A noncooperative elliptic system with p-Laplacians that preserves positivity

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Abstract

A nonlinear noncooperative elliptic system is shown to have a positivity preserving property. That is, there exists a uniform positive constant such that, whenever the noncooperative part is bounded by this constant, positivity of the source term implies that the solution is positive. The model operator is the p-laplacian with $1 < p < \infty$ on a one-dimensional domain. The source term appears in one of the equations.

1 Introduction and main result

We will study the positivity preserving property of the following nonlinear noncooperative elliptic system

$$\left\{ \begin{array}{rcll} -\Delta_p u(x) & = & f(x) - \lambda \phi_p(v(x)) & \text{for } x \in \Omega, \\ -\Delta_p v(x) & = & \phi_p(u(x)) & \text{for } x \in \Omega, \\ u(x) = v(x) & = & 0 & \text{for } x \in \partial\Omega, \end{array} \right. \tag{1}$$

where $\Omega = (-1, 1)$.

The following notation is used:

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- $\phi_p(u) = |u|^{p-2}u$, the inverse being denoted by ϕ_p^{inv},
- $\Delta_p u = \left(|u'|^{p-2}u'\right)'$,
- $0 < f$ meaning $0 \le f(x)$ for all $x \in \Omega$ and $f \not\equiv 0$.
- $0 \ll f$ meaning $0 < f(x)$ for all $x \in \Omega$.

The number p lies in $(1, \infty)$.

Problem (1) is interesting in higher dimensions as well. The crucial condition necessary for our proofs however we are able to prove just for one dimensional domains. Needless to say that we are eager to see a proof showing that the condition is satisfied for higher dimensional domains.

For $\lambda > 0$ the system is noncooperative and hence does not satisfy the assumptions used by Fleckinger e.a. in [2] that yield the maximum principle. However, it has been proven ([6],[4]) that for $p = 2$ the system above, with only a source term in the first component, is still positivity preserving for small $\lambda > 0$ on general domains Ω under some smoothness conditions on the boundary. Such type of result follows from the 3G-Theorem that is due to Zhao in [8]. We will prove that a similar result holds for the p-laplacian with $p \in (1, \infty)$ in one dimension.

For given f we solve for u (and v). Our main result is:

Theorem 1 *There exists $\lambda_p > 0$ such that for all $\lambda \in [0, \lambda_p)$ and $f \in C[-1, 1]$ with $f > 0$ the following holds. There exists at least one solution (u, v) of (1) and all solutions satisfy $u \gg 0$ and $v \gg 0$.*

Remarks:

i. We call (u, v) a solution if $u, v, \phi_p(u'), \phi_p(v') \in C^1[-1, 1]$ and (1) holds.

ii. We will obtain even strong positivity in the following sense. There exists $C_{f,\lambda,1}$ and $C_{f,\lambda,2}$ such that
$$\begin{aligned} u(x) &\ge C_{f,\lambda,1}(1 - |x|), \\ v(x) &\ge C_{f,\lambda,2}(1 - |x|). \end{aligned}$$

iii. We are not able to show uniqueness of the solution. We include a partial result in section 7.

iv. For $p = 2$ in one dimension the result follows from elementary estimates on the Green function. The solution is unique for $p = 2$.

2 An approximation

Let us denote by $G_p : C\left(\bar{\Omega}\right) \to C^1\left(\bar{\Omega}\right)$ the solution operator of

$$\begin{cases} -\Delta_p w = f & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

that is, $w = G_p(f)$ solves (2). Then the solution u of (1) satisfies

$$u = G_p\left(f - \lambda\phi_p(v)\right) = G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p(u)\right).$$

Now define the iteration

$$\begin{cases} u_0 &= 0, \\ u_{n+1} &= S_{p,\lambda,f}(u_n), \end{cases} \tag{3}$$

where

$$S_{p,\lambda,f}(w) := G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p(w)\right). \tag{4}$$

Note that G_p and ϕ_p are order preserving (see [7] and appendix). By the strong maximum principle one even finds $f > 0$ implies $G_p(f) \gg 0$. An operator with the last property is called strongly positive.

Lemma 2 *Suppose that $f \in C(\bar{\Omega})$ and let $\{u_n\}_{n=0}^{\infty}$ be defined by (3). If we have $f > 0$ and $G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p \circ G_p(f)\right) > 0$, then:*

i. $0 \le u_n \le G_p(f)$ *for all* $n \ge 0$;

ii. $\{u_{2n}\}_{n=0}^{\infty}$, *respectively* $\{u_{2n+1}\}_{n=0}^{\infty}$, *is strictly increasing, respectively decreasing, pointwise; hence $\underline{u}$ and $\bar{u}$ are well defined by:*

$$\begin{aligned} \underline{u}(x) &= \lim_{n\to\infty} u_{2n}(x), \\ \bar{u}(x) &= \lim_{n\to\infty} u_{2n+1}(x); \end{aligned} \tag{5}$$

iii. every solution (u, v) of (1) with $0 \le u \le G_p(f)$ satisfies

$$\underline{u} \le u \le \bar{u}.$$

Proof. Since G_p and ϕ_p are order preserving and since G_p is even strongly positive, we find that $w < v$ implies $S_{p,\lambda,f}(w) \gg S_{p,\lambda,f}(v)$. Since $f > 0$ it follows that $u_0 = 0 \ll G_p(f) = u_1$ and hence $u_1 = S_{p,\lambda,f}(u_0) \gg S_{p,\lambda,f}(u_1) = u_2$. By the assumption we have $u_2 = G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p \circ G_p(f)\right) > 0$.

