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Uniform Anti-Maximum Principle for Polyharmonic Boundary Value Problems

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Abstract. A uniform anti-maximum principle is obtained for iterated polyharmonic Dirichlet problems. The main tool, combined with regularity results for weak solutions, is an estimate for positive functions in negative Sobolev norms.

1. Introduction and statement of main results

Let us recall the situation for the second order boundary value problem

$$\begin{cases} -\Delta u &= \lambda u + f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$. It is well known that for $\lambda < \lambda_1$, with λ_1 the first eigenvalue, a sign preserving property holds: $f \geq 0$ implies that $u \geq 0$. In [**4**] it has been shown that when $\lambda - \lambda_1 > 0$ and small a sign reversing phenomenon occurs. The more precise statement for (1.1) of this so-called anti-maximum principle is as follows.

For $f \in L^p(\Omega)$ with $0 \neq f \geq 0$ and $p > n$ there exists $\lambda_f > \lambda_1$ such that for $\lambda \in (\lambda_1, \lambda_f)$ the solution u of (1.1) satisfies $u < 0$ in Ω.

In [**11**] it has been proven that the restriction on p, that is $p > n$, is genuine. Indeed, for Ω smooth there is $f \in L^n(\Omega)$, with $f > 0$, such that the solution u of (1.1) changes sign for all $\lambda > \lambda_1$.

A much stronger result would have been that the sign reversing result holds for $\lambda \in (\lambda_1, \lambda_1 + \delta)$ with $\delta > 0$ independent of f. Such a result could be called a uniform anti-maximum principle and in fact exists for another boundary condition. Indeed for the one-dimensional Neumann problem, $-u'' = \lambda u + f$ in Ω with $u' = 0$ on $\partial\Omega$, such a uniform anti-maximum principle was obtained in [**4**].

In order to get a better understanding when the anti-maximum principle holds uniformly, we consider some elliptic systems of higher order: iterated polyharmonic Dirichlet problems. Let m and k be fixed positive integers. Defining

$$\begin{aligned} D(A) &= H^{2m,2}(\Omega) \cap H_0^{m,2}(\Omega) \\ A &= (-\Delta)^m : D(A) \subset L^2(\Omega) \to L^2(\Omega), \end{aligned} \tag{1.2}$$

we will study for $f \in L^2(\Omega)$ sign properties for λ near λ_1 of

$$A^k u = \lambda u + f, \tag{1.3}$$

where λ_1 is the first eigenvalue of A^k. Equation (1.3) corresponds to the boundary value problem $(-\Delta)^{mk} u = \lambda u + f$ in Ω, and $\left(\frac{\partial}{\partial n}\right)^i (-\Delta)^{mj} u = 0$ on $\partial\Omega$ for $i = 0, \ldots, m-1$ and $j = 0, \ldots, k-1$. Here n denotes the outward normal.

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A necessary condition for sign-reversing and sign-preserving property for (1.3) to hold with λ near λ_1, is that the first eigenvalue is simple and that the corresponding eigenfunction has a fixed sign. On general domains polyharmonic operators with Dirichlet boundary conditions do not have a first eigenfunction with a fixed sign. Boggio ([**3**]) however proved that for $\Omega = B$ a ball, the Green function for $Au = f$ is positive and hence, by results of Jentszch ([**8**]) or Krein-Rutman (see [**9**]), the first eigenvalue is algebraically simple and has a strictly positive eigenfunction (see [**7**]). Although the eigenfunction remains positive under small perturbations of the domain ([**7**]) we will restrict ourselves to $\Omega = B$ when $m > 1$. Note that eigenfunctions of A and of A^m, $m > 1$, coincide.

THEOREM 1. *Let A be as in (1.2) and suppose that, either*

i. $m = 1$ *and* Ω *is a bounded domain in* $\mathbb{R}^n$ *with* $\partial\Omega \in C^\infty$, *or*
ii. $m > 1$ *and* $\Omega = \{x \in \mathbb{R}^n ; |x| < R\}$ *for some* $R > 0$.

