# Sharp estimates for iterated Green functions

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**Abstract**

Optimal pointwise estimates from above and below are obtained for iterated (poly)harmonic Green functions corresponding to zero Dirichlet boundary conditions. For second order elliptic operators these estimates hold true on bounded $C^{1,1}$ domains. For higher order elliptic operators we have to restrict ourselves to the polyharmonic operator on balls. We will also consider applications to noncooperatively coupled elliptic systems and to the lifetime of conditioned Brownian motion.

## 1 Introduction and main results

Roughly spoken the pointwise deflection of solutions of elliptic partial differential equations in bounded smooth domains $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions is submitted to two opposite influences. A positive source terms locally 'bends' the solution and hence urges it to increase globally while the zero boundary condition(s) pulls in the opposite direction. As is well known in second order elliptic equations such as $-\Delta$ the positive source term will dominate the sign of the solution everywhere in the domain. Indeed, this result is a consequence of the maximum principle. For higher order elliptic operators such a sign preserving property in general does not hold except when considering e.g. the polyharmonic operator $(-\Delta)^m$ with zero Dirichlet boundary conditions on a ball. These positivity preserving results are reflected in the fact that the kernel of the solution operator is of fixed sign. Sharp pointwise estimates of this Green function will show the balance between the two effects just mentioned.

For studying perturbations of those elliptic operators, both for second and for higher order operators, it also becomes crucial to find optimal pointwise estimates for the corresponding Green functions. And indeed, in recent years such estimates have been developed. Motivated by Schrödinger operators Zhao ([21], see also [4]) was in 1986 the first to prove a sharp estimate from below for the Green function of the Laplace operator. The estimate from above for the Laplacian had been proven earlier in 1967 by Widman ([20], see also [7, Theorem 4.6.11]). In fact, due to a result of Ancona [1], such estimates hold for quite general second order elliptic operators. For the explicit formula of those estimates, both for $n = 2$ and $n > 2$, we refer to [18].

The need of having pointwise estimates for iterated Green functions of polyharmonic operators became first obvious to the authors when studying in [11] positivity questions for perturbations of the polyharmonic Dirichlet problem on the (unit-) ball $B$ in $\mathbb{R}^n$ :

$$\begin{cases} (-\Delta)^m u = f & \text{in } B, \\ u = \frac{\partial}{\partial \boldsymbol{n}} u = \cdots = \frac{\partial^{m-1}}{\partial \boldsymbol{n}^{m-1}} u & \text{on } \partial B. \end{cases} \tag{1.1}$$

In that paper sharp estimates from above and below for the Green function for (1.1) were proven.

The main difficulty in [11] was to find the appropriate balance between the singularity in the Green function $G(x, y)$ when $x \to y$ (a point mass source term) and $G(x, y) \to 0$ when $x$ or $y \to \partial\Omega$ (the zero boundary condition). For more general operators, but without the effect of the boundary condition, estimates from above for the elliptic kernel are obtained by Krasovskiĭ in [15]. The higher order 'heat'-kernel $K(t, x, y)$ on $\mathbb{R}^n$ has been considered e.g. by Davies and Barbatis in [2] and [9]. There for $K(t, x, x)$, i.e. on the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$, they give even bounds from below. For a survey on recent results on higher order elliptic equations with emphasis on spectral theory we refer to [8].

## 1.1 Green function estimates

The starting point in [11] is an explicit expression for the Green function $G_{m,n}$ of (1.1), which was discovered by Boggio [3, p. 126] already at the beginning of the twentieth century:

$$G_{m,n}(x, y) = k_{m,n} |x - y|^{2m-n} \int_1^{\Phi(x,y)} \frac{\left(v^2 - 1\right)^{m-1}}{v^{n-1}} dv \tag{1.2}$$

where $k_{m,n}$ is a positive constant and

$$\Phi(x, y) = \frac{\left| x\,|y| - \frac{y}{|y|} \right|}{|x - y|}.$$

Since $\left| x\,|y| - \frac{y}{|y|} \right|^2 - |x - y|^2 = \left(1 - |x|^2\right)\left(1 - |y|^2\right) > 0$ one directly finds that $\Phi(x, y) > 1$ in $B^2$. This result is singular in so far as for general higher order elliptic boundary value problems the Green function is not positive in an arbitrary domain. For a survey on positivity preserving properties of higher order elliptic equations we refer to [12].

In order to develop a perturbation theory of positivity for Dirichlet problems like (1.1) one uses Neumann series, the first term of which is the unperturbed Green function for (1.1) itself. This term is the only one which one is sure to be positive and which consequently has to dominate the other terms of the Neumann series.

As a first step in [11] we derived from Boggio's formula the following two-sided estimate. There exists $C_{m,n} > 0$ such that

$$C_{m,n}^{-1} H_{m,n}(x,y) \leq G_{m,n}(x,y) \leq C_{m,n} H_{m,n}(x,y) \text{ for all } x, y \in B, \tag{1.3}$$

where

$$H_{m,n}(x,y) = \begin{cases} \left(|x-y|^2\right)^{m-\frac{1}{2}n} \left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^m & \text{for } m - \frac{1}{2}n < 0, \\ \log\left(1 + \left(\frac{d(x)d(y)}{|x-y|^2}\right)^m\right) & \text{for } m - \frac{1}{2}n = 0, \\ (d(x)\,d(y))^{m-\frac{1}{2}n} \left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^{\frac{1}{2}n} & \text{for } m - \frac{1}{2}n > 0, \end{cases} \tag{1.4}$$

with $a \wedge b = \min\{a, b\}$ and $d$ denoting the distance to the boundary $\partial B$ :

$$d(x) = 1 - |x|. \tag{1.5}$$

We also use the notation $a \vee b = \max\{a, b\}$.

As we mentioned, these estimates are appropriate for the polyharmonic Dirichlet problem on a ball. For second order elliptic equations, that is $m = 1$, one does not have this restriction. On arbitrary bounded domains the maximum principle implies a positivity preserving property and even the estimate in (1.3) holds in all bounded $C^{1,1}$-domains when $d(x)$ is defined by

$$d(x) = d(x, \partial\Omega) = \inf\{|x - x^*|\,; x^* \in \partial\Omega\}. \tag{1.6}$$

Obviously one should replace $C_{m,n}$ in (1.3) by some constant $C_{m,\Omega}$. We just remark that if $n = 2$ then for any polyharmonic operator also smooth domains may be considered, which are sufficiently close to the disk in a suitable sense (depending on the order of the operator), see [10].

Since our proofs do not distinguish between $B$ (for $m \geq 2$) or a $C^{1,1}$-domain (for $m = 1$) we consider $H_{m,n}$ defined on $\Omega^2$ with $d$ as in (1.6).

## 1.2 Boundary and internal behavior

The estimate (1.4) can be understood with help of distinguishing the cases $|x - y| \leq \frac{1}{2}(d(x) \vee d(y))$ and $|x - y| \geq \frac{1}{2}(d(x) \vee d(y))$. For later use we will formulate it as a lemma.

**Lemma 1.1** *Let $d(\,.\,)$ be as in (1.5) or (1.6). If $|x - y| \leq \frac{1}{2}(d(x) \vee d(y))$ then*

$$\tfrac{1}{2} d(x) \leq d(y) \leq 2d(x) \qquad \textit{and} \qquad 1 \leq \frac{d(x)\,d(y)}{|x-y|^2}. \tag{1.7}$$

*If $|x - y| \geq \frac{1}{2}(d(x) \vee d(y))$ then*

$$\frac{d(x)}{|x-y|} \leq 2, \qquad \frac{d(y)}{|x-y|} \leq 2 \qquad \textit{and} \qquad \frac{d(x)\,d(y)}{|x-y|^2} \leq 4. \tag{1.8}$$

*Proof.* Assume that $|x-y| \leq \frac{1}{2}(d(x) \vee d(y))$ and suppose $d(y) \leq d(x)$. Let $y^* \in \partial\Omega$ such that $d(y) = |y - y^*|$. Then $d(x) \leq |x-y^*| \leq |x-y| + d(y) \leq \frac{1}{2}d(x) + d(y)$ implying that $d(x) \leq 2d(y)$. The second case is immediate. □

In the first case both $x$ and $y$ are not as close to the boundary as to each other, and the term, which describes the boundary behavior, may be neglected; the singularity $|x-y|^{2m-n}$ (if $n > 2m$) is dominating. In the second case both $x$ and $y$ are closer to the boundary than to each other, and the boundary behavior is described by the term $(d(x)\,d(y))^m\,|x-y|^{-n}$.

## 1.3 Results

The second step in [11] in order to obtain perturbation results for positivity consists in estimating the modulus of higher terms in the Neumann series mentioned above by the first term. That means that we have to estimate iterated Green functions from above by the Green function itself. The estimates as needed for this purpose were derived in [11] without taking care e.g. of the fact that the singular character of iterated Green functions becomes weaker and weaker when the number of iterations increases.

A first goal of this paper is to derive precise estimates from above and below by the same simple function for iterated Green functions. Below, after presenting the main result we will also comment on a number of further applications.

For the iterated polyharmonic Dirichlet problem

$$\left\{\begin{array}{ll} \qquad (-\Delta)^{mk}\,u = f & \text{in } B, \\ \left.\begin{array}{l} u = \frac{\partial}{\partial \boldsymbol{n}} u = \dots = \frac{\partial^{m-1}}{\partial \boldsymbol{n}^{m-1}} u \\ (-\Delta)^m u = \frac{\partial}{\partial \boldsymbol{n}} (-\Delta)^m u = \dots = \frac{\partial^{m-1}}{\partial \boldsymbol{n}^{m-1}} (-\Delta)^m u \\ \vdots \\ (-\Delta)^{(k-1)m} u = \frac{\partial}{\partial \boldsymbol{n}} (-\Delta)^{(k-1)m} u = \dots = \frac{\partial^{m-1}}{\partial \boldsymbol{n}^{m-1}} (-\Delta)^{(k-1)m} u \end{array}\right. & \text{on } \partial B \end{array}\right\} \tag{1.9}$$

the Green function $G_{m,n}^{(k)}$ satisfies

$$G_{m,n}^{(k)}(x,y) = \int_B \dots \int_B G_{m,n}(x,z_1)\,G_{m,n}(z_1,z_2)\dots G_{m,n}(z_{k-1},y)\,dz_1\dots dz_{k-1}.$$

**Theorem 1.2** *Let $G_{m,n}^{(k)}$ be the Green function above. Then there exists $C_{k,m,n} > 0$ such that for $x, y \in B$*

$$C_{k,m,n}^{-1}\,H_m\left(mk - \tfrac{1}{2}n; x, y\right) \leq G_{m,n}^{(k)}(x,y) \leq C_{k,m,n}\,H_m\left(mk - \tfrac{1}{2}n; x, y\right) \tag{1.10}$$

*where*

$$H_m(\theta; x, y) = \begin{cases} |x-y|^{2\theta} \left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^m & \text{for } \theta < 0, \\ \log\left(1 + \left(\frac{d(x)d(y)}{|x-y|^2}\right)^m\right) & \text{for } \theta = 0, \\ \Big(d(x)\,d(y)\Big)^\theta \left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^{m-\theta} & \text{for } 0 < \theta < m, \\ \Big(d(x)\,d(y)\Big)^m \log\left(2 + \frac{1}{|x-y|^2 + d(x)d(y)}\right) & \text{for } \theta = m, \\ \Big(d(x)\,d(y)\Big)^m & \text{for } \theta > m. \end{cases}$$

*If $m = 1$ these estimates hold on bounded $C^{1,1}$-domains $\Omega$ with the obvious replacements. There exists a positive constant $C_{k,1,\Omega}$ such that for $x, y \in \Omega$*

$$C_{k,1,\Omega}^{-1} H_1\left(k - \tfrac{1}{2}n; x, y\right) \le G_{1,\Omega}^{(k)}(x, y) \le C_{k,1,\Omega} H_1\left(k - \tfrac{1}{2}n; x, y\right). \tag{1.11}$$

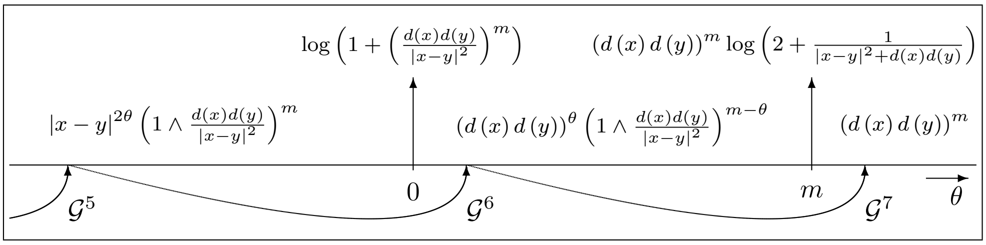
This theorem will be proved in Section 3.