We found that

$$0 = u_0 \ll u_2 \ll u_1 = G_p(f).$$

Since $u_0 < w < u_1$ implies that $S_{p,\lambda,f}(u_0) \gg S_{p,\lambda,f}(w) \gg S_{p,\lambda,f}(u_1)$ we have

$$u_1 \gg u_3 \gg u_2 \text{ and } u_1 \gg u \gg u_2.$$

The result follows by induction. □

Remark 1: Lemma 2 also holds when $\Omega = (-1,1)$ is replaced by a bounded smooth domain Ω in $\mathbb{R}^n$.

Remark 2: Suppose that the conditions of the Lemma are satisfied. Since the operator $S_{p,\lambda,f} : C[-1,1] \to C^1[-1,1] \subset C[-1,1]$ is compact and $[0, G_p(f)]$ is bounded, the two functions $\underline{u}$ and $\bar{u}$, satisfy

$$\begin{aligned} \underline{u} &= G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p(\bar{u})\right), \\ \bar{u} &= G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p(\underline{u})\right). \end{aligned} \tag{6}$$

Note that both $\underline{u}$ and $\bar{u}$ satisfy

$$u = G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p \circ G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p(u)\right)\right),$$

which corresponds to an 8^{th}-order equation.

If we would be able to show that $\underline{u} = \bar{u}$ then we have existence of a solution of (1). We would even have found, due to the 'inverse' ordering property of $S_{p,\lambda,f}$, that between 0 and $G_p(f)$ there exists just one solution.

3 An auxiliary system

In the previous section $G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p \circ G_p(f)\right)$ was found to play a crucial role. For the trivially coupled system

$$\begin{cases} -\Delta_p u = f - \lambda\phi_p(v) & \text{in } \Omega, \\ -\Delta_p v = \phi_p(w) & \text{in } \Omega, \\ -\Delta_p w = f & \text{in } \Omega, \\ u = v = w = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

one finds $u = G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p \circ G_p(f)\right)$.

We will show that for λ small the operator $T_{p,\lambda} : C[-1,1] \to C[-1,1]$ defined by

$$T_{p,\lambda}(f) = G_p\left(f - \lambda\phi_p \circ G_p \circ \phi_p \circ G_p(f)\right) \tag{8}$$

is positivity preserving. For (7) it means that there exists λ_p such that for all $\lambda \in [0, \lambda_p]$ one finds that $f > 0$ implies $u > 0$.

Since $T_{p,\lambda}(cf) = \phi_p^{inv}(c)\, T_{p,\lambda}(f)$ for all $c \in I\!R$ it will be sufficient to show positivity under some rescaling. The appropriate rescaling that we will use is the following. We suppose that

$$\max_{x \in [-1,1]} G_p(f)(x) = 1. \tag{9}$$

Lemma 3 *Let $f \in C[-1,1]$ with $f > 0$ satisfying (9). Then there exists $x_f \in (-1,1)$ such that $w = G_p(f)$ satisfies $w(x_f) = 1$, $w'(x_f) = 0$ and*

$$w(x) = \int_{-1}^{x} \phi_p^{inv}\left(\int_t^{x_f} f(s)\,ds\right) dt, \tag{10}$$

or similarly

$$w(x) = \int_{x}^{1} \phi_p^{inv}\left(\int_{x_f}^{t} f(s)\,ds\right) dt. \tag{11}$$

Moreover

$$w(x) \geq \min\left(\frac{1-x}{1-x_f}, \frac{1+x}{1+x_f}\right) \geq \tfrac{1}{2}(1-|x|). \tag{12}$$

Proof. Since w satisfies $(\phi_p(w'))' = -f \leq 0$ one finds that $\phi_p(w')$ is decreasing. Hence w' is decreasing and w is concave. Then (12) follows from condition (9). $\square$

Lemma 4 *Suppose that (9) is satisfied. Then for all $x \in [-1,1]$ we find*

$$\phi_p \circ G_p \circ \phi_p \circ G_p(f)(x) \leq (1-|x|)^{p-1}.$$

Proof. Since (9) holds we find by the maximum principle ([7], see also the appendix) that

$$\phi_p \circ G_p \circ \phi_p \circ G_p(f)(x) \leq \phi_p \circ G_p(1)(x) =$$
$$= \left(\int_{|x|}^{1}\left(\int_0^t 1\,ds\right)^{\frac{1}{p-1}} dt\right)^{p-1} = \left(\int_{|x|}^{1} t^{\frac{1}{p-1}}\,dt\right)^{p-1} =$$
$$= \left(\frac{1}{1+\frac{1}{p-1}}\left(1-|x|^{1+\frac{1}{p-1}}\right)\right)^{p-1} \leq (1-|x|)^{p-1}.$$

$\square$

4 Regularity

By direct calculus one finds that $f \in C[-1,1]$ implies $G_p(f) \in C^{1,\gamma}[-1,1]$ with $\gamma = \min\left(1, \frac{1}{p-1}\right)$. But we will need an exact control of the dependence of u' by the source f. To do so the cases $p \geq 2$ and $1 < p < 2$ need different approaches.

