On domains for which the clamped plate system is positivity preserving

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Abstract

Boggio proved in 1905 that the clamped plate equation is positivity preserving for a disk. It is known that on many other domains such a property fails. In this paper we will show that an affirmative result holds on still a large class of domains. We also survey the available methods in obtaining domains with such property.

1. Introduction

In 1905 Boggio in [1] gave an explicit Green formula for the clamped plate equation on a disk, that is, for the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \frac{\partial}{\partial\nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

with $\Omega = B = \left\{x \in \mathbb{R}^2; |x| < 1\right\}$. As a direct consequence of that formula one finds that (1) is positivity preserving:

$$f > 0 \text{ implies } u > 0. \tag{2}$$

Boggio and Hadamard conjectured that such a property holds on almost any (convex) domain. By now this conjecture has numerous counterexamples. Duffin [5] was the first one who in 1949 showed that on the infinite strip a positive f exists for which (1) has a sign-changing solution u. Garabedian [7] obtained a similar result for an elongated ellipse with axes having ratio 2. Other domains such as non-simply connected ones ([3]) and domains with corners ([14],[2]) followed. It was believed that most non-circular domains failed to have the sign preserving property, or as Hayman and Korenblum stated in [13]: we are tempted to conjecture that balls are the only domains in $\mathbb{R}^n$. But since they consider the sign not just for biharmonic but for all polyharmonic Green functions they could still be right. In this paper we will show that for the biharmonic there are many domains even quite different from the disk where the clamped plate problem is positivityy preserving.

In [8] a perturbation argument did show that on domains very close to the disk (2) remained. Next to showing some more domains for which (2) holds we aim to survey the limited methods to find such domains that we know to be available presently. In doing so we will also explain that the Möbius transformation plays a special role not only in higher dimensions but also for polyharmonic equations in 2 dimensions.

The three different ways of finding domains other than a disk and for which (2) holds will be addressed in the next sections. Although each of these three approaches are known, the combination has not been exploited. The perturbation that we state has a wider range than the version published in [8]. We will end with a section that states one ingredient for possible extensions of these results namely optimal estimates from above for the polyharmonic Green functions and its derivatives on general domains. Such estimates do not seem to be ready available in literature and do have some interest for their own sake.

2. Direct approaches

The example $\left(x^3 - x\right)'''' = 0$ immediately shows that for the biharmonic one cannot proceed to a positivity preserving property by way of the local maximum principle as for second order elliptic equations. A way out is to start from a domain with an explicitly known positive Green function and try to transform this to another domain. One may start from the results of Boggio mentioned above. By the way, Boggio in [1] not only derived the Green function for the clamped plate equation on the disk but did even so for any polyharmonic equation, $(-\Delta)^m u = f$, on a ball in any dimension under zero Dirichlet boundary conditions $u = \frac{\partial}{\partial n} u = \cdots = \left(\frac{\partial}{\partial n}\right)^{m-1} u = 0$. This Green function is as follows:

$$G_B(x,y) = c_{n,m} |x-y|^{2m-n} \int_0^{\frac{\left(1-|x|^2\right)\left(1-|y|^2\right)}{|x-y|^2}} w^{m-1} (1+w)^{-\frac{1}{2}n} \, dw,$$

with $c_{n,m}$ some explicit constants. The solution of (1) is $u(x) = \int_B G_B(x,y) f(y) dy$.

One might try to transfer this formula to other domains. A necessary condition that such a transformation h from B to Ω at least keeps the highest order terms polyharmonic, that is $(-\Delta)^m (w.(u \circ h)) = \tilde{w}.((-\Delta)^m u) \circ h + l.o.t.$, is that h is conformal. Without the conformality assumption the transformed differential equation would become nonisotropic. Let us shortly address such conformal mappings.

2.1. Conformal mappings in $\mathbb{R}^n$.

It is well known that in dimensions 3 and larger very few conformal mappings exists. Except so-called similarities, the only ones that exist are the Möbius transformations. This result is due to Liouville around 1850 for $n = 3$. For general dimensions $n \geq 3$ see Theorem 5.10 in [16].

A mapping ϕ is called a similarity if there are $c \in \mathbb{R}^+$, $a \in \mathbb{R}^n$ and an orthogonal matrix F such that $\phi(x) = a + cFx$. A Möbius transformation can be written as a finite combination of similarities and the inversion $j_0 : x \mapsto |x|^{-2} x$. In fact, see Corollary 4 on page 39 of [16], every Möbius transformation ϕ can be written as

$$\phi = \phi_1 \circ j_0 \circ \phi_2 \tag{3}$$

with ϕ_1, ϕ_2 similarities and $j_0(x) = |x|^{-2} x$.

Combining polyharmonic equations with similarity transformations give an obvious result. Let us address how one may combine biharmonic (and polyharmonic) equations with the inversion j_0. By the way, for $n = 2$ it is common to use notation in $\mathbb{C}$ and to consider the conjugate version $\bar{\mathbf{j}}_0(z) = z^{-1}$.

We shall see that there is only one obvious choice if we want to keep the same polyharmonic differential operator. Since pure powers of $|x|$ remain in this class both under j_0 and Δ it seems reasonable to try with power functions of $|x|$ only.