As far as only estimates from above for the singular behaviour of iterated Green functions are concerned, this partial result is well known and goes back to Hadamard (see e.g. [17, p. 39]). Here the crucial point is to give *twosided* estimates in particular for the *boundary behaviour*, resp. the interplay of singular and boundary behaviour.

Defining the operator $\mathcal{G}$ by

$$\mathcal{G}(f)(x) = \int_\Omega G_{m,n}(x, y) f(y) dy, \tag{1.12}$$

and hence $\mathcal{G}^k(f)(x) = \int_\Omega G_{m,n}^{(k)}(x, y) f(y) dy$, we find the following scheme.




$\mathcal{G}$ with $10m < n < 12m$

**Remark 1.3** *The dependence of $C_{k,m,n}$ (resp. $C_{k,m,\Omega}$) on $k$ in Theorem 1.2 can be made more explicit. Let $k_0$ be the smallest number such that $mk_0 - \frac{1}{2}n > m$. Since there exist constants $0 < c_{\varphi_{1,m}} < C_{\varphi_{1,m}}$ (see [5]) such that*

$$c_{\varphi_{1,m}} \varphi_{1,m}(x) \le d(x)^m \le C_{\varphi_{1,m}} \varphi_{1,m}(x),$$

*where $\varphi_{1,m}$ is a suitably normalized positive first eigenfunction of $(-\Delta)^m$ under Dirichlet boundary conditions, one concludes for $k \geq k_0$ from*

$$\begin{aligned} G^{(k)}_{m,n}(x,y) &= \underbrace{\int_B \cdots \int_B}_{(k-k_0)\text{-times}} G_{m,n}(x,z_1)\, G_{m,n}(z_1,z_2) \ldots G_{m,n}(z_{k-k_0-1}, z_{k-k_0}) \times \\ &\qquad\qquad \times G^{(k_0)}_{m,n}(z_{k-k_0}, y)\, dz_1 \ldots dz_{k-k_0} \end{aligned}$$

*that*

$$\tilde{C}^{-1}_{m,n}\, \lambda^{k-k_0}_{1,m} \Big( d(x)\, d(y) \Big)^m \leq G^{(k)}_{m,n}(x,y) \leq \tilde{C}_{m,n}\, \lambda^{k-k_0}_{1,m} \Big( d(x)\, d(y) \Big)^m$$

*with $\tilde{C}_{m,n} = C_{k_0,m,n} \frac{C_{\varphi_{1,m}}}{c_{\varphi_{1,m}}}$.*

**Remark 1.4** *Let $H_m(\Theta; x, y)$ be as in Theorem 1.2. If $\Theta \geq \theta$ then there exists a constant $c^{\Theta}_{\theta} \in \mathbb{R}^+$ such that*

$$H_m(\Theta; x,y) \leq c^{\Theta}_{\theta}\, H_m(\theta; x,y) \text{ for all } x, y \in \Omega. \tag{1.13}$$

*Moreover, if (1.13) holds for some constant $c^{\Theta}_{\theta} \in \mathbb{R}^+$, then $\theta \leq \Theta$ or $m < \Theta$.*

As a first immediate consequence of Theorem 1.2 and the preceding remark we have the so called '3-G-theorems' from [11]:

**Corollary 1.5** *There exists a constant $\hat{C}_{m,n} > 0$, such that for $x, z \in B$*

$$\int_B G_{m,n}(x,y)\, G_{m,n}(y,z)\, dy \leq \hat{C}_{m,n} G_{m,n}(x,z).$$

In the remainder we will use the following symbols for the sake of notation.

**Notation 1.6** *Let $f, g$ be nonnegative functions on $\Omega$ (resp. $\Omega^2$, $\Omega^3$). We say $f \preceq g$ on $\Omega$ (resp. $\Omega^2$, $\Omega^3$) if there exists a constant $c > 0$, such that*

$$f(x) \leq c\, g(x) \text{ for all } x \in \Omega \text{ (resp. } \Omega^2, \Omega^3).$$

*We say $f \sim g$ on $\Omega$ if $f \preceq g$ and $g \preceq f$ on $\Omega$ (resp. $\Omega^2$, $\Omega^3$).*

In what follows the constants in the estimates, which appear implicitly by the use of the symbols $\sim$ and $\preceq$, depend on the space dimension $n$ (resp. the domain $\Omega$), the order $m$ of the differential operator and on the number $k$ of iterations.

# 2 Applications

Another important motivation to find optimal estimates is the use of these estimates in necessary and sufficient conditions for uniform anti-maximum-principles to hold. The idea to use Green function estimates for anti-maximum-principles goes back to a paper by Takáč [19].

Anti-maximum-principles concern the resolvent for boundary value problems like (1.9), when the resolvent parameter $\lambda$ is beyond the first eigenvalue. Usually for sufficiently regular fixed right hand side $f$ there exists a small $\lambda$-interval where $f \geq 0$ implies $u \leq 0$. If this interval does not depend on $f$ the anti-maximum-principle is called uniform.

The estimates obtained here allow us to solve an open problem from [5]. This application will be treated in a separate paper [13].

## 2.1 Application to coupled elliptic systems

As studied in [16], noncooperatively coupled elliptic systems may still satisfy some positivity preserving property. For example for the system

$$\begin{cases} -\Delta u = f - \varepsilon^2 v & \text{in } \Omega, \\ -\Delta v = u & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

it holds true that $f \geq 0$ implies $(u,v) \geq 0$ when the operator $\mathcal{G}$ and $\mathcal{G} - \varepsilon\mathcal{G}^2$ are positive and $\sum_{k=0}^{\infty} (\varepsilon\mathcal{G})^{4k}$ converges. The result follows from $v = \mathcal{G}u$ and

$$u = (I + \varepsilon\mathcal{G}) \sum_{k=0}^{\infty} (\varepsilon\mathcal{G})^{4k} \left(\mathcal{G} - \varepsilon\mathcal{G}^2\right) f. \tag{2.1}$$

Here $\mathcal{G}$ is as in (1.12) with $m = 1$. In more involved systems, see [16, Theorem 6.3], it was necessary for proving this sign preserving property to have for $k$ large that

$$G^{(k)}(x,y) \sim G^{(k+1)}(x,y), \tag{2.2}$$

where we denote $G^{(k)}(x,y) = G^{(k)}_{1,\Omega}(x,y)$. Then, instead of considering an infinite series in the solution operator for $u$, it is possible to restrict oneself to a finite number of terms. Indeed in [16, page 272] it is proven that (2.2) holds for $k \geq k_n = \left[\frac{n+2}{2}\right] + 1$ with $[\,\cdot\,]$ the entier function. In fact it is shown that on $\Omega \times \Omega$ the following 'ordering' exists:

$$G \succeq G^{(2)} \succeq G^{(3)} \succeq \cdots \succeq G^{(k_n - 1)} \succeq G^{(k_n)} \sim G^{(k_n+1)} \sim G^{(k_n+2)} \sim \dots\ .$$

It was conjectured in that paper that $k_n$ is optimal in the sense that $G^{(k_n-1)} \nsim G^{(k_n)}$.

This conjecture holds true since from Theorem 1.2 and Remark 1.4 it follows that (2.2) holds if and only if $k - \frac{1}{2}n > 1$. Notice that for $k \in \mathbb{N}$ this bound can be rewritten as $k \geq \left[\frac{n+2}{2}\right] + 1$.

## 2.2 Application to Brownian motion

We assume that $n \geq 2$. Let $y \in \Omega$ and let $\{X(t)\}_{t\geq 0}$ denote a $G(\cdot, y)$–conditioned Brownian motion in $\Omega \backslash \{y\}$, normalized for $\Delta$ instead of $\frac{1}{2}\Delta$. See [4, Chapter 5]. This Brownian motion is conditioned to be killed at exiting $\Omega \backslash \{y\}$. Since $x \mapsto G(x,y)$ is harmonic in $\Omega \backslash \{y\}$ with $G(x,y) = 0$ for $x \in \partial\Omega$ it will be killed almost surely at $y$.

The lifetime is denoted by $\tau_{\Omega\backslash\{y\}}$. Denoting by $\mathbb{P}^x$ and $\mathbb{E}^x$ the probability respectively expectation for Brownian motion starting at $x \in \Omega$, that is $\mathbb{P}^x[X(0)=x]=1$, the expected lifetime $\mathbb{E}_y^x\left[\tau_{\Omega\backslash\{y\}}\right]$ of the $G(\cdot,y)$-conditioned Brownian motion equals

$$\begin{aligned}
\mathbb{E}_y^x\left[\tau_{\Omega\backslash\{y\}}\right] &:= \mathbb{E}_{G(\cdot,y)}^x\left[\tau_{\Omega\backslash\{y\}}\right] = -\int_{t=0}^{\infty} t\,\mathrm{d}\mathbb{P}_y^x\left[\tau_{\Omega\backslash\{y\}}>t\right] \\
&= \int_{t=0}^{\infty} \mathbb{P}_y^x\left[\tau_{\Omega\backslash\{y\}}>t\right] dt = \int_{t=0}^{\infty}\int_{z\in\Omega} \frac{G(z,y)}{G(x,y)} p(t,x,z)\, dz\, dt \\
&= \int_{z\in\Omega} \frac{G(z,y)}{G(x,y)} \int_{t=0}^{\infty} p(t,x,z)\, dt\, dz \\
&= \int_{z\in\Omega} \frac{G(z,y)}{G(x,y)} G(x,z)\, dz = \frac{G^{(2)}(x,y)}{G(x,y)}.
\end{aligned}$$

Here $p(t,x,z)$ is the kernel for the diffusion equation on $\Omega$ with Dirichlet boundary conditions.

A similar expression holds for the expectation of the moments of $\tau_{\Omega\backslash\{y\}}$.

**Lemma 2.1** *Denoting $G^{(k)}(x,y) = G_{1,\Omega}^{(k)}(x,y)$ with $\Omega$ as above one has:*

$$\mathbb{E}_y^x\left[\tau_{\Omega\backslash\{y\}}^k\right] = k!\frac{G^{(k+1)}(x,y)}{G(x,y)}.$$

**Remark 2.2** $\mathbb{E}_y^x\left[\tau_{\Omega\backslash\{y\}}^k\right]$ *is the expectation of the $k^{th}$ moment of the lifetime of $G(\,.\,,y)$–conditioned Brownian motion that starts in $x$ and stays inside $\Omega$ until it is killed when reaching $y$.*

*Proof.* For $k=1$ it is just the result mentioned before. A repeated integration by parts shows

$$\begin{aligned}
&\mathbb{E}_y^x\left[\tau_{\Omega\backslash\{y\}}^k\right] = -\int_{t=0}^{\infty} t^k\,\mathrm{d}\mathbb{P}_y^x\left[\tau_{\Omega\backslash\{y\}}>t\right] \\
&= k!\int_{t_1=0}^{\infty}\int_{t_2=t_1}^{\infty}\cdots\int_{t_k=t_{k-1}}^{\infty} \mathbb{P}_y^x\left[\tau_{\Omega\backslash\{y\}}>t_k\right] dt_1 dt_2\ldots dt_k \\
&= k!\int_{t_1=0}^{\infty}\int_{t_2=t_1}^{\infty}\cdots\int_{t_k=t_{k-1}}^{\infty}\int_{z_1\in\Omega} \frac{G(z_1,y)}{G(x,y)} p(t_k,x,z_1)\, dz_1\, dt_1 dt_2\ldots dt_k \\
&= \frac{k!}{G(x,y)}\int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty}\cdots\int_{t_k=0}^{\infty}\int_{z_1\in\Omega} p(t_1+t_2+\cdots+t_k,x,z_1)\times \\
&\qquad\qquad \times G(z_1,y)\, dz_1\, dt_1 dt_2\ldots dt_k. \qquad (2.3)
\end{aligned}$$

By Chapman-Kolmogorov one obtains

$$\begin{aligned}
&p(t_1+t_2+\cdots+t_k,x,z_1) \\
&= \int_{z_k\in\Omega}\cdots\int_{z_2\in\Omega} p(t_k,x,z_k)\, p(t_{k-1},z_k,z_{k-1})\ldots p(t_1,z_2,z_1)\, dz_2\ldots dz_k
\end{aligned}$$

and hence with Fubini-Tonelli

$$\begin{aligned}(2.3) \quad &= \frac{k!}{G(x,y)} \int_{z_k\in\Omega} \cdots \int_{z_1\in\Omega} G(x,z_k)\, G(z_k,z_{k-1}) \cdots \times \\ &\qquad \times G(z_2,z_1)\, G(z_1,y)\, dz_1 dz_2 \ldots dz_k \\ &= k! \frac{G^{(k+1)}(x,y)}{G(x,y)}.\end{aligned}$$

□

As a consequence of Theorem 1.2 we obtain the following estimates for $\mathbb{E}_y^x\left[\tau_{\Omega\setminus\{y\}}^k\right]$. This result improves and optimizes related ones in [18].

