Classical Solutions for Some Higher Order Semilinear Elliptic Equations under Weak Growth Conditions

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1 Introduction

Classical solvability of "positive" semilinear elliptic Dirichlet problems of higher order

$$(1)\qquad \begin{aligned} &Lu(x) + g(x,u) = f(x) \text{ in } \Omega,\\ &D^\alpha u|\partial\Omega = 0 \text{ for } |\alpha| \le m-1 \end{aligned}$$

is not yet very well understood. Here L is a uniformly elliptic operator of order $2m$, $\Omega \subset I\!R^n$ $(n \ge 2)$ a sufficiently smooth bounded domain. The linear operator L is assumed to be positive definite, that is, for all $u \in C^{2m}(\overline{\Omega})$ with $D^\alpha u|\partial\Omega = 0$ $(|\alpha| \le m-1)$ the following holds:

$$\int_\Omega Lu(x)\cdot u(x)\,dx \ge c_0 ||u||^2_{W^{m,2}(\Omega)},$$

where c_0 is a positive constant. The nonlinear term is subject to a sign condition

$$(2)\qquad g(x,t)\cdot t \ge 0 \text{ for all } t \in I\!R,\ x \in \Omega.$$

We want to explain the crucial difference between the second order $(m = 1)$ and higher order $(m > 1)$ case.

If $m = 1$ a very satisfactory result is known. Indeed, the sign condition (2) alone is sufficient to ensure classical solvability of the Dirichlet problem (1). There are two very strong devices in the theory of second order elliptic equations, which make the proof of the needed a-priori maximum estimate an easy exercise:

- A very general comparison principle: $Lu \ge 0$ in Ω, $u \ge 0$ on $\partial\Omega$ implies either $u > 0$ in Ω or $u \equiv 0$ in Ω.
- The restriction of u on level sets defines a new Dirichlet problem, e.g. on $\Omega^+ := \{x \in \Omega : u(x) > 0\}$.

On Ω^+ the term $f(x) - g(x,u(x))$ is bounded from above by $\sup_{\overline{\Omega}} f^+$, and the general comparison principle yields a bound for u^+. The negative part u^- is estimated in the same way.

Now let us turn to higher order equations. The nonlinear operator $u \mapsto Lu + g(.,u)$ is still coercive with respect to the $W^{m,2}$-norm, but things get substantially worse with respect to pointwise properties. If $m > 1$ the level set trick fails completely because we have to observe at least two boundary conditions simultaneously. Further we loose the comparison principle in its general form, too.

Up to now all existence results for classical solutions to (1) need some extra condition on g, usually some growth condition. So the known existence results are in the higher order case $m > 1$ considerably weaker than in the second order case. On the other hand we do not know examples showing that the existence results are already optimal.

The aim of this paper is to find out to what extent the authors' comparison result for equations of arbitrary order [GS] allow for more general or at least different conditions on g than the usual ones. Our main tool will be a rather general local maximum principle for differential inequalities of arbitrary order.

The plan of the present paper is as follows. In section 2 we briefly discuss the state of the art and our main result. In section 3 we give a precise formulation of our existence theorem. In section 4 the local maximum principle is presented. This is used in section 5 to prove the existence theorem.

2 Comparison with existing results

Usually, if $n > 2m$, the Dirichlet problem (1) is considered under controllable growth conditions

$$(3) \quad |g(x,t)| \leq C(1 + |t|^{\tau}),\ \tau = \frac{n+2m}{n-2m}.$$

If $n = 2m$ only polynomial growth is assumed, if $n < 2m$ the nonlinearity g may grow arbitrarily. Under some additional monotonicity assumption on g, classical solutions $u \in C^{2m,\mu}(\overline{\Omega})$ to the Dirichlet problem (1) exist, see e.g. [L], [W1], [W2]. For $n \to \infty$ only little more than linear growth is admissible. We presume that the restrictive growth condition (3) is caused by a very restricted knowledge of comparison principles for higher order equations.