For $p \geq 2$ we will use the space $C_0[-1,1] \cap C^1[-1,1]$ equipped with the norm $\|\cdot\|_{C^1}$ defined by $\|u\|_{C^1} = \|u'\|_\infty$. This space is a Banach space.

Proposition 5 *Let $p \geq 2$ and $f \in C[-1,1]$. Then we find for all $g \in C[-1,1]$ that*

$$\|G_p(f+g) - G_p(f)\|_{C^1} \leq 4 \|g\|_{L^1}^{\frac{1}{p-1}}. \tag{13}$$

Remark 3: As an immediate consequence we find

$$|G_p(f+g)(x) - G_p(f)(x)| \leq 4 \|g\|_{L^1}^{\frac{1}{p-1}} (1 - |x|). \tag{14}$$

For $1 < p < 2$ we cannot directly use the space $C_0[-1,1] \cap C^1[-1,1]$ in our estimate. Instead we will give pointwise estimates. The right hand side of the estimate that replaces (14) has a dependence on f but in a way that we are able to control.

Proposition 6 *Let $p \in (1,2)$ and $f \in C[-1,1]$. If we assume $f > 0$, then we find for all $g \in C[-1,1]$ and for every $\varepsilon \in (0,1]$ that*

$$|G_p(f+g)(x) - G_p(f)(x)| \leq \varepsilon\, G_p(f)(x) + 2^{\frac{4}{(p-1)^2}}\, \varepsilon^{-\frac{2-p}{p-1}}\, \|g\|_{L^1}^{\frac{1}{p-1}} \frac{G_p(f)(x)}{\|G_p(f)\|_\infty}. \tag{15}$$

Remark 4: If we assume (9) then (15) reduces to

$$|G_p(f+g)(x) - G_p(f)(x)| \leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}}\, \varepsilon^{-\frac{2-p}{p-1}}\, \|g\|_{L^1}^{\frac{1}{p-1}}\right) G_p(f)(x). \tag{16}$$

For $f \equiv 0$ one finds that for $1 < p \leq 2$:

$$|G_p(g)(x)| \leq \|g\|_{L^1}^{\frac{1}{p-1}} (1 - |x|). \tag{17}$$

Remark 5: Assuming (9) we are in fact using for $p \in (1,2)$ the space

$$C_e[-1,1] = \{v \in C_0[-1,1]\, ; \|v\|_e < \infty\},$$

where

$$\|v\|_e = \sup_{x \in (-1,1)} \frac{|v(x)|}{e(x)}, \text{ with } e(x) = G_p(f)(x).$$

For all $f \in C[-1,1]$ with $f > 0$ we obtain the same space $C_e[-1,1]$. But even with the normalization of such f by (9) there is no uniform estimate between all norms $\|\cdot\|_{e_1}$ and $\|\cdot\|_{e_2}$ coming from different f_1, f_2.

Since $1 \geq G_p(f)(x) > \frac{1}{2}(1 - |x|)$ the space $C_e[-1,1]$ satisfies

$$\left(C_0[-1,1] \cap C^1[-1,1]\right) \subset C_e[-1,1] \subset C_0[-1,1].$$

$C_e[-1,1]$ is a Banach space. A similar space has been used by Amann ([1]) in relation with positivity properties of the Laplace-Dirichlet problem.

We will show a technical lemma first.

Lemma 7 *For every $f \in C[-1,1]$ there is a unique constant c_f such that the function $u = G_p(f)$ satisfies*

$$u(x) = \int_{-1}^{x} \phi_p^{inv}\left(\int_t^1 f(s)\,ds - c_f\right) dt \tag{18}$$

Moreover, for $f, g \in C[-1,1]$ we find:

$$|c_f - c_g| \le \|f - g\|_{L^1}.$$

Proof. The expression in (18) follows immediately. By contradiction we show that $|c_f - c_g| \le \|f - g\|_{L^1}$. Indeed, suppose that $c_f - c_g > \|f - g\|_{L^1}$. Then

$$0 = \int_{-1}^{1} \phi_p^{inv}\left(\int_t^1 g(s)\,ds - c_g\right) dt =$$

$$= \int_{-1}^{1} \phi_p^{inv}\left(\int_t^1 f(s)\,ds - c_f + \int_t^1 (g(s) - f(s))\,ds + c_f - c_g\right) dt \ge$$

$$\ge \int_{-1}^{1} \phi_p^{inv}\left(\int_t^1 f(s)\,ds - c_f - \|f - g\|_{L^1} + c_f - c_g\right) dt >$$

$$> \int_{-1}^{1} \phi_p^{inv}\left(\int_t^1 f(s)\,ds - c_f\right) dt = 0,$$

a contradiction. By symmetry one finds $c_g - c_f \le \|f - g\|_{L^1}$. □

Proof of Proposition 5: We will denote

$$\begin{cases} u(x) &= G_p(f)(x), \\ v(x) &= G_p(f+g)(x), \\ F(x) &= \int_x^{x_f} f(s)\,ds. \end{cases} \tag{19}$$

