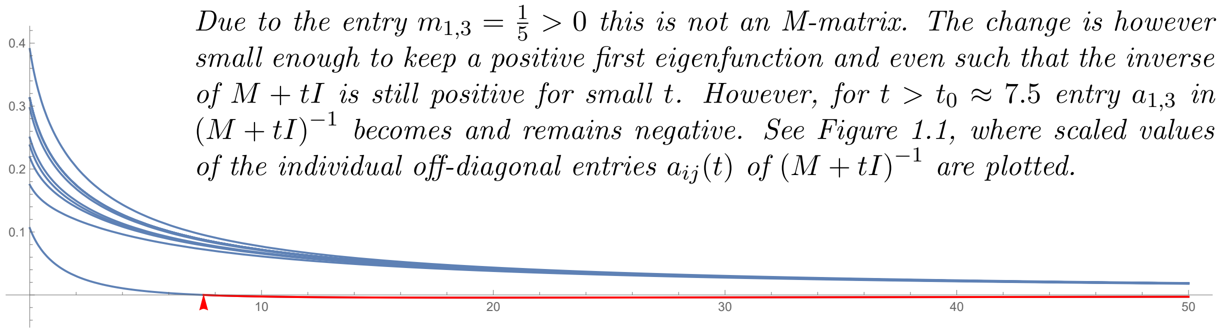


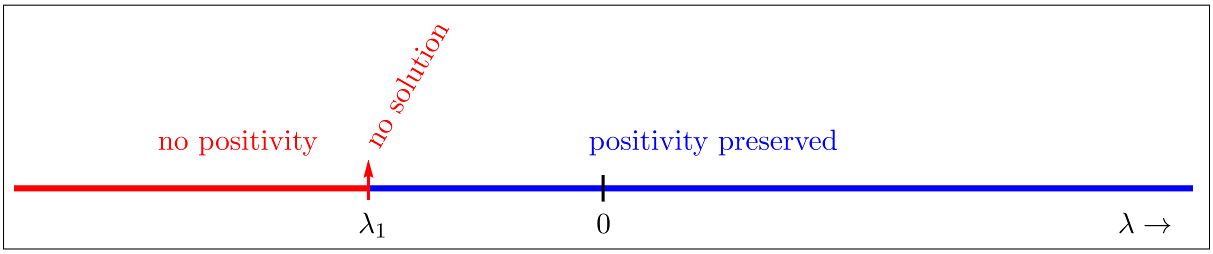
Figure 1.1: A graph of the off-diagonal entries $t\mapsto(1+t)\,a_{ij}(t)$ with M from (1.4).

Let us formulate the result concerning the positivity preserving property for M-matrices, in short PPP.

Theorem 1.1.12 (PPP for M-matrices) *Let M be an irreducible nonsingular M-matrix. Then the following holds.*

1. *M^{-1} has a positive eigenvector φ_1 with eigenvalue $\mu_1>0$. Set $\lambda_1=-\mu_1^{-1}$.*
2. *(a) if $\lambda>\lambda_1$ then $\lambda I+M$ is strongly inverse positive.*

   (b) if $\lambda=\lambda_1$ and $f\gneq 0$, then there is no solution u for $(\lambda I+M)\,u=f$.



(c) if $\lambda < \lambda_1$ and $f \gneq 0$, then there is no positive solution u for $(\lambda I + M)\,u = f$.

Remark 1.1.13 *Since $M = sI - B$, the equation $M^{-1}\varphi_1 = \mu_1\varphi_1$ is equivalent to $B\varphi_1 = (s+\lambda_1)\,\varphi_1$. Since B is also positive, one finds that $\lambda_1 = \rho(B) - s < 0$.*

Remark 1.1.14 *If M is symmetric, then*

$$\lambda_1 = -\min_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v \cdot Mv}{v \cdot v}.$$

Proof. Since M is inverse positive, the first item follows from the Perron-Frobenius Theorem for M^{-1}.

Suppose that $M = sI - B$. For $\lambda > \lambda_1 = \rho(B) - s$ the matrix $(\lambda + s)\,I - B$ satisfies the conditions 1., 2. and 3. of Definitions 1.1.3 and 1.1.4. Hence Theorem 1.1.10 implies that $\lambda I + M$ is inverse-positive. By the series 1.3 and the assumption of irreducibility one finds even strongly inverse-positivity.

Since also B^T has the same eigenvalues and similar properties, let $\tilde{\varphi}_1$ be the corresponding positive eigenfunction for $B^T\tilde{\varphi}_1 = \rho(B)\,\tilde{\varphi}_1$. Let $f \gneq 0$ and let u be the solution of $(\lambda I - B)\,u = f$. Then

$$0 < \tilde{\varphi}_1 \cdot f = \tilde{\varphi}_1 \cdot (\lambda I - B)\,u = \left(\lambda I - B^T\right)\tilde{\varphi}_1 \cdot u = (\lambda - \rho(B))\,\tilde{\varphi}_1 \cdot u \tag{1.5}$$

and a contradiction follows if $\lambda = \rho(B)$ or when $u \gneq 0$, since then $\tilde{\varphi}_1 \cdot u > 0$. ∎

One might wonder what goes wrong when the matrix is perturbed such that it is no longer an M-matrix but still close enough to one in the sense that the first eigenfunction is still positive. The generic picture is as in Figure 1.2.

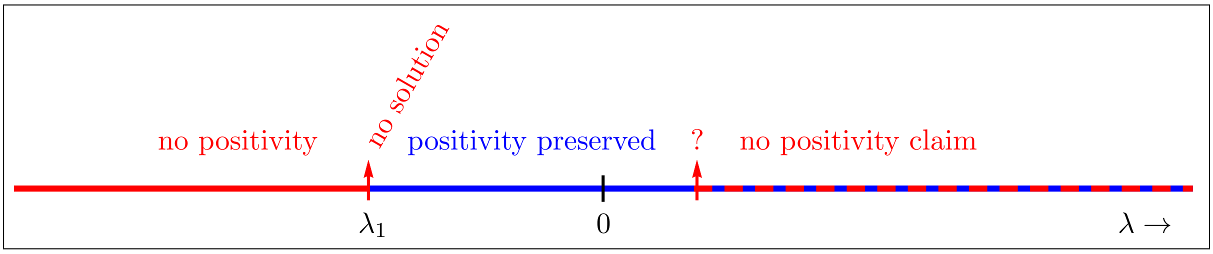


Figure 1.2: the generic case for an M, which is almost an M-matrix, but misses the definition by some small positive off-diagonal term(s), nevertheless such that the first eigenfunction is still positive. As in Example 1.1.11

Example 1.1.15 *Note that for $C^2(\mathbb{R})$-functions it holds that*

$$\lim_{h \downarrow 0} \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} = u''(x).$$

So one may discretize

$$\begin{cases} -u''(x) = f(x) & \text{for } x \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

with $n \in \mathbb{N}^+$ *through* $x_k = \frac{k}{n}$, *by*

$$\begin{cases} \dfrac{-u(x_{k+1}) + 2u(x_k) - u(x_{k-1})}{1/n^2} = f(x_k) & \text{for } k \in \{1, \dots, n-1\}, \\ u(x_0) = u(x_n) = 0. \end{cases}$$

In matrix form with $u_k = u\left(\frac{k}{n}\right)$ *this equals* $u_0 = u_n = 0$ *and*

$$n \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \frac{1}{n} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} \end{pmatrix}.$$

The matrix on the left is an irreducible nonsingular M-matrix. The inverse of this matrix can be explicitly computed: writing $M^{-1} = (a_{ij})_{i,j=1\dots n-1}$, *one finds*

$$a_{ij} = \begin{cases} \frac{j}{n}\left(1 - \frac{i}{n}\right) & \text{for } 1 \le i \le j \le n-1, \\ \frac{i}{n}\left(1 - \frac{j}{n}\right) & \text{for } 1 \le j < i \le n-1. \end{cases}$$

A graph one finds below in Figure 1.3.

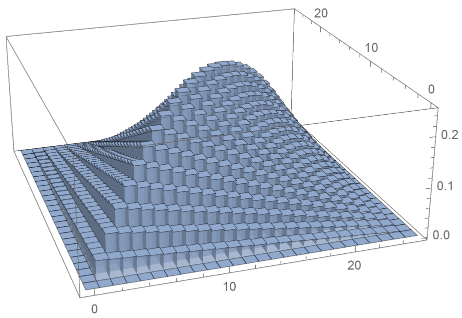


Figure 1.3: A graph of the matrix for $n = 25$ with the boundary points $i \in \{0, 25\}$ added, i.e. $(a_{ij})_{i,j=0\dots 25}$

So for the discretized boundary value problem we find

$$f \gneq 0 \implies u > 0.$$

Example 1.1.16 *Discretizing the Laplacian* $-\Delta$ *in two dimensions by finite differences one uses*

$$-\Delta u(x,y) = \lim_{h \to 0} \frac{4u(x,y) - u(x-h,y) - u(x+h,y) - u(x,y-h) - u(x,y+h)}{h^2}.$$

Inside the domain the corresponding matrix, skipping the h^2, *has* 4 *on the diagonal and on each row (at most) four times* -1. *Precisely four times when no boundary conditions are involved. Also this is an M-matrix.*

Example 1.1.17 *A more common numerical approximation uses finite elements. For the Laplacian, when using piecewise linear finite elements on a triangular mesh with only acute angles, the corresponding matrix turns out to be an M-matrix. See [5]. Also in three dimensions this holds true. Surprisingly, in higher dimensions ($n \geq 7$?) one may show that in general no acute triangularisation exists. See [18] and [17].*

1.2 The time dependent problem

Let us consider for $u : [0, \infty) \to \mathbb{R}^n$ the intial value problem:

$$\begin{cases} \left(\frac{\partial}{\partial t} + M\right) u(t) = 0 \text{ for } t > 0, \\ u(0) = u_0. \end{cases} \tag{1.6}$$

The solution for this problem is

$$u(t) = \exp(-Mt)\, u_0,$$

which is given by Euler-backwards approximation as

$$u(t) = \lim_{n \to \infty} \left(I + \frac{t}{n} M \right)^{-n} u_0.$$

From the second part of Theorem 1.1.10 it follows for $t > 0$ and n large enough, indeed then

$$\left(I + \frac{t}{n} M \right)^{-1} = \frac{n}{t} \left(M + \frac{n}{t} I \right)^{-1} > 0,$$

that $\left(I + \frac{t}{n} M\right)^{-n}$ is strongly positive. Hence we conclude:

Corollary 1.2.1 *Let M be an irreducible M-matrix and let $u_0 \gneq 0$. Then the solution $u : [0, \infty) \to \mathbb{R}^n$ of (1.6) satisfies:*

$$u_0 \gneq 0 \implies u(t) > 0 \textit{ for all } t > 0.$$

Remark 1.2.2 *For*

$$\begin{cases} \left(\frac{\partial}{\partial t} + M\right) u(t) = f(t) \textit{ for } t > 0, \\ u(0) = u_0. \end{cases} \tag{1.7}$$

one uses

$$u(t) = \exp(-Mt)\, u_0 + \int_0^t \exp(-M(t-s))\, f(s)\, ds$$

and finds from the positivity of $\exp(-Mt)$ *that*

$$f(t) \geq 0,\ u_0 \gneq 0 \implies u(t) > 0 \textit{ for all } t > 0.$$

Session 2

Maximum Principles

2.1 Positivity preserving and the Laplace operator

The Laplace operator is defined for functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ by

$$\Delta u = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} \right)^2 u.$$

The natural assumption, used throughout the literature, is that Ω is a domain, which means an open and connected set. In the present setting we will even restrict ourselves to bounded domains.

2.2 Classical maximum principles

2.2.1 Results that hold for general second order elliptic operators

In this section we state the result for the Laplace operator. The first three versions of the maximum principle also hold for general second order elliptic operators with nice enough coefficients:

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}.$$

One assumes without loss of generality that $a_{ij}(x) = a_{ji}(x)$ and the ellipticity condition,

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq c |\xi|^2 \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

is satisfied. The optimal conditions that the coefficients in L may satisfy we will not state, but can be found following the corresponding references in [12]. We just assume $a_{ij}, b \in C^\infty(\overline{\Omega})$.