Luckhaus [L] also shows, that the growth condition (3) implies regularity for every weak solution $u \in W_0^{m,2}(\Omega)$ to (1).

Several authors, see e.g. [K], [G1], [G2], [SM], [SP], [T], are interested in whether there are different or more general conditions on g than (3) that imply the existence of classical solutions. Tomi [T] e.g. treats fourth order equations ($m = 2$) with a nonlinear term $g(u)$. The function g is supposed to be monotone and L to be the square of a second order operator. The existence of a solution u is shown, which is classical in the interior and takes on the boundary values in a weak sense: $u \in C^{4,\mu}(\Omega) \cap W_0^{2,2}(\Omega)$. The papers [K] and [SP] also contain results in this direction.

The first author [G1], [G2] used the positivity of the Green function for $(-\Delta)^m$ under Dirichlet boundary conditions in balls, i.e. of a very weak comparison principle. Beside the sign condition (2) on $u \mapsto g(u)$ an asymmetrical condition is imposed:

$$(4) \quad -C(1 + |t|^{\sigma}) \leq g(t) \leq C(1 + |t|^{\tau}),$$

where $\frac{n+2m}{n-2m} \leq \tau \leq \infty$, $0 \leq \sigma < \frac{4m}{n-2m} + \frac{1}{\tau} \cdot \frac{n+2m}{n-2m}$ and vice versa. The condition "$\tau = \infty$" means arbitrarily strong growth of g^+. Unfortunately the linear principal part has to be very special, namely $L = (-\Delta)^m$. Then again there exists a solution in the class $C^{2m,\mu}(\Omega) \cap W_0^{m,2}(\Omega)$. Although the family of growth conditions (4) is rather flexible, e.g. $g(u) = \exp u - 1$ is admissible ($\tau = \infty$), it seems to be as strong as condition (3), if we put $\tau = \frac{n+2m}{n-2m}$.

For further references and comments see also [G1], [G2].

In our previous paper [GS] a rather general comparison result in balls has been found. This enables us to treat nonlinearities $u \mapsto g(u)$, $u \cdot g(u) \geq 0$, whose positive part may grow arbitrarily and whose negative part may grow linearly. Moreover, L is allowed to be rather general, only the leading term has to be a power of a second order elliptic operator with constant coefficients. The solution class is the same as above, i.e. $C^{2m,\mu}(\Omega) \cap W_0^{m,2}(\Omega)$.

Having our comment on condition (4) in mind, we think that for large dimensions n the restriction on the dependence of g on u is considerably weaker than the controllable growth condition (3).

3 The existence result

For the purpose of convenient reference we first introduce our main assumptions. Most of the notation is adopted from [GT].

Let $n, m \in I\!N$, $n \geq 2$, $\mu \in (0,1)$. We assume $n \geq 2m$, because in the case of small dimensions, our results would be contained in [W1].

Assumptions A :

I. *$\Omega \subset I\!R^n$ is a bounded domain of class $C^{2m,\mu}$.*

II. *The elliptic operator L is of the form*

$$Lu(x) = \left(-\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \right)^m u(x) + \sum_{|\alpha| \leq 2m-1} b_\alpha(x) D^\alpha u(x)$$

with $\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$, for some $0 < \lambda < \Lambda$, and $b_\alpha \in C^{|\alpha|,\mu}(\overline{\Omega})$.

III. *The operator L is coercive:*

$$(5) \quad \int_\Omega Lu(x) \cdot u(x)\, dx \geq c_0 ||u||^2_{W^{m,2}(\Omega)}.$$

for all $u \in C^{2m}(\overline{\Omega})$, $D^\alpha u|\partial\Omega = 0$ ($|\alpha| \leq m-1$).

