Optimal conditions for anti-maximum principles

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Abstract

The resolvent for some polyharmonic boundary value problems is positive for λ positive and less than the first eigenvalue. It is known that beyond this first eigenvalue a sign-reversing property exists. Such a result is called an anti-maximum principle. Depending on the boundary conditions, the dimension of the domain and the order of the operator, the result is uniform or not. In the non-uniform case the right hand side needs to be in $L^p(\Omega)$ with p large enough. Sharp estimates for iterated Green functions are used in order to prove that such restrictions are optimal both for the non-uniform and the uniform anti-maximum principle. We will also use these estimates to give an alternative proof of the (uniform) anti-maximum principle.

1 Introduction

1.1 Example

Let us consider the elliptic boundary value problem

$$\begin{cases} (-\Delta)^3 u = \lambda u + f & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$. For this system a first eigenvalue $\lambda_1 > 0$ exists such that for $\lambda \in [0, \lambda_1)$ a sign preserving property holds true: $f \geq 0$ implies that $u \geq 0$. A sign reversing property exists for $\lambda > \lambda_1$. Indeed, similarly as Clément and Peletier [4] proved for (general) second order elliptic differential equations with zero Dirichlet boundary condition, one may show that for smooth $f \gneqq 0$ there exists $\delta > 0$ such that $u < 0$ for $\lambda \in (\lambda_1, \lambda_1 + \delta)$. Such result is known as an anti-maximum principle. In [5] it is shown that for (1) with $n \geq 4$ such numbers δ can be found, which depend on f. There it is also proven that for $n \leq 3$ one could choose δ independently of f; a uniform anti-maximum principle.

Using the Green function estimates from the previous paper [12] it will be proven that the restrictions on (1) for a uniform anti-maximum principle to hold, and similar restrictions on more general systems, cannot be improved. These estimates will also provide an alternative proof of anti-maximum principles.

1.2 General setting

Let A be an elliptic operator of order $2m$ on a bounded smooth domain $\Omega \subset \mathbb{R}^n$, and let the boundary conditions B on $\partial\Omega$ be such that for $\lambda = 0$ the system

$$\begin{cases} Au &= \lambda u + f \quad \text{in } \Omega, \\ Bu &= 0 \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

is positively self-adjoint in some appropriate real Hilbert space of real-valued weakly differentiable functions. Moreover let us assume that the first eigenfunction is simple and positive. Denoting this first eigenvalue/function by (λ_1, φ_1) one expects the solution of (2) for $\lambda \neq \lambda_1$ near λ_1 to have the same sign as $\frac{1}{\lambda_1 - \lambda} \langle \varphi_1, f \rangle = \frac{1}{\lambda_1 - \lambda} \int_\Omega \varphi_1(x) f(x)\, dx$. And indeed, if the eigenfunctions $\{\varphi_i; i \in \mathbb{N}^+\}$ form a complete orthonormal system the solution operator for (2) can be written as

$$u = (A - \lambda)^{-1} f = \sum_{i=1}^{\infty} \frac{\langle \varphi_i, f \rangle}{\lambda_i - \lambda} \varphi_i$$

and one expects the pole in λ_1 to dominate the sign of u for λ near λ_1. That is, assuming $f > 0$, for $\lambda_1 - \lambda < 0$ but near to 0 the solution u will be negative. For $\lambda_1 - \lambda > 0$ and small the solution will be positive.

For this heuristic argument to become rigorous, observing that

$$\begin{aligned} (A - \lambda)^{-1} f &= (A - \lambda)^{-1} \left(f - \langle \varphi_1, f \rangle \varphi_1 + \langle \varphi_1, f \rangle \varphi_1\right) \\ &= \frac{1}{\lambda_1 - \lambda} \Big(\langle \varphi_1, f \rangle \varphi_1 + (\lambda_1 - \lambda)(A - \lambda)^{-1} \left(f - \langle \varphi_1, f \rangle \varphi_1\right) \Big), \end{aligned}$$

it is sufficient to prove that

$$|\lambda_1 - \lambda| \left| (A - \lambda)^{-1} \left(f - \langle \varphi_1, f \rangle \varphi_1\right)(x) \right| \leq \langle \varphi_1, f \rangle \; \varphi_1(x) \tag{3}$$

holds for nonnegative f and $|\lambda_1 - \lambda|$ small.

The result that u is negative for sufficiently good positive f and $\lambda \in (\lambda_1, \lambda_1 + \delta_f)$, where δ_f is some small positive constant depending in general on f, is called an *Anti-Maximum Principle*. Clément and Peletier in [4] obtained such results for general second order elliptic operators with Dirichlet boundary condition. They remarked that in dimension 1 and with Neumann boundary condition, the constant δ_f can be chosen independently of f. Such a stronger result is called a *Uniform Anti-Maximum Principle*. For short we will denote these results by AMP and UAMP.

In [18] it is shown that for $A = -\Delta$ and B the Dirichlet boundary condition the AMP is not uniform. In [5] and [6] UAMP's were recently proved for some higher order elliptic operators. In the present paper we use pointwise estimates for iterated Green functions to give an alternative proof of the result in [6]. Complementary, these estimates allow us to prove that the restrictions on the coefficients are indeed sharp as conjectured. For second order boundary value problems the idea to use Green function estimates for AMP's is due to Takáč [19]. AMP's for second order elliptic equations are also obtained by Pinchover in [16] and [17].