For maximum principles in all its variations one should have a look at the following books: [25], [10] and [26]. Most arguments in this section are borrowed from the first book.

Lemma 2.2.1 (Maximum Principle I) *Suppose that $u \in C^2(\Omega)$. If $\Delta u(x) > 0$ for $x \in \Omega$, then u has no interior maximum.*

Proof. If u has an interior maximum in x_0, then $\left(\frac{\partial}{\partial x_i} \right)^2 u(x_0) \leq 0$ for each $i \in \{1, \ldots, n\}$. ■

Lemma 2.2.2 (Maximum Principle II) *Suppose that $u \in C^2(\Omega)$. If $\Delta u(x) \geq 0$ for $x \in \Omega$, then for any ball $\overline{B_\varepsilon(x_0)} \subset \Omega$ one finds $u(x_0) \leq \max\{u(x); x \in \partial B_\varepsilon(x_0)\}$. Hence u cannot have a strict interior maximum.*

Proof. Suppose that for some $\varepsilon > 0$ and $\overline{B_\varepsilon(x_0)} \subset \Omega$ one finds

$$u(x_0) > \max\{u(x); x \in \partial B_\varepsilon(x_0)\}.$$

Set $m = \max\{u(x); x \in \partial B_\varepsilon(x_0)\}$ and we consider the auxiliary function

$$\tilde{u}(x) = u(x) + \tfrac{1}{2}\varepsilon^{-2}(u(x_0) - m)|x - x_0|^2.$$

One finds $\tilde{u}(x_0) = u(x_0)$ and for $|x - x_0| = \varepsilon$ it holds that

$$\tilde{u}(x) \leq m + \tfrac{1}{2}(u(x_0) - m) < u(x_0).$$

Hence $\tilde{u}$ has a maximum in some $\tilde{x} \in B_\varepsilon(x_0)$ and

$$\Delta\tilde{u}(\tilde{x}) = \Delta u(\tilde{x}) + n\varepsilon^{-2}(u(x_0) - m) > 0,$$

which contradicts Lemma 2.2.1. ■

Theorem 2.2.3 (The Strong Maximum Principle) *Suppose that $u \in C^2(\Omega)$. If $\Delta u(x) \geq 0$ for $x \in \Omega$ and u has a maximum in $x_0 \in \Omega$, then $u(x) = u(x_0)$ for all $x \in \Omega$.*

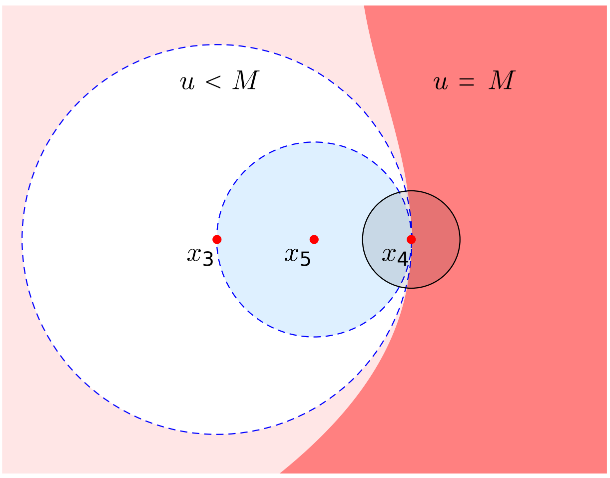


Figure 2.1: A sketch for the construction in the proof of Lemma 2.2.3.

Proof. If $u(x_1) < u(x_0) = M := \max\{u(x); x \in \Omega\}$, then there is a curve inside Ω connecting x_1 with x_0. Let x_2 be the first point on this curve such that $u(x_2) = M$. Now let $r_2 > 0$ be such that $B_{r_2}(x_2) \subset \Omega$. Then there is $x_3 \in \partial B_{r_2}(x_2)$ with $u(x_3) < M$. Let $r_3 \in (0, r_2]$ be the largest number such that $u(x) < M$ for $x \in B_{r_3}(x_3)$ and let $x_4 \in \partial B_{r_3}(x_3)$ satisfy $u(x_4) = M$. Set $x_5 = \frac{1}{2}(x_3 + x_4)$ and $r_5 = \frac{1}{2}r_3$. Next consider on $B_{r_4}(x_4)$ with $r_4 = \frac{1}{4}r_3$ the auxiliary function

$$\begin{aligned} \tilde{u}(x) &= u(x) + \varepsilon\left(|x - x_5|^{2-n} - r_5^{2-n}\right) && \text{for } n \geq 3, \\ \tilde{u}(x) &= u(x) + \varepsilon\left(\log\left(\tfrac{1}{|x-x_5|}\right) - \log\left(\tfrac{1}{r_5}\right)\right) && \text{for } n = 2. \end{aligned}$$

Set $m = \max\{u(x); x \in \partial B_{r_4}(x_4) \cap B_{r_5}(x_5)\} < M$ and take $\varepsilon > 0$ such that

$$\tilde{u}(x) < M \text{ for } x \in \partial B_{r_4}(x_4) \cap B_{r_5}(x_5).$$

Since $|x - x_5|^{2-n} - r_5^{2-n} \leq 0$ for $x \in \partial B_{r_4}(x_4) \setminus B_{r_5}(x_5)$ we find with such an ε that

$$\tilde{u}(x) < M \text{ for } x \in \partial B_{r_4}(x_4).$$

Since $\tilde{u}(x_4) = M$ and since

$$\Delta\tilde{u} = \Delta u \geq 0 \text{ on } B_{r_4}(x_4)$$

we find a contradiction by Lemma 2.2.2. ■

Finally let us state a maximum principle at boundary points.

Theorem 2.2.4 (Hopf's boundary point Lemma) *Suppose that $u \in C_0(\overline{\Omega}) \cap C^2(\Omega)$ is a solution of*

$$\begin{cases} -\Delta u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = u_0(x) & \text{for } x \in \partial\Omega, \end{cases} \tag{2.1}$$

with $f \geq 0$, $u_0 \geq 0$ and $(f, u_0) \neq (0,0)$. Then at each point $x^* \in \partial\Omega$ with $u_0(x^*) = 0$ and where Ω satisfies an interior sphere condition, we find for any outward pointing unit vector $\vec{\nu}$, that

$$\liminf_{t \downarrow 0} \frac{u(x^* - t\,\vec{\nu}) - u(x^*)}{t} \in (0, \infty].$$

If u is differentiable at x^* then

$$-\frac{\partial u}{\partial \vec{\nu}}(x^*) > 0.$$

Remark 2.2.5 *It is sufficient, instead of the interior sphere condition at x^*, that a Dini-smooth subdomain $\tilde{\Omega} \subset \Omega$ exists with $x^* \in \partial\tilde{\Omega}$. See [20]. Dini-smooth means that $|\nabla p|$ is Dini-continuous, where $\mathbb{R}^{n-1} \ni x' \mapsto (x', p(x')) \in \mathbb{R}^n$ parametrizes the boundary near x^*. In other words, for some $\varepsilon > 0$ it holds that*

$$\int_0^\varepsilon \frac{\omega(t)}{t} dt < \infty,$$

where ω is the modulus of continuity of $|\nabla p|$.

Proof. Let $B_r(x_0)$ be the ball such that $x^* \in \partial B_r(x_0)$ and $B_r(x_0) \subset \Omega$. By the strong maximum principle, Theorem 2.2.3, one finds that $u(x) > 0$ on $B_r(x_0)$. As in the proof of that theorem one takes $r_4 = \frac{1}{4}r$, $x_4 = x^*$ and $r_5 = \frac{1}{2}r$, $x_5 = \frac{1}{2}(x_0 + x^*)$. Next define similarly

$$\begin{aligned} \tilde{u}(x) &= \varepsilon\left(|x - x_5|^{2-n} - r_5^{2-n}\right) && \text{for } n \geq 3, \\ \tilde{u}(x) &= \varepsilon\left(\log\left(\tfrac{1}{|x-x_5|}\right) - \log\left(\tfrac{1}{r_5}\right)\right) && \text{for } n = 2, \end{aligned}$$

and note that $u(x) \geq \tilde{u}(x)$ on $B_{r_5}(x_5) \cap B_{r_4}(x_4)$, which implies our estimates. ■

The three proofs above can be generalised to more general second order elliptic problems.

2.2.2 Special proofs for the laplacian

For the Laplace operator, and mostly for the Laplace operator only, one may proceed in a more straightforward way. For the Laplacian the following holds:

Proposition 2.2.6 *Suppose that $u \in C^2(\Omega)$, $\overline{B_r(x_0)} \subset \Omega$. If $-\Delta u \geq 0$ on $\overline{B_r(x_0)}$, then*

$$u(x_0) \geq \fint_{\partial B_r(x_0)} u(x)\, d\sigma_x. \tag{2.2}$$

Remark 2.2.7 *Here we used a notation for the integral average:*

$$\fint_S v(t)\,dt = \int_S v(t)\,dt \Big/ \int_S 1dt\,.$$

Remark 2.2.8 *A function u that satisfies (2.2) for every closed ball $\overline{B_r(x_0)} \subset \Omega$, is called superharmonic on Ω. A function that satisfies the inverse inequality is called subharmonic on Ω.*

Corollary 2.2.9 *Suppose that $u \in C^2(\Omega)$ and $\Delta u \geq 0$ on Ω. If u has a maximum in $x_0 \in \Omega$, then $u(x) = u(x_0)$ for all $x \in \Omega$.*

Proof. Apply Proposition 2.2.6 to $-u$ on $B_r(x_0)$ and connect x_0 with any $x \in \Omega$ through a sequence of balls. See Figure 2.2. ■

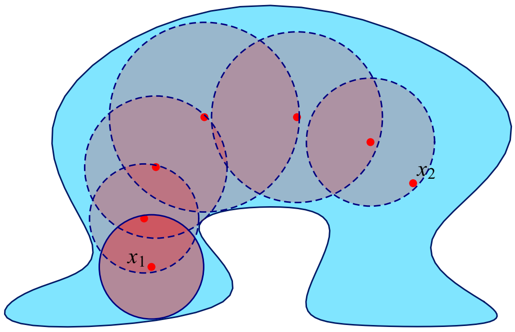


Figure 2.2: Connecting x_1 to x_2 through several balls inside Ω

Proof of Proposition 2.2.6. We write F_n for the fundamental solution of the Laplace operator in $\mathbb{R}^n$, that is

$$F_n(x) = \begin{cases} \frac{1}{\omega_n(n-2)} |x|^{2-n} & \text{for } n \geq 3, \\ \frac{1}{\omega_2} \log\left(\frac{1}{|x|}\right) & \text{for } n = 2, \end{cases}$$

with ω_n the (hyper)surface area of the unit n-sphere.

One obtains with Gauß' formula and the normal $n(x) = \frac{x-x_0}{|x-x_0|}$ that

$$\begin{aligned} 0 &\leq \int_{B_r(x_0)} (-\Delta u(x))\,(F_n(x-x_0) - F_n(r\hat{e}_1))\,dx \\ &= \lim_{\varepsilon \downarrow 0} \int_{\varepsilon<|x-x_0|<r} (-\Delta u(x))\,(F_n(x-x_0) - F_n(r\hat{e}_1))\,dx \\ &= \lim_{\varepsilon \downarrow 0} \left(-\int_{|x-x_0|=\varepsilon} \frac{\partial u(x)}{\partial n} (F_n(x-x_0) - F_n(r\hat{e}_1))\,d\sigma_x + \int_{|x-x_0|=r} u(x) \frac{\partial F_n(x-x_0)}{\partial n} d\sigma_x \right. \\ &\quad \left. - \int_{|x-x_0|=\varepsilon} u(x) \frac{\partial F_n(x-x_0)}{\partial n} d\sigma_x + \int_{\varepsilon<|x-x_0|<r} u(x)\,(-\Delta F_n(x-x_0))\,dx \right) \\ &= 0 - \fint_{\partial B_r(x_0)} u(x)\,d\sigma_x + u(x_0) + 0. \end{aligned}$$

Indeed

$$\frac{\partial F_n(x-x_0)}{\partial n} = \frac{-1}{\omega_n} |x-x_0|^{1-n}$$

and

$$\int_{|x-x_0|=\varepsilon} \frac{\partial u(x)}{\partial n}\left(F_n(x-x_0)-F_n(r\hat{e}_1)\right)d\sigma_x = \mathcal{O}(\varepsilon\log\varepsilon).$$

■

The next result is not a result that only works for the laplacian but it does need symmetry. We will state is for the laplacian but before being able to do so, we need some tools.