Let λ_1 be the smallest eigenvalue of A^k. If $n < 2m(k-1)$, then there exists $\delta > 0$ such that for all $\lambda \in (\lambda_1, \lambda_1 + \delta)$ and $f \in L^2(\Omega)$ with $0 \neq f \geq 0$ the solution u of (1.3), which belongs to $C^m(\bar{\Omega})$, satisfies

$$\left\{ \begin{array}{rcll} u(x) & < & 0 & \text{for all } x \in \Omega, \\ \left(\frac{\partial}{\partial n}\right)^i u(x) & = & 0 & \text{for all } x \in \partial\Omega \text{ and } i \in \{0, 1, \ldots, m-1\}, \\ \left(-\frac{\partial}{\partial n}\right)^m u(x) & < & 0 & \text{for all } x \in \partial\Omega. \end{array} \right. \tag{1.4}$$

The conditions of Theorem 1 guarantee that all eigenvalues are real and positive. Moreover, for $\lambda \in [0, \lambda_1)$ the system in (1.3) is sign preserving. See [**7**].

The inequality $n < 2m(k-1)$ is sharp. A proof of this fact will appear elsewhere.

The result in Theorem 1 coincides with that of Theorem 3 in [**5**]. In [**5**] more general, not necessarily self-adjoint, boundary value problems were considered using Sobolev spaces $H^{m,p}(\Omega)$ with $p \neq 2$. The non-Hilbert approach forces the proofs to be rather involved. An advantage of using L^p-type spaces is that one finds as an intermediate result a non-uniform anti-maximum principle. For the system in (1.3) it reads as:

PROPOSITION 2. *Let A and $\Omega \subset \mathbb{R}^n$ be as in Theorem 1.*
If $n < 2pm\left(k - \frac{1}{2}\right)$, then for $f \in L^p(\Omega)$ with $0 \neq f \geq 0$ there exists a $\lambda_f > \lambda_1$ such that for all $\lambda \in (\lambda_1, \lambda_f)$ the solution u of (1.3) which belongs to $C^m(\bar{\Omega})$, satisfies (1.4).

2. Solutions to $Au = f$

First we will recall and derive some properties of solutions to $Au = f$, that is, for the system

$$\left\{ \begin{array}{ll} (-\Delta)^m u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial n} u = \cdots = \left(\frac{\partial}{\partial n}\right)^{m-1} u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.1}$$

The spaces $H^{k,2}(\Omega)$ and $H_0^{k,2}(\Omega)$ for $k \in \mathbb{Z}$ that we use are defined in [**10**]. We recall that

$$H^{-s,2}(\Omega) = \left(H_0^{s,2}(\Omega)\right)' \text{ for } s \in \mathbb{R}. \tag{2.2}$$

For short notation we set $H^{s,2} = H^{s,2}(\Omega)$ etc.

2.1. Strong solutions. It follows from Theorem 8.4 of [**10**, page 196] that the operator A is *self-adjoint* in L^2. Indeed conditions (i), (ii) and (iii) of [**10**, page 148] are satisfied, the operator A is formally self-adjoint and the boundary operators C_j can be chosen equal to $B_j = \left(\frac{\partial}{\partial n}\right)^j$ in Theorem 8.4.

Next we show that $N(A) = \{0\}$. If $u \in D(A)$ satisfies $Au = 0$ then $\langle\langle u, u\rangle\rangle_m = \int_\Omega u\, Au\, dx = 0$, where the bilinear form $\langle\langle \cdot, \cdot\rangle\rangle_m$ is defined by

$$\langle\langle u, v\rangle\rangle_m = \begin{cases} \int_\Omega \Delta^{\frac{m}{2}} u\, \Delta^{\frac{m}{2}} v\, dx & \text{for } m \text{ even}, \\ \int_\Omega \nabla\left(\Delta^{\frac{m-1}{2}} u\right) \cdot \nabla\left(\Delta^{\frac{m-1}{2}} v\right) dx & \text{for } m \text{ odd}. \end{cases} \tag{2.3}$$

By a one-dimensional version of an inequality of Poincaré it follows for $v \in H_0^{1,2}$ with Ω bounded, that

$$\int_\Omega v^2 dx \le c_\Omega \int_\Omega \left(\frac{\partial v}{\partial x_i}\right)^2 dx \text{ for each } i \in \{1, \dots, n\}. \tag{2.4}$$