2 The main results

A genuine restriction for sign preserving and reversing is positivity of the first eigenfunction. It is well known that for second order elliptic operators with Dirichlet boundary conditions on bounded domains a maximum principle exists and in the self-adjoint setting it is relatively easy to show that the first eigenvalue is simple and has a positive eigenfunction. Such a general result does not hold for higher order elliptic operators on general domains. For two classes of problems, however, a kind of maximum principle holds and the first eigenfunction is positive. Namely, 1) higher order boundary value problems that allow a decomposition into second order boundary value problems, and 2) polyharmonic operators with Dirichlet boundary conditions with a ball as domain.

In order to define an appropriate boundary behavior of such eigenfunctions we need the distance d to the boundary

$$d(x) = \inf\{|x - x^\star|\,; x^\star \in \partial\Omega\}. \tag{4}$$

Let $m, k \in \mathbb{N}^+$. Defining $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$\begin{array}{rcl} D(A) & = & H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega) \\ A & = & (-\Delta)^m : D(A) \subset L^2(\Omega) \to L^2(\Omega), \end{array} \tag{5}$$

A is indeed positively self-adjoint. One finds that A^k, as well as A, has a simple first eigenvalue with a positive eigenfunction satisfying for some c_1, $c_2 > 0$

$$c_2\, d(x)^m \le \varphi_1(x) \le c_1\, d(x)^m, \tag{6}$$

if one the following conditions is satisfied:

Condition 1 *Suppose that either*

i. $m = 1$ *and* $\Omega \subset \mathbb{R}^n$ *is bounded with* $\partial\Omega \in C^\infty$, *or*

ii. $m \ge 2$ *and* $\Omega = \{x \in \mathbb{R}^n; |x| < R\}$ *for some* $R > 0$.

Remark 1 *Both conditions can be weakened. In the first case* $\partial\Omega \in C^2$ *is sufficient for an appropriate eigenfunction but the statement of the next theorem will be somewhat more involved. In the second case for* $n = 2$ *one may allow domains* Ω *that are close to but not necessarily equal to a disk. See [9].*

Indeed, Condition 1 guarantees the existence of a simple positive first eigenvalue with a positive eigenfunction, see e.g. [6], [7, Remark 1]. We recall that the estimate from below is obtained as follows. The first condition implies existence of an appropriate eigenfunction due to Hopf's boundary point lemma. The second condition guarantees that $(A - \lambda)^{-1}$ is positivity preserving for $\lambda \in [0, \lambda_1)$ as a consequence of Boggio's explicit formula for the Green function for A on balls in $\mathbb{R}^n$. Jentzsch's result [13] gives the eigenfunction. According to a very weak generalized Hopf lemma ([11, Theorem 3.2]) we have $\left(-\frac{\partial}{\partial\nu}\right)^m \varphi_1 > 0$ on ∂B with ν the outward normal. The estimate from above in (6) follows from $\left(\frac{\partial}{\partial\nu}\right)^j \varphi_1 | \partial\Omega = 0$ ($j = 0, \ldots, m-1$), since φ_1 is sufficiently smooth.

Using the estimates for iterated Green functions from the preceding paper [12] we may improve the main theorem in [6] and find that the condition $n < 2m(k-1)$ below is sharp for the Uniform Anti-Maximum Principle. Condition 1 is also sufficient to find the estimates for the (iterated) Green functions that will be used in the proofs.

Theorem 1 (UAMP) *Take* $k, m \in \mathbb{N}^+$. *Let* A *be as in (5) and suppose that Condition 1 holds. Let* λ_1 *be the smallest eigenvalue of* A^k *and consider*

$$A^k u = \lambda u + f. \tag{7}$$

Then the first eigenvalue is simple and the corresponding eigenfunction (taken positive) satisfies (6). Moreover, the following are equivalent:

a. $n < 2m(k-1)$;

b. There exists $\delta > 0$ *such that for all* $\lambda \in (\lambda_1, \lambda_1 + \delta)$ *and* $f \in L^2(\Omega)$ *with* $0 \not\equiv f \ge 0$ *the solution* u *of (7), which belongs to* $C^m(\bar{\Omega})$, *satisfies for some* $c = c(f) > 0$

$$\left\{ \begin{array}{rcll} u(x) & < & -c\ (d(x))^m & \text{for all } x \in \Omega, \\ \left(\frac{\partial}{\partial\nu}\right)^i u(x) & = & 0 & \text{for all } x \in \partial\Omega \text{ and } i \in \{0, 1, \ldots, m-1\}. \end{array} \right. \tag{8}$$

Remark 2 *The system in (7) corresponds to the boundary value problem*

$$\begin{cases} \qquad (-\Delta)^{mk} u &= & \lambda u + f & & \text{in } \Omega, \\ \left(\frac{\partial}{\partial\nu}\right)^i (-\Delta)^{mj} u &= & 0 & \begin{array}{l}\text{for } i = 0, \ldots, m-1 \\ \text{and } j = 0, \ldots, k-1\end{array} & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Remark 3 *Since δ does not depend on f, the UAMP above also holds for $f \in L^p(\Omega)$, with $0 \not\equiv f \geq 0$, for any $p \in (1,2]$. For $n < 2m(k-1)$ one has $W^{2mk,p}(\Omega) \hookrightarrow C^m(\bar{\Omega})$, and hence the result follows, because the Green function for (7) does not depend on p. Also the case $p = 1$ can be covered by using a kind of duality argument.*

Remark 4 *Let $m = 1$ and $\Omega \subset \mathbb{R}^n$ bounded with $\partial\Omega \in C^\infty$. Then for iterated Dirichlet Laplacians (i.e. polyharmonic operators under so-called Navier boundary conditions):*

$$\begin{cases} (-\Delta)^k u = \lambda u + f & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = \ldots = \Delta^{k-1} u = 0 & \text{on } \partial\Omega, \end{cases}$$

the Anti-Maximum Principle is uniform if and only if $n < 2(k-1)$.