**Proposition 2.3** *Let $\Omega$ be a bounded $C^{1,1}$-domain in $\mathbb{R}^n$ and let $\mathbb{E}_y^x\left[\tau_{\Omega\setminus\{y\}}^k\right]$ be as above with $k \in \mathbb{N}^+$.*

1. *If $n \geq 3$ then $\mathbb{E}_y^x\left[\tau_{\Omega\setminus\{y\}}^k\right] \sim \ldots$*

$$\begin{array}{lll} & \text{for } x,y\in\Omega: & \text{for } x\in\Omega,\ y\in\partial\Omega: \\ \text{for } k<\frac{n-2}{2}: & \cdots \sim |x-y|^{2k} & \sim |x-y|^{2k}, \\ \text{for } k=\frac{n-2}{2}: & \cdots \sim |x-y|^{n-2}\log\left(2+\frac{d(x)d(y)}{|x-y|^2}\right) & \sim |x-y|^{n-2}, \\ \text{for } k=\frac{n-1}{2}: & \cdots \sim |x-y|^{n-1}\left(1+\frac{d(x)d(y)}{|x-y|^2}\right)^{\frac{1}{2}} & \sim |x-y|^{n-1}, \\ \text{for } k=\frac{n}{2}: & \cdots \sim |x-y|^{n}\left(1+\frac{d(x)d(y)}{|x-y|^2}\right)\log\left(2+\frac{1}{|x-y|^2+d(x)d(y)}\right) & \\ & & \sim |x-y|^{n}\log\left(2+\frac{1}{|x-y|^2}\right), \\ \text{for } k>\frac{n}{2}: & \cdots \sim |x-y|^{n}\left(1+\frac{d(x)d(y)}{|x-y|^2}\right) & \sim |x-y|^{n}. \end{array}$$

2. *If $n = 2$ then $\mathbb{E}_y^x\left[\tau_{\Omega\setminus\{y\}}^k\right] \sim \ldots$*

$$\begin{array}{lll} & \text{for } x,y\in\Omega: & \text{for } x\in\Omega,\ y\in\partial\Omega: \\ \text{for } k=1: & \cdots \sim \left(|x-y|^2+d(x)\,d(y)\right)\dfrac{\log\left(2+\frac{1}{|x-y|^2+d(x)d(y)}\right)}{\log\left(2+\frac{d(x)d(y)}{|x-y|^2}\right)} & \\ & & \sim |x-y|^2\log\left(2+\frac{1}{|x-y|^2}\right), \\ \text{for } k>1: & \cdots \sim \left(|x-y|^2+d(x)\,d(y)\right)\dfrac{1}{\log\left(2+\frac{d(x)d(y)}{|x-y|^2}\right)} & \\ & & \sim |x-y|^2. \end{array}$$

*The constants in the two-sided estimates denoted by $\sim$ depend on $\Omega$ and $k$.*

**Remark 2.4** *For $y \in \partial\Omega$ one recovers the Brownian motion in $\Omega$ that is conditioned to exit $\Omega$ at $y$.*

**Remark 2.5** *The results in this proposition give local estimates for the lifetime with the explicit dependence on $d(x), d(y)$ and $|x-y|$. Global estimates have been studied intensively. See the book by Chung and Zhao [4]. For bounded Lipschitz domains there exists $C_\Omega < \infty$ such that for all $x \in \Omega$ and $y \in \bar{\Omega}$ the estimate $\mathbb{E}_y^x\left[\tau_{\Omega\setminus\{y\}}\right] \leq C_\Omega$ holds. For $n=2$ Cranston and McConnell [6] even proved that $C_\Omega = C \cdot m(\Omega)$ with $C$ an absolute constant and $m(\Omega)$ the Lebesgue measure of $\Omega$.*

*Proof of Proposition 2.3.* The proof is rather straightforward by using the expression in Lemma 2.1, Lemma 3.4 below and the estimates for $G^{(k)}(x,y)$ of Theorem 1.2. Consider for example the case $n \geq 3$ and $k = \frac{n-1}{2}$. Then

$$\begin{aligned}
\frac{G^{(k+1)}(x,y)}{G(x,y)} &\sim \frac{\left(d(x)\,d(y)\right)^{\frac{1}{2}}\left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^{\frac{1}{2}}}{|x-y|^{2-n}\left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)} \\
&\sim |x-y|^{n-1}\left(\frac{d(x)d(y)}{|x-y|^2}\right)^{\frac{1}{2}}\left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^{-\frac{1}{2}} \\
&\sim |x-y|^{n-1}\left(1 + \frac{d(x)d(y)}{|x-y|^2}\right)^{\frac{1}{2}},
\end{aligned}$$

where in the last step the cases $\frac{d(x)d(y)}{|x-y|^2} \geq 1$ and $\frac{d(x)d(y)}{|x-y|^2} \leq 1$ were distinguished. □

# 3 Proofs

The main estimates that we need will be stated in the following lemmata. We will split the estimates according to three types for $G_{m,n} = G_{m,n}^{(1)}$ over three lemmata which cover respectively the cases $n > 2m$, $n = 2m$ and $n < 2m$. The principal terms contain logarithms when $\frac{n}{2m} = k$ and $\frac{n}{2m} = k-1$, hence in dimensions $n$ that are a multiple of $2m$.

**Lemma 3.1** *Let $n, m \in \mathbb{N}^+$ with $2m < n$. Suppose that $\alpha, \delta \in \mathbb{N}^+$ with $1 \leq \alpha \leq n-2m$ and $1 \leq \delta < 2m$. Then on $\Omega^2$ holds:*

1.
$$\int_{y\in\Omega} |x-y|^{-\alpha}\left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^m |y-z|^{2m-n}\left(1 \wedge \frac{d(y)d(z)}{|y-z|^2}\right)^m dy \sim$$
$$\sim \begin{cases} |x-z|^{2m-\alpha}\left(1 \wedge \frac{d(x)d(z)}{|x-z|^2}\right)^m & \text{if } \alpha > 2m, \\ \log\left(1 + \left(\frac{d(x)d(z)}{|x-z|^2}\right)^m\right) & \text{if } \alpha = 2m, \\ \left(d(x)\,d(z)\right)^{m-\frac{1}{2}\alpha}\left(1 \wedge \frac{d(x)d(z)}{|x-z|^2}\right)^{\frac{1}{2}\alpha} & \text{if } \alpha < 2m. \end{cases} \tag{3.1}$$

2.
$$\int_{y\in\Omega}\left(d(x)\,d(y)\right)^{\frac{1}{2}\delta}\left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^{m-\frac{1}{2}\delta}|y-z|^{2m-n}\left(1 \wedge \frac{d(y)d(z)}{|y-z|^2}\right)^m dy \sim$$
$$\sim \left(d(x)\,d(z)\right)^m. \tag{3.2}$$

3.
$$\int_{y\in\Omega} \log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)|y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \sim$$
$$\sim \Big(d(x)\,d(z)\Big)^m \log\left(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\right). \tag{3.3}$$

4.
$$\int_{y\in\Omega} \Big(d(x)\,d(y)\Big)^m \log\left(2+\tfrac{1}{|x-y|^2+d(x)d(y)}\right)\ |y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \sim$$
$$\sim \Big(d(x)\,d(z)\Big)^m. \tag{3.4}$$

**Lemma 3.2** *Let $n=2m\in\mathbb{N}^+$. Then:*

1.
$$\int_{y\in\Omega} \log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)\log\left(1+\left(\tfrac{d(y)d(z)}{|y-z|^2}\right)^m\right) dy \sim$$
$$\sim \Big(d(x)\,d(z)\Big)^m \log\left(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\right). \tag{3.5}$$

2.
$$\int_{y\in\Omega} \Big(d(x)\,d(y)\Big)^m \log\left(2+\tfrac{1}{|x-y|^2+d(x)d(y)}\right)\log\left(1+\left(\tfrac{d(y)d(z)}{|y-z|^2}\right)^m\right) dy \sim$$
$$\sim \Big(d(x)\,d(z)\Big)^m. \tag{3.6}$$

**Lemma 3.3** *Let $m,n\in\mathbb{N}^+$ with $2m>n$. Suppose that $\delta\in\mathbb{N}^+$ with $1\le\delta<2m$. Then*

$$\int_{y\in\Omega} \Big(d(x)\,d(y)\Big)^{\frac12\delta}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^{m-\frac12\delta}\Big(d(y)\,d(z)\Big)^{m-\frac12 n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^{\frac12 n} dy \sim$$
$$\sim \Big(d(x)\,d(z)\Big)^m. \tag{3.7}$$

*Proof of Theorem 1.2.* The results stated in Theorem 1.2 immediately follow from the estimates in the lemmata above. Indeed if $k=1$ the estimates follow for $m,n\in\mathbb{N}^+$ from (1.4) since $H_m\left(m-\frac12 n;x,y\right)=H_{m,n}(x,y)$. For $k>1$ one should note that the results of the previous three lemmata can be condensed to

$$\int_\Omega H_m(\theta;x,y)\,H_m\left(m-\tfrac12 n;y,z\right)dy \sim H_m(\theta+m;x,z)$$

for all bounded $\theta$ of the form $\theta=\ell m-\frac12 n$, $\ell\in\mathbb{N}^+$. Repeating this step $(k-1)$ times the statement of Theorem 1.2 follows. □

First we recall and modify some estimates from [11].

**Lemma 3.4** *Let $\gamma \in \mathbb{N}^+$. For all $x, y, z \in \Omega$ the following holds:*

$$
\begin{array}{rrcl}
i. & \left(1 \wedge \frac{d(x)}{|x-y|}\right)\left(1 \wedge \frac{d(y)}{|x-y|}\right) & \sim & \left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right), \\
ii. & \left(1 \wedge \frac{d(x)}{|x-y|}\right) & \sim & \left(1 \wedge \frac{d(x)}{d(y)} \wedge \frac{d(x)}{|x-y|}\right), \\
iii. & \left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)\left(1 \wedge \frac{d(y)d(z)}{|y-z|^2}\right) & \preceq & \left(1 \wedge \frac{d(x)d(z)}{|x-z|^2}\right), \\
iv. & \log\left(1 + \left(\frac{d(x)d(y)}{|x-y|^2}\right)^\gamma\right) & \sim & \left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^{\gamma-1} \log\left(1 + \frac{d(x)d(y)}{|x-y|^2}\right), \\
v. & \log\left(1 + \left(\frac{d(x)d(y)}{|x-y|^2}\right)^\gamma\right) & \sim & \left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^{\gamma} \log\left(2 + \frac{d(x)}{|x-y|}\right), \\
vi. & \log\left(2 + \left(\frac{d(x)d(y)}{|x-y|^2}\right)^\gamma\right) & \sim & \log\left(2 + \frac{d(x)}{|x-y|}\right).
\end{array}
$$

**Remark 3.5** *We will also use the obvious estimate* $(1 \wedge s) \preceq \log(1+s)$ *for* $s \in [0, \infty)$.

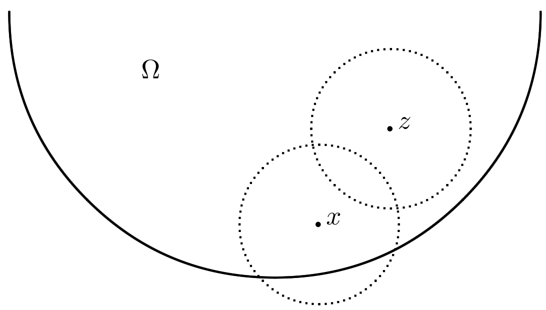
*Proof.* For $i$ and $ii$, respectively $iii$ see Lemma 3.2 and Lemma 4.3 of [11]. For the last three we distinguish two cases according to Lemma 1.1. Assume that $|x-y| \leq \frac{1}{2}(d(x) \vee d(y))$ and by using (1.7) and $\log(1+s) \sim \log(2+s) \sim \log(1+s^\gamma) \sim \log(2+s^\gamma)$ for $s \in [1,\infty)$ the estimates $iv$, $v$ and $vi$ follow. Next assume that $|x-y| \geq \frac{1}{2}(d(x) \vee d(y))$. Using (1.8) and $\log(1+s) \sim s$ respectively $\log(2+s) \sim 1$ for $s \in [0, 4^\gamma]$ we obtain the estimates $iv$, $v$ and $vi$ also in the latter case. $\square$

The proofs of Lemmata 3.1, 3.2 and 3.3 will be split into two parts, the estimates from above, respectively those from below.

## 3.1 Estimates from above

The lemmata will be proven in several steps. For the estimates from above we will split the domain of integration $\Omega \ni y$ in three parts, $\Omega = \mathcal{O}_x \cup \mathcal{O}_z \cup \mathcal{R}$, which are defined by

$$
\mathcal{O}_x = B_{\frac{2}{3}|x-z|}(x) \cap \Omega, \qquad \mathcal{O}_z = B_{\frac{2}{3}|x-z|}(z) \cap \Omega \quad \text{and} \qquad \mathcal{R} = \Omega \backslash (\mathcal{O}_x \cup \mathcal{O}_z).
$$




A similar distinction was made in [17, Lemma 4.1.1].