Lemma 1 *Let $\alpha, \beta, \gamma \in \mathbb{R}$ be such that for all $u \in C^4(\bar{\Omega})$ with $\Omega \subset \mathbb{R}^n \backslash \{0\}$ some open domain*

$$\Delta^k \left(|x|^\alpha (u \circ j_0)(x)\right) = \gamma \ |x|^\beta \left(\Delta^k u\right) \circ j_0(x) \ \text{ for } x \in j_0(\Omega),$$

then $\alpha = 2k - n$, $\beta = -2k - n$ and $\gamma = 1$.

Proof. By testing with $u(x) = |x|^\delta$ for $\delta \in \mathbb{R}$, using $\Delta_{rad} = r^{1-n} \partial_r r^{n-1} \partial_r$, one finds:

$$\Delta^k \left(|x|^\alpha |j_0(x)|^\delta\right) = \left(\prod_{m=0}^{k-1} (\alpha - \delta - 2m)(n - 2 + \alpha - \delta - 2m)\right) |x|^{\alpha - \delta - 2k}, \tag{4}$$

$$|x|^\beta \left(\Delta^k |y|^\delta\right)_{y = j_0(x)} = \left(\prod_{m=0}^{k-1} (\delta - 2m)(n - 2 + \delta - 2m)\right) |x|^{\beta - (\delta - 2k)}. \tag{5}$$

These two expressions are identical for all δ if and only if $\alpha - \delta - 2k = \beta - (\delta - 2k)$, and hence

$$\beta = \alpha - 4k.$$

This leaves us with two coefficients which are polynomials in δ and these are multiples of each other if and only if the roots coincide. For the largest root one finds $n - 2 + \alpha = 2(k-1)$ and hence

$$\alpha = 2k - n.$$

In fact now all roots coincide and one finds that $\gamma = 1$.

To show that

$$\Delta^k \left(|x|^{2k-n} (u \circ j_0)(x) \right) = |x|^{-2k-n} \left(\Delta^k u \right) \circ j_0 (x) \tag{6}$$

holds for all sufficiently smooth functions u we remark that $\Delta = r^{1-n} \partial_r r^{n-1} \partial_r + r^{-2} \Delta_\Gamma$ where Δ_Γ is the Laplace-Beltrami operator on the surface of the unit ball. Let φ denote these angular coordinates. Then

$$\Delta \left(r^\delta \Phi(\varphi) \right) = r^{\delta - 2} \left(\delta (\delta + n - 2) + \Delta_\Gamma \right) \Phi(\varphi).$$

So a similar computation as for (4-5) leads for $u = r^\delta \Phi(\varphi)$ to

$$\Delta^k \left(r^{2k-n-\delta} \Phi(\varphi) \right) = \left(\prod_{m=0}^{k-1} \left((2k - \delta - n - 2m)(2k - \delta - 2m - 2) + \Delta_\Gamma \right) \right) r^{-n-\delta} \Phi(\varphi),$$

$$r^{-2k-n} \left(\Delta^k s^\delta \Phi(\varphi) \right)_{s=r^{-1}} = \left(\prod_{m=0}^{k-1} \left((\delta - 2m)(\delta + n - 2m - 2) + \Delta_\Gamma \right) \right) r^{-n-\delta} \Phi(\varphi).$$

Both right hand sides are equal so (6) holds for a dense set of functions in $C^4\left(\bar{\Omega}\right)$ and hence for all $u \in C^4\left(\bar{\Omega}\right)$. ■

Corollary 2 *For any Möbius transformation h in $\mathbb{R}^n$ one finds:*

$$\Delta^k \left(J_h^{\frac{1}{2} - \frac{k}{n}} \; u \circ h \right) = J_h^{\frac{1}{2} + \frac{k}{n}} \; \left(\Delta^k u \right) \circ h, \tag{7}$$

where $J_h = \left| \det \left(\frac{\partial h_i}{\partial x_j} \right) \right|$ is the Jacobian.

Remark 2.1 Some special cases:

i. For $k = 2$ and $n = 2$:

$$\Delta^2 \left(J_h^{-\frac{1}{2}} \cdot (u \circ h) \right) = J_h^{\frac{3}{2}} \cdot \left(\Delta^2 u \circ h \right). \tag{8}$$

ii. For $k \geq 1$ and $n = 2k$:

$$\Delta^k (u \circ h) = J_h \cdot \left(\Delta^k u \circ h \right). \tag{9}$$

Proof. It is sufficient to show that (7) holds for each of the transformations involved. Since scaling, dilation, rotation and reflection give immediately the appropriate changes and since every Möbius transformation can be expressed as (3), we are left with the inversion j_0. For j_0 one finds

$$\frac{\partial j_0(x)}{\partial x_1, \ldots, x_n} = \frac{1}{|x|^2} I - \left(\frac{2x_i x_j}{|x|^4}\right)_{ij} = \frac{1}{|x|^2}\left[I - 2\left(\frac{x}{|x|}\right)^T \left(\frac{x}{|x|}\right)\right],$$

using column notation for x. Since the matrix between square brackets describes the reflection in the hyperplane through 0 perpendicular to x, it has determinant -1. Hence the Jacobian of j_0 satisfies:

$$J_{j_0}(x) = \left|\det\left(\frac{\partial j_0(x)}{\partial x_1 \ldots x_n}\right)\right| = \frac{1}{|x|^{2n}}.$$

■

A well-known property of Möbius transformations is that the image of a (generalized) sphere is again a (generalized) sphere (see Theorem 3.4 in [16]). Hence there is no conformal mapping in dimensions ≥ 3 available that could extend Boggio's result to other domains than generalized spheres. By the way, a generalized sphere is either a sphere or a hyperplane.