In what follows we have to interpret boundary value problems like (7) in various L^p-spaces. If $p \in (1,\infty)$ we define in analogy with (5)

$$\begin{aligned} D(A_p) &= H^{2m,p}(\Omega) \cap H_0^{m,p}(\Omega) \\ A_p &= (-\Delta)^m : D(A_p) \subset L^p(\Omega) \to L^p(\Omega). \end{aligned} \tag{10}$$

As a corresponding L^1-theory does not hold, in order to cover also the case $p = 1$ we consider the Green function $G_{m,n}^{(k)}$ for (7) with $\lambda = 0$. The Green operator $A_1^{-k} := \int_\Omega G_{m,n}^{(k)}(x,y) f(y)\, dy$ maps $L^1(\Omega) \to H^{2mk-1,q}(\Omega)$ for $q < \frac{n}{n-1}$. Solving boundary value problems has to be understood in a suitable weak sense. For regular values λ, i.e. λ not an eigenvalue, of A_1^{-k} a solution e.g. of (7) is given by $u = \left(I - \lambda A_1^{-k}\right)^{-1} A_1^{-k} f$, where A_1^{-k} is considered as bounded operator in $L^1(\Omega)$.

The estimates for the Green functions in [12] also give the optimal range of the nonuniform Anti-Maximum Principle. For the Dirichlet Laplacian the results of Clément and Peletier [4] imply that for every $0 < f \in L^p(\Omega)$ with $p > n$ a number $\delta_f > 0$ exists such that the solution u for $\lambda \in (\lambda_1, \lambda_1 + \delta_f)$ is negative. In [18] it is proven that $p > n$ is sharp: there is a positive $f \in L^n(\Omega)$ such that the solution u changes sign for all $\lambda \in (\lambda_1, \lambda_2)$.

Theorem 2 (AMP) *Take $k, m \in \mathbb{N}^+$ and $p \in (1,\infty)$. Suppose that Condition 1 is satisfied and again denote by λ_1 the smallest eigenvalue of A^k. Then the following holds:*

- *Suppose that $n \leq m(2k-1)$. If $f \in L^1(\Omega)$ and $\langle \varphi_1, f \rangle > 0$, then there exists $\delta_f > 0$ such that for all $\lambda \in (\lambda_1, \lambda_1 + \delta_f)$ the solution u_λ of (7) satisfies (8).*
- *Suppose that $n > m(2k-1)$.*
  - *i. If $f \in L^p(\Omega)$ and $\langle \varphi_1, f \rangle > 0$, with $p > \frac{n}{m(2k-1)}$, then there exists $\delta_f > 0$ such that for all $\lambda \in (\lambda_1, \lambda_1 + \delta_f)$ the solution u_λ of (7) satisfies (8).*
  - *ii. If $p = \frac{n}{m(2k-1)}$, then there exists $0 < f \in L^p(\Omega)$ such that for all regular $\lambda > \lambda_1$ the solution u_λ changes sign.*

Remark 5 *Note that $0 \leq f \not\equiv 0$ implies $\langle \varphi_1, f \rangle > 0$. The theorem by Clément and Peletier [4] used the stronger condition that $f > 0$. For their proof however, it is sufficient that, for a suitable projection P_1 on the first eigenspace, $P_1 f > 0$ holds, which simplifies in our situation to $\langle \varphi_1, f \rangle > 0$.*

Remark 6 *For the iterated Dirichlet Laplacians, as considered in Remark 4, we have*

	UAMP	*AMP for* $f\in L^1(\Omega)$	*AMP for* $f\in L^p(\Omega)$
for $k=1:$	*no*	$n=1$	$n\geq 2$ *with* $p>n;$
for $k=2:$	$n=1$	$n=2,3$	$n\geq 4$ *with* $p>\frac{n}{3};$
for $k=3:$	$n=1,2,3$	$n=4,5$	$n\geq 6$ *with* $p>\frac{n}{5}.$

Here, and also in general, a gap between the UAMP and the AMP with $p>1$ *is filled by AMP for* $p=1$. *This gap contains the dimensions* n *such that*

$$m(2k-2)\leq n\leq m(2k-1).$$

3 Proof for UAMP

We refer to [6] for the result that A^k is positively self-adjoint. By Condition 1 the first eigenvalue is simple and strictly positive, and has a positive eigenfunction that satisfies the estimate in (6). For $f\in L^2(\Omega)$ we split the solution operator as follows:

$$\left(A^k-\lambda\right)^{-1}=\frac{1}{\lambda_1-\lambda}P_1+\left(A^k-\lambda\right)^{-1}(I-P_1) \tag{11}$$

where, assuming $\langle\varphi_1,\varphi_1\rangle=1$,

$$P_1f=\langle\varphi_1,f\rangle\,\varphi_1. \tag{12}$$

Since λ_2 is the eigenvalue with least absolute value for A^k restricted to the subspace $\left\{f\in L^2(\Omega);\langle\varphi_1,f\rangle=0\right\}$, it follows that the series

$$\left(A^k-\lambda\right)^{-1}=\sum_{i=0}^{\infty}\lambda^i\,A^{-k(i+1)}$$

converges on $\left\{f\in L^2(\Omega);\langle\varphi_1,f\rangle=0\right\}$ for λ with $|\lambda|<\lambda_2$.