### 3.1.1 On $\mathcal{O}_x$

Note that for $y \in \mathcal{O}_x$ we have

$$|z-y| \sim |z-x|\,. \tag{3.8}$$

• **Lemma 3.1-1.** Using Lemma 3.4.iii and (3.8) we find

$$\begin{aligned}
&\int_{y\in\mathcal{O}_x} |x-y|^{-\alpha}\left(1\wedge \tfrac{d(x)d(y)}{|x-y|^2}\right)^m |y-z|^{2m-n}\left(1\wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \;\preceq \\
&\preceq \quad \left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m \int_{y\in\mathcal{O}_x} |x-y|^{-\alpha}\,|y-z|^{2m-n}\,dy \\
&\sim \quad |z-x|^{2m-n}\left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m \int_{y\in\mathcal{O}_x} |x-y|^{-\alpha}\,dy \\
&\sim \quad |x-z|^{2m-\alpha}\left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m \\
&\preceq \begin{cases} |x-z|^{2m-\alpha}\left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m & \text{if } \alpha>2m; \\ \Big(d\left(x\right)d\left(z\right)\Big)^{m-\frac12\alpha}\left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{\frac12\alpha} & \text{if } \alpha\le 2m. \end{cases}
\end{aligned}$$

• **Lemma 3.1-*2*.** Using Lemma 3.4.iii, i, ii and (3.8) we obtain

$$\begin{aligned}
&\int_{y\in\mathcal{O}_x} \Big(d\left(x\right)d\left(y\right)\Big)^{\frac12\delta}\left(1\wedge \tfrac{d(x)d(y)}{|x-y|^2}\right)^{m-\frac12\delta} |y-z|^{2m-n}\left(1\wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \;\preceq \\
&\preceq \quad d\left(x\right)^{\frac12\delta}\left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{m-\frac12\delta} |x-z|^{2m-n}\int_{y\in\mathcal{O}_x} d\left(y\right)^{\frac12\delta}\left(1\wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^{\frac12\delta} dy \\
&\preceq \quad d\left(x\right)^{\frac12\delta}\left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{m-\frac12\delta} |x-z|^{2m-n}\int_{y\in\mathcal{O}_x} d\left(y\right)^{\frac12\delta}\left(1\wedge \tfrac{d(z)}{|y-z|}\right)^{\frac12\delta} dy \\
&\preceq \quad d\left(x\right)^{\frac12\delta}\left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{m-\frac12\delta} |x-z|^{2m-n}\int_{y\in\mathcal{O}_x} d\left(y\right)^{\frac12\delta}\left(\tfrac{d(z)}{d(y)}\right)^{\frac12\delta} dy \\
&\preceq \quad d\left(x\right)^{\frac12\delta} d\left(z\right)^{\frac12\delta}\left(1\wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{m-\frac12\delta} |x-z|^{2m} \\
&\preceq \quad \Big(d\left(x\right)d\left(z\right)\Big)^m.
\end{aligned}$$

• **Lemma 3.1-*3*.** First we need the following auxiliary result:

$$\int_{y\in\mathcal{O}_x} |x-y|^{\beta}\log\left(2+\tfrac{s}{|x-y|}\right)dy \preceq |x-z|^{\beta+n}\log\left(2+\tfrac{s}{|x-z|}\right), \tag{3.9}$$

for any $s > 0$ and $\beta > -n$. The constant in (3.9) only depends on $\beta + n > 0$. Indeed one has with $\tau > 0$ :

$$\begin{aligned} \int_{r=0}^{\tau} \log\left(2 + \tfrac{s}{r}\right) r^{\beta+n-1} dr &= \left[\frac{r^{\beta+n}}{\beta+n} \log\left(2 + \tfrac{s}{r}\right)\right]_0^{\tau} + \int_{r=0}^{\tau} \frac{r^{\beta+n-1}}{\beta+n} \frac{s}{2r+s} dr \\ &\leq \tfrac{1}{\beta+n} \left(\tau^{\beta+n} \log\left(2 + \tfrac{s}{\tau}\right) + \tfrac{1}{\beta+n} \tau^{\beta+n}\right) \\ &\leq \left(\tfrac{1}{\beta+n} + \tfrac{2}{(\beta+n)^2}\right) \tau^{\beta+n} \log\left(2 + \tfrac{s}{\tau}\right). \end{aligned} \tag{3.10}$$

Next we employ Lemma 3.4.v, iii, vi and (3.8)

$$\int_{y \in \mathcal{O}_x} \log\left(1 + \left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right) |y-z|^{2m-n} \left(1 \wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \ \preceq$$

$$\begin{aligned} &\preceq \int_{y \in \mathcal{O}_x} \left(1 \wedge \tfrac{d(x)d(y)}{|x-y|^2}\right)^m \log\left(2 + \tfrac{d(x)}{|x-y|}\right) |y-z|^{2m-n} \left(1 \wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \\ &\preceq \left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m |x-z|^{2m-n} \int_{y \in \mathcal{O}_x} \log\left(2 + \tfrac{d(x)}{|x-y|}\right) dy \\ &\preceq \left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m |x-z|^{2m} \log\left(2 + \tfrac{d(x)}{|x-z|}\right) \\ &\preceq \left(|x-z|^2 \wedge d(x)\, d(z)\right)^m \log\left(2 + \left(\tfrac{d(x)d(z)}{|x-z|^2}\right)^m\right) \\ &\preceq \left(d(x)\, d(z)\right)^m. \end{aligned}$$

In the last step we distinguished the cases as in Lemma 1.1.

- **Lemma 3.1-*4*.** By Lemma 3.4.i, $d(y) \preceq 1$, again (3.8) and (3.9):

$$\int_{y \in \mathcal{O}_x} \left(d(x)\, d(y)\right)^m \log\left(2 + \tfrac{1}{|x-y|^2 + d(x)d(y)}\right) \ |y-z|^{2m-n} \left(1 \wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \ \preceq$$

$$\begin{aligned} &\preceq \left(d(x)\, d(z)\right)^m |x-z|^{m-n} \int_{y \in \mathcal{O}_x} \log\left(2 + \tfrac{1}{|x-y|}\right) dy \\ &\preceq \left(d(x)\, d(z)\right)^m |x-z|^{m-n} |x-z|^n \log\left(2 + \tfrac{1}{|x-z|}\right) \\ &\preceq \left(d(x)\, d(z)\right)^m. \end{aligned}$$

- **Lemma 3.2-*1*.** We have $n = 2m$. Suppose $|x-z| \leq \frac{1}{2}\left(d(x) \vee d(z)\right)$. Then $d(x) \sim d(z)$ and with (3.8) and (3.9)

$$\int_{y \in \mathcal{O}_x} \log\left(1 + \left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right) \log\left(1 + \left(\tfrac{d(y)d(z)}{|y-z|^2}\right)^m\right) dy \ \preceq$$

$$\begin{aligned}
&\preceq \int_{y\in\mathcal{O}_x} \left(1\wedge \tfrac{d(x)}{|x-y|}\right)^m \log\left(2+\tfrac{d(x)}{|x-y|}\right)\left(1\wedge \tfrac{d(z)}{|y-z|}\right)^m \log\left(2+\tfrac{d(z)}{|y-z|}\right) dy \\
&\preceq \log\left(2+\tfrac{d(z)}{|x-z|}\right)\int_{y\in\mathcal{O}_x} \log\left(2+\tfrac{d(x)}{|x-y|}\right) dy \\
&\preceq \log\left(2+\tfrac{d(z)}{|x-z|}\right)|x-z|^n \log\left(2+\tfrac{d(x)}{|x-z|}\right) \\
&\preceq \tfrac{d(z)}{|x-z|}\,|x-z|^{2m}\,\tfrac{d(x)}{|x-z|} \\
&\preceq \Big(d\,(x)\,d\,(z)\Big)^m .
\end{aligned}$$

If $|x-z|\geq \frac{1}{2}\Big(d\,(x)\vee d\,(z)\Big)$ then

$$\int_{y\in\mathcal{O}_x} \log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)\log\left(1+\left(\tfrac{d(y)d(z)}{|y-z|^2}\right)^m\right) dy \ \preceq$$

$$\begin{aligned}
&\preceq \int_{y\in\mathcal{O}_x} \left(1\wedge \tfrac{d(x)}{|x-y|}\right)^m \log\left(2+\tfrac{d(x)}{|x-y|}\right)\left(1\wedge \tfrac{d(z)}{|y-z|}\right)^m \log\left(2+\tfrac{d(z)}{|y-z|}\right) dy \\
&\preceq \int_{y\in\mathcal{O}_x} \left(1\wedge \tfrac{d(x)}{|x-y|}\right)^m \log\left(2+\tfrac{d(x)}{|x-y|}\right)\left(1\wedge \tfrac{d(z)}{|x-z|}\right)^m \log\left(2+\tfrac{d(z)}{|x-z|}\right) dy \\
&\preceq \left(\tfrac{d(z)}{|x-z|}\right)^m \int_{y\in\mathcal{O}_x} \left(1\wedge \tfrac{d(x)}{|x-y|}\right)^m \log\left(2+\tfrac{d(x)}{|x-y|}\right) dy \\
&\preceq \left(\tfrac{d(z)}{|x-z|}\right)^m \left(\int_0^{\frac{1}{4}d(x)} \log\left(2+\tfrac{d(x)}{r}\right) r^{n-1}dr\, + \right. \\
&\qquad\qquad \left. +\, d\,(x)^m \int_{\frac{1}{4}d(x)}^{|x-z|} r^{-m} r^{n-1} dr\right),
\end{aligned} \tag{3.11}$$

and using (3.10) we get

$$(3.11) \preceq \left(\tfrac{d(z)}{|x-z|}\right)^m \Big(d\,(x)^n \log 6 + d\,(x)^m\,|x-z|^m\Big) \preceq \Big(d\,(x)\,d\,(z)\Big)^m .$$

• **Lemma 3.2-*2*.** By Lemma 3.4.v, $d\,(y)\preceq 1$, (3.8), (3.9) and $|x-z|\preceq 1$ :

$$\int_{y\in\mathcal{O}_x} \Big(d\,(x)\,d\,(y)\Big)^m \log\left(2+\tfrac{1}{|x-y|^2+d(x)d(y)}\right)\log\left(1+\left(\tfrac{d(y)d(z)}{|y-z|^2}\right)^m\right) dy \ \preceq$$

$$\begin{aligned}
&\preceq d\,(x)^m \int_{y\in\mathcal{O}_x} \log\left(2+\tfrac{1}{|x-y|^2}\right)\left(\tfrac{d(z)}{|x-z|}\right)^m \log\left(2+\tfrac{1}{|x-z|^2}\right) dy \\
&\preceq \Big(d\,(x)\,d\,(z)\Big)^m \log\left(2+\tfrac{1}{|x-z|^2}\right)|x-z|^{-m}\int_{y\in\mathcal{O}_x} \log\left(2+\tfrac{1}{|x-y|^2}\right) dy \\
&\preceq \Big(d\,(x)\,d\,(z)\Big)^m \left(\log\left(2+\tfrac{1}{|x-z|^2}\right)\right)^2 |x-z|^{n-m} \\
&\preceq \Big(d\,(x)\,d\,(z)\Big)^m ,
\end{aligned}$$

where we used that $n-m=m>0$.