First recall that $W_0^{1,2}(\Omega)$ is a Hilbert space with norm

$$\|u\|_{W_0^{1,2}(\Omega)} = \left(\int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2}.$$

Lemma 2.2.10 *If $u \in W_0^{1,2}(\Omega)$, then also $|u| \in W_0^{1,2}(\Omega)$.*

Remark 2.2.11 *It follows that $u^+ = \max(0,u) = \frac{1}{2}(u+|u|)$ and $u^- = \max(0,-u)$ lie in $W_0^{1,2}(\Omega)$.*

Proof. The difficult part is to prove that the weak derivative of $|u|$ is appropriate.

1) For $u \in C^1(\Omega)$ one has for $\varphi \in C_0^\infty(\Omega)$ that

$$\begin{aligned}\int_\Omega \nabla|u|\ \varphi dx &\underset{\text{Def.}}{=} -\int_\Omega |u|\ \nabla\varphi dx = -\int_{[u>0]} u\ \nabla\varphi dx + \int_{[u<0]} u\ \nabla\varphi dx \\ &\underset{\text{i.b.p.}}{=} \int_{[u>0]} \nabla u\ \varphi dx - \int_{[u<0]} \nabla u\ \varphi dx = \int_\Omega \operatorname{sign}(u)\nabla u\ \varphi dx.\end{aligned}$$

So let us set for $u \in W^{1,1}(\Omega)$:

$$\nabla|u| = \operatorname{sign}(u)\nabla u, \tag{2.3}$$

which is formally well-defined since $\operatorname{sign}(u) \in L^\infty(\Omega)$. If $C^1(\Omega) \ni u_m \to u$ in $W^{1,1}(\Omega)$ one finds

$$\begin{aligned}&\left|\int_\Omega \operatorname{sign}(u)\nabla u\ \varphi dx - \int_\Omega \operatorname{sign}(u_m)\nabla u_m\ \varphi dx\right| \le \\ &\left|\int_\Omega (\operatorname{sign}(u)-\operatorname{sign}(u_m))\nabla u\ \varphi dx\right| + \left|\int_\Omega \operatorname{sign}(u_m)(\nabla u - \nabla u_m)\ \varphi dx\right| \le \\ &\int_\Omega |\operatorname{sign}(u)-\operatorname{sign}(u_m)|\,|\nabla u|\ |\varphi|\, dx + \int_\Omega |\nabla u - \nabla u_m|\ |\varphi|\, dx\end{aligned} \tag{2.4}$$

and $\int_\Omega |\nabla u - \nabla u_m|\ |\varphi|\, dx \to 0$ for $m \to \infty$. The other term does not converge in general, but since $u_m \to u$ in $L^1(\Omega)$ there exists a subsequence that converges almost everywhere. Taking the subsequence that converges a.e. and the dominated convergence theorem implies

$$\lim_{m\to\infty}\int_\Omega |\operatorname{sign}(u)-\operatorname{sign}(u_m)|\,|\nabla u|\ |\varphi|\, dx = 0,$$

concludes that (2.4) goes to 0 for a subsequence and hence confirms (2.3).

2) In order to show that $u \in W_0^{1,2}(\Omega)$ implies $|u| \in W_0^{1,2}(\Omega)$ one defines for $\varepsilon > 0$

$$f_\varepsilon(u) = \sqrt{\varepsilon^2+u^2} - \varepsilon.$$

Note that for nice(?) u one finds $\nabla f_\varepsilon(u) = \frac{u}{\sqrt{\varepsilon^2+u^2}}\nabla u$ "$\to$" $\nabla|u|$ for $\varepsilon \downarrow 0$.

More precisely, one finds $f_\varepsilon(u) \in W_0^{1,2}(\Omega)$ and it follows that

$$\int_\Omega |\nabla f_\varepsilon(u)|^2\, dx \le \int_\Omega |\nabla u|^2\, dx.$$

The dominated convergence theorem gives us for some sequence $\{n_k\} \subset \mathbb{N}$ that

$$\int_\Omega |\nabla |u||^2 \, dx = \int_\Omega \lim_{k\to\infty} \left|\nabla f_{1/n_k}(u)\right|^2 dx = \lim_{k\to\infty} \int_\Omega \left|\nabla f_{1/n_k}(u)\right|^2 dx \le \int_\Omega |\nabla u|^2 \, dx.$$

■

Theorem 2.2.12 *Let $\lambda \ge 0$ and suppose that $f \in L^2(\Omega)$. Let $u \in W_0^{1,2}(\Omega)$ be a weak solution of (3.12), i.e. u satisfies (3.13). Then the following holds*

$$f \gneq 0 \implies u \gneq 0.$$

Proof. For $f \ge 0$ and taking $\varphi = u^-$, which lies in $W_0^{1,2}(\Omega)$ by Lemma 2.2.10, one finds that

$$(\lambda u - f)\, u^- = -\lambda \left(u^-\right)^2 - f u^- \le 0$$

and

$$0 = \int_\Omega \left(\nabla u \cdot \nabla u^- + (\lambda u - f)\, u^-\right) dx \le -\int_\Omega \left|\nabla u^-\right|^2 dx \le 0.$$

So $\nabla u^- = 0$ holds and since $u^- = 0$ on $\partial\Omega$ this implies $u^- = 0$ on Ω. The fact, that $u = 0$ is contradicted by $f \gneq 0$, is direct. ■

Session 3

Positivity and elliptic bvp

To consider inverse-positivity for the problem

$$Mu = f$$

in the setting of partial differential equations, one has to include boundary conditions. For the Laplace operator under homogeneous Dirichlet boundary conditions, i.e.

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

the maximum principle guarantees uniqueness of a solution but not yet existence. So we may a priori only say, that *if the solution operator* $(-\Delta)_0^{-1}$ *for (3.1) exists*, then it will be strongly positive:

$$f \gneq 0 \implies u > 0.$$

In any case we still have to specify what these inequalities will mean.

For a sharp statement concerning inverse-positivity we have to have the existence of such a solution operator $(-\Delta)_0^{-1}$. We will address this question in the next sections.

3.1 Second order elliptic

3.1.1 Introduction

We will focus on the homogeneous Dirichlet boundary value problem on $\Omega \subset \mathbb{R}^n$, namely

$$\begin{cases} (\lambda - L)\, u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

and obtain a result as in Theorem 1.1.12. As before

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}\,(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i\,(x) \frac{\partial}{\partial x_i}.$$

The same results concerning the positivity preserving property can be derived for the Neumann, intermediate or mixed boundary conditions except for a shift of λ_1.

But let us first fix the following positivity notations for functions $u : \Omega \to \mathbb{R}$:

- $u \geq 0$ means $u\,(x) \geq 0$ for all $x \in \Omega$,
- $u \gneq 0$ means $u\,(x) \geq 0$ for all $x \in \Omega$ and $u \not\equiv 0$,

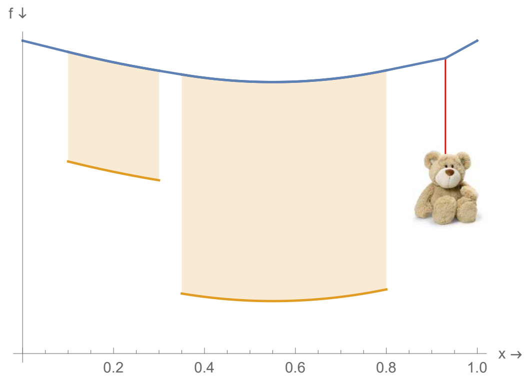
- $u > 0$ means $u(x) > 0$ for all $x \in \Omega$.

The simplest case is one-dimensional. The next example will deal with the corresponding homogeneous Dirichlet problem.

Example 3.1.1 *Consider*

$$\begin{cases} -u''(x) = f(x) \text{ for } x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.3}$$

This boundary value problem models laundry and other things hanging from a line under tension.



We may find an explicit solution by straightforward integration. Starting with

$$u(x) = -\int_0^x \int_0^y f(s)\, ds dy + c_1 + c_2 x,$$

using that $u(0) = 0$ implies $c_1 = 0$, and with an integration by parts it follows that

$$\begin{aligned} u(x) &= -\left[y \int_0^y f(s)\, ds \right]_0^x + \int_0^x y f(y)\, dy + c_2 x \\ &= -x \int_0^x f(s)\, ds + \int_0^x y f(y)\, dy + c_2 x \\ &= \int_0^x (y - x) f(y)\, dy + c_2 x. \end{aligned}$$

From $u(1) = 0$ we find $c_2 = -\int_0^1 (y-1) f(y)\, dy$ and hence

$$\begin{aligned} u(x) &= \int_0^x (y - x) f(y)\, dy - x \int_0^1 (y-1) f(y)\, dy \\ &= \int_0^x (y - xy) f(y)\, dy - x \int_x^1 (y-1) f(y)\, dy \\ &= \int_0^x (1-x)\, y \; f(y)\, dy + \int_x^1 x(1-y) \; f(y)\, dy. \end{aligned}$$

So one may write

$$u(x) = \int_0^1 g(x,y) f(y)\, dy \text{ with } g(x,y) = \begin{cases} (1-x)\, y & \text{for } 0 \le y \le x \le 1, \\ (1-y)\, x & \text{for } 0 \le x < y \le 1. \end{cases} \tag{3.4}$$

The function g is called a Green function for (3.5). Since

$$g(x,y) > 0 \text{ for } x, y \in (0,1),$$

the expression shows that

$$f \gneqq 0 \implies u > 0.$$

A graph of this function one finds in Figure 3.1.

One may compare with the discretised version in Figure 1.3.

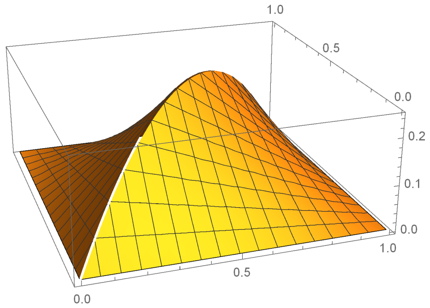


Figure 3.1: Sketch of the Green function

Example 3.1.2 *For $\lambda > 0$ one may imagine a model where the line, from which the laundry hangs, is connected to a ceiling by a lot of uniformly distributed elastic ropes. That would lead to*

$$\begin{cases} \lambda u(x) - u''(x) = f(x) & \text{for } x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{3.5}$$

Mathematically one might even consider $\lambda < 0$ but it will be hard to conceive a model for that case. Formally one may solve this problem through a Green function:

$$g_\lambda(x,y) = \begin{cases} \dfrac{\sin\left(\sqrt{-\lambda}\,(1-\max(x,y))\right)\sin\left(\sqrt{-\lambda}\min(x,y)\right)}{\sqrt{-\lambda}\sin\left(\sqrt{-\lambda}\right)} & \text{for } \lambda < 0, \\[2ex] (1-\max(x,y))\min(x,y) & \text{for } \lambda = 0, \\[2ex] \dfrac{\sinh\left(\sqrt{\lambda}\,(1-\max(x,y))\right)\sinh\left(\sqrt{\lambda}\min(x,y)\right)}{\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\right)} & \text{for } \lambda > 0. \end{cases}$$

One directly checks that for $\lambda > -\pi^2$, which corresponds to the first eigenvalue, one finds that

$$g_\lambda(x,y) > 0 \text{ for all } x,y \in (0,1).$$

With a more careful check one notices that $g_\lambda(x,y)$ is not defined for $\lambda = -k^2\pi^2$ with any $k \in \mathbb{N}^+$ and even that for all other $\lambda < -\pi^2$ the function $g_\lambda(x,y)$ will be sign changing. In Figure 3.2 are some sketches of two g_λ.