First we will show that we may restrict ourselves to a finite sum instead of this infinite series.

Lemma 1 *Let* $\varepsilon>0$. *There is* $i_1\in\mathbb{N}$ *and* $C>0$ *such that for all* λ *with* $|\lambda|\leq\lambda_2-\varepsilon$ *and* $f\in L^2(\Omega)$ *with* $f\geq 0$

$$\left|\sum_{i=i_1}^{\infty}\lambda^i\left(A^{-k(i+1)}(I-P_1)f\right)(x)\right|\leq C\,\langle\varphi_1,f\rangle\,\varphi_1(x).$$

Proof. Since P_1 and A^{-1} commute we find that for $i_2,i_3\in\mathbb{N}$, with $i_2+i_3=i_1$,

$$\sum_{i=i_1}^{\infty}\lambda^iA^{-k(i+1)}(I-P_1)f=\lambda^{i_1}A^{-ki_2}\left(A^k-\lambda\right)^{-1}(I-P_1)A^{-ki_3}f.$$

Take i_2 such that $2mki_2>m+\frac{n}{2}$. By regularity theory we find that for all $v\in L^2(\Omega)$ the function $A^{-ki_2}v\in H^{2mki_2}(\Omega)\cap H_0^m(\Omega)$ and

$$\left\|A^{-ki_2}v\right\|_{H^{2mki_2}(\Omega)}\leq c_1\|v\|_{L^2(\Omega)}. \tag{13}$$

By Sobolev imbedding it follows that $A^{-ki_2}v\in C^m\left(\bar\Omega\right)$, $\left(\frac{\partial}{\partial\nu}\right)^j\left(A^{-ki_2}v\right)(x)=0$ on $\partial\Omega$ for $j=0,\ldots,m-1$ and

$$\left\|A^{-ki_2}v\right\|_{C^m(\bar\Omega)}\leq c_2\left\|A^{-ki_2}v\right\|_{H^{2mki_2}(\Omega)}. \tag{14}$$

As $A^{-ki_2}v$ vanishes of order m on $\partial\Omega$ we proceed with

$$\begin{aligned}\left|\left(A^{-ki_2}v\right)(x)\right| &\leq c_3 \left\|A^{-ki_2}v\right\|_{C^m(\bar{\Omega})} d(x)^m \\ &\leq c_4 \left\|A^{-ki_2}v\right\|_{C^m(\bar{\Omega})} \varphi_1(x),\end{aligned} \tag{15}$$

where (6) is used in the last step.

Due to (13), (14) and the fact that $\left(A^k-\lambda\right)^{-1}(I-P_1)$ is a bounded operator on $L^2(\Omega)$, with a norm uniformly bounded for $\lambda \leq \lambda_2 - \varepsilon$, there is $c_5 > 0$ such that

$$\begin{aligned}\left\|A^{-ki_2}\left(A^k-\lambda\right)^{-1}(I-P_1)A^{-ki_3}f\right\|_{C^m(\bar{\Omega})} &\leq c_1 c_2 \left\|\left(A^k-\lambda\right)^{-1}(I-P_1)A^{-ki_3}f\right\|_{L^2(\Omega)} \\ &\leq c_5 \left\|A^{-ki_3}f\right\|_{L^2(\Omega)}\end{aligned} \tag{16}$$

(the constant $c_5 = c_5(\varepsilon) \uparrow \infty$ as $\varepsilon \downarrow 0$).

Taking $mki_3 - \frac{n}{2} > m$ it follows by Theorem 2 of [12] that for $f \in L^1(\Omega)$ with $f \geq 0$ it holds that

$$\left(A^{-ki_3}f\right)(x) \leq c_6 \int_\Omega (d(x)\,d(y))^m f(y)\,dy$$

and hence by (6)

$$\left(A^{-ki_3}f\right)(x) \leq c_7\, \varphi_1(x) \int_\Omega \varphi_1(y) f(y)\,dy, \tag{17}$$

implying

$$\left\|A^{-ki_3}f\right\|_{L^2(\Omega)} \leq c_7 \langle \varphi_1, f\rangle. \tag{18}$$

Finally, combining (15), (16) and (18), we find

$$\begin{aligned}&\left|\left(\lambda^{i_1}A^{-ki_2}\left(A^k-\lambda\right)^{-1}(I-P_1)A^{-ki_3}f\right)(x)\right| \\ &\leq c_4\, |\lambda_2|^{i_1} \left\|A^{-ki_2}\left(A^k-\lambda\right)^{-1}(I-P_1)A^{-ki_3}f\right\|_{C^m(\bar{\Omega})} \varphi_1(x) \\ &\leq c_4\, c_5\, |\lambda_2|^{i_1} \left\|A^{-ki_3}f\right\|_{L^2(\Omega)} \varphi_1(x) \leq c_4\, c_5\, c_7\, |\lambda_2|^{i_1} \langle \varphi_1, f\rangle\, \varphi_1(x).\end{aligned}$$