• **Lemma 3.3.** We have to estimate

$$\begin{aligned}
&\int_{y\in\mathcal{O}_x}\Big(d\left(x\right)d\left(y\right)\Big)^{\frac{1}{2}\delta}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^{m-\frac{1}{2}\delta}\Big(d\left(y\right)d\left(z\right)\Big)^{m-\frac{1}{2}n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^{\frac{1}{2}n}dy\\
&\quad=\quad d\left(x\right)^{\frac{1}{2}\delta}d\left(z\right)^{m-\frac{1}{2}n}\ \times\\
&\qquad\qquad\times\int_{y\in\mathcal{O}_x}d\left(y\right)^{m-\frac{1}{2}(n-\delta)}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^{m-\frac{1}{2}\delta}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^{\frac{1}{2}n}dy. \qquad (3.12)
\end{aligned}$$

If $\delta\geq n$ then Lemma 3.4.ii implies

$$\begin{aligned}
(3.12)\quad &\preceq\quad d\left(x\right)^{\frac{1}{2}\delta}d\left(z\right)^{m-\frac{1}{2}n}\int_{y\in\mathcal{O}_x}d\left(y\right)^{m+\frac{1}{2}(\delta-n)}\left(\tfrac{d(x)}{d(y)}\right)^{m-\frac{1}{2}\delta}\left(\tfrac{d(z)}{d(y)}\right)^{\frac{1}{2}n}dy\\
&\sim\quad\Big(d\left(x\right)d\left(z\right)\Big)^m\int_{y\in\mathcal{O}_x}d\left(y\right)^{\delta-n}dy\\
&\preceq\quad\Big(d\left(x\right)d\left(z\right)\Big)^m\left|x-z\right|^n\ \preceq\ \Big(d\left(x\right)d\left(z\right)\Big)^m.
\end{aligned}$$

If $\delta<n$ then

$$\begin{aligned}
(3.12)\quad &\preceq\quad d\left(x\right)^{\frac{1}{2}\delta}d\left(z\right)^{m-\frac{1}{2}n}\int_{y\in\mathcal{O}_x}d\left(y\right)^{m-\frac{1}{2}(n-\delta)}\left(\tfrac{d(x)}{d(y)}\right)^{m-\frac{1}{2}n}\times\\
&\qquad\qquad\times\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^{\frac{1}{2}(n-\delta)}\left(\tfrac{d(z)}{d(y)}\right)^{\frac{1}{2}\delta}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^{\frac{1}{2}(n-\delta)}dy\\
&\preceq\quad\Big(d\left(x\right)d\left(z\right)\Big)^{m-\frac{1}{2}(n-\delta)}\int_{y\in\mathcal{O}_x}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^{\frac{1}{2}(n-\delta)}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^{\frac{1}{2}(n-\delta)}dy\\
&\preceq\quad\Big(d\left(x\right)d\left(z\right)\Big)^{m-\frac{1}{2}(n-\delta)}\int_{y\in\mathcal{O}_x}\left(\tfrac{d(x)}{|x-y|}\right)^{\frac{1}{2}(n-\delta)}\left(\tfrac{d(z)}{|y-z|}\right)^{\frac{1}{2}(n-\delta)}dy\\
&\preceq\quad\Big(d\left(x\right)d\left(z\right)\Big)^m\left|x-z\right|^{-\frac{1}{2}(n-\delta)}\int_0^{|x-z|}r^{\frac{1}{2}(\delta-n)+n-1}dr\\
&\preceq\quad\Big(d\left(x\right)d\left(z\right)\Big)^m\left|x-z\right|^{\delta}\ \preceq\ \Big(d\left(x\right)d\left(z\right)\Big)^m.
\end{aligned}$$

### 3.1.2 On $\mathcal{O}_z$

Notice that $y\in\mathcal{O}_z$ implies

$$\left|x-y\right|\sim\left|x-z\right|. \qquad (3.13)$$

• **Lemma 3.1-*1*.** The proof is similar as on $\mathcal{O}_x$. Using Lemma 3.4.iii we find

$$\int_{y\in\mathcal{O}_z}\left|x-y\right|^{-\alpha}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\left|y-z\right|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy\ \preceq \qquad (3.14)$$

$$
\begin{aligned}
&\preceq \left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m \int_{y\in\mathcal{O}_z} |x-y|^{-\alpha}\,|y-z|^{2m-n}\,dy \\
&\sim |z-x|^{-\alpha}\left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m \int_{y\in\mathcal{O}_z} |y-z|^{2m-n}\,dy \\
&\sim |x-z|^{2m-\alpha}\left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m \\
&\preceq \begin{cases} |x-z|^{2m-\alpha}\left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m & \text{if } \alpha > 2m; \\ \Big(d\left(x\right)d\left(z\right)\Big)^{m-\frac{1}{2}\alpha}\left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{\frac{1}{2}\alpha} & \text{if } \alpha \le 2m. \end{cases}
\end{aligned}
$$

• **Lemma 3.1-*2*.** Using Lemma 3.4.iii, i and ii we conclude in an almost similar way as for $\mathcal{O}_x$ :

$$
\int_{y\in\mathcal{O}_z} \Big(d\left(x\right)d\left(y\right)\Big)^{\frac{1}{2}\delta}\left(1 \wedge \tfrac{d(x)d(y)}{|x-y|^2}\right)^{m-\frac{1}{2}\delta}|y-z|^{2m-n}\left(1 \wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \;\preceq
$$

$$
\begin{aligned}
&\preceq d\left(x\right)^{\frac{1}{2}\delta}\left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{m-\frac{1}{2}\delta}\int_{y\in\mathcal{O}_z} d\left(y\right)^{\frac{1}{2}\delta}|y-z|^{2m-n}\left(\tfrac{d(z)}{d(y)}\right)^{\frac{1}{2}\delta} dy \\
&\preceq \Big(d\left(x\right)d\left(z\right)\Big)^{\frac{1}{2}\delta}\left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{m-\frac{1}{2}\delta}|x-z|^{2m} \\
&\preceq \Big(d\left(x\right)d\left(z\right)\Big)^m .
\end{aligned}
$$

• **Lemma 3.1-*3*.** Using Lemma 3.4.v and iii we obtain similarly as on $\mathcal{O}_x$ using here (3.13).

$$
\int_{y\in\mathcal{O}_z} \log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)|y-z|^{2m-n}\left(1 \wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \;\preceq
$$

$$
\begin{aligned}
&\preceq \left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m \log\left(2+\tfrac{d(x)}{|x-z|}\right)\int_{y\in\mathcal{O}_z} |y-z|^{2m-n}\,dy \\
&\preceq \left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m \log\left(2+\left(\tfrac{d(x)d(z)}{|x-z|^2}\right)^m\right)|x-z|^{2m} \\
&\preceq \Big(d\left(x\right)d\left(z\right)\Big)^m .
\end{aligned}
$$

• **Lemma 3.1-*4*.** With (3.13) and $d\left(y\right) \preceq 1$

$$
\int_{y\in\mathcal{O}_z} \Big(d\left(x\right)d\left(y\right)\Big)^m \log\left(2+\tfrac{1}{|x-y|^2+d(x)d(y)}\right)\;|y-z|^{2m-n}\left(1 \wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \;\preceq
$$

$$
\begin{aligned}
&\preceq \Big(d\left(x\right)d\left(z\right)\Big)^m \log\left(2+\tfrac{1}{|x-z|^2}\right)\int_{y\in\mathcal{O}_z} |y-z|^{m-n}\,dy \\
&\preceq \Big(d\left(x\right)d\left(z\right)\Big)^m \log\left(2+\tfrac{1}{|x-z|^2}\right)|x-z|^m \\
&\preceq \Big(d\left(x\right)d\left(z\right)\Big)^m .
\end{aligned}
$$

• **Lemma 3.2-*1*.** This statement is symmetric in $x$ and $z$ and hence the same result as on $\mathcal{O}_x$ holds.
• **Lemma 3.2-*2*.** Again we observe (3.13), (3.9) and $|x-z| \preceq 1$:

$$\int_{y\in\mathcal{O}_z} \Big(d\,(x)\,d\,(y)\Big)^m \log\Big(2+\tfrac{1}{|x-y|^2+d(x)d(y)}\Big)\ \log\Big(1+\Big(\tfrac{d(y)d(z)}{|y-z|^2}\Big)^m\Big)\,dy\ \preceq$$

$$\begin{aligned}
&\preceq\ \Big(d\,(x)\,d\,(z)\Big)^m \log\Big(2+\tfrac{1}{|x-z|^2}\Big)\int_{y\in\mathcal{O}_z}|y-z|^{-m}\log\Big(2+\tfrac{1}{|y-z|}\Big)\,dy\\
&\preceq\ \Big(d\,(x)\,d\,(z)\Big)^m\Big(\log\Big(2+\tfrac{1}{|x-z|^2}\Big)\Big)^2|x-z|^m\\
&\preceq\ \Big(d\,(x)\,d\,(z)\Big)^m.
\end{aligned}$$

• **Lemma 3.3.** Similarly as for $\mathcal{O}_x$ we get

$$\begin{aligned}
&\int_{y\in\mathcal{O}_z}\Big(d\,(x)\,d\,(y)\Big)^{\frac12\delta}\Big(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\Big)^{m-\frac12\delta}\ \times\\
&\qquad\qquad\times\ \Big(d\,(y)\,d\,(z)\Big)^{m-\frac12 n}\Big(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\Big)^{\frac12 n}dy\ \preceq
\end{aligned}\tag{3.15}$$

If $\delta\geq n$ then Lemma 3.4.ii implies

$$\begin{aligned}
(3.15)\ &\preceq\ d\,(x)^{\frac12\delta}\,d\,(z)^{m-\frac12 n}\int_{y\in\mathcal{O}_z}d\,(y)^{m+\frac12(\delta-n)}\Big(\tfrac{d(x)}{d(y)}\Big)^{m-\frac12\delta}\Big(\tfrac{d(z)}{d(y)}\Big)^{\frac12 n}dy\\
&\preceq\ \Big(d\,(x)\,d\,(z)\Big)^m\int_{y\in\mathcal{O}_z}d\,(y)^{\delta-n}\,dy\\
&\preceq\ \Big(d\,(x)\,d\,(z)\Big)^m|x-z|^n\ \preceq\ \Big(d\,(x)\,d\,(z)\Big)^m.
\end{aligned}$$

If $\delta<n$ then by (3.13)

$$\begin{aligned}
(3.15)\ &\preceq\ d\,(x)^{\frac12\delta}\,d\,(z)^{m-\frac12 n}\int_{y\in\mathcal{O}_z}d\,(y)^{m-\frac12(n-\delta)}\Big(\tfrac{d(x)}{d(y)}\Big)^{m-\frac12 n}\Big(\tfrac{d(x)}{|x-y|}\Big)^{\frac12(n-\delta)}\ \times\\
&\qquad\qquad\times\Big(\tfrac{d(z)}{d(y)}\Big)^{\frac12\delta}\Big(\tfrac{d(z)}{|y-z|}\Big)^{\frac12(n-\delta)}dy\\
&\preceq\ \Big(d\,(x)\,d\,(z)\Big)^m\int_{y\in\mathcal{O}_z}|x-y|^{-\frac12(n-\delta)}\,|y-z|^{-\frac12(n-\delta)}\,dy\\
&\preceq\ \Big(d\,(x)\,d\,(z)\Big)^m|x-z|^{-\frac12(n-\delta)}\int_0^{|x-z|}r^{\frac12(\delta-n)+n-1}dr\\
&\preceq\ \Big(d\,(x)\,d\,(z)\Big)^m|x-z|^\delta\ \preceq\ \Big(d\,(x)\,d\,(z)\Big)^m.
\end{aligned}$$

### 3.1.3 On $\mathcal{R}$

Here we have $|y-z|\geq\frac23|x-z|$ implying that $|x-y|\leq|x-z|+|z-y|\leq\frac52|y-z|$. By symmetry $|y-x|\geq\frac23|x-z|$ implies $|y-z|\leq\frac52|x-y|$ and hence

$$|x-y|\sim|y-z|.\tag{3.16}$$

Let us denote

$$D = 2\,\mathrm{diam}\,(\Omega)\,. \tag{3.17}$$

• **Lemma 3.1-*1*.** We have by (3.16)

$$\int_{y\in\mathcal{R}} |x-y|^{-\alpha}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^m |y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \;\preceq$$

$$\begin{aligned}
&\preceq \int_{y\in\mathcal{R}} |x-y|^{2m-n-\alpha}\left(1\wedge\tfrac{d(x)}{|x-y|}\right)^m\left(1\wedge\tfrac{d(z)}{|x-y|}\right)^m dy\\
&\preceq \int_{|x-z|}^{D} r^{2m-\alpha-1}\left(1\wedge\tfrac{d(x)}{r}\right)^m\left(1\wedge\tfrac{d(z)}{r}\right)^m dr.
\end{aligned} \tag{3.18}$$

Notice that we may replace $\frac{2}{3}\,|x-z|$ by $|x-z|$, as $D = 2\,\mathrm{diam}\,(\Omega)\,.$

Assuming that $|x-z|^2 \le d\,(x)\,d\,(z)$ we find

$$\begin{aligned}
(3.18) \;\preceq\;& \int_{|x-z|}^{\sqrt{d(x)d(z)}} r^{2m-\alpha-1}dr + \Big(d\,(x)\,d\,(z)\Big)^m \int_{\sqrt{d(x)d(z)}}^{D} r^{-\alpha-1}dr\\
\preceq\;& \begin{cases}
|x-z|^{2m-\alpha} \;+\; \Big(d\,(x)\,d\,(z)\Big)^{m-\frac{1}{2}\alpha} & \text{if } \alpha > 2m;\\
\log\frac{\sqrt{d(x)d(z)}}{|x-z|} \;+\; 1 & \text{if } \alpha = 2m;\\
\Big(d\,(x)\,d\,(z)\Big)^{m-\frac{1}{2}\alpha} \;+\; \Big(d\,(x)\,d\,(z)\Big)^{m-\frac{1}{2}\alpha} & \text{if } \alpha < 2m;
\end{cases}\\
\preceq\;& \begin{cases}
|x-z|^{2m-\alpha}\left(1\wedge\frac{d(x)d(z)}{|x-z|^2}\right)^m & \text{if } \alpha > 2m;\\
\log\left(1+\left(\frac{d(x)d(z)}{|x-z|^2}\right)^m\right) & \text{if } \alpha = 2m;\\
\Big(d\,(x)\,d\,(z)\Big)^{m-\frac{1}{2}\alpha}\left(1\wedge\frac{d(x)d(z)}{|x-z|^2}\right)^{\frac{1}{2}\alpha} & \text{if } \alpha < 2m.
\end{cases}
\end{aligned} \tag{3.19}$$