Hence there exists $c_{\Omega,m} > 0$ such that

$$\int_\Omega u^2\, dx \le c_{\Omega,m} \langle\langle u, u\rangle\rangle_m \text{ for all } u \in H_0^{m,2}. \tag{2.5}$$

Therefore $Au = 0$ implies $u = 0$. By applying Theorem 5.4 of [**10**, page 165] we obtain:

LEMMA 1. *Let $s \ge 0$. For each $f \in H^{s,2}$ there exists a unique solution $u \in H_0^{m,2} \cap H^{2m+s,2}$ of (2.1). Moreover, there exists $c_{\Omega,m,s} > 0$ such that*

$$\|u\|_{H^{2m+s,2}} \le c_{\Omega,m,s} \|f\|_{H^{s,2}} \text{ for all } f \in H^{s,2}. \tag{2.6}$$

2.2. Weak solutions. By duality we may extend the estimate in (2.6) for $s < 0$. Note that [**10**, Theorem 8.3, page 195] extends the estimate in (2.6) to $s \in [-m, 0)$ with $s + \frac{1}{2} \notin \mathbb{Z}$. For $s \in [-2m, -m)$ the same estimate is no longer true due to the fact that $H^{s,2} = (H_0^{-s,2})'$ has to be replaced with $(H_0^{m,2} \cap H^{-s,2})'$. For sake of short notation we will use for $\kappa \in \mathbb{N}$

$$\|f\|_{-m,-\kappa} = \|f\|_{\left(H_0^{m,2} \cap H^{m+\kappa,2}\right)'} = \tag{2.7}$$

$$= \sup\left\{ |\langle \varphi, f\rangle| ; \varphi \in H_0^{m,2} \cap H^{m+\kappa,2} \text{ with } \|\varphi\|_{H^{m+\kappa,2}} \le 1 \right\},$$

where $\langle \varphi, f\rangle$ denotes the value of the functional f at φ. For $f, g \in L^2$ we will also use the notation $\langle f, g\rangle = \int_\Omega f\, g\, dx$.

Notice that for $m, \kappa > 0$ the norm $\|\cdot\|_{-m-\kappa,0}$ (resp. $\|\cdot\|_{-m,-\kappa}$) is strictly weaker than the norm $\|\cdot\|_{-m,-\kappa}$ (resp. $\|\cdot\|_{0,-m-\kappa}$).

According to Lemma 1 the operator $A_0 : H_0^{m,2} \cap H^{2m,2} \to L^2$ defined by $A_0 u = Au$ is an isomorphism. Hence $A_{-1} : L^2 \to (H_0^{m,2} \cap H^{2m,2})'$ defined by $A_{-1} = A_0'$ is an isomorphism, that is, for all $v \in L^2$ there is unique $f \in (H_0^{m,2} \cap H^{2m,2})'$ with $A_{-1} v = f$ in the sense that $\int_\Omega v\, A\varphi\, dx = \langle \varphi, f\rangle$ for all $\varphi \in H_0^{m,2} \cap H^{2m,2}$. From the self-adjointness of A it follows that A_{-1} is an extension of A_0. We get the following scheme:

$$H_0^{m,2} \cap H^{2m,2} \underset{\hookrightarrow}{\xrightarrow{A_0}} L_2 \simeq (L_2)' \underset{\hookrightarrow}{\xrightarrow{A_{-1}}} \left(H_0^{m,2} \cap H^{2m,2}\right)'. \tag{2.8}$$

Defining for $\kappa \in \mathbb{N}^+$ the spaces

$$\begin{aligned} B_\kappa &= \left\{ u \in H^{2m\kappa,2}; A^j u \in H_0^{m,2} \text{ for } 0 \leq j \leq \kappa - 1 \right\}, \\ B_{-\kappa} &= (B_\kappa)' \end{aligned} \tag{2.9}$$

and $B_0 = L^2 \simeq (L_2)'$, we obtain the following scale of Hilbert spaces with restrictions and extensions of A_0 :