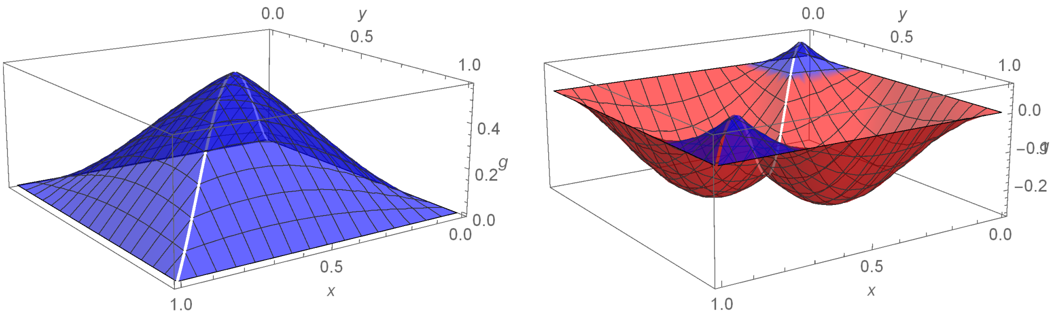


Figure 3.2: Sketch of g_λ for $\lambda = -5 > -\pi^2$ and for $\lambda = -18 < -\pi^2$.

3.1.2 Solutions

In one dimension the regularity of the solution directly follows from the expression in (3.4) involving f only. It does not depend on the boundary.

In higher dimensions the boundary plays a crucial role which regularity the solution can obtain. Indeed the regularity of the boundary determines the maximal global regularity that a solution may obtain and one has to choose the space accordingly. Assuming more for the boundary one increases the settings possible. The different settings will have their own advantages and disadvantages. So let us first address the main notions of solution.

Definition 3.1.3 *A function $u : \bar{\Omega} \to \mathbb{R}$ is called*

- *a classical solution of (3.2), when $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies (3.2);*
- *a weak solution of (3.2), when $u \in W_0^{1,2}(\Omega)$ satisfies*

$$\int_\Omega \left(\lambda u\varphi + \sum_{i,j=1}^{n} a_{ij}\left(\partial_j u\right)\left(\partial_i \varphi\right) - \sum_{i=1}^{n} b_i \left(\partial_i u\right)\varphi - f\varphi \right) dx = 0 \text{ for all } \varphi \in W_0^{1,2}(\Omega). \tag{3.6}$$

- *a strong solution of (3.2) in L^p-sense, when $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for some $p > 1$ satisfies the differential equation in (3.2) a.e.*

Remark 3.1.4 *A strong solution in L^2-sense is a weak solution and this follows through an integration by parts. A weak solution is in general not a classical solution and vice versa.*

Remark 3.1.5 *Note that $C^2(\Omega) \cap C(\overline{\Omega})$ does not have a norm for which it becomes a Banach space. A space like $C^{2,\gamma}(\overline{\Omega})$ does, but also needs quite a regular boundary such that solutions, even for $f \in C_0^\infty(\Omega)$ will lie in that space. On the other hand, the space $W_0^{1,2}(\Omega)$ does not need any regularity of the boundary. For any $f \in L^2(\Omega)$ Riesz' Representation Theorem gives the existence of a weak solution. But since no second derivatives are defined, arguments that use these will fail.*

3.1.2.1 The classical and the strong setting

As just mentioned, $C^2(\Omega) \cap C(\overline{\Omega})$ does not have a norm for which it becomes a Banach space. A space like $C^{2,\gamma}(\overline{\Omega})$ does, but also needs quite a regular boundary in order that solutions, even for $f \in C_0^\infty(\Omega)$ will lie in that space.

How to obtain a solution of (3.1) for example for $f \in C(\overline{\Omega})$? If we extend f by 0 outside of Ω to $\bar{f}$ we get at most $\bar{f} \in L^\infty(\mathbb{R}^n)$. Taking the convolution with the fundamental solution F_n one obtains by $F_n * \bar{f} \in W^{2,p}_{loc}(\mathbb{R}^n)$ for all $p \in (1,\infty)$. The weak second derivatives exist and one may show that

$$-\Delta\left(F_n * \bar{f}\right) = \bar{f} \text{ in } L^p(\mathbb{R}^n)\text{-sense.}$$

By Sobolev imbedding it holds that $F_n * \bar{f} \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in [0,1)$. But in general one does not obtain $C^2(\overline{\Omega})$. Also for $f \in C^\gamma(\overline{\Omega})$ one does not find a better regularity on $\overline{\Omega}$. The interior regularity however improves for such f, that is $\left(F_n * \bar{f}\right)_{|\Omega} \in C^{2,\gamma}(\Omega)$. For this interior regularity see [12].

So $u_1 = F_n * \bar{f}$ solves the differential equation but what about the boundary conditions?

It remains to find a solution u_2 of

$$\begin{cases} -\Delta u_2(x) = 0 & \text{for } x \in \Omega, \\ u_2(x) = u_1(x) & \text{for } x \in \partial\Omega. \end{cases} \tag{3.7}$$

hen $u = u_1 - u_2$ will solve (3.1). Here is where Perron's method sets in. First we need a definition.

Definition 3.1.6 *The function $w \in C(\overline{\Omega})$ is a barrier function at $x^* \in \partial\Omega$, whenever*

1. *w is superharmonic in Ω;*

2. *$w > 0$ on $\overline{\Omega} \setminus \{x^*\}$ and $w(x^*) = 0$.*

If Ω satisfies a uniform exterior cone condition, then there exists a barrier function at each boundary point.

Theorem 3.1.7 (Perron) *The function*

$$u(x) = \sup\{v(x); v \text{ is subharmonic and } v(x) \le \varphi(x) \text{ on } \partial\Omega\}$$

is harmonic in Ω. Moreover, if each boundary point has a barrier function and $\varphi \in C(\partial\Omega)$, then $u \in C(\overline{\Omega})$ and $u = \varphi$ on $\partial\Omega$.

For a complete proof see [12, Section 2.8]. A crucial step is the so-called harmonic lifting of a subharmonic function u on a ball $B_r(x_0)$ with $\overline{B_r(x_0)} \subset \Omega$:

$$\bar{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega \setminus B_r(x_0), \\ \displaystyle\int_{\partial B_r(x_0)} K_{B_r(x_0)}(x,y)\, u(y)\, d\sigma_y & \text{for } x \in B_r(x_0). \end{cases}$$

with $K_{B_r(x_0)} : B_r(x_0) \times \partial B_r(x_0) \to \mathbb{R}^+$ the Poisson kernel for $B_r(x_0)$ defined by

$$K_{B_r(x_0)}(x,y) = \frac{r^2 - |x - x_0|^2}{n\omega_n r\, |x-y|^n}.$$

One may check that $\bar{u}$ solves

$$\begin{cases} -\Delta \bar{u}(x) = 0 & \text{for } x \in B_r(x_0), \\ \bar{u}(x) = u(x) & \text{for } x \in \partial B_r(x_0). \end{cases}$$

3.1.2.2 Solution operators in a C-setting

Coming back to (3.2) with $\lambda = 0$, that is

$$\begin{cases} -Lu(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{3.8}$$

one may prove the following result.

Theorem 3.1.8 *Let Ω be a bounded domain in $\mathbb{R}^n$.*

1. *If Ω satisfies a uniform exterior cone condition, then for each $f \in C(\overline{\Omega})$ there exists a unique strong solution $u \in C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ of (3.8) with $p \in [n, \infty)$.*

2. *If* $\partial\Omega \in C^{1,1}$*, then for each* $f \in C(\overline{\Omega})$ *there exists a unique strong solution* $u \in C^1(\overline{\Omega}) \cap W^{2,p}(\Omega)$ *of (3.8) with* $p \in (1,\infty)$*.*

Proof. A real proof is outside the scope of these lecture notes and we will just give references.

With $p \geq n$ the existence and uniqueness of a solution $u \in C(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ can be found in [12, Theorem 9.30], which confirms the first claim.

The second result can be found in [12, Theorem 9.15], that is, the existence of a unique strong solution in $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ for any $p \in (1,\infty)$. We may use any p since $C^1(\overline{\Omega}) \subset L^p(\Omega)$. For domains satisfying a uniform interior cone condition, which holds for $\partial\Omega \in C^{1,1}$, $W^{2,p}(\Omega)$ imbeds in $C^1(\overline{\Omega})$ for $p > n$. ■

We have for a bounded domain $\Omega \subset \mathbb{R}^n$ the following hierarchy:

$$\partial\Omega \in C^{1,1}$$
$$\Downarrow$$
$$\Omega \text{ satisfies uniform exterior sphere condition}$$
$$\Downarrow$$
$$\Omega \text{ satisfies uniform exterior cone condition}$$

- $\partial\Omega \in C^{1,1}$ means that the boundary can be covered by finitely many open blocks on each of which $\partial\Omega$ can be written as the graph of a $C^{1,1}$-function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$.
- A uniform exterior sphere condition means that $r > 0$ exists, such that for each $x^* \in \partial\Omega$ there is $B_r(y) \subset \Omega^c$ with $x^* \in \partial B_r(y)$.
- A uniform cone condition means that for some $c > 0$ and $\varepsilon > 0$ there is a finite cone $C = \{(x', x_n)\,; c\,|x'| < x_n < \varepsilon\}$ which can be rotated by R such that for each $x^* \in \partial\Omega$ one finds $x^* + RC \subset \Omega^c$.

3.1.3 Finding the first eigenfunction

Considering again

$$\begin{cases} (\lambda - \Delta)\, u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{3.9}$$

the weak setting is very convenient to find the first eigenvalue. Indeed, the first eigenvalue, with our sign convention, is defined by the Rayleigh quotient:

$$\lambda_1 = -\inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega u^2 dx} < 0$$

and the corresponding eigenfunction $\varphi_1 \in W_0^{1,2}(\Omega)$ is such that

$$\int_\Omega (\lambda_1 \varphi_1 \varphi + \nabla\varphi_1 \nabla\varphi)\, dx = 0 \text{ for all } \varphi \in W_0^{1,2}(\Omega)\,.$$

All other eigenvalues λ_i are real and satisfy $\lambda_i < \lambda_1$.

If one replaces Δ in (3.9) by the general L these arguments are no longer valid. For a sharp result of the positivity preserving property the existence of a first eigenvalue is necessary and the usual argument that replaces Perron-Frobenius for the matrix case is a result by Krein and Rutman.

3.1.3.1 Versions of Krein-Rutman

Before stating the result we need to fix a few notions which are used. Necessary is that the space of functions is an ordered Banach space. Both $C(\overline{\Omega})$ and $L^p(\Omega)$ are a priori suitable.

Definition 3.1.9 *The cone of positive elements P in the Banach space X is total, whenever*

$$\overline{P-P} = X.$$

A version that can be found in [1] is as follows.

Theorem 3.1.10 (Krein-Rutman I) *Let X be an ordered Banach space with total positive cone P. Suppose that $T \in \mathcal{L}(X)$ is compact, satisfies $T(P\setminus\{0\}) \subset P^\circ$ and has a positive spectral radius $\rho(T)$. Then $\rho(T)$ is an eigenvalue of T and of the dual operator T^* with eigenvectors in P and in P^*, respectively.*

Remark 3.1.11 *One may show that all other eigenvalues μ_i satisfy $|\mu_i| < \mu_1$ and that there is no other positive eigenfunction.*

A problem in the assumptions of this last version is P°. In $L^p(\Omega)$ the open cone is empty. In $C(\overline{\Omega})$ the Dirichlet boundary conditions prevent that $T(P\setminus\{0\}) \subset P^\circ$, since

$$P^\circ = \left\{ u \in C(\overline{\Omega}); \min_{x\in\Omega} u(x) > 0 \right\}.$$

A way out is to define

$$\begin{array}{c} C_e(\overline{\Omega}) = \left\{ u \in C_0(\overline{\Omega}); \exists c>0 \text{ such that } |u(x)| \le c\,d(x,\partial\Omega) \right\} \\ \text{with } \|u\|_e = \sup \frac{|u(x)|}{d(x,\partial\Omega)} \text{ and } d(x,\partial\Omega) = \inf_{x^*\in\partial\Omega} |x-x^*| \end{array}$$

with $d(\cdot,\partial\Omega)$ the distance to boundary. In order that this approach works, we need Hopf's boundary point Lemma.