2.2. Conformal mappings in two dimensions

For the second order Laplace equation in two dimensions Riemann's Mapping Theorem allows us to solve

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{10}$$

at least for simply connected Ω by way of a Green function for the disk. There exists a bijective conformal $h : B \to \Omega$ and it holds that

$$\Delta(u \circ h) = |\mathbf{h}'|^2 (\Delta u) \circ h \tag{11}$$

where $\mathbf{h}(x+iy) = h_1(x,y) + ih_2(x,y)$. Indeed, since $|\mathbf{h}'(x+iy)|^2 = J_h(x,y)$, the Jacobian of h, it follows that one solves (10) by

$$u(x) = \int_\Omega G_B\left(h^{-1}(x), h^{-1}(y)\right) f(y)\, dy.$$

For the biharmonic equation we could try to mimic this approach even if we have to add weight functions as in (8). Boldface is used for the complex alternative.

Lemma 3 *Let $h \in C^1\left(\bar{A}; \mathbb{R}^2\right)$ be a conformal mapping from A to $\Omega \subset \mathbb{R}^2$ and suppose h is not a similarity. Then there exists a meromorphic function $\mathbf{f}$ and a number c such that for all $u \in C^4\left(\bar{\Omega}\right)$:*

$$\Delta^2\left(|\mathbf{f}|^2 (u \circ h)\right) = c\,|\mathbf{f}|^2 |\mathbf{h}'|^4 \left(\Delta^2 u\right) \circ h$$

if and only if h is a Möbius transformation, $c = 1$ and $|\mathbf{f}|^2 |\mathbf{h}'|$ is constant.

Proof. Using the complex notation and new independent variables $z = x + iy$ and $\bar{z} = x - iy$ the notations will simplify. Setting $U(x + iy, x - iy) = u(x, y)$ we find

$$\Delta u = 4\partial_{\bar{z}}\partial_z U.$$

Notice that formally $\overline{\mathbf{h}(z)} = \bar{\mathbf{h}}(\bar{z})$ and hence $\partial_{\bar{z}}\overline{\mathbf{h}(z)} = \overline{\mathbf{h}'(z)}$. With $\mathbf{h} = \mathbf{h}(z)$ and $\bar{\mathbf{h}} = \bar{\mathbf{h}}(\bar{z})$ and a tedious computation:

$$\partial_{\bar{z}}\partial_z\partial_{\bar{z}}\partial_z \Big(\mathbf{f}(z)\,\bar{\mathbf{f}}(\bar{z})\;\; U\big(\mathbf{h}(z), \bar{\mathbf{h}}(\bar{z})\big)\Big) = \tag{12}$$

$$\begin{aligned} = \quad & \mathbf{f}''\ \bar{\mathbf{f}}''\ U + \bar{\mathbf{f}}''\left(2\mathbf{f}'\ \mathbf{h}' + \mathbf{f}\ \mathbf{h}''\right)\ U_1 + \mathbf{f}''\left(2\bar{\mathbf{f}}'\ \bar{\mathbf{h}}' + \bar{\mathbf{f}}\ \bar{\mathbf{h}}''\right)\ U_2 + \\ & + \mathbf{f}\ \bar{\mathbf{f}}''\ \left(\mathbf{h}'\right)^2\ U_{11} + \mathbf{f}''\ \bar{\mathbf{f}}\ \left(\bar{\mathbf{h}}'\right)^2\ U_{22} + \left(2\bar{\mathbf{f}}'\ \bar{\mathbf{h}}' + \bar{\mathbf{f}}\ \bar{\mathbf{h}}''\right)\left(2\mathbf{f}'\ \mathbf{h}' + \mathbf{f}\ \mathbf{h}''\right)\ U_{12} + \\ & + \mathbf{f}\ \left(2\bar{\mathbf{f}}'\ \bar{\mathbf{h}}' + \bar{\mathbf{f}}\ \bar{\mathbf{h}}''\right)\left(\mathbf{h}'\right)^2\ U_{121} + \bar{\mathbf{f}}\ \left(2\mathbf{f}'\ \mathbf{h}' + \mathbf{f}\ \mathbf{h}''\right)\left(\bar{\mathbf{h}}'\right)^2\ U_{212} + \\ & + \mathbf{f}\ \bar{\mathbf{f}}\ \left(\mathbf{h}'\right)^2\left(\bar{\mathbf{h}}'\right)^2\ U_{1212}. \end{aligned}$$

In order for the lower order coefficients to cancel we need $\mathbf{f}'' = 0$ and hence find

$$\mathbf{f}(z) = \alpha + \beta z.$$

Plugging this result in we may see that (12) simplifies to

$$\begin{aligned} = \quad & \Big(2\bar{\beta}\ \bar{\mathbf{h}}' + \left(\bar{\alpha} + \bar{\beta}\bar{z}\right)\ \bar{\mathbf{h}}''\Big)\Big(2\beta\ \mathbf{h}' + (\alpha + \beta z)\ \mathbf{h}''\Big)\, U_{12} + \\ & + (\alpha + \beta z)\Big(2\bar{\beta}\ \bar{\mathbf{h}}' + \left(\bar{\alpha} + \bar{\beta}\bar{z}\right)\ \bar{\mathbf{h}}''\Big)\left(\mathbf{h}'\right)^2\ U_{121}\ + \\ & + \left(\bar{\alpha} + \bar{\beta}\bar{z}\right)\Big(2\beta\ \mathbf{h}' + (\alpha + \beta z)\ \mathbf{h}''\Big)\left(\bar{\mathbf{h}}'\right)^2\ U_{212} + \\ & + |\alpha + \beta z|^2\ \left(\mathbf{h}'\right)^2\left(\bar{\mathbf{h}}'\right)^2\ U_{1212}. \end{aligned}$$