If $|x-z|^2 \ge d\,(x)\,d\,(z)$ we estimate

$$\begin{aligned}
(3.18) \;\preceq\;& \Big(d\,(x)\,d\,(z)\Big)^m \int_{|x-z|}^{D} r^{-\alpha-1}dr\\
\preceq\;& \Big(d\,(x)\,d\,(z)\Big)^m |x-z|^{-\alpha} \preceq (3.19)\,.
\end{aligned}$$

• **Lemma 3.1-*2*.** By Lemma 3.4.i and ii we find, again using (3.16), that

$$\int_{y\in\mathcal{R}} \Big(d\,(x)\,d\,(y)\Big)^{\frac{1}{2}\delta}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^{m-\frac{1}{2}\delta} |y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \preceq$$

$$\preceq \quad d(x)^{\frac{1}{2}\delta}\int_{y\in\mathcal{R}} d(y)^{\frac{1}{2}\delta}\left(1\wedge\tfrac{d(x)}{|x-y|}\right)^{m-\frac{1}{2}\delta}|x-y|^{2m-n}\left(1\wedge\tfrac{d(z)}{|x-y|}\right)^{m-\frac{1}{2}\delta}\left(\tfrac{d(z)}{d(y)}\right)^{\frac{1}{2}\delta}dy$$

$$\preceq \quad \Big(d(x)\,d(z)\Big)^{\frac{1}{2}\delta}\int_{\frac{2}{3}|x-z|}^{D}\left(1\wedge\tfrac{d(x)}{r}\right)^{m-\frac{1}{2}\delta}r^{2m-1}\left(1\wedge\tfrac{d(z)}{r}\right)^{m-\frac{1}{2}\delta}dr$$

$$\preceq \quad \Big(d(x)\,d(z)\Big)^{m}\int_{\frac{2}{3}|x-z|}^{D}r^{\delta-1}dr\preceq\Big(d(x)\,d(z)\Big)^{m}. \tag{3.20}$$

- **Lemma 3.1-*3*.** With Lemma 3.4.i and v and (3.16)

$$\int_{y\in\mathcal{R}}\log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)|y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy\preceq$$

$$\preceq \quad \int_{y\in\mathcal{R}}\left(1\wedge\tfrac{d(x)}{|x-y|}\right)^m\log\left(2+\tfrac{d(x)}{|x-y|}\right)|x-y|^{2m-n}\left(1\wedge\tfrac{d(z)}{|x-y|}\right)^m dy$$

$$\preceq \quad \int_{\frac{2}{3}|x-z|}^{D}\left(1\wedge\tfrac{d(x)}{r}\right)^m\log\left(2+\tfrac{d(x)}{r}\right)r^{2m-1}\left(1\wedge\tfrac{d(z)}{r}\right)^m dr. \tag{3.21}$$

Assuming that $|x-z|\le\frac{1}{2}\Big(d(x)\vee d(z)\Big)$ we may use as in (1.7) that $d(x)\sim d(z)$ to proceed by

$$
\begin{aligned}
(3.21)\quad &\preceq \int_{\frac{2}{3}|x-z|}^{\sqrt{d(x)d(z)}}\log\left(2+\frac{\sqrt{d(x)d(z)}}{r}\right)r^{2m-1}dr\\
&\quad+\Big(d(x)\,d(z)\Big)^m\int_{\sqrt{d(x)d(z)}}^{D}\log\left(2+\frac{\sqrt{d(x)d(z)}}{r}\right)r^{-1}dr\\
&\preceq \Big(d(x)\,d(z)\Big)^m\left(\int_0^1\log\left(2+\tfrac{1}{s}\right)s^{2m-1}ds+\int_1^{\frac{D}{\sqrt{d(x)d(z)}}}s^{-1}ds\right)\\
&\preceq \Big(d(x)\,d(z)\Big)^m\left(1+\log\left(\tfrac{D}{\sqrt{d(x)d(z)}}\right)\right)\\
&\preceq \Big(d(x)\,d(z)\Big)^m\log\left(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\right).
\end{aligned}
$$

If $|x-z|\ge\frac{1}{2}\Big(d(x)\vee d(z)\Big)$ we have $r=|x-y|\ge\frac{2}{3}|x-z|\ge\frac{1}{3}d(x)$ and hence

$$
\begin{aligned}
(3.21)\quad &\preceq \Big(d(x)\,d(z)\Big)^m\int_{\frac{2}{3}|x-z|}^{D}r^{-1}dr \quad\preceq\quad \Big(d(x)\,d(z)\Big)^m\log\left(\tfrac{3}{2}\tfrac{D}{|x-z|}\right)\\
&\preceq \Big(d(x)\,d(z)\Big)^m\log\left(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\right).
\end{aligned}
$$

- **Lemma 3.1-*4*.** With $d(y)\preceq 1$ and (3.16)

$$\int_{y\in\mathcal{R}}\Big(d(x)\,d(y)\Big)^m\log\left(2+\tfrac{1}{|x-y|^2+d(x)d(y)}\right)\;|y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy\preceq$$

$$\begin{aligned}
&\preceq \Big(d(x)\,d(z)\Big)^m \int_{y\in\mathcal{R}} \log\Big(2+\tfrac{1}{|x-y|^2}\Big)\,|x-y|^{m-n}\,dy \\
&\preceq \Big(d(x)\,d(z)\Big)^m \int_0^D \log\big(2+\tfrac{1}{r^2}\big)\,r^{m-1}dr \\
&\preceq \Big(d(x)\,d(z)\Big)^m .
\end{aligned}$$

• **Lemma 3.2-*1*.** We have $n=2m$ and need to estimate

$$\int_{y\in\mathcal{R}} \log\Big(1+\Big(\tfrac{d(x)d(y)}{|x-y|^2}\Big)^m\Big)\log\Big(1+\Big(\tfrac{d(y)d(z)}{|y-z|^2}\Big)^m\Big)\,dy \preceq$$

$$\preceq \int_{\frac{2}{3}|x-z|}^D \Big(1\wedge\tfrac{d(x)}{r}\Big)^m \log\Big(2+\tfrac{d(x)}{r}\Big)\Big(1\wedge\tfrac{d(z)}{r}\Big)^m \log\Big(2+\tfrac{d(z)}{r}\Big)\,r^{n-1}dr. \quad (3.22)$$

Assuming that $|x-z|\le\frac{1}{2}\Big(d(x)\vee d(z)\Big)$ we take from (1.7) that $d(x)\sim d(z)$ in order to find

$$\begin{aligned}
(3.22)\quad &\preceq \int_{\frac{2}{3}|x-z|}^{\sqrt{d(x)d(z)}} \log\Big(2+\frac{\sqrt{d(x)d(z)}}{r}\Big)^2 r^{n-1}dr \\
&\qquad + \Big(d(x)\,d(z)\Big)^m \int_{\sqrt{d(x)d(z)}}^D \log\Big(2+\frac{\sqrt{d(x)d(z)}}{r}\Big)^2 r^{-1}dr \\
&\preceq \Big(d(x)\,d(z)\Big)^m \left(\int_0^1 \log\big(2+\tfrac{1}{s}\big)^2 s^{2m-1}ds + \int_1^{\frac{D}{\sqrt{d(x)d(z)}}} s^{-1}ds\right) \\
&\preceq \Big(d(x)\,d(z)\Big)^m \left(1+\log\Big(\frac{D}{\sqrt{d(x)d(z)}}\Big)\right) \\
&\preceq \Big(d(x)\,d(z)\Big)^m \log\Big(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\Big).
\end{aligned}$$

If $|x-z|\ge\frac{1}{2}\Big(d(x)\vee d(z)\Big)$ we have as above $|x-y|\ge\frac{1}{3}d(x)$, $|x-y|\ge\frac{1}{3}d(z)$ and consequently

$$\begin{aligned}
(3.22)\quad &\preceq \Big(d(x)\,d(z)\Big)^m \int_{\frac{2}{3}|x-z|}^D r^{-1}dr \;\preceq\; \Big(d(x)\,d(z)\Big)^m \log\Big(\tfrac{3}{2}\tfrac{D}{|x-z|}\Big) \\
&\preceq \Big(d(x)\,d(z)\Big)^m \log\Big(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\Big).
\end{aligned}$$

• **Lemma 3.2-*2*.** Since $2m=n$ it follows that

$$\int_{y\in\mathcal{R}} \Big(d(x)\,d(y)\Big)^m \log\Big(2+\tfrac{1}{|x-y|^2+d(x)d(y)}\Big)\;\log\Big(1+\Big(\tfrac{d(y)d(z)}{|y-z|^2}\Big)^m\Big)\,dy \preceq$$

$$\begin{aligned}
&\preceq \Big(d(x)\,d(z)\Big)^m \int_{y\in\mathcal{R}} \log\Big(2+\tfrac{1}{|x-y|^2}\Big)\,|y-z|^{-m}\log\Big(2+\tfrac{1}{|y-z|^2}\Big)\,dy \\
&\preceq \Big(d(x)\,d(z)\Big)^m \int_{\frac{2}{3}|x-z|}^D \big(\log\big(2+\tfrac{1}{r^2}\big)\big)^2 r^{m-1}dr \\
&\preceq \Big(d(x)\,d(z)\Big)^m .
\end{aligned}$$

• **Lemma 3.3.** As in the first steps of the case on $\mathcal{O}_x$ we have

$$\int_{y\in\mathcal{R}} \Big(d(x)\,d(y)\Big)^{\frac{1}{2}\delta}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^{m-\frac{1}{2}\delta}\Big(d(y)\,d(z)\Big)^{m-\frac{1}{2}n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^{\frac{1}{2}n} dy \preceq$$

$$\preceq d(x)^{\frac{1}{2}\delta}\,d(z)^{m-\frac{1}{2}n}\int_{y\in\mathcal{R}} d(y)^{m-\frac{1}{2}n+\frac{1}{2}\delta}\left(1\wedge\tfrac{d(x)}{|x-y|}\right)^{m-\frac{1}{2}\delta}\left(1\wedge\tfrac{d(z)}{|y-z|}\right)^{\frac{1}{2}n} dy. \quad (3.23)$$

If $\delta\geq n$ then by Lemma 3.4.ii

$$\begin{aligned}(3.23) \;\;&\preceq\;\; d(x)^{\frac{1}{2}\delta}\,d(z)^{m-\frac{1}{2}n}\int_{y\in\mathcal{R}} d(y)^{m-\frac{1}{2}n+\frac{1}{2}\delta}\left(\tfrac{d(x)}{d(y)}\right)^{m-\frac{1}{2}\delta}\left(\tfrac{d(z)}{d(y)}\right)^{\frac{1}{2}n} dy\\ &\preceq\;\; \Big(d(x)\,d(z)\Big)^m\int_{y\in\mathcal{R}} d(y)^{\delta-n}\,dy \;\preceq\; \Big(d(x)\,d(z)\Big)^m.\end{aligned}$$

If $\delta<n$ then by Lemma 3.4.ii and (3.16) we also have

$$\begin{aligned}(3.23) \;\;&\preceq\;\; d(x)^{\frac{1}{2}\delta}\,d(z)^{m-\frac{1}{2}n}\int_{y\in\mathcal{R}} d(y)^{m-\frac{1}{2}n+\frac{1}{2}\delta}\left(\tfrac{d(x)}{|x-y|}\right)^{\frac{1}{2}(n-\delta)}\left(\tfrac{d(x)}{d(y)}\right)^{m-\frac{1}{2}n}\times\\ &\qquad\qquad\times\left(\tfrac{d(z)}{|y-z|}\right)^{\frac{1}{2}(n-\delta)}\left(\tfrac{d(z)}{d(y)}\right)^{\frac{1}{2}\delta} dy\\ &\preceq\;\; \Big(d(x)\,d(z)\Big)^m\int_{\frac{2}{3}|x-z|}^{D} r^{\delta-n+n-1}dr \;\preceq\; \Big(d(x)\,d(z)\Big)^m,\end{aligned}$$

which completes the proofs for the estimates from above.