Since $p \ge 2$ we have that $\frac{1}{p-1} \le 1$ holds and we may use (see Lemma B) that

$$(a+b)^{\frac{1}{p-1}} - a^{\frac{1}{p-1}} \le b^{\frac{1}{p-1}} \text{ for all } a, b \ge 0.$$

Indeed, for $0 \le F(x)$:

$$\phi_p^{inv}\left(F(x) + 2\|g\|_{L^1}\right) - \phi_p^{inv}(F(x)) =$$

$$= (F(x) + 2\|g\|_{L^1})^{\frac{1}{p-1}} - (F(x))^{\frac{1}{p-1}} \leq (2\|g\|_{L^1})^{\frac{1}{p-1}};$$

for $-2\|g\|_{L^1} < F(x) < 0$:

$$\phi_p^{inv}(F(x) + 2\|g\|_{L^1}) - \phi_p^{inv}(F(x)) \leq 2(2\|g\|_{L^1})^{\frac{1}{p-1}};$$

and for $F(x) \leq -2\|g\|_{L^1}$:

$$\phi_p^{inv}(F(x) + 2\|g\|_{L^1}) - \phi_p^{inv}(F(x)) =$$

$$= (-F(x) - 2\|g\|_{L^1} + 2\|g\|_{L^1})^{\frac{1}{p-1}} - (-F(x) - 2\|g\|_{L^1})^{\frac{1}{p-1}} \leq$$

$$\leq (2\|g\|_{L^1})^{\frac{1}{p-1}}.$$

Then we have

$$v'(x) - u'(x) =$$

$$= \phi_p^{inv}\left(F(x) + \int_x^1 g(s)\,ds - c_{f+g} + c_f\right) - \phi_p^{inv}(F(x)) \leq$$

(by Lemma 7)

$$\leq \phi_p^{inv}(F(x) + \|g\|_{L^1} + \|g\|_{L^1}) - \phi_p^{inv}(F(x)) \leq$$

$$\leq 2(2\|g\|_{L^1})^{\frac{1}{p-1}} \leq 4\|g\|_{L^1}^{\frac{1}{p-1}}.$$

In a similar way one obtains the estimate from below. Together they imply

$$|v'(x) - u'(x)| \leq 4\|g\|_{L^1}^{\frac{1}{p-1}}.$$

$\square$

Proof of Proposition 6: We will use u, v and F as in (19). Now we have $\frac{1}{p-1} > 1$ and hence $\frac{2-p}{p-1} > 0$. In the estimate from above we apply that for $a, b \geq 0$ (see Lemma B):

$$(a+b)^{\frac{1}{p-1}} - a^{\frac{1}{p-1}} \leq 2^{\frac{1}{p-1}}\left(a^{\frac{2-p}{p-1}}b + b^{\frac{1}{p-1}}\right).$$

Indeed, for $0 \leq F(t)$ we have

$$\phi_p^{inv}(F(t) + 2\|g\|_{L^1}) - \phi_p^{inv}(F(t)) \leq$$

$$\leq 2^{\frac{1}{p-1}}\left(2\|g\|_{L^1}(F(t))^{\frac{2-p}{p-1}} + (2\|g\|_{L^1})^{\frac{1}{p-1}}\right) \leq$$

$$\leq 2^{\frac{2}{p-1}} \left((F(t))^{\frac{2-p}{p-1}} \|g\|_{L^1} + \|g\|_{L^1}^{\frac{1}{p-1}} \right). \tag{20}$$

For $F(t) \geq 0$ we can use (46) from Lemma B to find that for any $\delta > 0$

$$(F(t))^{\frac{2-p}{p-1}} \leq \left(\delta^{-1} \|g\|_{L^1}\right)^{\frac{2-p}{p-1}} + \left(\delta^{-1} \|g\|_{L^1}\right)^{-1} \ (F(t))^{\frac{1}{p-1}}. \tag{21}$$

Since $F(t) \geq 0$ for $x \leq x_f$ we get by (20-21) that

$$v(x) - u(x) \leq$$

$$\leq \int_{-1}^{x} \left(\phi_p^{inv}(F(t) + 2\|g\|_{L^1}) - \phi_p^{inv}(F(t))\right) dt \leq$$

$$\leq 2^{\frac{2}{p-1}} \int_{-1}^{x} \left((F(t))^{\frac{2-p}{p-1}} \|g\|_{L^1} + \|g\|_{L^1}^{\frac{1}{p-1}} \right) dt \leq$$

$$\leq 2^{\frac{2}{p-1}} \int_{-1}^{x} \left(\delta \ (F(t))^{\frac{1}{p-1}} + \left(1 + \delta^{-\frac{2-p}{p-1}}\right) \|g\|_{L^1}^{\frac{1}{p-1}} \right) dt =$$

$$= \ 2^{\frac{2}{p-1}} \left(\delta \ u(x) + \left(1 + \delta^{-\frac{2-p}{p-1}}\right) \|g\|_{L^1}^{\frac{1}{p-1}} (1 + x) \right). \tag{22}$$