$$\ldots \overset{A_\kappa}{\underset{\hookrightarrow}{\longrightarrow}} B_\kappa \overset{A_{\kappa-1}}{\underset{\hookrightarrow}{\longrightarrow}} \ldots \overset{A_1}{\underset{\hookrightarrow}{\longrightarrow}} B_1 \overset{A_0}{\underset{\hookrightarrow}{\longrightarrow}} B_0 \overset{A_{-1}}{\underset{\hookrightarrow}{\longrightarrow}} B_{-1} \overset{A_{-2}}{\underset{\hookrightarrow}{\longrightarrow}} \ldots \overset{A_{-\kappa}}{\underset{\hookrightarrow}{\longrightarrow}} B_{-\kappa} \overset{A_{-\kappa-1}}{\underset{\hookrightarrow}{\longrightarrow}} \ldots \tag{2.10}$$

See [**2**, Chapter V]. The operators $A_\kappa : B_{\kappa+1} \to B_\kappa$ for $\kappa \in \mathbb{Z}$ are isomorphisms. Finally, we introduce the (complex) interpolation spaces $B_{\kappa+\frac{1}{2}}$ defined by

$$B_{\kappa+\frac{1}{2}} = [B_\kappa, B_{\kappa+1}]_{\frac{1}{2}}. \tag{2.11}$$

LEMMA 2. *Let $\kappa \in \mathbb{Z}$.*

i. *If $\kappa \geq 0$, then $B_{\kappa+\frac{1}{2}} = \left\{ u \in H^{2m\kappa+m,2}; A^j u \in H_0^{m,2} \text{ for } 0 \leq j \leq \kappa \right\}$.*

ii. *If $\kappa < 0$, then $B_{\kappa+\frac{1}{2}} = \left(B_{-\kappa-\frac{1}{2}} \right)'$.*

PROOF. It is known, see [**13**, Thm 4.3.3, p.321], that for $\kappa > 0$ that the space $[B_\kappa, B_{\kappa+1}]_{1/2}$ is the subspace of $H^{2m\kappa+m,2} = [H^{2m(\kappa+1),2}, H^{2m\kappa,2}]_{1/2}$ submitted to exactly all boundary conditions of $B_{\kappa+1}$ that are of order less then $2m\kappa + m - \frac{1}{2}$. The second result follows from $[X', Y']_\theta = [X, Y]'_\theta$, see [**10**, Thm 6.2, p.29]. □

3. Positivity and simplicity of the first eigenfunction

We will need that the first eigenvalue be simple, with the corresponding eigenfunction be positive and having an appropriate behavior at the boundary.

Let d denote the distance to $\partial\Omega$:

$$d(x, \partial\Omega) = \inf \{|x - y| ; y \in \partial\Omega\}. \tag{3.1}$$

LEMMA 3. *Let Ω be as in Theorem 1. Then the first eigenvalue $\mu_{m,1}$ of A is strictly positive and simple. Moreover, the corresponding eigenfunction $\varphi_{m,1}$, chosen positive, satisfies for some $c_m, C_m > 0$*

$$c_m \, d(x, \partial\Omega)^m \leq \varphi_{m,1}(x) \leq C_m \, d(x, \partial\Omega)^m \text{ for all } x \in \Omega. \tag{3.2}$$

PROOF. If $m = 1$ and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary the eigenfunction for the first eigenvalue is positive. The estimate in (3.2) follows from Hopf's boundary point Lemma and $\varphi_{1,1} \in C^1\left(\bar{\Omega}\right) \cap C_0\left(\bar{\Omega}\right)$.

Now suppose that $m > 1$. Then $\Omega = \{x \in \mathbb{R}^n; |x| < R\}$ and the explicit formula of the Green function by Boggio [**3**, (48), page 126] guarantees that the solution u of $(-\Delta)^m u = f \in C\left(\bar{\Omega}\right)$ with $0 \neq f \geq 0$ with the Dirichlet boundary conditions $\left(\frac{\partial}{\partial n}\right)^j u = 0$ on $\partial\Omega$, $j = 0, \ldots, m-1$, satisfies for some $c_f > 0$

$$u(x) \geq c_f \, d(x, \partial\Omega)^m \text{ for all } x \in \Omega. \tag{3.3}$$