Since $\mathbf{h}' \neq 0$ it follows that the remaining lower order terms vanish if and only if

$$2\beta\ \mathbf{h}' + (\alpha + \beta z)\ \mathbf{h}'' = 0,$$

which implies $\mathbf{h}'' = \beta = 0$ or $\mathbf{h}' = \gamma(\alpha + \beta z)^{-2}$. The first possibility gives the similarities $\mathbf{h}(z) = \gamma_1 + \gamma_2 z$ and the second one the Möbius transformations

$$\mathbf{h}(z) = \frac{-\gamma/\beta}{\alpha + \beta z} + \delta.$$

Also note that $\mathbf{h}' = \gamma\mathbf{f}^{-2}$. ■

Corollary 4 *Suppose that $A, \Omega \subset \mathbb{R}^2$ are bounded domains such that there exists a Möbius transformation from A to Ω. Then (1) is positivity preserving for Ω if and only if (1) is positivity preserving for A.*

Proof. Let G_Ω, G_A be the respective Green functions and let us call the Möbius transformstion h. A direct computation shows that

$$\sqrt{J_h(x)\, J_h(y)}\ G_A(x, y) = G_\Omega(h(x), h(y)). \tag{13}$$

Indeed, if u is the solution of (1) on Ω, then $\Delta^2 \left(J_h(x)^{-\frac{1}{2}} (u \circ h)(x) \right) = J_h(x)^{\frac{3}{2}} (f \circ h)(x)$ and since also the boundary conditions go over nicely in the case of zero Dirichlet type:

$$\begin{aligned} (u \circ h)(x) &= J_h(x)^{\frac{1}{2}} \int_A G_A(x,y)\, J_h(y)^{\frac{3}{2}} (f \circ h)(y) dy = \\ &= \int_A J_h(x)^{\frac{1}{2}} J_h(y)^{\frac{1}{2}} G_A(x,y) (f \circ h)(y) J_h(y)\, dy. \end{aligned}$$

On the other hand

$$u(h(x)) = \int_\Omega G_\Omega(h(x), \eta)\, f(\eta) d\eta = \int_A G_\Omega(h(x), h(y)) (f \circ h)(y) J_h(y)\, dy.$$

The claim follows from (13). ■

3. A perturbation argument

In [8] it has been shown that small perturbations of the disk do not destroy property (2). However, the perturbed domains for which $f > 0$ implies $u > 0$ that were allowed needed a small C^2-bound for the difference between Ω and B. The result of [8] in fact is not restricted the small perturbations of the disk; only the appropriate estimates for the Green function on the specific domain are needed. Let us give the precise statement.

Here α is a multi-index of nonnegative integers, $|\alpha| = \sum \alpha_i$ and $\partial_x^\alpha = \prod \dfrac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$.

Proposition 5 *Suppose that the Green function for (1) on Ω satisfies the following estimates:*

i. from below: $\exists c_\Omega > 0 \forall x, y \in \Omega$

$$G_\Omega(x,y) \geq c_\Omega d(x)\, d(y) \min\left\{1, \frac{d(x)\, d(y)}{|x-y|^2}\right\}, \tag{14}$$

ii. from above: $\exists c_{i,\Omega} \forall x, y \in \Omega :$

$$\begin{aligned} & |G_\Omega(x,y)| \leq c_{0,\Omega}\ d(x)\, d(y) \min\left\{1, \tfrac{d(x)d(y)}{|x-y|^2}\right\}, \\ |\alpha| = 1: \quad & |\partial_x^\alpha G_\Omega(x,y)| \leq c_{1,\Omega}\ d(y) \min\left\{1, \tfrac{d(x)d(y)}{|x-y|^2}\right\}, \\ |\alpha| = 2: \quad & |\partial_x^\alpha G_\Omega(x,y)| \leq c_{2,\Omega}\ \log\left(1 + \tfrac{d(y)^2}{|x-y|^2}\right), \\ |\alpha| = 3: \quad & |\partial_x^\alpha G_\Omega(x,y)| \leq c_{3,\Omega}\ \tfrac{1}{|x-y|} \min\left\{1, \tfrac{d(y)^2}{|x-y|^2}\right\}. \end{aligned}$$

Then there exists $\varepsilon > 0$ such that the following holds.

If there is a conformal map $\mathbf{h}$ from $\mathbf{A}$ to $\mathbf{\Omega}$ with $\|\mathbf{h} - \mathbf{Id}\|_{C^2(\bar{\mathbf{A}})} \leq \varepsilon$, then (1) is also positivity preserving for Ω replaced by A.

Here we used the identity $\mathbf{Id}$ on $\mathbb{C}$: $\mathbf{Id}\,(z)=z$.

Remark 5.1 As in [8] one may show that it is sufficient that there is a $C^{3,\gamma}$-diffeomorphism from A to Ω close to the identity.

Remark 5.2 For the estimate from above to hold we expect to need a more regular boundary than just the C^2 from the conformal map. The estimates above are based on results of Krasovskii that would use a C^{12} boundary in the present setting.