For $x > x_f$ we find similarly by using (11) that

$$v(x) - u(x) \leq$$

$$\leq \ 2^{\frac{2}{p-1}} \left(\delta \ u(x) + \left(1 + \delta^{-\frac{2-p}{p-1}}\right) \|g\|_{L^1}^{\frac{1}{p-1}} (1 - x) \right).$$

Now we will use that f is such that (9) holds. From (12) in Lemma 3 we find that for $x \in [-1, x_f]$:

$$v(x) - u(x) \leq$$

$$\leq 2^{\frac{2}{p-1}} \left(\delta \ u(x) + \left(1 + \delta^{-\frac{2-p}{p-1}}\right) \|g\|_{L^1}^{\frac{1}{p-1}} (1 + x_f) \, u(x) \right). \tag{23}$$

and for $x \in [x_f, 1]$:

$$v(x) - u(x) \leq$$

$$\leq 2^{\frac{2}{p-1}} \left(\delta \ u(x) + \left(1 + \delta^{-\frac{2-p}{p-1}}\right) \|g\|_{L^1}^{\frac{1}{p-1}} (1 - x_f) \, u(x) \right). \tag{24}$$

Combining (23) and (24) yields

$$v(x) - u(x) \leq 2^{\frac{2}{p-1}} \left(\delta + 2 \left(1 + \delta^{-\frac{2-p}{p-1}} \right) \|g\|_{L^1}^{\frac{1}{p-1}} \right) u(x) .$$

By setting $\delta = 2^{\frac{-2}{p-1}} \varepsilon$ and fixing $\varepsilon \in (0,1]$ we find that

$$2^{\frac{2}{p-1}} \cdot 2 \left(1 + \delta^{-\frac{2-p}{p-1}} \right) \leq 2^{\frac{4}{(p-1)^2}} \varepsilon^{-\frac{2-p}{p-1}}$$

which shows

$$G_p(f+g)(x) - G_p(f)(x) \leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}} \varepsilon^{-\frac{2-p}{p-1}} \|g\|_{L^1}^{\frac{1}{p-1}} \right) G_p(f)(x) .$$

Set $m = \|G_p(f)\|_\infty$. Without rescaling we find

$$G_p(f+g)(x) - G_p(f)(x) = m \left(G_p\left(\frac{f+g}{m^{p-1}} \right)(x) - G_p\left(\frac{f}{m^{p-1}} \right)(x) \right) \leq$$

$$\leq m \left(\varepsilon + 2^{\frac{4}{(p-1)^2}} \varepsilon^{-\frac{2-p}{p-1}} \left\| \frac{g}{m^{p-1}} \right\|_{L^1}^{\frac{1}{p-1}} \right) G_p\left(\frac{f}{m^{p-1}} \right)(x) =$$

$$\leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}} \varepsilon^{-\frac{2-p}{p-1}} \frac{\|g\|_{L^1}^{\frac{1}{p-1}}}{\|G_p(f)\|_\infty} \right) G_p(f)(x)$$

which completes the estimate from above.

For the estimate from below we use that for $a, b \geq 0$ (see Lemma B) holds:

$$\phi_p^{inv}(a-b) - \phi_p^{inv}(a) \geq -\frac{1}{p-1} \left(b a^{\frac{2-p}{p-1}} + b^{\frac{1}{p-1}} \right),$$

which implies instead of (20) that

$$\phi_p^{inv}\left(F(t) - 2\|g\|_{L^1}\right) - \phi_p^{inv}(F(t)) \geq$$

$$\geq -\frac{2^{\frac{1}{p-1}}}{p-1} \left((F(t))^{\frac{2-p}{p-1}} \|g\|_{L^1} + \|g\|_{L^1}^{\frac{1}{p-1}} \right). \qquad (25)$$

We may continue as in the part where we estimated from above and since

$$\frac{2^{\frac{1}{p-1}}}{p-1} < 2^{\frac{2}{p-1}}$$

we find (15). □

5 Positivity for the auxiliary system

Lemma 8 *There exists* $\lambda_p^* > 0$ *such that for all* $\lambda \in [0, \lambda_p^*)$ *the operator* $T_{p,\lambda}$ *defined in (8) satisfies for all* $f \in C[-1,1]$:

$$f > 0 \Rightarrow T_{p,\lambda}(f) \gg 0.$$

Remark 6: In the proof we will obtain rough estimates from below for λ_p^*. We find

$$\begin{array}{rcll} \lambda_p^* & \geq & \frac{1}{4}\left(\frac{1}{32}\right)^{p-1} & \text{for} \quad p \geq 2, \\ \lambda_p^* & \geq & \frac{1}{16}\left(\frac{1}{16}\right)^{\frac{1}{p-1}} & \text{for} \quad p \in (1,2). \end{array}$$

These estimates for λ_p^* are rather small. For $p = 2$ one finds $\lambda_2^* = 3\frac{3}{14}$.