Since the Green function is positive Jentszch' Theorem ([**8**]), or Krein-Rutman, implies that the first eigenvalue $\lambda_{m,1}$ is algebraically simple and that the corresponding eigenfunction $\varphi_{m,1}$ is positive. Using (3.3) and the fact that $\varphi_{m,1} \in C^m\left(\bar{\Omega}\right) \cap C_0^{m-1}\left(\bar{\Omega}\right)$ there are $c_m, C_m > 0$ such that

$$c_m \, d(x, \partial\Omega)^m \leq \varphi_{m,1}(x) \leq C_m \, d(x, \partial\Omega)^m \text{ for all } x \in \Omega, \tag{3.4}$$

which completes the proof. □

4. Solving by eigenfunctions

Recall that the unbounded operator $A : D(A) \subset L^2 \to L^2$ is positive self-adjoint (see 2.5). Since the imbedding of $H^{2m,2}$ in L^2 is compact we find that $A^{-1} : L^2 \to L^2$ is compact and positive symmetric. Hence L^2 has a complete orthonormal system consisting of eigenfunctions of A. Let us denote these eigenfunctions by $\{\varphi_{m,i}\}_{i=1}^{\infty}$ and the corresponding eigenvalues by $\{\mu_{m,i}\}_{i=1}^{\infty}$, that is, for $i, j \in \mathbb{N}^+$

$$\left\{ \begin{array}{lcll} (-\Delta)^m \varphi_{m,i} & = & \mu_{m,i}\varphi_{m,i} & \text{in } \Omega, \\ \left(\frac{\partial}{\partial n}\right)^{\kappa} \varphi_{m,i} & = & 0 & \text{on } \partial\Omega \text{ for } \kappa = 0, 1, \ldots, m, \\ \langle \varphi_{m,i}, \varphi_{m,j} \rangle & = & \delta_{ij}. & \end{array} \right. \tag{4.1}$$

Using (2.6) repeatedly one finds that $\varphi_{m,i} \in H^{k,2}$ for all $k \in \mathbb{N}$ and hence $\varphi_{m,i} \in C^{\infty}(\bar{\Omega})$. An equivalent norm on the space B_κ defined in (2.9,2.11) for $\kappa \in \frac{1}{2}\mathbb{Z}$, is given by

$$\|u\|_{B_\kappa} = \left(\textstyle\sum_{i=1}^{\infty} \mu_{m,i}^{2\kappa} \langle \varphi_{m,i}, u \rangle^2 \right)^{\frac{1}{2}}. \tag{4.2}$$

We may use these eigenfunctions to solve (1.3). Note that $\lambda_1 = \mu_{m,1}^k$.

Lemma 4. *There exist $C_{k,m,\Omega} > 0$ and $\delta > 0$ such that the following holds. Let $\lambda \in \mathbb{R}$ with $\left|\lambda - \mu_{m,1}^k\right| < \delta$. For all $f \in B_{-\frac{1}{2}k}$ with $\langle \varphi_{m,1}, f \rangle = 0$ there exists a unique weak solution $u_\lambda \in B_{\frac{1}{2}k}$ to*

$$\left\{ \begin{array}{lcl} \left(A^k - \lambda\right) u & = & f, \\ \langle \varphi_{m,1}, u \rangle & = & 0, \end{array} \right.$$

and moreover

$$\|u_\lambda\|_{B_{\frac{1}{2}k}} \leq C_{k,m,\Omega} \|f\|_{B_{-\frac{1}{2}k}}. \tag{4.3}$$

Proof. The lemma is an immediate consequence of $\langle \varphi_{m,1}, f \rangle = 0$, the solution formula

$$u_\lambda = \sum_{i=2}^{\infty} \frac{1}{\mu_{m,i}^k - \lambda} \langle \varphi_{m,i}, f \rangle \varphi_{m,i} \tag{4.4}$$

and choosing $\delta \in \left(0, \mu_{m,2}^k - \mu_{m,1}^k\right)$. □

5. A weighted C-space

Let us define $C_{d^m}(\bar{\Omega}) = \left\{ u \in C(\bar{\Omega}) ; \|u\|_{d^m} < \infty \right\}$ where

$$\|u\|_{d^m} = \sup \left\{ \left| \frac{u(x)}{d(x, \partial\Omega)^m} \right| ; x \in \Omega \right\}. \tag{5.1}$$