From [19] one may find a more convenient version.

Theorem 3.1.12 (Krein-Rutman II) *Let X be an ordered Banach space with a total positive cone. Suppose that $T \in \mathcal{L}(X)$ is compact, positive and such that for some $u_0 \in X$ with $u_0 \gneqq 0$ and $r > 0$ it holds that $Tu_0 \ge ru_0$.*

Then $\mu_1 := \rho(T) \ge r$ is an eigenvalue with a positive eigenfunction φ_1.

Remark 3.1.13 *In the case that $T = (-\Delta)_0^{-1}$ one may take for u_0 a nontrivial nonnegative function with compact support.*

Remark 3.1.14 *This theorem still needs something like irreducibility to conclude that the positive eigenfunction is 'unique' and that all other eigenvalues μ_i satisfy $|\mu_i| < \mu_1$. A sufficient condition that is used, is the following:*

- *for every $f \in X$ with $f \gneqq 0$ there exists $c_f > 0$ such that $Tf \ge c_f u_0$.*

which can be weakened to:

- *there exists $k \in \mathbb{N}^+$ such that for every $f \in X$ with $f \gneqq 0$ there exists $c_f > 0$ such that $T^k f \ge c_f u_0$.*

For functions f, g in $L^p(\Omega)$ or $C(\overline{\Omega})$ the infimum and supremum

$$(f \vee g)(x) = \max(f(x), g(x)) \text{ and } (f \wedge g)(x) = \min(f(x), g(x))$$

are well-defined in $L^p(\Omega)$ or $C(\overline{\Omega})$. Such ordered spaces are called lattices. One defines

$$|f| = (f \vee 0) + (-f \vee 0).$$

If $|f| \leq |g|$ implies $\|f\| \leq \|g\|$, then the space is called a normed lattice. If a Banach space is also a normed lattice, then it is called a Banach lattice. Notice that $W^{1,2}(\Omega)$ is not a Banach lattice, since $|f| \leq |g|$ does not imply $\|f\|_{W^{1,2}(\Omega)} \leq \|g\|_{W^{1,2}(\Omega)}$.

Combining with a result in [24] one finds the following version in [27].

Theorem 3.1.15 (Krein-Rutman-de Pagter) *Let X be a Banach lattice with $\dim(X) > 1$. If $T \in \mathcal{L}(X)$ is compact, positive and irreducible, then $\mu_1 := \rho(T) > 0$ is an eigenvalue with a positive eigenfunction φ_1. Moreover, every nonnegative eigenfunction is a multiple of φ_1. Every other eigenvalue μ_i of T satisfies $|\mu_i| < \mu_1$.*

Remark 3.1.16 *T is irreducible, if X and $\{0\}$ are the only T-invariant lattice ideals. The subspace $D \subset X$ is a lattice ideal, if $|f| \leq |d|$ and $d \in D$ implies that $f \in D$.*

This version does not need Hopf's boundary point Lemma, and hence can deal with less regular boundaries, and can be directly applied to so-called fully coupled cooperative systems.

3.1.4 PPP for the Dirichlet problem

Although we state the result for

$$\begin{cases} (\lambda - \Delta) u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{3.10}$$

the next theorem also holds for more general second order elliptic operators and for more general homogeneous boundary conditions.

Theorem 3.1.17 (PPP for the Dirichlet problem) *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{1,1}$.*

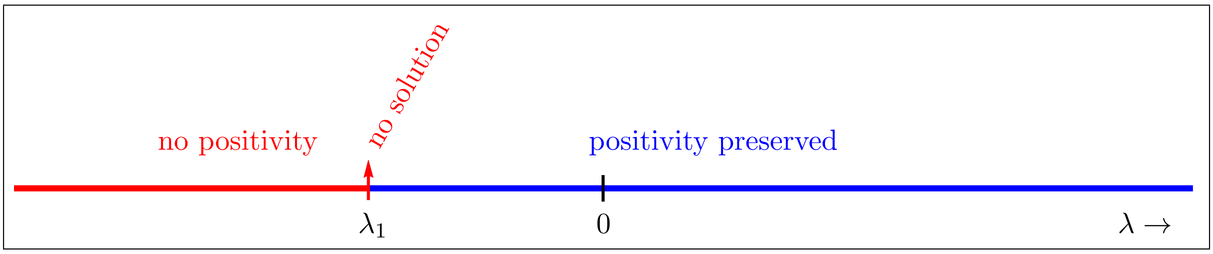
1. *Then there is a first eigenfunction $\varphi_1 \in C^1(\overline{\Omega}) \cap C^2(\Omega)$, i.e.*

$$\begin{cases} (\lambda_1 - \Delta) \varphi_1(x) = 0 & \text{for } x \in \Omega, \\ \varphi_1(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{3.11}$$

   and there are no eigenvalues λ with $\lambda > \lambda_1$. It holds that $\lambda_1 \in \mathbb{R}^-$ and assuming φ_1 is normalised by $\max_{x \in \Omega} \varphi_1 = 1$ one finds $\varphi_1 > 0$.

2. (a) *If $\lambda > \lambda_1$ then for every $0 \lneq f \in C(\overline{\Omega})$ problem (3.10) has a solution $u > 0$.*
   (b) *If $\lambda = \lambda_1$ then for every $0 \lneq f \in C(\overline{\Omega})$ problem (3.10) has no solution.*
   (c) *If $\lambda < \lambda_1$ then for every $0 \lneq f \in C(\overline{\Omega})$ problem (3.10) has no nonnegative solution u (either u changes sign or doesn't exist).*

Remark 3.1.18 *For this theorem it is sufficient that $T = (\Delta)_0^{-1} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is positive and compact. So one may weaken the boundary smoothness. The irreducibility still follows from the strong maximumm principle.*



Proof. We assume that there is a solution operator $(-\Delta)_0^{-1}$ from $C(\overline{\Omega})$ to $C^1(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$ which is continuous as $(-\Delta)_0^{-1} : C(\overline{\Omega}) \to C^1(\overline{\Omega})$. Consider $A := I \circ (-\Delta)_0^{-1} : C(\overline{\Omega}) \to C(\overline{\Omega})$ with $I : C^1(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})$ the imbedding. Since the imbedding is compact, so is A. The maximum principle shows that A is positive and even strongly positive. Hence, if u_0 is a function with compact support, then $Au_0 \geq ru_0$ for some $r > 0$. The above version of the Krein-Rutman theorem then gives the existence of a first eigenvalue $\mu_1 > 0$ and eigenfunction $\varphi_1 \in C(\overline{\Omega})$ for A, i.e.

$$A\varphi_1 = \mu_1 \varphi_1$$

with $\mu_1 = \rho(A)$. The eigenspace is spanned by a strictly positive eigenfunction φ_1 and all other eigenvalues μ_i satisfy $|\mu_i| < \mu_1$. Moreover, any positive eigenfunction is a multiple of φ_1.

We have $\lambda_1 = -\mu_1^{-1}$ and $\varphi_1 = -\lambda_1 (-\Delta)_0^{-1} \varphi_1 \in C^1(\overline{\Omega}) \cap W^{2,p}_{loc}(\Omega)$. Since eigenvalues λ_i for (3.11) give eigenvalues for A with eigenvalue $\mu_i = -\lambda_i^{-1}$ the first item is proven.

For the second item notice that for $\lambda \in (\lambda_1, 0)$ we have

$$\lambda A u + u = Af$$

and

$$u = (I - (-\lambda) A)^{-1} Af.$$

Since $\rho(-\lambda A) = |\lambda/\lambda_1| < 1$ we find that

$$u = \sum_{k=0}^{\infty} (-\lambda A)^k Af$$

converges and since A and hence $-\lambda A$ are strongly positive operators, that $f \gneq 0$ implies $u > 0$. The last two remaining items use the argument in (1.5). ■

3.1.4.1 The weak setting

The space $W_0^{1,2}(\Omega)$ does not need any regularity of the boundary. Indeed, for every $f \in L^2(\Omega)$ there is a weak solution u. Existence follows by Riesz' representation theorem for $-\Delta$ and by Lax-Milgram for more general L. Indeed $W_0^{1,2}(\Omega)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{W_0^{1,2}(\Omega)} = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx.$$

Example 3.1.19 *Consider $\Omega = B_1(0) \subset \mathbb{R}^n$ and $u(x) = \sqrt[3]{1 - |x|^2}$. Obviously $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Since*

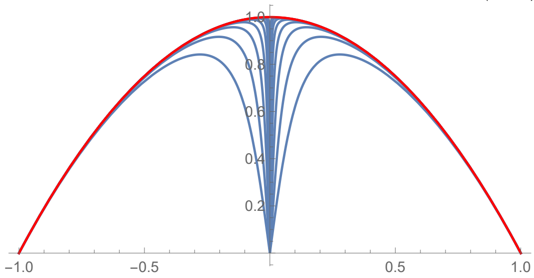
$$\nabla u(x) = (1 - |x|)^{-\frac{2}{3}} \left(-\tfrac{2}{3} (1 + |x|)^{-\frac{2}{3}} x \right)$$

and since $2 \times \frac{-2}{3} < -1$, the function u does not belong to $W_0^{1,2}(\Omega)$. The function u is a classical solution of (3.1) with some ugly right hand side.

Example 3.1.20 *Consider $\Omega = B_1(0) \setminus \{0\} \subset \mathbb{R}^n$ with $n \geq 3$ and $u(x) = 1 - |x|^2$. One may show that u_ε defined by*

$$u_\varepsilon(x) = \frac{|x|}{\sqrt{\varepsilon^2 + |x|^2}} u(x)$$

lies in $C^2(\Omega) \cap C^{0,1}(\overline{\Omega}) \cap W_0^{1,2}(\Omega)$ and moreover $u_\varepsilon \to u$ in $W_0^{1,2}(\Omega)$. Note that u does not satisfy the boundary condition in 0 pointwisely, but is a weak solution of (3.1) on Ω for $f(x) = 2n$.



Weak solutions of

$$\begin{cases} (-\Delta + \lambda) u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{3.12}$$

are functions $u \in W_0^{1,2}(\Omega)$, which satisfy

$$\int_\Omega (\nabla u \cdot \nabla\varphi + (\lambda u - f)\varphi)\, dx = 0 \text{ for all } \varphi \in W_0^{1,2}(\Omega). \tag{3.13}$$

3.1.4.2 A symmetric setting

When the problem is symmetric, or more precisely self-adjoint, the weak setting is very convenient to find a start with existence and positivity. The first eigenvalue, with our sign convention, is defined by by the Rayleigh quotient:

$$\lambda_1 = -\inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\, dx}{\int_\Omega u^2 dx} < 0. \tag{3.14}$$

One show that the infimum is in fact a minimum and a function $\varphi_1 \in W_0^{1,2}(\Omega)$ for which the minimum is assumed is an eigenfunction. Next one find that φ_1 is such that

$$\int_\Omega (\lambda_1 \varphi_1 \varphi + \nabla\varphi_1 \nabla\varphi)\, dx = 0 \text{ for all } \varphi \in W_0^{1,2}(\Omega).$$

Using regularity estimates and bootstrapping results in $\varphi_1 \in C^\infty(\Omega)$. In order to show that φ_1 has a fixed sign, one assumes that φ_1 changes sign and defines

$$\Omega^+ = \{x \in \Omega; \varphi_1(x) > 0\} \text{ and } \Omega^- = \{x \in \Omega; \varphi_1(x) < 0\}.$$

Since

$$\begin{aligned} -\lambda_1 &= \frac{\int_{\Omega^+} |\nabla\varphi_1|^2\, dx + \int_{\Omega^-} |\nabla\varphi_1|^2\, dx}{\int_{\Omega^+} \varphi_1^2 dx + \int_{\Omega^-} \varphi_1^2 dx} \\ &= \theta \frac{\int_{\Omega^+} |\nabla\varphi_1|^2\, dx}{\int_{\Omega^+} \varphi_1^2 dx} + (1-\theta) \frac{\int_{\Omega^-} |\nabla\varphi_1|^2\, dx}{\int_{\Omega^-} \varphi_1^2 dx} \end{aligned}$$

with

$$\theta = \frac{\int_{\Omega^+} \varphi_1^2 dx}{\int_{\Omega^+} \varphi_1^2 dx + \int_{\Omega^-} \varphi_1^2 dx} \in (0,1),$$

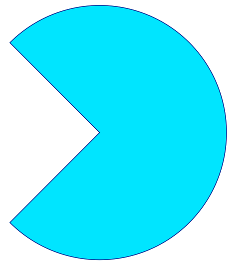
it follows from (3.14) that

$$-\lambda_1 = \frac{\int_{\Omega^+} |\nabla \varphi_1|^2 \, dx}{\int_{\Omega^+} \varphi_1^2 dx} = \frac{\int_{\Omega^-} |\nabla \varphi_1|^2 \, dx}{\int_{\Omega^-} \varphi_1^2 dx}.$$

Hence also $\max(\varphi_1, 0)$ and $-\min(\varphi_1, 0)$ are eigenfunctions. Such eigenfunctions however contradict Corollary 2.2.9.