IV. *The nonlinearity $g \in C^\mu(\overline{\Omega} \times I\!R)$ satisfies the sign condition:*

$$(6) \quad g(x,t) \cdot t \geq 0, \text{ for all } t \in I\!R,\ x \in \Omega,$$

and a one-sided growth condition:

$$(7) \quad g(x,t) \geq -C_0(1 + |t|^\sigma),\ t < 0,\ x \in \Omega;$$

where

$$\sigma \begin{cases} = 1, & \text{if } n \geq 6m, \\ < \frac{4m}{n-2m}, & \text{if } 6m > n > 2m, \\ < \infty, & \text{if } n = 2m. \end{cases}$$

Remarks. 1) Instead of (7) we may assume that g^+ grows linearly and g^- arbitrarily.
2) The formally adjoint operator L^* also satisfies **A.II** and **A.III**.

Theorem 1 (Existence of classical solutions) *Let the assumptions* **A.I-IV** *be satisfied. Then for any $f \in C^{0,\mu}(\overline{\Omega})$ the Dirichlet problem*

$$(8) \qquad \begin{aligned} &Lu(x) + g(x,u) = f(x) \ \text{in } \Omega, \\ &D^\alpha u|\partial\Omega = 0 \ \text{for } |\alpha| \le m-1 \end{aligned}$$

has a solution $u \in C^{2m,\mu}(\Omega) \cap W_0^{m,2}(\Omega)$.

Remark. If Ω is a ball, $a_{ij} = \delta_{ij}$ and if the coefficients b_α are "small", then u is globally smooth, i.e. in $C^{2m,\mu}(\overline{\Omega})$.

The crucial tool for the proof of Theorem 1 is a local maximum principle for linear differential *in*equalities, see Theorem 2 in Section 3 below.

With help of this local maximum principle we derive in Section 4 local a-priori maximum estimates for smooth solutions of Dirichlet problems like (8). These local a-priori estimates and an approximation procedure are used to show the existence of locally smooth solutions.

4 A local maximum principle

In what follows $B_\rho(x_0)$ denotes the open ball in $I\!R^n$ with radius ρ, centered at x_0. $B := B_1(0)$.

In our previous paper [GS] Green's functions for higher order operators have been estimated. We extensively exploited the fact, that Green's function for $(-\Delta)^m$ with Dirichlet boundary conditions is known explicitly in balls [B]. The main result is summarized in the following lemma, which is essential for the proof of our local maximum principle for solutions of higher order differential *in*equalities.

Lemma 1 *Let $Lu(x) := (-\Delta)^m u(x) + \sum_{|\alpha| \le 2m-1} b_\alpha(x) D^\alpha u(x)$, and assume the coefficients to be smooth: $b_\alpha \in C^{|\alpha|,\mu}(\overline{B})$.*

There is a number $M_0 = M_0(n,m) > 0$ and a constant $C = C(M_0)$ such that the following holds.

If one has $\sum\limits_{|\beta| \le |\alpha|} ||D^\beta b_\alpha||_{C^0(\overline{B})} \le M_0$ for all $|\alpha| \le 2m-1$, then the Green function G_L for the Dirichlet problem

$$\begin{aligned} &Lu(x) = f(x) \ \text{for } \ x \in B, \\ &D^\alpha u|\partial B = 0 \ \text{for } |\alpha| \le m-1 \end{aligned}$$

exists and is positive. We have:

$$(9) \quad 0 \le G_L(x,y) \le C \cdot G_{(-\Delta)^m}(x,y) \ \text{for } \ x, y \in B,$$

$$(10) \quad \left|\frac{\partial^{|\alpha|}}{\partial y^\alpha} G_L(0,y)\right| \le C \ \text{for } y \in \partial B, \ |\alpha| \le 2m-1.$$

Proof. We put $M := \sup_{0\le|\alpha|\le 2m-1} \sum_{|\beta|\le|\alpha|} ||D^\beta b_\alpha||_{C^0(\overline{B})}$.