Proof. Proceeding as in (12) with $\mathbf{f}=1$ one obtains since $\Delta\left|\mathbf{h}'\right|^2=4\left|\mathbf{h}''\right|^2$ that

$$
\begin{aligned}
\Delta^2(u\circ h) &= \Delta\left(\left|\mathbf{h}'\right|^2(\Delta u)\circ h\right)= \\
&= \left|\mathbf{h}'\right|^2\Delta\left((\Delta u)\circ h\right)+2\nabla\left|\mathbf{h}'\right|^2\cdot\nabla\left((\Delta u)\circ h\right)+\Delta\left|\mathbf{h}'\right|^2\left((\Delta u)\circ h\right) \\
&= \left|\mathbf{h}'\right|^4\left(\left((\Delta^2 u)\circ h\right)+\frac{2\nabla\left|\mathbf{h}'\right|^2}{\left|\mathbf{h}'\right|^4}\cdot(\partial_i h_j)\left((\nabla\Delta u)\circ h\right)+\frac{4\left|\mathbf{h}''\right|^2}{\left|\mathbf{h}'\right|^4}\left((\Delta u)\circ h\right)\right). \qquad (15)
\end{aligned}
$$

If the L^∞-norms of α,β and $\left|\mathbf{h}''\right|^2\left|\mathbf{h}'\right|^{-2}$ are sufficiently small then we may use the Green function estimates from [9, Theorem 5.1] to find that the modified fourth order operator in (15) on Ω has a positive Green function. And indeed, these L^∞-norms become as small as one likes by choosing the ε-bound for the C^2-difference of $\mathbf{h}$ and the identity. For the disk such an approach is found in [8]. If (15) with Dirichlet boundary conditions satisfies (2) on Ω then (1) is positivity preserving on A. ∎

Remark 5.3 The obvious guess would be to proceed by considering $\Delta^2(\left|\mathbf{h}'\right|^{-1}u\circ h)$ instead of $\Delta^2(u\circ h)$. However this approach gives lower order terms that contain third order derivatives of $\mathbf{h}$. Indeed one finds

$$
\Delta^2\left(\left|\mathbf{h}'\right|^{-1}u\circ h\right)=\left|\mathbf{h}'\right|^3(\Delta^2 u)\circ h+\tfrac{1}{2}\alpha\left(u_{xx}-u_{yy}\right)+\beta u_{xy}+\left(\alpha^2+\beta^2\right)u.
$$

Here $\alpha(x,y)=\operatorname{Re}(\mathbf{w}(x+iy))$ and $\beta(x,y)=\operatorname{Im}(\mathbf{w}(x+iy))$ with

$$
\mathbf{w}=\left(\tfrac{3}{4}\left(\mathbf{h}'\right)^{-\frac{5}{2}}\left(\mathbf{h}''\right)^2-\tfrac{1}{2}\left(\mathbf{h}'\right)^{-\frac{3}{2}}\ \mathbf{h}'''\right)\left(\mathbf{h}'\right)^{-\frac{3}{2}}.
$$

One would need C^3-closeness of $\mathbf{h}$ to the identity in order to apply the results of [9].

Remark 5.4 The three main ingredients of the proof of this proposition are 1) a conformal mapping near the identity from B to Ω, 2) estimates for the Green function from below, and 3) estimates from above for the Green function and its derivatives. By the way, the estimates for the derivatives of the Green function are not the necessary ones for the proposition but are the ones that come out of the Green function itself.

Let us focus on the three ingredients mentioned in the remark above.

Using conformal mappings other than Möbius will restrict us to 2-dimensional domains. If estimates would be available for small perturbations in the leading order this could be overcome. We are not aware if such exist in dimensions higher than 2. In two dimensions small perturbations in the leading order terms are allowed since such a differential equation can be transformed to one with bi(poly)harmonic leading order on a disk. See [8].

The estimates for the Green function from below all come from an explicit formula and the perturbation arguments as in [8], [9].

The estimates from above however are available on rather general domains by starting from the kernel estimates of Krasovskii in [15]. Since the consequences for the Dirichlet problem of Krasovskii's estimates do not seem to be available in the literature we will state these in the last section.

4. Some domains with an explicit Green formula

Boggio's explicit formula for the Green function of (1) on a ball directly implied the positivity preserving property. Another type of domains for which an explicit Green function of (1) is available are limacons. Hadamard in [12] gave such a formula. Proceeding by direct arguments and not going through this tedious formula, he claimed that all limacons had property (2). However his proof started with an erroneous argument. Only recently it was proven ([4]) that some limacons do have the property in (2) and some don't. Let us recall the result from [4].