A similar space $C_e(\bar{\Omega})$ has been used by Amann in [**1**], where e is the solution to $-\Delta e = 1$ in Ω and $e = 0$ on $\partial\Omega$. For $\partial\Omega \in C^{1,\gamma}$ the spaces $C_e(\bar{\Omega})$ and $C_{d^1}(\bar{\Omega})$ coincide. Note that

$$C_0^{m-1}(\bar{\Omega}) \cap C^m(\bar{\Omega}) \hookrightarrow C_{d^m}(\bar{\Omega}) \hookrightarrow C_0(\bar{\Omega}). \tag{5.2}$$

5.1. An imbedding. The next lemma is a consequence of the imbedding

$$H_0^m \cap H^{m+\kappa} \hookrightarrow C_0^{m-1}\left(\bar{\Omega}\right) \cap C^m\left(\bar{\Omega}\right) \text{ for } n < 2\kappa. \tag{5.3}$$

LEMMA 5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^\infty$ and suppose that $n < 2\kappa$. Then there exists $c_{\Omega,m,\kappa} > 0$ such that for all $u \in H_0^m \cap H^{m+\kappa}$*

$$\|u\|_{d^m} \leq c_{\Omega,m,\kappa} \|u\|_{H^{m+\kappa}}. \tag{5.4}$$

PROOF. Since $2\kappa > n$ the Rellich-Kondrachov Theorem shows that there exists a $c_{\Omega,\kappa} > 0$ such that $\|u\|_{C^m(\bar{\Omega})} \leq c_{\Omega,\kappa} \|u\|_{H^{m+\kappa}}$. If $v \in C^1\left(\bar{\Omega}\right) \cap H_0^1$ then $v = 0$ on $\partial\Omega$ and hence (an inequality of Poincaré)

$$|v(x)| \leq \|v\|_{C^1(\bar{\Omega})}\, d(x, \partial\Omega) \text{ for all } x \in \Omega. \tag{5.5}$$

A repeated use of the last inequality shows that $u \in C^m\left(\bar{\Omega}\right) \cap H_0^m$ satisfies

$$|u(x)| \leq \frac{1}{m!} \|u\|_{C^m(\bar{\Omega})}\, d(x, \partial\Omega)^m \text{ for all } x \in \Omega, \tag{5.6}$$

and hence (5.4) follows with $c_{\Omega,m,\kappa} = c_{\Omega,\kappa}/m!$. □

By the previous lemma and the definition of the norm in (2.7) for the dual space we find:

COROLLARY 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in C^\infty$ and suppose that $n < 2\kappa$. Then there exists $c_{\Omega,m,\kappa} > 0$ such that for all $f \in L^2$*

$$\|f\|_{-m,-\kappa} \leq c_{\Omega,m,\kappa} \sup\left\{|\langle\varphi, f\rangle|\,; \varphi \in C_{d^m}\left(\bar{\Omega}\right) \text{ with } \|\varphi\|_{d^m} \leq 1\right\}. \tag{5.7}$$

6. An estimate for positive functions in a negative Sobolev space

PROPOSITION 4. *Let $\varphi_{m,1}$ be the first eigenfunction of (4.1) satisfying (3.4) and normalized by $\langle\varphi_{m,1}, \varphi_{m,1}\rangle = 1$. Let $2\kappa > n$. Then there exists $c > 0$ such that for all $f \in L^2$ with $f \geq 0$ the following estimate holds*

$$\|f - \langle\varphi_{m,1}, f\rangle \varphi_{m,1}\|_{-m,-\kappa} \leq c \langle\varphi_{m,1}, f\rangle. \tag{6.1}$$

PROOF. Let us define $f_e = f - \langle\varphi_{m,1}, f\rangle \varphi_{m,1}$. In view of (5.7) it is sufficient to prove that there exists a constant C such that

$$\langle\varphi, f_e\rangle \leq C \langle\varphi_{m,1}, f\rangle. \tag{6.2}$$

for all $\varphi \in C_{d^m}\left(\bar{\Omega}\right)$ with $\|\varphi\|_{d^m} \leq 1$.