3.1.4.3 An exceptional solution

Consider the function u, defined in radial coordinates on a pacman-shaped domain



$$\Omega = \left\{(r, \varphi); 0 < r < 1, \ |\varphi| < \tfrac{1}{2}\alpha\right\}$$

with $\alpha \in (\pi, 2\pi)$ as follows:

$$u(r, \varphi) = \left(r^{-\frac{\pi}{\alpha}} - r^{\frac{\pi}{\alpha}}\right) \cos\left(\tfrac{\pi}{\alpha}\varphi\right). \tag{3.15}$$

One finds that

- $\Delta u = 0$,
- $u = 0$ a.e. on $\partial\Omega$,
- $u > 0$ in Ω,
- $u \in L^p(\Omega)$ for $p < \frac{2\alpha}{\pi}$, which includes $p = 2$ for the pacman domain.
- $u \in W_0^{1,p}(\Omega)$ for $p < \frac{2\alpha}{\pi+\alpha}$, which ranges from $p = 1$ till $p = \frac{4}{3}$ for α above.

Doesn't the maximum principle forbid the existence of such a solution?

One may convince oneself that this function is not in $C(\overline{\Omega}) \cap C(\Omega)$, nor in $W^{1,2}(\Omega)$. Nevertheless such a function plays a crucial role in the case of a hinged plate. See (4.5).

3.2 Cooperative Second Order Systems

Cooperative second order elliptic systems are of the following type

$$\begin{pmatrix} -\Delta & 0 & \cdots & 0 \\ 0 & -\Delta & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} + \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1m} \\ h_{21} & h_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1} & \cdots & \cdots & h_{mm} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{pmatrix},$$

more general written as

$$Lu + Hu = f,$$

where

- L is a diagonal matrix of uniformly second order elliptic operators with suufficiently smooth coefficients;
- H is a matrix with nonpositive off-diagonal terms.

Concerning inequalities for a vectorvalued function $u : \Omega \to \mathbb{R}^m$ we will use:

- $u \geq 0$ when $\forall i \in \{1, \ldots, m\}$ $\forall x \in \Omega :$ $u_i(x) \geq 0$,
- $u \gneq 0$ when $u \geq 0$ and $\exists i \in \{1, \ldots, m\}$ $\exists x \in \Omega :$ $u_i(x) > 0$,
- $u > 0$ means $\forall i \in \{1, \ldots, m\}$ $\forall x \in \Omega :$ $u_i(x) > 0$.

As in [27] let us consider the boundary value problem for $u, f : \Omega \to \mathbb{R}^m$ defined by

$$\begin{cases} \lambda u + Lu + Hu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Theorem 3.2.1 (PPP for cooperative second order elliptic systems) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^{1,1}$-boundary and let L be a diagonal matrix of uniformly elliptic second order operators, i.e.*

$$L_k = -\sum_{ij=1}^{n} \frac{\partial}{\partial x_i} a_{ijk}(x) \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} b_{jk}(x) \frac{\partial}{\partial x_j} + c_k(x)$$

with $a_{ijk}, b_{jk} \in C^1(\overline{\Omega})$ and for some $c > 0$:

$$\sum_{ij=1}^{n} a_{ijk}(x)\, \xi_i \xi_j \geq c\, |\xi|^2 \text{ for all } x \in \overline{\Omega},\ \xi \in \mathbb{R}^m.$$

Let $H \in M^{m \times m}\left(C(\overline{\Omega})\right)$ be such that

- *all off-diagonal terms are nonpositive (i.e. $-H$ is cooperative);*
- $\tilde{H} = \left(\|h_{ij}(x)\|_\infty\right)_{i,j=1}^{n}$ *is irreducible.*

Then the following holds:

1. *there is a first eigenfunction $\varphi_1 \in C^1(\overline{\Omega}; \mathbb{R}^m) \cap C^2(\Omega; \mathbb{R}^m)$, i.e.*

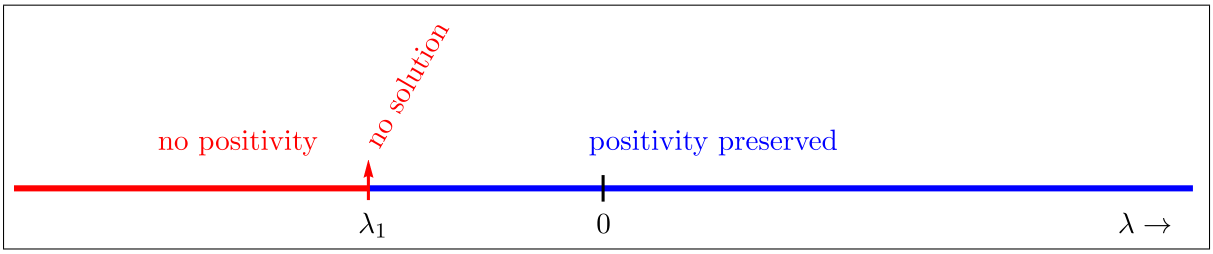
$$\begin{cases} (\lambda_1 I + L + H)\, \varphi_1 = 0 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.17}$$

   and there are no eigenvalues λ with $\lambda > \lambda_1$. Assuming φ_1 is normalised by

$$\max_{\substack{x \in \Omega \\ 1 \leq k \leq m}} \varphi_{1,k} = 1$$

   one finds $\varphi_1 > 0$. Any nonnegative eigenfunction is a multiple of φ_1.

2. (a) *If $\lambda > \lambda_1$ then for every $0 \lneq f \in C(\overline{\Omega}; \mathbb{R}^n)$ problem (3.16) has a solution $u > 0$.*
   (b) *If $\lambda = \lambda_1$ then for every $0 \lneq f \in C(\overline{\Omega}; \mathbb{R}^n)$ problem (3.16) has no solution.*
   (c) *If $\lambda < \lambda_1$ then for every $0 \lneq f \in C(\overline{\Omega}; \mathbb{R}^n)$ problem (3.16) has no nonnegative solution u (either u changes sign or doesn't exist).*



Proof. For λ sufficiently large,

$$(\lambda I + L)_0^{-1} : C(\overline{\Omega}; \mathbb{R}^n) \to C_0(\overline{\Omega}; \mathbb{R}^n) \cap C^1(\overline{\Omega}; \mathbb{R}^n) \cap W^{2,p}(\Omega; \mathbb{R}^n)$$

is a well defined nonnegative diagonal operator. The diagonal elements are strongly positive. So for λ large problem (3.16) can be rewritten to

$$u = ((\lambda + c)\, I + L)_0^{-1} \, ((cI - H)\, u + f)\,,$$

where we take $c \geq 0$ such that $cI - H$ has only nonnegative entries. Now fix $\tilde{\lambda} > 0$ large enough such that for $\lambda \geq \tilde{\lambda}$

$$R_\lambda := \mathcal{I}\,((\lambda + c)\, I + L)_0^{-1}\,(cI - H) : C(\overline{\Omega}; \mathbb{R}^n) \to C_0(\overline{\Omega}; \mathbb{R}^n)$$

one has

$$\rho\,(R) < 1.$$

Hence

$$((\lambda + c)\, I + L + H)_0^{-1} = \sum_{k=0}^{\infty} (R_\lambda)^k \, ((\lambda + c)\, I + L)^{-1}$$

Note that $cI - H$ and $((\lambda + c)\, I + L)_0^{-1}$ are both nonnegative. More precisely,

1. If $u \geq 0$ and $u_k \gneq 0$, then $((\lambda + c)\, I + L)_0^{-1}\, u \gneq 0$ and
$$\left(((\lambda + c)\, I + L)_0^{-1}\, u \right)_k > 0.$$

2. If $u \geq 0$ and $u_k > 0$, then $(cI - H)\, u \gneq 0$, $((cI - H)\, u)_k > 0$ and for all ℓ with $h_{\ell k} \neq 0$:
$$((cI - H)\, u)_\ell \gneq 0.$$

   Combining these two results one finds that

3. If $u \geq 0$ and $u_k \gneq 0$, then for all ℓ with $h_{\ell k} \neq 0$ and for $\ell = k$ we have
$$(R_\lambda u)_\ell > 0.$$

   Since we assumed that $\tilde{H}$ is irredudible, it follows that

4. If $u \gneq 0$, then
$$R_\lambda^m u > 0.$$

So for $\tilde{\lambda}$ large enough

$$\left(\left(\tilde{\lambda} + c \right) I + L + H \right)_0^{-1} = \sum_{k=0}^{\infty} (R_\lambda)^k \left(\left(\tilde{\lambda} + c \right) I + L \right)_0^{-1}$$

is a strongly positive operator. Strongly positive implies irreducibility. One may see that

$$\left(\left(\tilde{\lambda}+c\right)I+L\right)_0^{-1}:C(\overline{\Omega};\mathbb{R}^n)\to C_0(\overline{\Omega};\mathbb{R}^n)$$

is compact and hence also $\left(\left(\tilde{\lambda}+c\right)I+L+H\right)_0^{-1}$. By the Krein-Rutman Theorem it follows that there exists $\mu_1>0$ and a corresponding eigenfunction $\varphi_1>0$, such that

$$\left(\left(\tilde{\lambda}+c\right)I+L+H\right)_0^{-1}\varphi_1=\mu_1\varphi_1. \tag{3.18}$$

All other eigenvalues μ_i of $\left(\left(\tilde{\lambda}+c\right)I+L+H\right)_0^{-1}$ satisfy $|\mu_i|<\mu_1$. Setting

$$\lambda_1=\tilde{\lambda}+c-\frac{1}{\mu_1}$$

one finds that (λ_1,φ_1) satisfies (3.17). Since this holds for all $\tilde{\lambda}$ large enough, we find from $|\mu_i|<\mu_1$ that all other eigenvalues satisfy

$$\mathrm{Re}\lambda_i<\mathrm{Re}\lambda_1.$$

This completes the first part.

For the second part we may conclude from the above that $(\lambda I+L+H)_0^{-1}$ exists for all $\lambda>\lambda_1$ and that this operator is strongly positive for $\lambda\geq\hat{\lambda}$ with $\hat{\lambda}$ large enough. What about $\lambda\in\left(\lambda_1,\hat{\lambda}\right)$? Here we may use a similar series as above:

$$(\lambda I+L+H)_0^{-1}=\sum_{k=0}^{\infty}\left(\left(\hat{\lambda}I+L+H\right)_0^{-1}\left(\hat{\lambda}-\lambda\right)\right)^k(\lambda I+L+H)_0^{-1}.$$

Since $\rho\left(\left(\hat{\lambda}I+L+H\right)_0^{-1}\right)=\left(\hat{\lambda}-\lambda_1\right)^{-1}$ and since $\frac{\hat{\lambda}-\lambda}{\hat{\lambda}-\lambda_1}<1$, the series converges. Since the series consists of strongly positive operators, also $(\lambda I+L+H)_0^{-1}$ is positive. This proves the first claim of item 2. The remaining claims use φ_1^* for the formally adjoint operator $\left(\lambda I+L^*+H^T\right)_0^{-1}$ similar as in the proof of ■

Session 4

Positivity and real higher order

4.1 Noncooperative Second Order Systems

4.1.1 Similar to cooperative

In the case that the diagonal operator L has identical second order elliptic operators on the diagonal, one may look for other cones than just the positive cone, say that we replace $u \geq 0$ by $Su \geq 0$, where S is some constant transformation matrix. Then

$$(\lambda I + L + H)\, u = f$$

is replaced by

$$\left(\lambda I + L + SHS^{-1}\right) Su = Sf.$$

Indeed, for example if $L_i = -\Delta$, then $SLS^{-1} = L$.