We denote the corresponding Green operators by $\mathcal{G}_L$ and $\mathcal{G}_{(-\Delta)^m}$,

$$(\mathcal{G}_L f)(x) := \int_B G_L(x,y)f(y)\,dy,\ (\mathcal{G}_{(-\Delta)^m} f)(x) := \int_B G_{(-\Delta)^m}(x,y)f(y)\,dy.$$

We take from [GS] that $\mathcal{G}_L : C^{0,\mu}(\overline{B}) \to C^{2m,\mu}(\overline{B})$ exists for M sufficiently small. Moreover, positivity of G_L, the estimate (9) and the following representation formula were shown:

$$(11)\quad \mathcal{G}_L = \sum_{\nu=0}^{\infty}(-1)^\nu \Big(\sum_{|\beta|\le 2m-1} \mathcal{G}_{(-\Delta)^m} b_\beta(.)D^\beta\Big)^\nu \mathcal{G}_{(-\Delta)^m}.$$

In order to show (10) we have to consider the formally adjoint operator

$$L^* v = (-\Delta)^m v + \sum_{|\beta|\le 2m-1} (-1)^{|\beta|} D^\beta(b_\beta(.)v) =: (-\Delta)^m v + \sum_{|\beta|\le 2m-1} b^*_\beta(x)v.$$

Due to our strong smoothness assumptions on b_β, L^* has smooth coefficients too, bounded by a small factor $C\cdot M$. If M is small enough, we have in analogy with (11):

$$\mathcal{G}_{L^*} = \sum_{\nu=0}^{\infty}(-1)^\nu \Big(\sum_{|\beta|\le 2m-1} \mathcal{G}_{(-\Delta)^m} b^*_\beta(.)D^\beta\Big)^\nu \mathcal{G}_{(-\Delta)^m},$$

$G_{L^*}(x,y)$ is well defined and positive. In order to find an estimate for $\frac{\partial^{|\alpha|}}{\partial x^\alpha} G_{L^*}(x,y)$, $|\alpha| \le 2m-1$, we have to control terms like

$$(12)\quad \begin{aligned} &(C_1 M)^\nu \int_B \cdots \int_B |x-z_1|^{2m-|\alpha|-n}\cdot|z_1-z_2|^{2m-|\beta_1|-n}\cdot\ldots\cdot|z_\nu-y|^{2m-|\beta_\nu|-n}\,dz_1\ldots dz_\nu \\ &\le (C_2 M)^\nu \int_B \cdots \int_B |x-z_1|^{1-n}\cdot|z_1-z_2|^{1-n}\cdot\ldots\cdot|z_\nu-y|^{1-n}\,dz_1\ldots dz_\nu; \end{aligned}$$

the constants C_1, C_2 and C_3 below only depend on n and m; they do *not* depend on ν. Using $\int_B |\xi-z|^{1-n}\cdot|z-\eta|^{1-n}\,dz \le C|\xi-\eta|^{1-n}$, we may estimate (12) by

$$(C_3 M)^\nu |x-y|^{1-n}.$$

For $|x|=1$, $|\alpha|\le 2m-1$ and $M\le M_0$, M_0 small enough we find

$$\Big|\frac{\partial^{|\alpha|}}{\partial x^\alpha} G_{L^*}(x,0)\Big| \le C(M_0).$$

Observing $G_L(x,y) = G_{L^*}(y,x)$, (10) follows. ∎

Theorem 2 (Local maximum principle) *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, $K\subset\Omega$ a compact subset, let L satisfy assumption* **A.II** *on $\overline{\Omega}$. In particular, there is a number $\tilde{M}$, such that for any $|\alpha|\le 2m-1$ one has $\sum_{|\beta|\le|\alpha|} ||D^\beta b_\alpha||_{C^0(\overline{\Omega})} \le \tilde{M}$. Let $q > \frac{n}{2m}$, $q\ge 1$.*