## 3.2 Estimates from below

Let us start with the simplest estimates. Since $H_m(\theta;x,y)\succeq\Big(d(x)\,d(y)\Big)^m$ for all $\theta$ we immediately find that

$$\begin{aligned}&\int_\Omega H_m(\theta;x,y)\,H_m\left(m-\tfrac{1}{2}n;y,z\right)dy\\ \succeq\;\;&\Big(d(x)\,d(z)\Big)^m\int_\Omega d(y)^{2m}\,dy\succeq\Big(d(x)\,d(z)\Big)^m.\end{aligned}$$

This implies the estimate from below for Lemma 3.1-*2* and *4*, Lemma 3.2-*2* and Lemma 3.3. In the remaining cases, i.e. for $\theta\leq 0$, we will exploit the following idea. The fact that the $H_m(\theta;x,y)$ are positive allows us to identify for each of these cases a region $\mathcal{R}_i\subset\Omega$ such that

$$\int_{\mathcal{R}_i} H_m(\theta;x,y)\,H_m\left(m-\tfrac{1}{2}n;y,z\right)dy\succeq H_m(m+\theta;x,z). \quad (3.24)$$

The main difficulty will be to choose $\mathcal{R}_i$ such that it is large enough to find (3.24) but nice enough to simplify the formula.

### 3.2.1 Some auxiliary geometrical results

We will start with some geometric properties of $C^{1,1}$-domains. Let us first define a family of cones for $\varepsilon \in (0,1)$ and $e \in \mathbb{R}^n \backslash \{0\}$ :

$$\mathcal{K}(\varepsilon, e) = \{x \in \mathbb{R}^n; x \cdot e > \varepsilon |x| |e|\}. \tag{3.25}$$

Without proof we state a lemma related to a cone condition, observe that bounded $C^{1,1}$-domains satisfy uniform interior sphere conditions.

**Lemma 3.6** *For every bounded $C^{1,1}$-domain $\Omega$ and $\varepsilon \in (0,1)$ there is a constant $C = C(\varepsilon, \Omega) > 0$ such that:*

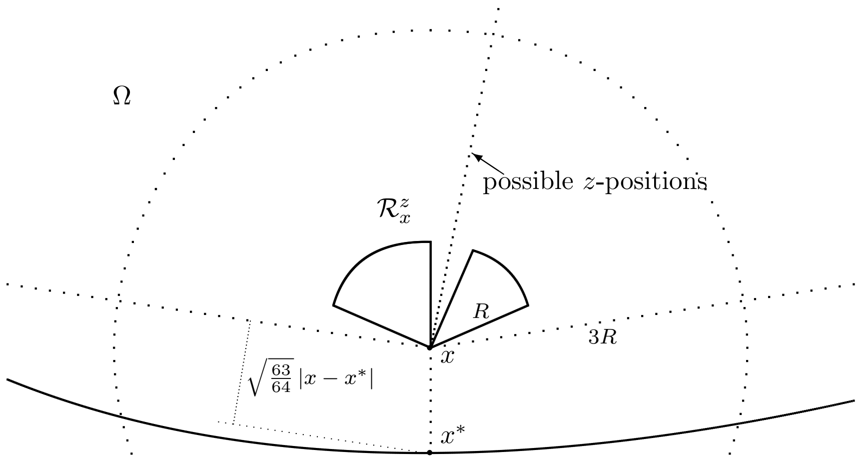
1. *if $x \in \partial\Omega$ and $n_x$ denotes the inward normal at $x$, then*
$$x + \Big(\mathcal{K}(\varepsilon, n_x) \cap B_C(0)\Big) \subset \Omega;$$
2. *if $x \in \Omega$ and $x^* \in \partial\Omega$ is such that $d(x) = |x - x^*|$, then*
$$x + \Big(\mathcal{K}(\varepsilon, x - x^*) \cap B_C(0)\Big) \subset \Omega.$$

*All $C(\varepsilon, \Omega)$ satisfy $C(\varepsilon, \Omega) \leq \frac{1}{2}\mathrm{diam}(\Omega)$.*

Let $x \neq z \in \Omega$ and fix a closest point $x^* \in \partial\Omega$. We take $\varepsilon = \frac{1}{8}$, set $R = \frac{1}{3}C\left(\frac{1}{8}, \Omega\right)$ and define

$$\mathcal{R}_x^z = x + \left(\mathcal{K}\left(\tfrac{1}{4}, n_{x^*}\right) \backslash \mathcal{K}\left(\tfrac{3}{4}, z - x\right)\right) \cap B_R(0). \tag{3.26}$$

Two crucial estimates that hold true on these subsets are given in the next lemma.




**Lemma 3.7** *Let $R$ be as above. Then there is a constant $C_1 = C_1(\Omega, R)$ such that for all $x \neq z \in \Omega$ and $y \in \mathcal{R}_x^z$ the following holds:*

$$d(y) \geq C_1 \Big(|x - y| + d(x)\Big) \quad \textit{and} \quad |z - y| \geq C_1 \, |x - y|.$$

*Proof.* We use the following definition of the distance between two sets $A, B \subset \mathbb{R}^n$:

$$d(A,B) = \inf\{|a-b|\,; a \in A,\, b \in B\}. \tag{3.27}$$

Let $y \in \mathcal{R}_x^z$. By Lemma 3.6 $\mathcal{O} := x + \left(\mathcal{K}\left(\frac{1}{8}, x - x^*\right) \cap B_{2R}(0)\right) \subset \Omega$. Since $y \in \mathcal{O} \subset \Omega \subset \mathbb{R}^n$ it follows that

$$d(y) \geq d(y, \partial\mathcal{O}) + d(\mathcal{O}, \partial\Omega). \tag{3.28}$$

The distance between $\mathcal{O}$ and $\partial\Omega$ satisfies

$$d(\mathcal{O},\partial\Omega) \geq \min\left(\sqrt{\tfrac{63}{64}}d(x), R\right) \geq \min\left(\sqrt{\tfrac{63}{64}}d(x), R\frac{d(x)}{D}\right) \geq \frac{R}{D}d(x), \tag{3.29}$$

where we have set

$$D = 2\operatorname{diam}(\Omega). \tag{3.30}$$

To estimate $d(y,\partial\mathcal{O})$ let $\tilde{y} \in \partial\mathcal{O}$ be such that $|y - \tilde{y}| = d(\partial\mathcal{O}, y)$. One has either $\tilde{y} \in \partial B_{2R}(x)$ or $\tilde{y} \in \partial\left(x + \mathcal{K}\left(\frac{1}{8}, x - x^*\right)\right)$. In the first case

$$|y - \tilde{y}| \geq |x - \tilde{y}| - |y - x| \geq 2R - R = R \geq |x - y|.$$

In the second case let $y_p$ denote the projection of $y$ on the hyperplane through $x$ perpendicular to $x - x^*$, that is

$$y_p = y - \frac{(x - x^*)\cdot(y - x)}{|x - x^*|^2}(x - x^*),$$

and let $\tilde{y}_p$ denote the intersection of $\partial\left(x + \mathcal{K}\left(\frac{1}{8}, x - x^*\right)\right)$ with the line through $y$ and $y_p$. Then

$$|y - \tilde{y}| = \sqrt{\tfrac{63}{64}}\,|y - \tilde{y}_p|$$

and

$$\begin{aligned} |y - \tilde{y}_p| &\geq |y - y_p| - |y_p - \tilde{y}_p| = \left|\frac{(x - x^*)\cdot(y - x)}{|x - x^*|}\right| - \tfrac{1}{8}|x - \tilde{y}_p| \\ &\geq \tfrac{1}{4}|y - x| - \tfrac{1}{8}|y - x| = \tfrac{1}{8}|y - x|. \end{aligned}$$

One concludes that

$$d(y, \partial\mathcal{O}) = |y - \tilde{y}| \geq \min\left(\tfrac{\sqrt{63}}{64}, 1\right)|x - y|. \tag{3.31}$$

The first claim follows from (3.28), (3.29) and (3.31).

For the second claim notice that $y \notin x + \mathcal{K}\left(\frac{3}{4}, z - x\right)$ implies

$$\begin{aligned} |x - y|^2 + |x - z|^2 &= |y - z|^2 + 2(x - z)\cdot(x - y) \\ &\leq |y - z|^2 + \tfrac{3}{2}|x - z|\,|x - y| \\ &\leq |y - z|^2 + |x - z|^2 + \tfrac{9}{16}|x - y|^2 \end{aligned}$$

and hence $\frac{7}{16}|x-y|^2 \le |y-z|^2$. $\square$

The following two sets will be used:

$$\mathcal{R}_1 = B_{\frac{1}{2}|x-z|}(x). \tag{3.32}$$

and

$$\mathcal{R}_2 = \mathcal{R}_x^z \backslash B_{\frac{R}{D}|x-z|}(x). \tag{3.33}$$

The first will only be considered under the assumption $|x-z| \le \frac{1}{2}(d(x) \vee d(z))$. We will list some estimates for later use.

**Case I** *Inequalities for $y \in \mathcal{R}_1$ assuming that $|x-z| \le \frac{1}{2}(d(x) \vee d(z))$ :*

*i. by assumptions and (1.7)*: $|x-y| \le \frac{1}{2}|x-z| \le \frac{1}{4}\big(d(x) \vee d(z)\big) \le \frac{1}{2}d(x)$,

*ii. from the previous and (1.7)*: $|y-z| \le |x-y| + |x-z| \le \frac{3}{2}|x-z| \le \frac{3}{2}d(z)$,

*iii. since $|y-x| \le \frac{1}{2}|x-z|$ :* $|y-z| \ge |x-z| - |x-y| \ge \frac{1}{2}|x-z|$,

*iv. from i*: $d(y) \ge d(x) - |x-y| \ge |x-y|$,

*v. and from i, (1.7) and ii*: $d(y) \ge d(x) - |x-y| \ge \frac{1}{2}d(x) \ge \frac{1}{2}|x-z| \ge$
$\ge \frac{1}{3}|y-z|$.

A similar list for $\mathcal{R}_2$ :

**Case II** *Inequalities for $y \in \mathcal{R}_2$* :

*i. by Lemma 3.7*: $d(y) \ge C_1 |x-y|$,

*ii. and also*: $|y-z| \ge C_1 |x-y|$,

*iii. since $y \notin B_{\frac{R}{D}|x-z|}(x)$ :* $|y-z| \le |x-z| + |x-y| \le \left(\frac{D}{R}+1\right)|x-y|$,

*iv. from i and iii*: $d(y) \ge C_1 |x-y| \ge C_1 \frac{R}{D+R}|y-z|$.

### 3.2.2 Integral estimates when $|x-z| \le \frac{1}{2}(d(x) \vee d(z))$

The estimates

$$\frac{1}{2}d(x) \le d(z) \le 2d(x) \tag{3.34}$$

by Lemma 1.1 will be used throughout this section. First suppose that $d(x)$ is bounded from below by $\frac{1}{2}R$. Hence $d(z) \ge \frac{1}{4}R$ by (3.34) and the proof of the estimates simplifies. The right hand sides of Lemma 3.1-*1* for $\alpha < 2m$, Lemma 3.1-*3* and Lemma 3.2-*1* reduce to $(d(x)\,d(z))^{m-(\alpha/2)}$ and $(d(x)\,d(z))^m$ respectively. Left over is the case Lemma 3.1-*1* for $n-2m \ge \alpha \ge 2m$. If $\alpha > 2m$ then we integrate over $\mathcal{R}_1$. By the inequalities iv and i of Case I we find $\left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right) \sim 1$ and ii, v imply $\left(1 \wedge \frac{d(y)d(z)}{|y-z|^2}\right) \sim 1$. These equivalence relations with ii, iii yield

$$\int_{y\in\mathcal{R}_1} |x-y|^{-\alpha}\left(1 \wedge \frac{d(x)d(y)}{|x-y|^2}\right)^m |y-z|^{2m-n}\left(1 \wedge \frac{d(y)d(z)}{|y-z|^2}\right)^m dy \;\sim$$

$$\begin{aligned}
&\sim \int_{y\in\mathcal{R}_1} |x-y|^{-\alpha}\ |x-z|^{2m-n}\ dy \\
&\sim |x-z|^{2m-n} \int_0^{\frac{1}{2}|x-z|} r^{n-\alpha-1} dr \sim |x-z|^{2m-\alpha} \\
&\sim |x-z|^{2m-\alpha} \left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^m . \qquad (3.35)
\end{aligned}$$

If $\alpha = 2m$ then integrating over $\mathcal{R}_2$ yields with i, iv, ii and iii of Case II, as $d(x)$, $d(z)$ are here assumed to be bounded from below:

$$\int_{y\in\mathcal{R}_2} |x-y|^{-\alpha} \left(1 \wedge \tfrac{d(x)d(y)}{|x-y|^2}\right)^m |y-z|^{2m-n} \left(1 \wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \ \sim$$

$$\begin{aligned}
&\sim \int_{y\in\mathcal{R}_2} |x-y|^{-\alpha} \left(1 \wedge \tfrac{d(x)}{|x-y|}\right)^m |y-z|^{2m-n} \left(1 \wedge \tfrac{d(z)}{|y-z|}\right)^m dy \qquad (3.36) \\
&\sim \int_{\mathcal{R}_2} |x-y|^{-\alpha} |y-z|^{2m-n} dz \sim \int_{\mathcal{R}_2} |x-y|^{-\alpha} |x-y|^{2m-n} dz \\
&\sim \int_{\frac{R}{D}|x-z|}^{R} r^{-\alpha+2m-1} dr \sim \log\left(\tfrac{D}{|x-z|}\right) \sim \log\left(1 + \left(\tfrac{d(x)\,d(z)}{|x-z|^2}\right)^m\right).
\end{aligned}$$

In the remaining part for the case $|x-z| \leq \frac{1}{2}\left(d\left(x\right) \vee d\left(z\right)\right)$ we may restrict ourselves to $d\left(x\right), d\left(z\right) \leq R$.