Remark 7: The number λ_p^* is estimated from above by $(\lambda_{1,p})^2$ where $\lambda_{1,p}$ is the first eigenvalue of the Dirichlet problem for the p-laplacian:

$$\begin{cases} -\Delta_p \varphi &= \lambda \phi_p(\varphi) \quad \text{in } (-1,1), \\ \varphi &= 0 \quad \text{on } \{-1,1\}. \end{cases} \tag{26}$$

Let $\varphi_{1,p}$ denote the first eigenfunction of (26). For $\lambda > (\lambda_{1,p})^2 = \left(\frac{\pi_p}{2}\right)^{2p}$, with π_p defined in [5], one uses $f = \phi_p(\varphi_{1,p})$ to find a non-positive $u = T_{p,\lambda}(f)$. Indeed one finds

$$u = \phi_p^{inv}\left(\left(\lambda_{1,p}^2 - \lambda\right)\lambda_{1,p}^{-3}\right)\ \varphi_{1,p}.$$

From [5] one can obtain

$$\pi_p = (p)^{-p^*}\ (p^*)^{-p}\ \Gamma\left(p^{-1}\right)\ \Gamma\left((p^*)^{-1}\right) \tag{27}$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$ and Γ is denoting the usual gamma function. For comparison: $(\lambda_{1,2})^2 = \frac{\pi^4}{16} = 6.088\ldots$.

Proof for $p \geq 2$: For $p \geq 2$ we have by Proposition 5 that

$$\|T_{p,\lambda}(f) - G_p(f)\|_{C^1} \leq$$

$$\leq 4\,\|\lambda\phi_p \circ G_p \circ \phi_p \circ G_p(f)\|_{L^1}^{\frac{1}{p-1}} = 4\ \lambda^{\frac{1}{p-1}}\,\|G_p \circ \phi_p \circ G_p(f)\|_{L^{p-1}} \leq$$

$$\leq 4\ (2\lambda)^{\frac{1}{p-1}}\,\|G_p \circ \phi_p \circ G_p(f)\|_{C^1} \leq$$

(again using Proposition 5)

$$\leq 16\ (2\lambda)^{\frac{1}{p-1}}\,\|\phi_p \circ G_p(f)\|_{L^1}^{\frac{1}{p-1}} = 16\ (2\lambda)^{\frac{1}{p-1}}\,\|G_p(f)\|_{L^{p-1}} \leq$$

$$\leq 16 \ (4\lambda)^{\frac{1}{p-1}} \left\| G_p(f) \right\|_\infty .$$

We may assume that (9) holds. Then with $\lambda \in \left[0, \frac{1}{4}\left(\frac{1}{32}\right)^{p-1}\right)$ we find for $x \in (-1,1)$ that

$$T_{p,\lambda}(f)(x) - G_p(f)(x) \geq -(1-|x|) \left\| T_{p,\lambda}(f) - G_p(f) \right\|_{C^1} >$$

$$> -\tfrac{1}{2}(1-|x|) \left\| G_p(f) \right\|_\infty = -\tfrac{1}{2}(1-|x|) .$$

Hence, using (12), we find for $x \in (-1,1)$ that

$$T_{p,\lambda}(f)(x) > G_p(f)(x) - \tfrac{1}{2}(1-|x|) \geq 0.$$

The case $1 < p < 2$**:** Again we assume that (9) holds. We find by (16) that

$$\left| T_{p,\lambda}(f)(x) - G_p(f)(x) \right| \leq$$

$$\leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}} \varepsilon^{-\frac{2-p}{p-1}} \left\| \lambda \phi_p \circ G_p \circ \phi_p \circ G_p(f) \right\|_{L^1}^{\frac{1}{p-1}} \right) G_p(f)(x) \leq$$

$$\leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}} \varepsilon^{-\frac{2-p}{p-1}} \lambda^{\frac{1}{p-1}} 2^{\frac{1}{p-1}} \left\| G_p \circ \phi_p \circ G_p(f) \right\|_\infty \right) G_p(f)(x) = (\star)$$

By Hölder's inequality we have ($\frac{1}{p-1} > 1$) that

$$\left\| G_p \circ \phi_p(w) \right\|_\infty = \sup_{x \in [-1,1]} \left| \int_{-1}^{x} \phi_p^{inv} \left(\int_t^{x_f} \phi_p(w(s))\, ds \right) dt \right| \leq$$

$$\leq \int_{-1}^{1} \left| \int_t^{x_f} |w(s)|^{p-1} ds \right|^{\frac{1}{p-1}} dt \leq$$

$$\leq \int_{-1}^{1} 2^{\frac{2-p}{p-1}} \left| \int_t^{x_f} |w(s)|\, ds \right| dt \leq 2^{\frac{1}{p-1}} \left\| w \right\|_{L^1} . \tag{28}$$

Hence we may continue by

$$(\star) \leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2} + \frac{2}{p-1}} \varepsilon^{-\frac{2-p}{p-1}} \ \lambda^{\frac{1}{p-1}} \left\| G_p(f) \right\|_{L^1} \right) G_p(f)(x) \leq$$

$$\leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2} + \frac{2}{p-1} + 1} \varepsilon^{-\frac{2-p}{p-1}} \ \lambda^{\frac{1}{p-1}} \right) G_p(f)(x) .$$