Then there exists a constant $C = C(n,m,\lambda,\Lambda,q,\tilde{M},\operatorname{dist}(K,\partial\Omega))$, such that for every subsolution $u\in C^{2m}(\overline{\Omega})$, $f\in C^0(\overline{\Omega})$ of

$$Lu \le f,$$

it follows that

$$(13)\quad \sup_K u \le C\{||f^+||_{L^q(\Omega)} + ||u||_{W^{m-1,1}(\Omega)}\}.$$

Proof. In order to apply Lemma 1, we need the coefficients to be small. This can be obtained by using a suitable scaling.

After a linear coordinate transformation we may assume $a_{ij} = \delta_{ij}$; i.e. the principal part of L is $(-\Delta)^m$.

Let $x_0 \in K$, without loss of generality we may write $x_0 = 0$. We put

$$(14) \quad \rho_0 := \min\{1, \frac{1}{2}\mathrm{dist}(K, \partial\Omega), \frac{M_0}{\tilde{M}}\} > 0.$$

Here M_0 is the small positive number of Lemma 1.

For $\rho \in (0, \rho_0]$ we define the following functions $\overline{B} \to I\!R$:

$$\begin{aligned} u_\rho(x) &= u(\rho x), \\ f_\rho(x) &= \rho^{2m} f(\rho x), \\ b_{\alpha,\rho}(x) &= \rho^{2m-|\alpha|} b_\alpha(\rho x). \end{aligned}$$

On $\overline{B}$ we get the differential inequality

$$(15) \quad (-\Delta)^m u_\rho(x) + \sum_{|\alpha| \le 2m-1} b_{\alpha,\rho}(x) D^\alpha u_\rho(x) \le f_\rho(x).$$

Furthermore we have for $x \in \overline{B}$, $|\alpha| \le 2m-1$ that by (14)

$$\sum_{|\beta| \le |\alpha|} |D^\beta b_{\alpha,\rho}(x)| \le \sum_{|\beta| \le |\alpha|} \rho^{2m-|\alpha|+|\beta|} |(D^\beta b_\alpha)(\rho x)| \le \rho \sum_{|\beta| \le |\alpha|} ||D^\beta b_\alpha||_{C^0(\overline{\Omega})} \le \rho_0 \tilde{M} \le M_0.$$

Hence we may apply Lemma 1. Let G_ρ denote Green's function for the operator L_ρ in (15). We remark that the estimates (9), (10) hold uniformly in $\rho \in (0, \rho_0]$. We use the representation formula for solutions of $L_\rho v = h$; beside $D^\alpha u$, $|\alpha| \le m-1$ the boundary integrals contain factors like $\frac{\partial^{|\alpha|}}{\partial y^\alpha} G(0, y)$, $m \le |\alpha| \le 2m-1$, and some of the coefficients b_α and some of their derivatives of order $\le m$. With estimates (9) and (10) of Lemma 1, we find that there are constants $C_i(\tilde{M}, \rho_0)$ such that:

$$\begin{aligned} u(0) &= u_\rho(0) \le \int_{|y| \le 1} G_\rho(0, y) f_\rho^+(y)\, dy + C_1(\tilde{M}, \rho_0) \sum_{|\alpha| \le m-1} \int_{|y|=1} |D^\alpha u_\rho(y)|\, d\omega(y) \\ &\le C_2(\tilde{M}, \rho_0)\{||f_\rho^+||_{L^q(B_1)} + \sum_{|\alpha| \le m-1} \rho^{|\alpha|} \int_{|y|=1} |(D^\alpha u)(\rho y)|\, d\omega(y)\} \\ &\le C_3(\tilde{M}, \rho_0)\{\rho^{2m-(n/q)} ||f^+||_{L^q(B_\rho)} + \sum_{|\alpha| \le m-1} \rho^{|\alpha|-n+1} \int_{|y|=\rho} |D^\alpha u(y)|\, d\omega(y)\}. \end{aligned}$$