• **Lemma 3.1-*1*.** If $\alpha > 2m$, then we use again $\mathcal{R}_1 = B_{\frac{1}{2}|x-z|}\left(x\right)$ and proceed precisely as in (3.35).

If $\alpha \leq 2m$, then we integrate on $\mathcal{R}_2$. It follows from the definition of $R$ that $\frac{R}{D} < \frac{1}{4}$, see (3.30). Beginning as in (3.36) and using (3.34)

$$\int_{y\in\mathcal{R}_2} |x-y|^{-\alpha} \left(1 \wedge \tfrac{d(x)d(y)}{|x-y|^2}\right)^m |y-z|^{2m-n} \left(1 \wedge \tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \ \sim$$

$$\begin{aligned}
&\sim \int_{y\in\mathcal{R}_2} |x-y|^{-\alpha} \left(1 \wedge \tfrac{d(x)}{|x-y|}\right)^m |y-z|^{2m-n} \left(1 \wedge \tfrac{d(z)}{|y-z|}\right)^m dy \\
&\sim \int_{y\in\mathcal{R}_2} |x-y|^{2m-\alpha-n} \left(1 \wedge \tfrac{d(x)}{|x-y|}\right)^{2m} dy \\
&\sim \int_{\frac{R}{D}|x-z|}^{d(x)} r^{2m-\alpha-1} dr + d\left(x\right)^{2m} \int_{d(x)}^{R} r^{-\alpha-1} dr \succeq \int_{\frac{R}{D}|x-z|}^{d(x)} r^{2m-\alpha-1} dr. \qquad (3.37)
\end{aligned}$$

If $\alpha = 2m$, then

$$\begin{aligned}
(3.37) \quad &\sim \quad \log\left(\tfrac{D}{R}\right) + \log\left(\tfrac{d(x)}{|x-z|}\right) \sim \log\left(1 + \tfrac{d(x)d(z)}{|x-z|^2}\right) \\
&\sim \quad \log\left(1 + \left(\tfrac{d(x)d(z)}{|x-z|^2}\right)^m\right).
\end{aligned}$$

If $\alpha < 2m$, then we use $\frac{R}{D}|x-z| < \frac{1}{4}\, d(x)$ and obtain

$$(3.37) \quad \sim \quad d\left(x\right)^{2m-\alpha} \sim \left(d\left(x\right) d\left(z\right)\right)^{m-\frac{1}{2}\alpha} \sim \left(d\left(x\right) d\left(z\right)\right)^{m-\frac{1}{2}\alpha} \left(1 \wedge \tfrac{d(x)d(z)}{|x-z|^2}\right)^{\frac{1}{2}\alpha}.$$

• **Lemma 3.1-*3*.** We will use the same set $\mathcal{R}_2$ from (3.33). By Lemma 3.4.v one estimates

$$\int_{y\in\mathcal{R}_2} \log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)|y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \ \sim$$

$$\sim\int_{y\in\mathcal{R}_2}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\log\left(2+\tfrac{d(y)}{|x-y|}\right)|y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \quad (3.38)$$

and continues with the estimates i, iv in Case II, (3.34), and ii, iii again in Case II to find

$$\begin{aligned}
(3.38) \ &\succeq \int_{y\in\mathcal{R}_2}\left(1\wedge\tfrac{d(x)}{|x-y|}\right)^m|y-z|^{2m-n}\left(1\wedge\tfrac{d(z)}{|y-z|}\right)^m dy\\
&\sim \int_{y\in\mathcal{R}_2}|x-y|^{2m-n}\left(1\wedge\tfrac{d(x)}{|x-y|}\right)^{2m} dy\\
&\sim \int_{\frac{R}{D}|x-z|}^{d(x)} r^{2m-1}dr + d(x)^{2m}\int_{d(x)}^{R} r^{-1}dr\\
&\sim d(x)^{2m}\left(1+\log\tfrac{R}{d(x)}\right)\sim d(x)^{2m}\log\left(2+\tfrac{1}{d(x)}\right)\\
&\sim \left(d(x)\,d(z)\right)^m\log\left(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\right).
\end{aligned}$$

• **Lemma 3.2-*1*.** Here we have $n=2m$. Again using the set $\mathcal{R}_2$ from (3.33) we obtain by Lemma 3.4.v that

$$\int_{y\in\mathcal{R}_2}\log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)\log\left(1+\left(\tfrac{d(y)d(z)}{|y-z|^2}\right)^m\right)dy \ \sim$$

$$\sim\int_{y\in\mathcal{R}_2}\left(1\wedge\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\log\left(2+\tfrac{d(x)}{|x-y|}\right)\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m\log\left(2+\tfrac{d(z)}{|y-z|}\right)dy. \ (3.39)$$

We again continue with the estimates i, iv in Case II, (3.34), and ii, iii in Case II to see that

$$\begin{aligned}
(3.39) \ &\succeq \int_{y\in\mathcal{R}_2}\left(1\wedge\tfrac{d(x)}{|x-y|}\right)^m\log\left(2+\tfrac{d(x)}{|x-y|}\right)\left(1\wedge\tfrac{d(z)}{|y-z|}\right)^m\log\left(2+\tfrac{d(z)}{|y-z|}\right)dy\\
&\sim \int_{y\in\mathcal{R}_2}\left(1\wedge\tfrac{d(x)}{|x-y|}\right)^{2m}\left(\log\left(2+\tfrac{d(x)}{|x-y|}\right)\right)^2 dy\\
&\sim \int_{\frac{R}{D}|x-z|}^{d(x)}\left(\log\left(2+\tfrac{d(x)}{r}\right)\right)^2 r^{2m-1}dr + d(x)^{2m}\int_{d(x)}^{R} r^{-1}dr\\
&\succeq d(x)^{2m}+d(x)^{2m}\log\left(\tfrac{R}{d(x)}\right)\\
&\sim d(x)^{2m}\log\left(2+\tfrac{1}{d(x)}\right)\\
&\sim \left(d(x)\,d(z)\right)^m\log\left(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\right).
\end{aligned}$$

### 3.2.3 Integral estimates when $|x-z| \geq \frac{1}{2}(d(x) \vee d(z))$

Now the second part of Lemma 1.1 applies and we find

$$d(x) \leq 2|x-z| \text{ and } d(z) \leq 2|x-z|.$$

Again we consider $\mathcal{R}_2 = \mathcal{R}_x^z \backslash B_{\frac{R}{D}|x-z|}(x)$. In addition to the inequalities in Case II we have that

$$d(x) \leq 2|x-z| \leq \tfrac{2D}{R}|x-y| \text{ and } d(z) \leq 2|x-z| \leq \tfrac{2D}{R}|x-y|. \tag{3.40}$$

- **Lemma 3.1-*1*.** By the estimates i, iv in Case II, respectively the estimates ii, iii in Case II and finally (3.40) it follows that

$$\int_{y\in\mathcal{R}_2} |x-y|^{-\alpha}\left(1\wedge \tfrac{d(x)d(y)}{|x-y|^2}\right)^m |y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \sim$$

$$\begin{aligned}
&\sim \int_{y\in\mathcal{R}_2} |x-y|^{-\alpha}\left(1\wedge \tfrac{d(x)}{|x-y|}\right)^m |y-z|^{2m-n}\left(1\wedge\tfrac{d(z)}{|y-z|}\right)^m dy\\
&\sim \int_{y\in\mathcal{R}_2} |x-y|^{2m-n-\alpha}\left(1\wedge \tfrac{d(x)}{|x-y|}\right)^m \left(1\wedge\tfrac{d(z)}{|x-y|}\right)^m dy\\
&\sim \Big(d(x)\,d(z)\Big)^m \int_{y\in\mathcal{R}_2} |x-y|^{-n-\alpha}\,dy\\
&\sim \Big(d(x)\,d(z)\Big)^m \int_{\frac{R}{D}|x-z|}^{R} r^{-1-\alpha}dr\\
&\sim \Big(d(x)\,d(z)\Big)^m |x-z|^{-\alpha}.
\end{aligned} \tag{3.41}$$

Distinguishing the three cases of $\alpha$ we conclude by $\frac{d(x)d(z)}{|x-z|^2} \preceq 1$ that

$$(3.41) \sim \begin{cases}
\text{if } \alpha > 2m: & |x-z|^{2m-\alpha}\left(\tfrac{d(x)d(z)}{|x-z|^2}\right)^m \sim\\
& \qquad \sim |x-z|^{2m-\alpha}\left(1\wedge\tfrac{d(x)d(z)}{|x-z|^2}\right)^m;\\
\text{if } \alpha = 2m: & \left(\tfrac{d(x)d(z)}{|x-z|^2}\right)^m \sim \log\left(1+\left(\tfrac{d(x)d(z)}{|x-z|^2}\right)^m\right);\\
\text{if } \alpha < 2m: & \Big(d(x)\,d(z)\Big)^{m-\frac{1}{2}\alpha}\left(\tfrac{d(x)d(z)}{|x-z|^2}\right)^{\frac{1}{2}\alpha} \sim\\
& \qquad \sim \Big(d(x)\,d(z)\Big)^{m-\frac{1}{2}\alpha}\left(1\wedge\tfrac{d(x)d(z)}{|x-z|^2}\right)^{\frac{1}{2}\alpha}.
\end{cases}$$

- **Lemma 3.1-*3*.** By similar steps as in the previous item we find

$$\int_{y\in\mathcal{R}_2} \log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)|y-z|^{2m-n}\left(1\wedge\tfrac{d(y)d(z)}{|y-z|^2}\right)^m dy \succeq$$

$$\begin{aligned}
&\succeq \int_{y\in\mathcal{R}_2} \log\left(1+\left(\tfrac{d(x)}{|x-y|}\right)^m\right)|y-z|^{2m-n}\left(1\wedge\tfrac{d(z)}{|y-z|}\right)^m dy\\
&\sim \int_{y\in\mathcal{R}_2} \log\left(1+\left(\tfrac{d(x)}{|x-y|}\right)^m\right)|x-y|^{2m-n}\left(1\wedge\tfrac{d(z)}{|x-y|}\right)^m dy\\
&\sim \int_{y\in\mathcal{R}_2} |x-y|^{2m-n}\left(\tfrac{d(x)}{|x-y|}\right)^m\left(\tfrac{d(z)}{|x-y|}\right)^m dy\\
&\sim \Big(d(x)\,d(z)\Big)^m \int_{\frac{R}{D}|x-z|}^{R} r^{-1}dr\\
&\sim \Big(d(x)\,d(z)\Big)^m \log\left(\tfrac{D}{|x-z|}\right) \sim \Big(d(x)\,d(z)\Big)^m \log\left(2+\tfrac{1}{|x-z|^2}\right). \qquad (3.42)
\end{aligned}$$

Using again $d(x)\,d(z) \preceq |x-z|^2$ we conclude that

$$(3.42) \sim \Big(d(x)\,d(z)\Big)^m \log\left(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\right). \qquad (3.43)$$

• **Lemma 3.2-*1*.** Again by similar steps and as $n = 2m$

$$\int_{y\in\mathcal{R}_2} \log\left(1+\left(\tfrac{d(x)d(y)}{|x-y|^2}\right)^m\right)\log\left(1+\left(\tfrac{d(y)d(z)}{|y-z|^2}\right)^m\right)dy \ \succeq$$

$$\begin{aligned}
&\succeq \int_{y\in\mathcal{R}_2} \log\left(1+\left(\tfrac{d(x)}{|x-y|}\right)^m\right)\log\left(1+\left(\tfrac{d(z)}{|y-z|}\right)^m\right)dy\\
&\sim \int_{y\in\mathcal{R}_2} \left(\tfrac{d(x)}{|x-y|}\right)^m\left(\tfrac{d(z)}{|x-y|}\right)^m dy\\
&\sim \Big(d(x)\,d(z)\Big)^m \int_{\frac{R}{D}|x-z|}^{R} r^{-1}dr\\
&\sim \Big(d(x)\,d(z)\Big)^m \log\left(2+\tfrac{1}{|x-z|^2+d(x)d(z)}\right).
\end{aligned}$$

□

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