Take $\varepsilon = \frac{1}{2}$. We have for $\lambda \in \left[0, \frac{1}{16}\left(\frac{1}{2}\right)^{\frac{4}{p-1}}\right)$ that

$$T_{p,\lambda}(f) - G_p(f) > -G_p(f)$$

and hence that $T_{p,\lambda}$ preserves positivity. $\square$

6 A positive solution for the main system

Existence of a positive solution follows by an application of Schauder's Fixed Point Theorem. Let us denote

$$[0, G_p(f)] = \{u \in C_0[-1,1]\,; 0 \le u(x) \le G_p(f)(x) \text{ for all } x \in [-1,1]\}.$$

Lemma 9 *Let $f \in C[-1,1]$ with $f > 0$ and $\lambda \in [0, \lambda_p^*)$ with λ_p^* as in Lemma 8. Then (1) has a positive solution (u,v).*

Proof. Let the operator $S_{p,\lambda,f}$ be as in (4). By Lemma 8 we find that

$$S_{p,\lambda,f}([0, G_p(f)]) \subset [0, G_p(f)].$$

Since $[0, G_p(f)]$ is a convex set in the Banach space $C[-1,1]$ and $S_{p,\lambda,f}$ is compact, Schauder's Fixed Point Theorem gives the existence of a solution u in $[0, G_p(f)]$. From

$$S_{p,\lambda,f}([0, G_p(f)]) \subset [G_p(f - \lambda\phi_p \circ G_p \circ \phi_p \circ G_p(f)), G_p(f)]$$

it follows that $u > 0$ (and hence $v > 0$). $\square$

Lemma 10 *There exists $\lambda_p^\bullet$ such that for all $\lambda \in [0, \lambda_p^\bullet)$, $f \in C[-1,1]$ with $f > 0$, and every solution (u,v) of (1) we have*

$$\tfrac{1}{2} G_p(f)(x) \le u(x) \le G_p(f)(x). \tag{29}$$

Remark 8: Again we do obtain some rough estimates from below, namely $\lambda_p^\bullet \ge \frac{1}{4}\left(\frac{1}{64}\right)^{p-1}$ when $p \ge 2$, and $\lambda_p^\bullet \ge \frac{1}{64}\left(\frac{1}{16}\right)^{\frac{1}{p-1}}$ when $1 < p < 2$.

Proof. We assume (9). Note that u satisfies

$$u = G_p(f - \lambda\phi_p \circ G_p \circ \phi_p(u)). \tag{30}$$

The case $p \ge 2$. We find by using (14) that

$$|u(x) - G_p(f)(x)| =$$

$$= |G_p(f - \lambda\phi_p \circ G_p \circ \phi_p(u))(x) - G_p(f)(x)| \le$$

$$\le 4\,\lambda^{\frac{1}{p-1}}\,(1 - |x|)\,\|\phi_p \circ G_p \circ \phi_p(u)\|_{L^1}^{\frac{1}{p-1}} \le$$

$$\leq 4\ (2\lambda)^{\frac{1}{p-1}}\ (1-|x|)\,\|\phi_p \circ G_p \circ \phi_p(u)\|_{\infty}^{\frac{1}{p-1}} \leq$$

$$\leq 4\ (2\lambda)^{\frac{1}{p-1}}\ (1-|x|)\,\|G_p \circ \phi_p(u)\|_{C^1} \leq$$

(now use (13) with $f=0$)

$$\leq 16\ (2\lambda)^{\frac{1}{p-1}}\ (1-|x|)\,\|\phi_p(u)\|_{L^1}^{\frac{1}{p-1}} = 16\ (2\lambda)^{\frac{1}{p-1}}\ (1-|x|)\,\|u\|_{L^{p-1}} \leq$$

$$\leq 16\ (4\lambda)^{\frac{1}{p-1}}\ (1-|x|)\,\|u\|_{\infty}.$$

And hence by (12)

$$|u(x) - G_p(f)(x)| \leq 32\ (4\lambda)^{\frac{1}{p-1}}\ \|u\|_{\infty}\ G_p(f)(x). \tag{31}$$

Take $\lambda < \frac{1}{4}\left(\frac{1}{64}\right)^{p-1}$. It follows that (we assume (9))

$$\|u\|_{\infty} \leq \|G_p(f)\|_{\infty} + \|u - G_p(f)\|_{\infty} \leq 1 + \tfrac{1}{2}\|u\|_{\infty}$$

implying $\|u\|_{\infty} \leq 2$. Then (31) shows that

$$u(x) \geq \left(1 - \tfrac{1}{2}\|u\|_{\infty}\right) G_p(f)(x) \geq 0 \tag{32}$$

and

$$u(x) \leq \left(1 + \tfrac{1}{2}\|u\|_{\infty}\right) G_p(f)(x) \leq 2G_p(f)(x). \tag{33}$$

But for $u \geq 0$ we find by (30) and the maximum principle that $u \leq G_p(f)$ holds. Hence $\|u\|_{\infty} \leq 1$ and from (32) we have (29).