There is a constant $C = C(\tilde{M}, q, \rho_0, n, m)$, such that for $\rho \in [\frac{1}{2}\rho_0, \rho_0]$, one has:

$$u(0) \le C\{||f^+||_{L^q(\Omega)} + \sum_{|\alpha| \le m-1} \int_{|y|=\rho} |D^\alpha u(y)|\, d\omega(y)\}.$$

We remark, that C behaves like ρ_0^{1-n} as $\rho_0 \searrow 0$.

The estimate (13) follows by integration with respect to $\rho \in [\frac{1}{2}\rho_0, \rho_0]$. ∎

5 Proof of the existence theorem

We first prove a local a-priori-maximum-estimate for solutions of Dirichlet problems like (8). This estimate is an immediate consequence of the local maximum principle, Theorem 2.

Lemma 2 *Let the assumptions* **A.I-IV** *be fulfilled. Let $K \subset \Omega$ be a compact subset. Moreover we assume that there exists a smooth function $G : I\!R \to I\!R$, $G \geq 0$, $G' \geq 0$ such that we have*

$$g(x,t) \leq G(t), \ x \in \Omega, \ t \in I\!R.$$

Let $\tilde{M}$ be a number, such that for any $|\alpha| \leq 2m-1$ one has

$$\sum_{|\beta| \leq |\alpha|} ||D^\beta b_\alpha||_{C^0(\overline{\Omega})} \leq \tilde{M}.$$

Let $u \in C^{2m,\mu}(\overline{\Omega})$, $f \in C^{0,\mu}(\overline{\Omega})$ solve

$$(16) \quad \begin{aligned} &Lu(x) + g(x,u) = f(x) \textit{ for } x \in \Omega, \\ &D^\alpha u|\partial\Omega = 0 \textit{ for } |\alpha| \leq m-1. \end{aligned}$$

Then there exists a constant

$$C = C(n, m, \lambda, \Lambda, \tilde{M}, c_0, C_0, \sigma, G, ||f||_{C^0(\overline{\Omega})}, \Omega, \operatorname{dist}(K, \partial\Omega))$$

such that

$$\sup_K |u| \leq C.$$

Proof. We introduce another compact subset $\hat{K} \subset \Omega$ such that $K \subset \hat{K} \subset \Omega$, $\operatorname{dist}(K, \partial\hat{K}) \geq \frac{1}{3}\operatorname{dist}(K, \partial\Omega)$, $\operatorname{dist}(\hat{K}, \partial\Omega) \geq \frac{1}{3}\operatorname{dist}(K, \partial\Omega)$.

First of all testing (16) with u and making use of assumption (6) and of Poincaré's inequality we obtain:

$$||u||_{W^{m,2}(\Omega)} \leq C(c_0, \Omega)||f||_{L^2(\Omega)}.$$

Now (16) implies a differential inequality on Ω:

$$(17) \quad \begin{aligned} Lu(x) &\leq ||f||_{C^0(\overline{\Omega})} - g(x, u(x)) \\ &\leq ||f||_{C^0(\overline{\Omega})} + 1_{\{y\,:\,u(y)\leq 0\}}(x) \cdot C_0(1 + |u|^\sigma). \end{aligned}$$

We want to apply the local maximum principle. Due to the imbedding $W^{m,2}(\Omega) \hookrightarrow L^{2n/(n-2m)}$, in the case $n < 6m$ the right hand side may be interpreted as an exterior force:

$$||1 + |u(.)|^\sigma||_{L^q} \leq C(1 + ||u||^\sigma_{L^{\sigma q}}) \leq C(1 + ||u||^\sigma_{W^{m,2}}) \leq C(1 + ||f||^\sigma_{L^2}),$$

where $q = \frac{1}{\sigma} \cdot \frac{2n}{n-2m} > \frac{n}{2m}$, if $6m > n > 2m$, and $q > 1$, if $n = 2m$. Theorem 2 yields:

$$(18) \quad \sup_{\hat{K}} u \leq C_1(n, m, \lambda, \Lambda, \tilde{M}, c_0, C_0, \sigma, ||f||_{C^0(\overline{\Omega})}, \Omega, \operatorname{dist}(K, \partial\Omega)).$$