□

Let i_1 be as in the previous Lemma and split the solution operator, developing (11), as follows

$$\begin{aligned}\left(A^k-\lambda\right)^{-1} &= \frac{1}{\lambda_1-\lambda}P_1 + \sum_{i=0}^{i_1-1}\lambda^i A^{-k(i+1)}(I-P_1) + \sum_{i=i_1}^{\infty}\lambda^i A^{-k(i+1)}(I-P_1) \\ &= \frac{1}{\lambda_1-\lambda}P_1 + \left(\sum_{i=0}^{i_1-1}\lambda^i A^{-k(i+1)} - \sum_{i=0}^{i_1-1}\frac{\lambda^i}{\lambda_1^{i+1}}P_1\right) + \lambda^{i_1}A^{-ki_1}\sum_{i=0}^{\infty}\lambda^i A^{-k(i+1)}(I-P_1) \\ &= \frac{1}{\lambda_1-\lambda}\left(\frac{\lambda}{\lambda_1}\right)^{i_1} P_1 + R_1(\lambda) + R_2(\lambda),\end{aligned} \tag{19}$$

where

$$\begin{aligned}R_1(\lambda) &= \sum_{i=0}^{i_1-1}\lambda^i A^{-k(i+1)}, \\ R_2(\lambda) &= \lambda^{i_1}A^{-ki_1}\left(A^k-\lambda\right)^{-1}(I-P_1).\end{aligned}$$

Proposition 3 *The following are equivalent:*

i. $n < 2m(k-1)$;

ii. there exists $C_1 > 0$ *such that for all* $0 \leq f \in L^2(\Omega)$:

$$C_1^{-1} \langle \varphi_1, f \rangle \varphi_1(x) \leq \left(A^{-k} f\right)(x) \leq C_1 \langle \varphi_1, f \rangle \varphi_1(x) \qquad \text{for all } x \in \Omega;$$

iii. for every $i \in \mathbb{N}^+$ *there exists* $C_i > 0$ *such that for all* $0 \leq f \in L^2(\Omega)$

$$C_i^{-1} \langle \varphi_1, f \rangle \varphi_1(x) \leq \left(A^{-ki} f\right)(x) \leq C_i \langle \varphi_1, f \rangle \varphi_1(x) \qquad \text{for all } x \in \Omega.$$

Proof. Theorem 2 of [12] states that $C_{k,m,n} > 0$ exists such that

$$C_{k,m,n}^{-1} (d(x)\, d(y))^m \leq G_{k,m,n}(x,y) \leq C_{k,m,n} (d(x)\, d(y))^m \qquad \text{for all } x, y \in \Omega$$

if and only if $mk - \frac{1}{2}n > m$. Hence statements i and ii are equivalent. Similarly the estimates in iii hold if and only if $mki - \frac{1}{2}n > m$ for all $i \in \mathbb{N}^+$. Hence it is necessary and sufficient that the inequality holds for $i = 1$. □

It remains to show the equivalence of the two statements in Theorem 1.

Proof of Theorem 1. Let the resolvent be split in 3 parts as in (19) and assume that $\lambda \neq \lambda_1$ and $\lambda \leq \lambda_2 - \varepsilon$. The first part satisfies

$$\frac{1}{\lambda_1 - \lambda} \left(\frac{\lambda}{\lambda_1}\right)^{i_1} P_1 f = \frac{1}{\lambda_1 - \lambda} \left(\frac{\lambda}{\lambda_1}\right)^{i_1} \langle \varphi_1, f \rangle \varphi_1.$$

The other two parts can be estimated as follows. According to Lemma 1 there exist positive constants c_1 and c_2 such that for all $\lambda \in [0, \lambda_2 - \varepsilon]$ and $0 \leq f \in L^2(\Omega)$:

$$\begin{array}{rcccl} c_1^{-1} A^{-k} f & \leq & R_1(\lambda) f & \leq & c_1 A^{-k} f, \\ -c_2 \langle \varphi_1, f \rangle \varphi_1 & \leq & R_2(\lambda) f & \leq & c_2 \langle \varphi_1, f \rangle \varphi_1. \end{array}$$

For the Anti-Maximum Principle to hold uniformly for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and $\delta > 0$, it is hence necessary and sufficient that there exists $c > 0$ such that for all $0 \leq f \in L^2(\Omega)$:

$$A^{-k} f \leq c \langle \varphi_1, f \rangle \varphi_1.$$

This condition is equivalent with the existence of a constant $C_{k,m,n} > 0$ such that

$$G_{k,m,n}(x,y) \leq C_{k,m,n} (d(x)\, d(y))^m \text{ for all } x, y \in \Omega,$$

which holds by the estimates in [12] if $mk - \frac{1}{2}n > m$.

The claim that the solution u lies in $C^m\left(\bar{\Omega}\right)$ uses standard regularity arguments. The estimate of u from above in (8) follows from the property (6) of the first eigenfunction. □

4 Proof for AMP

First notice that the spectrum of A_p does not depend on p. One has further $A_p^* = A_q$ for $\frac{1}{p} + \frac{1}{q} = 1, p \in (1, \infty)$. Also a projection on the first eigenfunction is well defined by $P_1 f = \langle \varphi_1, f \rangle \varphi_1 = \left(\int_\Omega f \cdot \varphi_1\right) \varphi_1$ for any $p \in [1, \infty)$. We will also use that P_1 and A_p commute.

The following result can also be found in [5]. Here we will sketch an alternative proof using Green function estimates.

Lemma 2 *Let $f \in L^p(\Omega)$ with $p \in [1,\infty)$ satisfy $0 \le f \not\equiv 0$, and assume that $p > \frac{n}{m(2k-1)}$, when $n > m(2k-1)$. Let $\varepsilon > 0$. Then there exists $c_f > 0$ such that for all $\lambda \in [0, \lambda_2 - \varepsilon]$*

$$\left| \left(A_p^k - \lambda I\right)^{-1} (I - P_1) f(x) \right| \le c_f \, d(x)^m .$$

Proof. The space $R(I - P_1)$ reduces the operator $I - \lambda A_p$ to a boundedly invertible one for λ in a neighborhood of λ_1. Set

$$\tilde{u} = \left(A_p^k - \lambda I\right)^{-1} (I - P_1) f = A_p^{-k} \left(I + \lambda \left(A_p^k - \lambda\right)^{-1} \right) (I - P_1) f.$$

If $p = 1$ this equation has to be interpreted in a suitable weak sense as explained in Section 2 before Theorem 2.