A so-called 'Limaçon de Pascal' is the image of $\mathbf{B} = \{z \in \mathbb{C}; |z| < 1\}$ under the conformal mapping:

$$\mathbf{h}_a(z) = z + a\, z^2,$$

where $a \in \left[0, \frac{1}{2}\right]$, that is

$$\Omega_a = \left\{(\rho\cos\varphi, \rho\sin\varphi) \in \mathbb{R}^2; 0 \le \rho < 1 + 2a\cos\varphi\right\}.$$

The explicit Green formula from [11, Supplement] for (1) is given as follows for $x, y \in \Omega_a$:

$$G_a(x,y) = \tfrac{1}{2}a^2 s^2 r^2 \left[\log\left(\tfrac{r^2}{r_1^2}\right) + \tfrac{r_1^2}{r^2} - 1 - \tfrac{a^2}{1-2a^2}\tfrac{r^2}{s^2}\left(\tfrac{r_1^2}{r^2} - 1\right)^2\right], \tag{16}$$

where, with $\eta, \xi \in B$ such that $x = h_a(\eta)$ and $y = h_a(\xi)$, the r, r_1 and s are given by

$$r^2 = |\eta - \xi|^2, \quad r_1^2 = \left|1 - \eta\bar{\xi}\right|^2, \quad s^2 = \left|\eta + \xi + \tfrac{1}{a}\right|^2. \tag{17}$$

Proposition 6 ([4]) *The limacons with $a \in [0, \frac{1}{6}\sqrt{6}]$ are the ones for which property (2) holds.*

Remark 6.1 One could view this result as a perturbation argument but only the explicit formula allows us to come up with the explicit number $\frac{1}{6}\sqrt{6}$ that is large enough to allow nonconvex domains. A small C^2-bound on the perturbation from the unit disk gives a small bound for the curvature κ, namely that $|1 - \kappa|$ should be small. Note that $\kappa \ge 0$ means convex.

In fact the claim that (1) is positivity preserving for all limacons goes back to Hadamard in [12] (and was quoted in [18]). Hadamard claimed his result for all such domains but his proof starts from an erroneous assumption. Nevertheless, as just mentioned, for some limacons the property in (2) holds. For $a \in (\frac{1}{4}, \frac{1}{6}\sqrt{6}]$ such a limacon is not convex and nevertheless (2) holds.

Other publications concerning the clamped plate equation on limacon are [17] and [6]. Sen considered explicit formula's for the clamped plate equation on other domains in $\mathbb{R}^2$ bounded by fourth order polynomials but only for constant right hand side f. Sen proceeded directly with no hint at Hadamard's result. Also Dube does not refer to Hadamard's explicit formula for the limacon nor does he consider positivity.

5. Using combinations

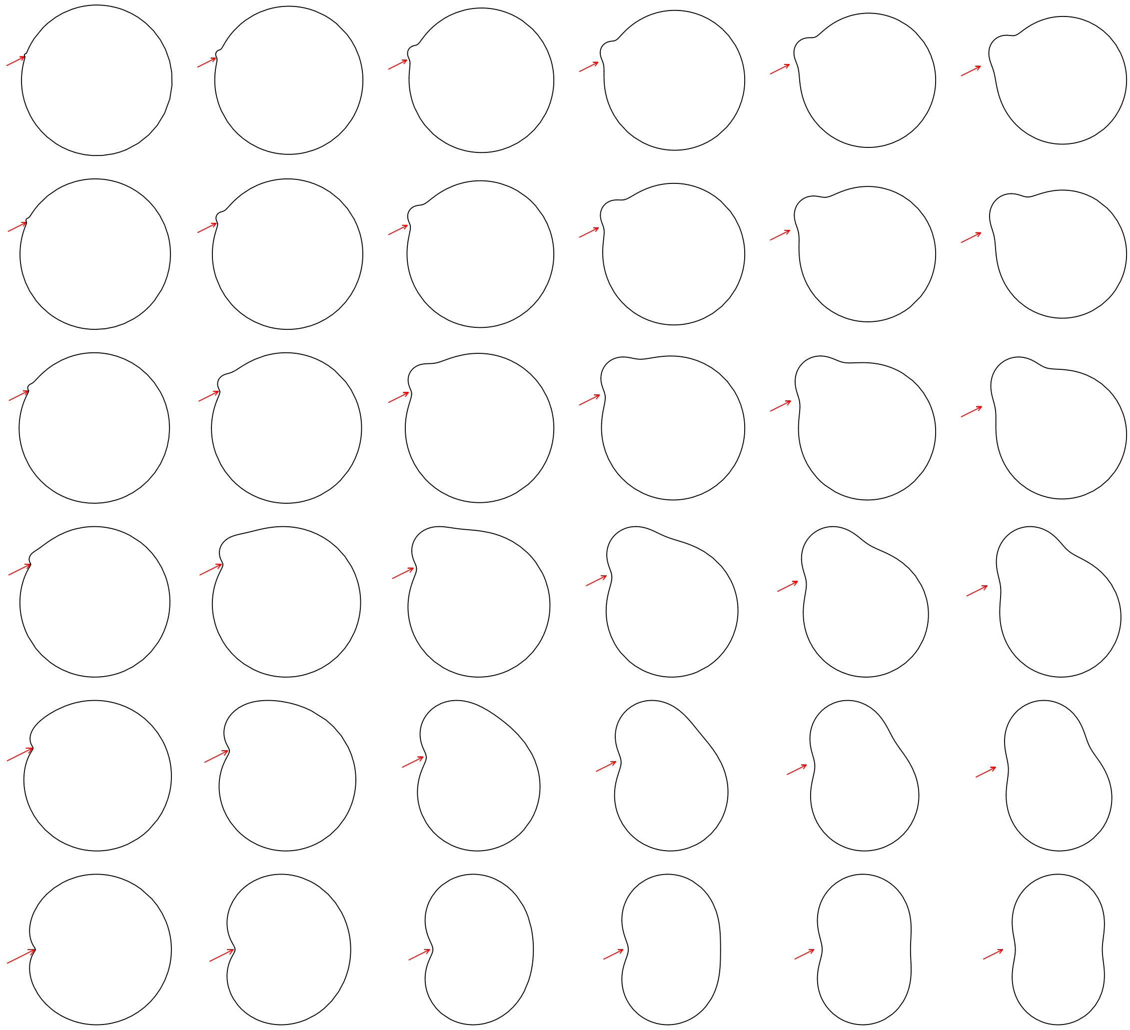

Figure 1: Domains for which the clamped plate system is positivity preserving.