> Is there a simple condition on H such that a transformation S exists, that turns SHS^{-1} into an M-matrix?

Weinberger in [29] studied this question. For 2 dimensions and

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $c > 0$ the condition is $\frac{1}{4}(a-d)^2 + bc$, which is nothing but the condition for H having real eigenvalues. For larger dimensions the condition becomes a mess.

4.1.2 Strictly noncooperative

For a system like

$$\begin{cases} -\Delta u - \varepsilon v = f & \text{in } \Omega, \\ -\Delta v + \varepsilon u = g & \text{in } \Omega, \\ \quad u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

the eigenvalues, expressed in ν_k those of the Dirichlet Laplace, are

$$\lambda_{ik,\pm} = \nu_k \pm \varepsilon i$$

and there is no hope for a preserved cone. Nevertheless, setting

$$\mathcal{G} = (-\Delta)_0^{-1}$$

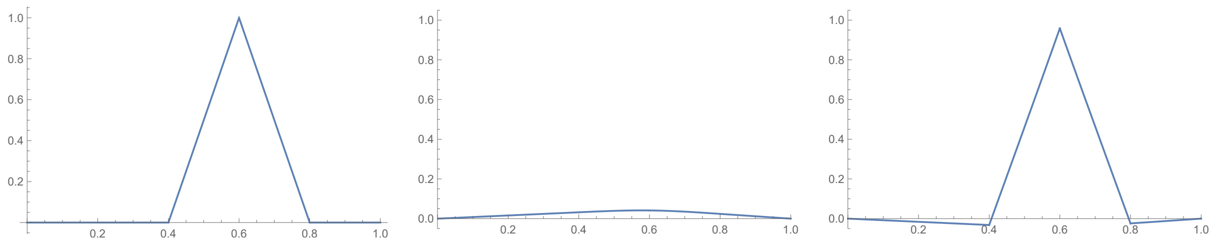


Figure 4.1: For $\Omega = (0,1) \in \mathbb{R}$ a sketch of a function f, $\mathcal{G}f$ and $f - \mathcal{G}f$.

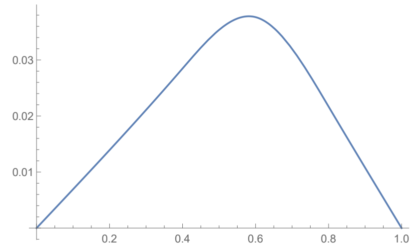


Figure 4.2: For $\Omega = (0,1) \in \mathbb{R}$ a sketch for the f in Figure 4.1 of $\mathcal{G}f - \mathcal{G}^2 f$.

one finds that

$$u = \varepsilon \mathcal{G} v + \mathcal{G} f \text{ and } v = -\varepsilon \mathcal{G} u + \mathcal{G} g.$$

Hence

$$u = -\varepsilon^2 \mathcal{G}^2 u + \varepsilon \mathcal{G}^2 g + \mathcal{G} f$$

and since $\rho(\mathcal{G}) = \nu_1^{-1}$ it follows for $\varepsilon \in (0, \nu_1)$ that

$$\begin{aligned} u &= \left(I + \varepsilon^2 \mathcal{G}^2\right)^{-1} \left(\varepsilon \mathcal{G}^2 g + \mathcal{G} f\right) = \sum_{k=0}^{\infty} \left(-\varepsilon^2 \mathcal{G}^2\right)^k \left(\varepsilon \mathcal{G}^2 g + \mathcal{G} f\right) \\ &= \left(I - \varepsilon^2 \mathcal{G}^2\right) \mathcal{G} \left(\sum_{k=0}^{\infty} \left(\varepsilon^4 \mathcal{G}^4\right)^k \right) \left(\varepsilon \mathcal{G} g + f\right) \\ &= \left(I - \varepsilon \mathcal{G}\right) \mathcal{G} \left(I + \varepsilon \mathcal{G}\right) \left(\sum_{k=0}^{\infty} \left(\varepsilon^4 \mathcal{G}^4\right)^k \right) \left(\varepsilon \mathcal{G} g + f\right). \end{aligned}$$

Most factors in this operator are positive. The only bad one is

$$I - \varepsilon \mathcal{G}$$

and positivity cannot be saved by taking ε small.

However, the combination of $I - \varepsilon \mathcal{G}$ with $\mathcal{G}$ might do the trick.

In order to show that $(I - \varepsilon \mathcal{G}) \mathcal{G}$ is a positive operator we may the integral expression for the Green function. Indeed, the operator $(-\Delta)_0^{-1}$ can be formally written as an integral operator:

$$(\mathcal{G} f)(x) := \left((-\Delta)_0^{-1} f \right)(x) = \int_{\Omega} G_{\Omega}(x, y) f(y) \, dy.$$

For some special domains explicit Green functions are known:

$$G_{B_1(0)}(x, y) = \begin{cases} \frac{1}{4\pi} \ln \left(1 + \frac{\left(1 - |x|^2\right)\left(1 - |y|^2\right)}{|x - y|^2} \right) & \text{when } n = 2, \\ \frac{1}{n\omega_n} \left(|x - y|^{2-n} - \left| x\,|y| - \frac{y}{|y|} \right|^{2-n} \right) & \text{when } n \geq 3. \end{cases}$$

The Green function $G_\Omega : \overline{\Omega} \times \overline{\Omega} \to [0, \infty]$ satisfies (formally) for each $y \in \Omega$

$$\begin{cases} G_\Omega(x,y) = G_\Omega(y,x) & \text{for } x \in \Omega, \\ -\Delta_x G_\Omega(x,y) = \delta_y(x) & \text{for } x \in \Omega, \\ G_\Omega(x,y) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

We are interested in

$$\begin{aligned} (I - \varepsilon\mathcal{G})\,\mathcal{G} f(x) &= \int_\Omega G_\Omega(x,y)\, f(y)\, dy - \varepsilon \int_\Omega \int_\Omega G_\Omega(x,z)\, G_\Omega(z,y)\, f(y)\, dy dz \\ &= \int_\Omega G_\Omega(x,y) \left(1 - \varepsilon \frac{\int_\Omega G_\Omega(x,z)\, G_\Omega(z,y)\, dz}{G_\Omega(x,y)}\right) f(y)\, dy. \end{aligned}$$

Theorem 4.1.1 (Cranston-Fabes-Zhao [6]) *Suppose that Ω is a bounded Lipschitz domain. Then there is $M_\Omega \in \mathbb{R}^+$ such that*

$$\sup_{x,y \in \Omega} \frac{\int_\Omega G_\Omega(x,z)\, G_\Omega(z,y)\, dz}{G_\Omega(x,y)} \le M_\Omega.$$

Corollary 4.1.2 *If $\varepsilon < M_\Omega^{-1}$, then $(I - \varepsilon\mathcal{G})\,\mathcal{G} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is strongly positive.*

Remark 4.1.3 *The expectation of the lifetime of a conditioned Brownian motion, namely Brownian motion starting at x, killed at the boundary $\partial\Omega$ and conditioned to converge to y, is given by*

$$\mathbb{E}_x^y(\tau_\Omega) = \frac{\int_\Omega G_\Omega(x,z)\, G_\Omega(z,y)\, dz}{G_\Omega(x,y)}.$$

4.2 Fourth order models, a hinged plate

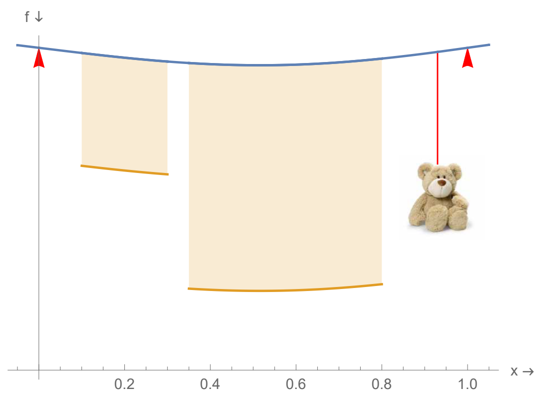
In one dimension $-u'' = f$ is a simple differential equation for hanging weight on a line. Going to two dimensions the differential equation $-\Delta u = f$ is used for the vertical deviation of a membrane under a weight distribution f. When fixing the line and the membrane at the boundary, that is, setting for example $u = 0$ t the boundary, the positivity preserving property translates into: the line/membrane moves in the direction that it is pushed:

$$f \ge 0 \implies u \ge 0.$$

One may think of other models, where one may expect such behaviour.

Example 4.2.1 *When we hang our laundry on a tube instead on a line, one obtains the following problem*

$$\begin{cases} u''''(x) = f(x) \text{ for } x \in (0,1), \\ u(0) = u''(0) = u(1) = u''(1) = 0. \end{cases} \tag{4.1}$$



This models laundry and other things hanging from a beam, which position is fixed at both ends.

We may find an explicit solution by straightforward integration. The formula is a rather ugly one, but one can see that also this Green function is positive.

This result should not be surprising, since one may recognise an iterated Dirichlet Laplace.

Also the two dimensional problem makes sense. A thin polygonal shaped plate for which one fixes the position at the boundary, but not the angle, results in the following problem.

$$\begin{cases} \Delta^2 u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = \Delta u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{4.2}$$

Guessing that we may use the positivity of the iterated Laplacian is ok on convex polygons. Indeed, setting $v = -\Delta u$ one obtains

$$\begin{cases} -\Delta v(x) = f(x) & \text{for } x \in \Omega, \\ -\Delta u(x) = v(x) & \text{for } x \in \Omega, \\ u(x) = v(x) = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{4.3}$$

and the problem splits nicely. So, if

$$u = \left((-\Delta)_0^{-1}\right)^2 f, \tag{4.4}$$

then one finds for (4.2) that

$$f \gneq 0 \implies u > 0.$$

However, when the domain has a nonconvex corner, the result is no longer that obvious. The problem in 4.2 has a well-defined solution $u_1 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and also a solution u_2 with $u_2, \Delta u_2 \in W_0^{1,2}(\Omega)$. These two solutions are not the same. It seems that the first one is physically more relevant. Only the second one, that is u as in (4.4), has the sign preserving property. In case of a pacman domain the difference between these two solutions is precisely a multiple of $(-\Delta)_0^{-1} u$ with u as in (3.15). See [22]. It is shown that

$$u_1 - u_2 = c(-\Delta)_0^{-1} u. \tag{4.5}$$

One should mention that Davini was the first to consider the supported plate with corners from a thorough analytical point of view. See [8].

One may consider the problem above with a feedback through an elastic medium and one obtains the additional λ at the positiin as before:

$$\begin{cases} \lambda u(x) + \Delta^2 u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = \Delta u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{4.6}$$

Even on smooth domains the splitting is not so nice, since one becomes

$$\begin{cases} (-\Delta + i\sqrt{\lambda})v = f & \text{in } \Omega, \\ (-\Delta - i\sqrt{\lambda})u = v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

which becomes complex unless $\lambda \le 0$.

Help comes from the from the following observation. Let

$$(t, x, y) \mapsto p(t, x, y) : \mathbb{R}^+ \times \Omega \times \Omega \to [0, \infty)$$

be the heat kernel for the heat equation on Ω and set

$$H(\lambda, t) := \begin{cases} \text{for } \lambda > 0 & \frac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}}, \\ \text{for } \lambda = 0 & t, \\ \text{for } \lambda < 0 & \frac{\sinh(\sqrt{-\lambda} t)}{\sqrt{-\lambda}}. \end{cases}$$

One may show that the solution of 4.6 can formally be written as

$$u(x) = \int_\Omega \left(\int_{t=0}^\infty H(\lambda, t)\ p(t, x, y)\ dt \right) f(y)\ dy.$$

Since $p(t, x, y) \simeq \exp(\mu_1 t)$ with $\mu_1 < 0$ the first eigenvalue for the Dirichlet Laplacian, one obtains:

- when $\lambda < -\mu_1^2$ divergence of the integral;
- when $-\mu_1^2 < \lambda \leq 0$ convergence of the integral and positivity;
- when $\lambda > 0$ convergence of the integral but positivity maybe only for small λ.