Let φ be such a function. Hence by (3.2) we have

$$|\varphi(x)| \leq c_m^{-1}\varphi_{m,1}(x) \text{ for all } x \in \Omega. \tag{6.3}$$

Since $\langle\varphi_{m,1}, f_e\rangle = 0$ it follows that

$$\langle\varphi, f_e\rangle = \left\langle c_m^{-1}\varphi_{m,1} - \varphi, -f_e\right\rangle \tag{6.4}$$

where (6.3) shows that $c_m^{-1}\varphi_{m,1} - \varphi \geq 0$. From $f \geq 0$ we obtain $-f_e \leq \langle\varphi_{m,1}, f\rangle \varphi_{m,1}$ and hence

$$\left\langle c_m^{-1}\varphi_{m,1} - \varphi, -f_e\right\rangle \leq \left\langle c_m^{-1}\varphi_{m,1} - \varphi, \langle\varphi_{m,1}, f\rangle \varphi_{m,1}\right\rangle. \tag{6.5}$$

Since $\langle\varphi_{m,1}, f\rangle \varphi_{m,1} \geq 0$ and since (6.3) also implies that $c_m^{-1}\varphi_{m,1} + \varphi \geq 0$, we find

$$\left\langle c_m^{-1}\varphi_{m,1} - \varphi, \langle\varphi_{m,1}, f\rangle \varphi_{m,1}\right\rangle \leq \left\langle 2c_m^{-1}\varphi_{m,1}, \langle\varphi_{m,1}, f\rangle \varphi_{m,1}\right\rangle. \tag{6.6}$$

Combining (6.4), (6.5), (6.6) and $\langle\varphi_{m,1}, \varphi_{m,1}\rangle = 1$ we obtain (6.2) with $C = 2c_m^{-1}$. □

Proposition 4 can be reformulated as an imbedding of $L^1(\Omega, \varphi_{m,1}dx)$ into $(H_0^{m,2} \cap H^{m+\kappa,2})'$.

Corollary 5. *Let $\varphi_{m,1}$ and κ be as in Proposition 4. Then there exists $c_1 > 0$ such that*

$$\|f\|_{-m,-\kappa} \le c_1 \int_\Omega |f|\, \varphi_{m,1}\, dx \qquad \text{for all } f \in L^1(\Omega, \varphi_{m,1}dx). \tag{6.7}$$

Proof. Since L^2 is dense in $L^1(\Omega, \varphi_{m,1}dx)$ it is sufficient to show (6.7) for $f \in L^2$. Set $f^+ = \frac{1}{2}(|f| + f)$. By Proposition 4 we find

$$\begin{aligned} \left\|f^+\right\|_{-m,-\kappa} &\le \left\|f^+ - \left\langle \varphi_{m,1}, f^+\right\rangle \varphi_{m,1}\right\|_{-m,-\kappa} + \left\|\left\langle \varphi_{m,1}, f^+\right\rangle \varphi_{m,1}\right\|_{-m,-\kappa} \le \\ &\le \left(c + c_{\varphi_{m,1}}\right) \int_\Omega f^+\, \varphi_{m,1}\, dx. \end{aligned}$$

With the estimate for $f^- = \frac{1}{2}(|f| - f)$ we find (6.7) for $c_1 = c + c_{\varphi_{m,1}}$. □

7. Proof of the main result

In order to prove Theorem 1 we consider $S_\lambda : B_{\frac{1}{2}k} \to B_{-\frac{1}{2}k}$ defined by $S_\lambda = A^k - \lambda$. For $f \in B_{-\frac{1}{2}k}$ and $\lambda \neq \mu_{m,i}^k = \lambda_1$ the solution u satisfies

$$u = \frac{1}{\mu_{m,1}^k - \lambda} \langle \varphi_{m,1}, f\rangle\, \varphi_{m,1} + u_{e,\lambda} \tag{7.1}$$

where $u_{e,\lambda} = (S_\lambda)^{-1} f_e$ with $f_e = f - \langle \varphi_{m,1}, f\rangle \varphi_{m,1}$. By Lemma 4 we have

$$\|u_{e,\lambda}\|_{B_{\frac{1}{2}k}} \le C_1 \|f_e\|_{B_{-\frac{1}{2}k}} \tag{7.2}$$