Combining the Green function for a limacon of Proposition 6 with an inversion that has its 'center' just outside of this limacon one obtains a new domain on which (1) is positivity preserving. Indeed this result follows from Corollary 4. The drawings in Figure 1 are transforms of the limacon in the extreme case $a = \frac{1}{6}\sqrt{6}$ and taking the inversion center just outside that limacon. Both the angular position and the distance to the limacon of the inversion center are varied. All graphs are scaled back to unit size. The arrow denotes the inversion center.

Note that for these domains no extra perturbation is allowed since $a = \frac{1}{6}\sqrt{6}$ is critical for positivity. Taking $a < \frac{1}{6}\sqrt{6}$ the resulting domains would allow a (small) perturbation argument without destroying property (2).

6. Sharp estimates for the Green function

As mentioned before one needs two types of estimates for the Green functions in order to use the perturbation argument. First an estimate for the Green function itself from below and secondly, estimates from above for the Green function and its derivatives. The estimate from below that holds both for the disk and the limacon with $a \in \left(0, \frac{1}{6}\sqrt{6}\right)$ is as follows (note that $\Omega_0 = B$).

Proposition 7 **([4])** *For all $a \in \left[0, \frac{1}{6}\sqrt{6}\right)$ there is $c_a > 0$ such that for all $x, y \in \Omega_a$*

$$G_a(x,y) \geq c_a d(x)\, d(y) \min\left\{1, \frac{d(x)\, d(y)}{|x-y|^2}\right\}, \tag{18}$$

where d is the distance to the boundary:

$$d(x) = \inf\left\{|x - \tilde{x}|\,; \tilde{x} \notin \Omega_a\right\}.$$

Remark 7.1 In [4] it is shown that one may take $c_a = c\left(\frac{1}{6}\sqrt{6} - a\right)$ for some uniform c.

Remark 7.2 For optimal estimates from below for the polyharmonic Green function for zero Dirichlet boundary values on the ball in $\mathbb{R}^n$ see [10].

The estimates from above for the Green are known to hold in a much wider range. Indeed, such estimates exists for all polyharmonic systems under zero Dirichlet boundary data, at least when this boundary has sufficient regularity. Let $G_{\Omega,m}$ denote the Green function for

$$\begin{cases} \Delta^m u = f & \text{in } \Omega, \\ \left(\frac{\partial}{\partial\nu}\right)^k u = 0 \text{ for all } 0 \leq \nu \leq m-1 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

Proposition 8 *Suppose that Ω is a bounded domain with $\partial\Omega \in C^\infty$. Let $G_{m,\Omega}(x,y)$ be the Green function for (19). There exist $c_{\Omega,\alpha,m,n} > 0$ such that for all $x, y \in \Omega$*

i. if $|\alpha| \geq m$,

(a) and $2m - n - |\alpha| < 0$, then

$$|\partial_x^\alpha G_{m,\Omega}(x,y)| \leq c_{\Omega,\alpha,m,n}\ |x-y|^{2m-n-|\alpha|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^m,$$

(b) and $2m - n - |\alpha| = 0$, then

$$|\partial_x^\alpha G_{m,\Omega}(x,y)| \leq c_{\Omega,\alpha,m,n}\ \log\left(1 + \frac{d(y)^m}{|x-y|^m}\right)$$

(c) and $2m - n - |\alpha| > 0$, then

$$|\partial_x^\alpha G_{m,\Omega}(x,y)| \leq c_{\Omega,\alpha,m,n}\ d(y)^{2m-n-|\alpha|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{n+|\alpha|-m},$$

ii. if $|\alpha| < m$,

(a) and $2m-n-|\alpha|<0,$ *then*

$$|\partial_x^\alpha G_{m,\Omega}(x,y)| \leq c_{\Omega,\alpha,m,n}\ \ |x-y|^{2m-n-|\alpha|} \min\left\{1, \frac{d(x)}{|x-y|}\right\}^{m-|\alpha|} \min\left\{1, \frac{d(y)}{|x-y|}\right\}^{m},$$

(b) and $2m-n-|\alpha|=0,$ *then*

$$|\partial_x^\alpha G_{m,\Omega}(x,y)| \leq c_{\Omega,\alpha,m,n}\ \log\left(1+\frac{d(y)^m\, d(x)^{m-|\alpha|}}{|x-y|^{2m-|\alpha|}}\right),$$

(c) and $2m-n-|\alpha|>0,$

i. and $m-\frac{1}{2}n \leq |\alpha|$, *then*

$$\left|\partial_x^{|\alpha|} G_{m,\Omega}(x,y)\right| \leq c_{\Omega,\alpha,m,n}\ d(y)^{2m-n-|\alpha|} \min\left\{1, \tfrac{d(x)}{|x-y|}\right\}^{m-|\alpha|} \min\left\{1, \tfrac{d(y)}{|x-y|}\right\}^{n-m+|\alpha|},$$

ii. and $|\alpha|<m-\frac{1}{2}n,$ *then*

$$|\partial_x^\alpha G_{m,\Omega}(x,y)| \leq c_{\Omega,\alpha,m,n}\ d(y)^{m-\frac{n}{2}}\, d(x)^{m-\frac{n}{2}-|\alpha|} \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\}^{\frac{n}{2}}.$$

Remark 8.1 These estimates from above have been established for balls in [9].