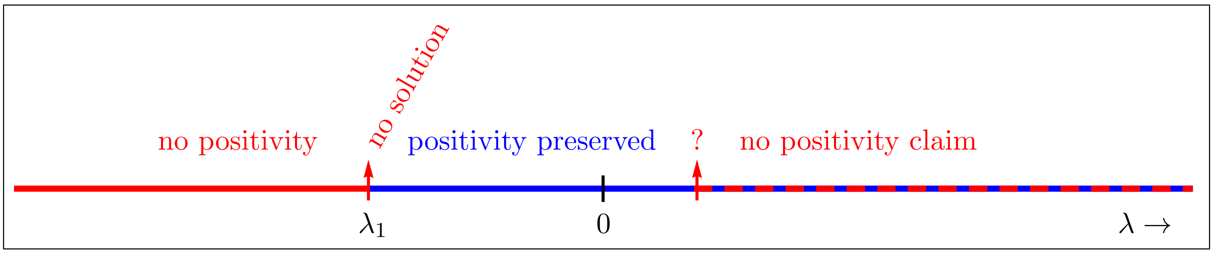


Figure 4.3: Concerning positivity the above picture describes the case for (4.6)

For more information see [28].

4.3 Fourth order models, a clamped plate

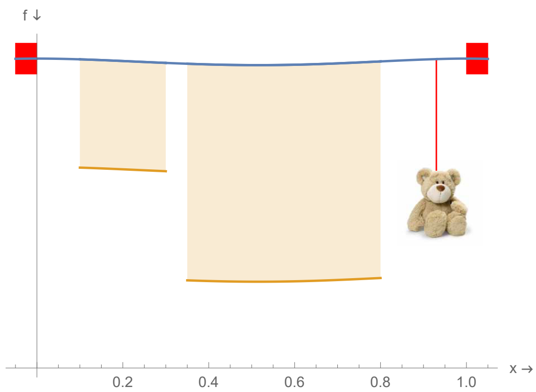
In the next example there is no obvious to use iterated second order problems, since the boundary conditions do not split nicely.

Example 4.3.1 *Here we consider the case, where laundry hangs from a tube or beam, of which we fix both the position and the angle at the boundary. This is the so-called clamped boundary condition*

$$\begin{cases} u''''(x) = f(x) & \textit{for } x \in (0,1), \\ u(0) = u'(0) = u(1) = u'(1) = 0. \end{cases} \tag{4.8}$$

This models laundry and other things hanging from a beam, which is clamped at both ends. That means that both the position and the first derivative is fixed. Here we fixed both to be zero.

Also here we may find an explicit solution.



In one dimension one finds again that $f > 0$ implies $u > 0$. When generalizing this to two or higher dimensions one obtains

$$\begin{cases} \Delta^2 u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = \frac{\partial}{\partial n} u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{4.9}$$

The positivity preserving property for (4.9) is lost in generically. For a long time the conjecture named after Boggio-Hadamard that the clamped problem is positivity preserving qt least for convex domains was open. Duffin in 1946 gave a counterexample. By now, one believes that, except on very few domains, such as balls in $\mathbb{R}^n$, PPP remains. Analytical results are however rare. In two dimensions small perturbations of the disk are allowed without ruining the positivity.

In any case, the part where the solution becomes negative when the force f is positive, seems to be extremely small. There is evidence from numerics that for $f > 0$ one generically would find

$$\left\|u^-\right\|_\infty \leq 10^{-4} \left\|u^+\right\|_\infty .$$

Also such a result seems quite hard to verify analytically.

The explicit Green function for (4.9) the case the ball is known since Boggio around 1900. With the function $d(\cdot)$, that measures the distance to the boundary

$$d(x) := d(x, \partial\Omega) = \inf \{|x - x^*| \,;\, x^* \in \partial\Omega\},$$

his explicit formula gives the following estimates:

- for $n \geq 5$
$$G_B(x,y) \approx g_n(x,y) := |x-y|^{4-n} \min\left(1, \frac{d(x)^2 d(y)^2}{|x-y|^4}\right);$$

- for $n = 4$
$$G_B(x,y) \approx g_n(x,y) := \log\left(1 + \frac{d(x)^2 d(y)^2}{|x-y|^4}\right);$$

- for $n \leq 3$
$$G_B(x,y) \approx g_n(x,y) := (d(x)d(y))^{2-n/2} \min\left(1, \frac{d(x)d(y)}{|x-y|^2}\right)^{n/2}.$$

Here $a \approx b$ means that there are uniform constants $c_1, c_2 \in \mathbb{R}^+$, such that $c_1 \leq \frac{a}{b} \leq b$.

Remark 4.3.2 *In the estimates one should recognize the contribution of the fundamental solution, that is* $|x-y|^{4-n}$ *for* $n \geq 5$*, the quadratic boundary behaviour* $d(x)^2$ *due to the clamped boundary conditions as well as the symmetry in* x *and* y *for Green functions.*

For more arbitrary domains one cannot expect positivity, so the estimate from below cannot hold true. Nevertheless the Green function is close and the following result has been proven in [13]:

Theorem 4.3.3 *Let* $\Omega \subset \mathbb{R}^n$ *be bounded and with a smooth domain. Then there exists constants* $c_1, c_2, C \in \mathbb{R}^+$ *such that*

$$c_1 g_n(x,y) \leq G_\Omega(x,y) + C d(x)^2 d(y)^2 \leq c_2 g_n(x,y) \text{ for all } x, y \in \Omega.$$

Remark 4.3.4 *The addition of a constant times* $d(x)^2 d(y)^2$ *is the lowest possible perturbation that cannot be improved in general. One may try to find the optimal constant* C *as a function of the domain but that seems a very tough problem.*

4.4 Other positivity questions

4.4.1 Szegö's claim

Whenever the solution operator, or say the Green function, is positive, then one may conclude that the first eigenfunction is positive. Indeed, this is what Szeg{ö) stated assuming that the Green function is positive. Since we now know that the assumption is false, one may ask:

When is the first eigenfunction for the clamped problem positive?

Also this question is largely open and it is not even clear, which kind of answer one may get. Coffmann and Duffin could show that on an annulus with a small hole, the first eigenfunction is sign-changing and has multiplicity 2. Also domains with narrow passages or close to corners seem to have a sign-changing first eigenfunction.

4.4.2 A question by Filoche and Mayboroda

In [9] some arguments simplify, if the solution of

$$\begin{cases} \Delta^2 u = 1 & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.10}$$

is positive. Of course a positive Green function $G_\Omega(\cdot,\cdot)$ for (4.9) also implies that the solution for a uniform weight, i.e. $f \equiv 1$, is positive. For the solution of (4.10) to be positive it would however be sufficient that

$$\int_\Omega G_\Omega(x,y)\,dy \geq 0 \text{ for all } x \in \Omega.$$

So the question would be:

Is the solution of the clamped plate with a constant weight of fixed sign?

Also this question has a negative answer: There are domains for which the solution

$$u(x) = \int_\Omega G_\Omega(x,y)\,dy$$

of (4.10) changes sign. See [16].

4.4.3 The real supported plate

Maybe due to sloppy translations between Russian and English the hinged plate is often referred to by supported plate. Indeed, when the hinged plate is positivity preserving, this would not be a bad guess. The real supported model however has a unilateral boundary condition.

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega, \\ \sigma\Delta u + (1-\sigma)\partial_n^2 u = 0 & \text{on } \partial\Omega, \\ u(x) = 0 \text{ or } \partial_n\left(\Delta u(x) + (1-\sigma)\partial_\tau^2 u(x)\right) = 0 & \text{for } x \in \partial\Omega. \end{cases} \tag{4.11}$$

If $u = 0$ on the boundary and the boundary consists of straight lines, we return to (4.2).

One may show that if $f < 0$, i.e. the weight pushing down, which seems more natural when thinking of gravity, that the solution changes sign near every corner. See the excerpt from [23] in Figure 4.4.

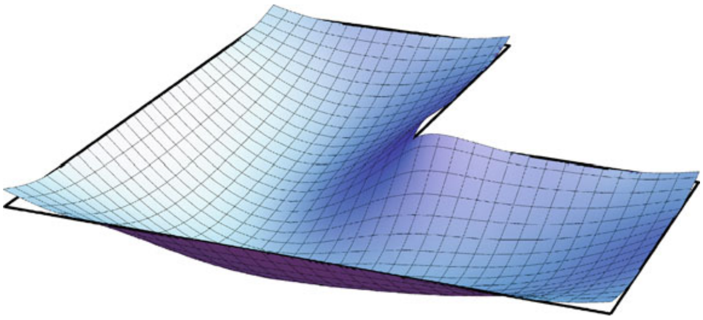


FIG. 6. An L-shaped plate with a uniform load ($f = -1$) leads to a solution that moves upwards near all corner points

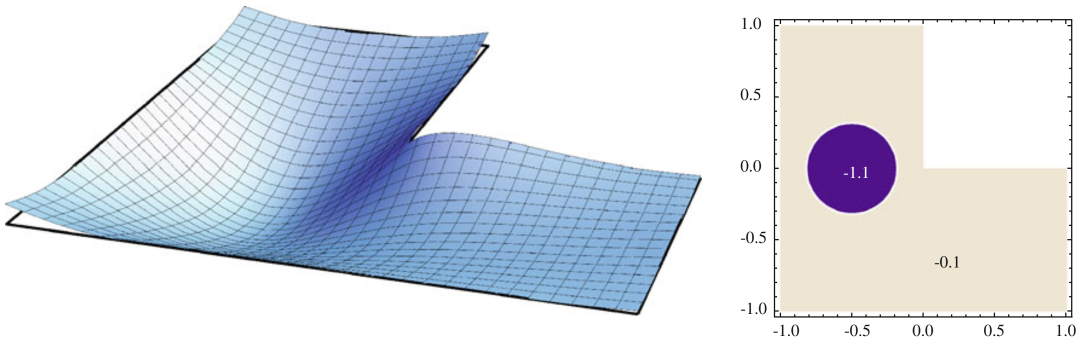


FIG. 7. An L-shaped plate with small load ($f = -.1$) everywhere except for a local heavier load ($f = -1.1$) on the *dark circular area*. On the *right* this force density

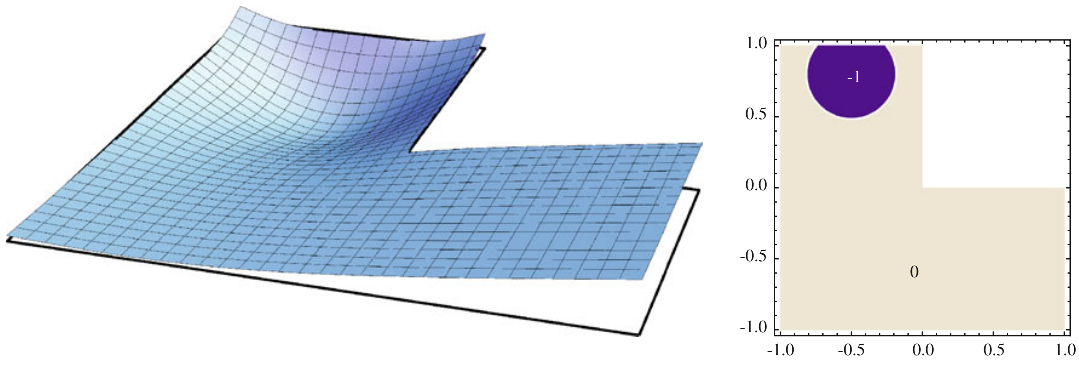


FIG. 8. An L-shaped plate loaded locally on one side results to a large free boundary on the other side

Figure 4.4: A copy taken from [23]

5

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