Let now $n \geq 6m$, $\sigma = 1$. We introduce a C^∞-function $h : I\!R \to I\!R$, $0 \leq h \leq 1$,

$$h(t) = \begin{cases} 1, & \text{if } t \leq -1, \\ 0, & \text{if } t \geq 0. \end{cases}$$

Then writing $\tilde{b}_0(x) = C_0 \cdot h(u(x))$ it follows from (17) that:

$$Lu(x) + \tilde{b}_0(x)u(x) \leq ||f||_{C^0(\overline{\Omega})} + 2C_0.$$

Replacing L by $L + \tilde{b}$, $\tilde{M}$ by $\tilde{M} + C_0$, application of Theorem 2 gives the estimate (18) from above also in the case $n \geq 6m$.

In order to estimate u from below we consider a differential inequality on $\hat{K}$:

$$\begin{aligned} Lu(x) &\geq -||f||_{C^0(\overline{\Omega})} - g(x, u(x)) \geq -||f||_{C^0(\overline{\Omega})} - G(u(x)) \\ &\geq -||f||_{C^0(\overline{\Omega})} - G(C_1), \end{aligned}$$

C_1 is taken from (18). Applying Theorem 2 to $(-u)$, K, $\hat{K}$, the proof of Lemma 2 is complete. ■

Proof of Theorem 1. We apply a well known approximation procedure, see e.g. [T].

In a first step modified Dirichlet problems are solved. Let $h : I\!R \to I\!R$ be a C^∞-function, $0 \leq h \leq 1$,

$$h(t) = \begin{cases} 1, & \text{if } t \leq -1, \\ 0, & \text{if } t \geq 0. \end{cases}$$

For $k \in I\!N$ we consider

$$g_k(x, u) = g(x, u) \cdot h(u - k)$$

and the corresponding Dirichlet problem

$$(19) \qquad \begin{aligned} & Lu(x) + g_k(x, u) = f(x) \text{ in } \Omega, \\ & D^\alpha u | \partial\Omega = 0 \text{ for } |\alpha| \leq 1. \end{aligned}$$

Due to the boundedness of g_k^+ and the weak growth of g_k^- it follows (see [T], [W1]) that (19) has a classical solution $u_k \in C^{2m,\mu}(\overline{\Omega})$.

In a second step we discuss the convergence properties the sequence (u_k). As g_k satisfies the sign condition (6), (u_k) is uniformly bounded in $W_0^{m,2}(\Omega)$. Obviously there exists a majorizing function G as described in Lemma 2:

$$-C_0(1 + |t|^\sigma) \leq g_k(x, t) \leq G(t).$$

So for every compact subset $\hat{K} \subset \Omega$, the sequence (u_k) is bounded in $C^0(\hat{K})$. Passing to a smaller compact subset K, applying interior Schauder estimates [DN] and interpolation of weighted Hölder seminorms (see e.g. [GT]), we find that (u_k) is bounded in $C^{2m,\mu}(K)$ for any compact subset $K \subset \Omega$.

Exhausting Ω by compact subsets, applying the Arzela-Ascoli theorem and a diagonal procedure we find a subsequence, which converges locally in C^{2m} and weakly in $W_0^{m,2}(\Omega)$ to a solution $u \in C^{2m,\mu}(\Omega) \cap W_0^{m,2}(\Omega)$ of $Lu(x) + g(x, u(x)) = f(x)$. ■

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