There exists $c_{p,\varepsilon} > 0$ such that for all $\lambda \in [0, \lambda_2 - \varepsilon]$:

$$\left\| \left(I + \left(A_p^k - \lambda\right)^{-1} \right) (I - P_1) f \right\|_p \le c_{p,\varepsilon} \|f\|_p .$$

First assume that $n > 2km$. By virtue of [12, Theorem 2] the following estimate holds for the Green function of A_p^{-k}:

$$0 \le G_{A^{-k}}(x,y) \le c_1 |x-y|^{2\left(mk - \frac{1}{2}n\right)} \left(1 \wedge \frac{d(x)\, d(y)}{|x-y|^2} \right)^m \le c_2 d(x)^m |x-y|^{m(2k-1)-n} . \tag{20}$$

Here we used the estimate $\left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right) \le c_3 \frac{d(x)}{|x-y|}$ from [12, Lemma 10]. The same estimate follows for $n = 2km$:

$$0 \le G_{A^{-k}}(x,y) \le c_1 \log \left(1 + \left(\frac{d(x)\, d(y)}{|x-y|^2} \right)^m \right) \le c_2 d(x)^m |x-y|^{-m} . \tag{21}$$

If $(2k-1)\, m \le n < 2km$ we use [12, Lemma 10] to find

$$\begin{aligned} 0 \ \le\ & G_{A^{-k}}(x,y) \le c_1 \left(d(x)\, d(y)\right)^{mk - \frac{1}{2}n} \left(1 \wedge \frac{d(x)\, d(y)}{|x-y|^2} \right)^{\frac{1}{2}n - m(k-1)} \\ \le\ & c_2 \left(d(x)\, d(y)\right)^{mk - \frac{1}{2}n} \left(\frac{d(x)}{d(y)} \right)^{mk - \frac{1}{2}n} \left(\frac{d(x)}{|x-y|} \right)^{n - m(2k-1)} \\ \le\ & c_2 d(x)^m |x-y|^{m(2k-1)-n} . \end{aligned} \tag{22}$$

Since $\left\| |x - .|^{m(2k-1)-n} \right\|_{L^q(\Omega)}$ is bounded, uniformly with respect to $x \in \bar{\Omega}$, if and only if $q < \frac{n}{n - m(2k-1)}$, for such q there is a uniform (i.e. independent of x) constant $c_q > 0$ such that

$$\|G_{A^{-k}}(x, \cdot)\|_q \le c_3 \, c_q \, d(x)^m .$$

With $\frac{1}{p} + \frac{1}{q} = 1$ the condition $q < \frac{n}{n - m(2k-1)}$ is identical with $p > \frac{n}{m(2k-1)}$. By Hölders inequality we find that

$$|\tilde{u}(x)| \le c_{p,\varepsilon} \|G_{A^{-k}}(x, \cdot)\|_q \|f\|_p \le c_3 \, c_q \, c_{p,\varepsilon} \, d(x)^m \|f\|_p .$$

The claim follows for $c_f = c_3 \, c_q \, c_{p,\varepsilon} \, \|f\|_p$.

For $n \le (2k-1)\, m$ we may use the 'ordering' of the Green functions to find by (22) that

$$0 \le G_{A^{-k}}(x,y) \le c_1 d(x)^m$$

which implies that

$$|\tilde{u}(x)| \le c_1 c_{p,\varepsilon} d(x)^m \|f\|_1 . \qquad \square$$

The first claim of Theorem 2, concerning sign reversing for appropriately integrable f, follows from the observation that for $\lambda > \lambda_1$

$$\frac{1}{\lambda_1-\lambda} P_1 f(x) = \frac{1}{\lambda_1-\lambda} \langle f,\varphi_1\rangle \varphi_1(x) \le c_2 \frac{1}{\lambda_1-\lambda} \langle f,\varphi_1\rangle d(x)^m ,$$

with c_2 as in (6), and hence with Lemma 2

$$\left(\left(A_p^k-\lambda\right)^{-1} f\right)(x) \le \left(c_2 \frac{\langle f,\varphi_1\rangle}{\lambda_1-\lambda} + c_f\right) d(x)^m \le -\frac{c_f}{2} d(x)^m < 0$$

for $\lambda_1 - \lambda < 0$ and sufficiently close to 0.

For the second claim in the case $n > m(2k-1)$, concerning the necessity of the condition $p > \frac{n}{m(2k-1)}$, we first make the following simple observation.