and since $B_{-\frac{1}{2}k} \supset (H_0^m \cap H^{km})'$ it follows that

$$\|f_e\|_{B_{-\frac{1}{2}k}} \le C_2 \|f_e\|_{-m,-(k-1)m}. \tag{7.3}$$

By Proposition 4 we find if $2(k-1)m > n$ that

$$\|f_e\|_{-m,-(k-1)m} \le C_3 \langle \varphi_{m,1}, f\rangle. \tag{7.4}$$

Since $u_{e,\lambda} \in B_{\frac{1}{2}k} \subset H_0^m \cap H^{km}$ we have

$$\|u_{e,\lambda}\|_{H_0^m \cap H^{km}} \le C_4 \|u_{e,\lambda}\|_{B_{\frac{1}{2}k}} \tag{7.5}$$

and assuming $2(k-1)m > n$ it follows by Lemma 5 with $\kappa = (k-1)m$ that

$$\|u_{e,\lambda}\|_{d^m} \le C_5 \|u_{e,\lambda}\|_{H_0^m \cap H^{km}}. \tag{7.6}$$

Hence for all λ with $\left|\lambda - \mu_{m,1}^k\right| < \delta$ and δ as in Lemma 4 we find by using (7.6), (7.5), (7.2), (7.3), and (7.4):

$$|u_{e,\lambda}(x)| \le C_6 \langle \varphi_{m,1}, f\rangle\, d(x, \partial\Omega)^m. \tag{7.7}$$

Finally, using the estimate in Lemma 3 for $\varphi_{m,1}$ we find that if $0 < \lambda - \mu_{m,1}^k < \delta_1$ with $\delta_1 = \min\left\{\delta, \frac{c_m}{C_6}\right\}$ that

$$u(x) \le \left(\frac{1}{\mu_{m,1}^k - \lambda} \langle \varphi_{m,1}, f\rangle + \frac{C_6}{c_m} \langle \varphi_{m,1}, f\rangle\right) \varphi_{m,1}(x) < 0 \tag{7.8}$$

Since $-\varphi_{m,1}$ satisfies the estimates in (1.4) the solution u satisfies these same estimates.

Observe that for $-\delta_1 < \lambda - \mu_{m,1}^k < 0$ a similar estimate from below recovers the maximum principle.

References

[1] Amann, H., Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976), no. 4, 620–709.

[2] Amann, H., Linear and quasilinear parabolic problems, Vol. I. Abstract linear theory, Monographs in Mathematics 89, Birkhäuser Verlag, Basel, 1995.

[3] Boggio, T., Sulle funzioni di Green d'ordine m, *Rend. Circ. Mat. Palermo* **20** (1905), 97-135.

[4] Clément, Ph., and Peletier, L.A., An anti-maximum principle for second order elliptic operators, *J. Differ. Equations* **34** (1979), 218-229.

[5] Clément, Ph., and Sweers, G., Uniform anti-maximum principles, submitted.

[6] Grunau, H.-Ch. and Sweers, G., Positivity for equations involving polyharmonic elliptic operators with Dirichlet boundary conditions, *Math. Ann.* **307** (1997), 589-626.

[7] Grunau, H.-Ch. and Sweers, G., The maximum principle and positive principal eigenfunctions for polyharmonic equations, in Reaction Diffusion systems, Marcel Dekker Inc., New York 1997, p 163-182.

[8] Jentszch, P., Über Integralgleichungen mit positivem Kern, *J. Reine Angew. Math.* 141 (1912), 235-244.

[9] Krasnosel'skij, M.A., Lifshits, Je.A., and Sobolev, A.V., "Positive Linear Systems", Heldermann Verlag, Berlin, 1989.

[10] Lions, J.L. and Magenes, E., "Non-homogeneous Boundary Value Problems and Applications I", Springer, Berlin, 1972.

[11] Sweers, G., L^N is sharp for the antimaximum principle, *J. Differential Equations* **134** (1997), 148-153.

[12] Takáč, P., An abstract form of maximum and anti-maximum principles of Hopf's type, *J. Math. Anal. Appl.* **201** (1996), no. 2, 339–364.

[13] Triebel, H., "Interpolation Theory, Function Spaces, Differential Operators", North-Holland, Amsterdam, 1978.

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