All such estimates can derived from the pointwise estimates of Krasovskii in [15] for general elliptic boundary value problems. For an elliptic equation of order $2m$, such as the m^{th}-polyharmonic, with appropriate but general boundary conditions on a bounded smooth domain in $\mathbb{R}^n$ his result can be rephrased as

$$\left|\partial_x^\alpha \partial_y^\beta G_{m,\Omega}(x,y)\right| \leq c_{\alpha,\beta,n,m} \begin{cases} 1 & \text{if } |\alpha|+|\beta|<2m-n, \\ \log\left(\frac{2diam\Omega}{|x-y|}\right) & \text{if } |\alpha|+|\beta|=2m-n, \\ |x-y|^{2m-n-|\alpha|-|\beta|} & \text{if } |\alpha|+|\beta|>2m-n. \end{cases}$$

Here $diam\Omega = \sup\{|x-y|\,; x,y\in\Omega\}$. Krasovskii uses rather strong but explicit C^M-conditions, with $M=M(\alpha,\beta,2m,n)$, on the regularity of $\partial\Omega$ so for each of the estimates above seperately one may weaken the C^∞-condition.

The proof of the estimates in Proposition 8 that now do involve the distance to the boundary, are obtained by a tedious repeated integration of the ones by Krasovskii inwards from the boundary and using the zero Dirichlet boundary conditions. The actual proof will appear elsewhere.

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References

[1] Boggio, T.: *Sulle funzioni di Green d'ordine m*, Rend. Circ. Mat. Palermo, **20,** (1905) 97-135. 3, 4

[2] Coffman, C.V., Duffin, R. J.: On the structure of biharmonic functions satisfying the clamped plate conditions on a right angle. Adv. in Appl. Math. 1 (1980), no. 4, 373–389. 3

[3] Coffman, C.V., Duffin, R. J., Shaffer, D.H.: *The fundamental mode of vibration of a clamped annular plate is not of one sign.* in: Constructive approaches to mathematical models, Academic Press, New York 1979, 267-277. 3

[4] Dall'Acqua, A., Sweers, G.; *The clamped plate equation for the Limaçon*, submitted. 10, 12

[5] Duffin, R.J.: *On a question of Hadamard concerning super-biharmonic functions*, J. Math. Phys., **27**, (1949) 253-258. 3

[6] Dube, R.S.: *Green's function of an elastic plate in the shape of Pascal's Limacon and clamped along the boundary*, Indian J. Pure Appl. Math., **2**, (1971) 59-68. 10

[7] Garabedian, P. R.: A partial differential equation arising in conformal mapping. Pacific J. Math. 1, (1951). 485–524. 3

[8] Grunau, H.C., Sweers, G.: *Positivity for perturbations of polyharmonic operators with Dirichlet boundary conditions in two dimensions*, Math. Nachr., **179**, (1996) 89-102. 3, 8, 9

[9] Grunau, H.C., Sweers, G.: *Positivity for equations involving polyharmonic operators with Dirichlet boundary condition*, Math. Ann., **307**, (1997) 589-626. 9, 13

[10] Grunau, H.C., Sweers, G.: *Sharp estimates for iterated Green function*, Proc. Roy. Soc. Edinburgh Sect. A, **132**, (2002) 91-120. 12

[11] Hadamard, J.: *Memoire sur le probleme d'analyse relatif a l'equilibrie des plaques elastiques encastrees*, in: Œuvres de Jacques Hadamard, Tomes II, Éditions du Centre National de la Recherche Scientifique, Paris 1968, 515-641, reprint of: *Mèmoire couronne par l'Acadèmie des Sciences (Prix Vaillant)*, Mèm. Sav. Etrang., **33**, 1907. 10

[12] Hadamard, J.: *Sur certains cas interessants du probleme biharmonique*, in: Œuvres de Jacques Hadamard, Tomes III, Éditions du Centre National de la Recherche Scientifique, Paris 1968, 1297-1299, reprint of: IV Congr. Math., (1908) Rome. 10

[13] Hayman, W.K., Korenblum, B.: *Representation and uniqueness theorems for polyharmonic functions*, J. d' Analyse Math., **60**, (1993) 113-133. 3

[14] Osher, S.: *On Green's function for the biharmonic equation in a right angle wedge*, J. Math. Anal. Appl. **43** (1973), 705-716. 3

[15] J.P. Krasovskii, Isolation of Singularities of the Green Function, Math. USSR - Izvestija, Vol.1, No.5 (1967), 935-966. 10, 13

[16] Reshetnyak, Yu. G.: *Space mappings with bounded distortion*, AMS, Translations of Mathematical Monographs 73, 1989, Providence. 4, 6

[17] Sen, B.: *Note on the bending of thin uniformly loaded plates bounded by cardioids, lemniscates and certain other quartic curves*, Z. Angew. Math. Mech., **20**, (1940) 99–103. 10

[18] Sweers, G.: *When is the first eigenfunction for the clamped plate equation of fixed sign?*, Electron J. Differ. Equ., **6**, (2001) 285-296. 10