Lemma 3 *Assume $p > 1$ and let $f \in L^p(\Omega)$ satisfy $0 \le f \not\equiv 0$, let $\lambda > \lambda_1$ and assume that $\left(A_p^k - \lambda I\right) u = f$. Then the solution u is somewhere negative.*

Proof. It is sufficient to show that $\langle \varphi_1, u\rangle < 0$. A straightforward argument by testing with φ_1 shows that

$$\langle \varphi_1, u\rangle = \frac{1}{\lambda_1}\left\langle A_q^k \varphi_1, u\right\rangle = \frac{1}{\lambda_1}\left\langle \varphi_1, A_p^k u\right\rangle = \frac{1}{\lambda_1}\langle \varphi_1, \lambda u + f\rangle = \frac{\lambda}{\lambda_1}\langle \varphi_1, u\rangle + \frac{1}{\lambda_1}\langle \varphi_1, f\rangle$$

and hence

$$\langle \varphi_1, u\rangle = -\frac{1}{\lambda-\lambda_1}\langle \varphi_1, f\rangle < 0. \qquad \square$$

It remains to show that there exists $f \in L^{p^*}(\Omega)$, with $p^* = \frac{n}{m(2k-1)}$, such that for all $\lambda - \lambda_1 > 0$ and small the solution u is somewhere positive in Ω. Fix $z \in \partial\Omega$ and set $\xi(r) = \left(\log\left(\frac{D}{r}\right)\right)^{-1}$ with $D := 2\,\mathrm{diam}(\Omega)$. We will take

$$f^*(x) = |x-z|^{-m(2k-1)} \xi(|x-z|). \tag{23}$$

Note that $0 \le \xi(|x-z|) \le (\log 2)^{-1}$.

Lemma 4 *Let f^* be as in (23). Then the following holds:*

i. $f^* \in L^{p^*}(\Omega)$ *for* $p^* = \frac{n}{m(2k-1)}$;

ii. The solution u_0 of (9), with $\lambda = 0$ and f^* as right hand side, satisfies for some positive constants $c = c(f^*) > 0$, $\varepsilon = \varepsilon(\Omega, f^*) > 0$

$$u_0(x) \ge c\,\varphi_1(x) \log\left(\log \frac{2D}{|x-z|}\right) \qquad \text{for } x \in [z, z-\varepsilon\nu_z] \cap \Omega, \tag{24}$$

where ν_z is the exterior normal at z and $[z,\tilde z] = \{\theta z + (1-\theta)\tilde z; 0 \le \theta \le 1\}$.

Proof. By straightforward calculus one finds since $p^* > 1$ that

$$\int_0^{D/2} \left(r^{-m(2k-1)} \xi(r)\right)^{\frac{n}{m(2k-1)}} r^{n-1} dr = \int_0^{D/2} r^{-1} (\xi(r))^{p^*} dr = \int_0^{(\log 2)^{-1}} \xi^{p^*-2}\, d\xi < \infty,$$

which implies $f^* \in L^{p^*}(\Omega)$.

Since $f^* > 0$ in Ω the Green function estimate immediately shows that $u(x) \geq c\,\varphi_1(x)$ for some $c > 0$ and all $x \in \Omega$. To obtain the estimate in (24) we first recall some notations and auxiliary results from section 3.2.1 of the previous paper [12]: There exists some number $R = R(\Omega) > 0$, such that

$$\mathcal{R}_x^z := x + \left(\mathcal{K}\left(\tfrac{1}{4}, -\nu_z\right) \cap B_R(0)\right) \subset \Omega,$$

where $\mathcal{K}(\sigma, -\nu_z) = \{y \in \mathbb{R}^n; -y \cdot \nu_z > \sigma|y|\}$ is a cone in direction $-\nu_z$ with opening angle arccos σ. We remark that for ε sufficiently small, one has $d(x) = |x - z| \leq R/2$ and z is the closest point to x in $\partial\Omega$, i.e. $z = x^*$ in the notation of [12, section 3.2.1]. As there we use the set $\mathcal{R}_2 = \mathcal{R}_x^z \backslash B_{\frac{R}{D}|x-z|}(x)$. The c_i will denote positive constants. For $y \in \mathcal{R}_2$ the following inequality holds by using the estimates in [12, Case II, section 3.2.1] and $d(x) = |x - z|$ in all three cases $n > 2mk$, $n = 2mk$ and $m(2k-1) < n < 2mk$:

$$G_{m,n}^{(k)}(x, y) \geq c_1 |x - y|^{2mk-n} \left(\frac{d(x)}{|x - y|}\right)^m .$$

Hence, using that $c^{-1}|x - y| \leq |y - z| \leq c\,|x - y|$ on $\mathcal{R}_2$ we have:

$$\begin{aligned}
u_0(x) &\geq \int_{\mathcal{R}_2} G_{m,n}^{(k)}(x, y)\ f^*(y)\,dy = \int_{\mathcal{R}_2} G_{m,n}^{(k)}(x, y)\ |y - z|^{-m(2k-1)}\,\xi(|y - z|)\,dy \\
&\geq c_2 \int_{\mathcal{R}_2} |x - y|^{2mk-n-m}\, d(x)^m\, |y - x|^{-m(2k-1)}\,\xi(|y - x|)\ dy \\
&\geq c_3\, d(x)^m \int_{\frac{R}{D}d(x)}^{R} r^{2mk-n-m-m(2k-1)}\,\xi(r)\ r^{n-1}dr \\
&= c_3\, d(x)^m \int_{\frac{R}{D}d(x)}^{R} r^{-1}\xi(r)\,dr = c_3\, d(x)^m \left[-\log\left(\log\frac{D}{r}\right)\right]_{\frac{R}{D}d(x)}^{R} \\
&\geq c_4\, d(x)^m \log\left(\log\left(\frac{D}{R}\cdot\frac{D}{d(x)}\right)\right) \geq c_4\, d(x)^m \log\left(\log\frac{2D}{|x - z|}\right).
\end{aligned}$$

□

The last claim of Theorem 2 follows from Lemma 3 and the next lemma.

Lemma 5 *For all regular $\lambda > \lambda_1$ there exists $\varepsilon > 0$ such that the solution u_λ of (9), with $f = f^*$ as right hand side, satisfies $u_\lambda(x) > 0$ for $x \in (z, z - \varepsilon\,\nu_z)$.*

Proof. We will split $\left(A_{p^*}^k - \lambda\right)^{-1}$ as in (19) with $i_1 = 1$:

$$\left(A_{p^*}^k - \lambda\right)^{-1} f^* = \left(A_{p^*}^k\right)^{-1} f^* + \frac{\lambda/\lambda_1}{\lambda_1 - \lambda} P_1 f^* + \lambda\left(A_{p^*}^k - \lambda\right)^{-1}(I - P_1)\left(A_{p^*}^k\right)^{-1} f^*.$$

Since $\left(A_{p^*}^k\right)^{-1} f^* \in L^\infty(\Omega)$ one has for $\lambda \neq \lambda_i$, with $i \geq 2$,

$$\left|\lambda\left(A_{p^*}^k - \lambda\right)^{-1}(I - P_1)\left(A_{p^*}^k\right)^{-1} f^*(x)\right| \leq c_{1,\lambda,f^*}\,\varphi_1(x) \ \text{ for } x \in \Omega.$$

A similar estimate for $\lambda \neq \lambda_1$ holds for the second term:

$$\left|\frac{\lambda/\lambda_1}{\lambda_1 - \lambda} P_1 f^*(x)\right| \leq c_{2,\lambda,f^*}\,\varphi_1(x) \ \text{ for } x \in \Omega.$$

Since Lemma 4 implies that

$$\left(A_{p^*}^k\right)^{-1} f^*(x) \geq c_{3,f^*}\varphi_1(x) \log\left(\log\frac{2D}{|x - z|}\right) \ \text{ for } x \in [z, z - \varepsilon_1\nu_z] \cap \Omega,$$

with some $\varepsilon_1 = \varepsilon_1(\Omega) > 0$, and since $\lim_{r\downarrow 0}\log\left(\log\frac{2D}{r}\right) = \infty$, there is for all regular $\lambda > \lambda_1$ an $\varepsilon_\lambda > 0$ such that the sign of $\left(A_{p^*}^k - \lambda\right)^{-1} f^*$ on $[z, z - \varepsilon_1\nu_z]$ is determined by the sign of $\left(A_{p^*}^k\right)^{-1} f^*$. That last sign is positive. □

References

[1] R.A. Adams, "Sobolev Spaces", Academic Press, New York etc., 1975.

[2] I. Birindelli, *Hopf's lemma and anti-maximum principle in general domains*, J. Differ. Equations **119** (1995), 450–472.

[3] T. Boggio, *Sulle funzioni di Green d'ordine m*, Rend. Circ. Mat. Palermo **20** (1905), 97–135.

[4] Ph. Clément – L.A. Peletier, *An anti-maximum principle for second order elliptic operators*, J. Differ. Equations **34** (1979), 218–229.

[5] Ph. Clément – G. Sweers, *Uniform anti-maximum principles*, J. Differential Equations **164** (2000), 118–154.

[6] Ph. Clément – G. Sweers, *Uniform anti-maximum principles for polyharmonic operators*, Proc. Amer. Math. Soc. **129** (2001), 467–474.

[7] F. Gazzola – H.-Ch. Grunau, *Critical dimensions and higher order Sobolev inequalities with remainder terms*, to appear in: Nonl. Differ. Equ. Appl. NoDEA.

[8] H.-Ch. Grunau – G. Sweers, *Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions*, Math. Ann. **307** (1997), 589–626.

[9] H.-Ch. Grunau – G. Sweers, *Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions*, Math. Nachr. **179** (1996), 89–102.

[10] H.-Ch. Grunau – G. Sweers, *The maximum principle and positive principal eigenfunctions for polyharmonic equations*, In: G. Caristi, E. Mitidieri (eds.), "Reaction Diffusion Systems", Marcel Dekker Inc., New York, Lecture Notes in Pure and Applied Mathematics **194** (1998), 163–182.

[11] H.-Ch. Grunau – G. Sweers, *Positivity properties of elliptic boundary value problems of higher order*, Proc. 2nd World Congress of Nonlinear Analysts, Nonlinear Analysis, T.M.A., **30** (1997), 5251–5258.

[12] H.-Ch. Grunau – G. Sweers, *Sharp estimates for iterated Green functions*, to appear in: Proc. Roy. Soc. Edinburgh Sect. A.

[13] P. Jentzsch, *Über Integralgleichungen mit positivem Kern*, J. Reine Angew. Math. **141** (1912), 235–244.

[14] M.A. Krasnosel'skij – Je.A. Lifshits – A.V. Sobolev, A.V., "Positive Linear Systems – The Method of Positive Operators", Heldermann Verlag, Berlin, 1989.

[15] J.L. Lions – E. Magenes, "Non-homogeneous Boundary Value Problems and Applications I", Springer, Berlin etc., 1972.

[16] Y. Pinchover, *Maximum and anti-maximum principles and eigenfunctions estimates via perturbation theory of positive solutions of elliptic equations*, Math. Ann. **314** (1999), 555–590.

[17] Y. Pinchover, *On the maximum and anti-maximum principles*, Differential equations and mathematical physics (Birmingham, AL, 1999), 323–338, AMS/IP Stud. Adv. Math., 16, Amer. Math. Soc., Providence, RI, 2000.

[18] Sweers, *L^N is sharp for the antimaximum principle*, J. Differ. Equations **134** (1997), 148–153.

[19] P. Takáč, *An abstract form of maximum and anti-maximum principles of Hopf's type*, J. Math. Anal. Appl. **201** (1996), 339–364.