i. **The case** $1 < p < 2$. Now $\frac{1}{p-1} > 1$. We obtain by (16) that

$$|u(x) - G_p(f)(x)| = |G_p(f - \lambda\phi_p \circ G_p \circ \phi_p(u))(x) - G_p(f)(x)| \leq$$

$$\leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}}\varepsilon^{-\frac{2-p}{p-1}}\ \lambda^{\frac{1}{p-1}}\ \|\phi_p \circ G_p \circ \phi_p(u)\|_{L^1}^{\frac{1}{p-1}}\right) G_p(f)(x) \leq$$

$$\leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}+\frac{1}{p-1}}\varepsilon^{-\frac{2-p}{p-1}}\ \lambda^{\frac{1}{p-1}}\ \ \|G_p \circ \phi_p(u)\|_{\infty}\right) G_p(f)(x) \leq$$

(using (28))

$$\leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}+\frac{2}{p-1}}\varepsilon^{-\frac{2-p}{p-1}}\ \lambda^{\frac{1}{p-1}}\ \ \|u\|_{L^1}\right) G_p(f)(x) \leq$$

$$\leq \left(\varepsilon + 2^{\frac{4}{(p-1)^2}+\frac{2}{p-1}+1}\varepsilon^{-\frac{2-p}{p-1}}\ \lambda^{\frac{1}{p-1}}\ \ \|u\|_\infty\right) G_p(f)(x). \tag{34}$$

Fix $\varepsilon = \frac{1}{4}$ and take $\lambda \in \left[0, \frac{1}{32}\left(\frac{1}{16}\right)^{\frac{1}{p-1}}\right]$ which yields $2^{\frac{4}{(p-1)^2}+\frac{2}{p-1}+1} 4^{\frac{2-p}{p-1}}\ \lambda^{\frac{1}{p-1}} \leq \frac{1}{4}$. Then

$$\|u\|_\infty \leq \|u - G_p(f)\|_\infty + \|G_p(f)\|_\infty \leq$$

$$\leq \left(\tfrac{1}{4} + \tfrac{1}{4}\|u\|_\infty + 1\right)\|G_p(f)\|_\infty \leq \tfrac{5}{4} + \tfrac{1}{4}\|u\|_\infty$$

and we find that $\|u\|_\infty \leq \frac{5}{3}$. If $\|u\|_\infty \leq \frac{5}{3}$ then we find from (34) that

$$|u(x) - G_p(f)(x)| \leq \left(\tfrac{1}{4} + \tfrac{1}{4}\tfrac{5}{3}\right) G_p(f)(x) = \tfrac{2}{3} G_p(f)(x).$$

Hence we have

$$\tfrac{1}{3} G_p(f)(x) \leq u(x) \leq \tfrac{5}{3} G_p(f)(x).$$

If $u > 0$ then $u(x) \leq G_p(f)(x)$ which implies that $\|u\|_\infty \leq 1$ and shows that

$$|u(x) - G_p(f)(x)| \leq \left(\tfrac{1}{4} + \tfrac{1}{4}\right) G_p(f)(x) = \tfrac{1}{2} G_p(f)(x).$$

The estimate in (29) follows. □

Proof of the main result. Set $\lambda_p = \min\left(\lambda_p^*, \lambda_p^\bullet\right)$. The lemmas 9 and 10 give the result of the Theorem 1**.** □

7 Some remarks on uniqueness

Uniqueness for the p-laplacian is in general much harder to prove. We recall that for the Dirichlet problem $-\Delta_p u = \lambda\phi_p(u) + f$ with $p > 2$ and $0 < \lambda < \lambda_{p,1}$ there are f (with sign change) such that there is no unique solution. This in contrary to the case $p = 2$. See the paper by Del Pino e.a., [5].

A similar problem appears for systems. In the case of the cooperative system that is studied by Fleckinger e.a. in [2], uniqueness of the positive solution has not been shown. And also we are not able to prove that there exists a unique solution for the problem we study. The only rather incomplete result we obtain, is the following.

Proposition 11 *Let $p > 2$ and assume that $\lambda \in [0, \lambda_p]$. Assume that $f \in C[-1,1]$. There cannot exist solutions (u_1, v_1) and (u_2, v_2) such that either*

$$\{x \in [-1,1]; u_1(x) = u_2(x)\} \ \text{ or } \ \{x \in [-1,1]; v_1(x) = v_2(x)\} \tag{35}$$

is finite.

Proof. We will proceed by contradiction. Suppose that there exist two solutions (u_1, v_1) and (u_2, v_2) with both the sets in (35) being finite. Let us denote $U = u_1 - u_2$ and $V = v_1 - v_2$. Note that $U, V \in C^1[-1, 1]$.

i) U cannot be of fixed sign. Suppose that $u_1 \geq u_2$. Then the maximum principle used for the second equation of (1) implies that $v_1 \geq v_2$. The maximum principle used in the first equation shows that $u_1 \leq u_2$, a contradiction. Similarly one finds that v_1 and v_2 are not ordered.

We will study the set of zeros of U (and of V) where a sign change occurs. Let us denote

$$\begin{aligned}\mathcal{Z}_U = \{-1,1\} \cup \; \{\alpha \in (-1,1)\,; \forall \varepsilon > 0 \; \exists x_1, x_2 \text{ with} \\ \alpha - \varepsilon < x_1 < \alpha < x_2 < \alpha + \varepsilon \\ \text{such that } U(x_1)\, U(x_2) < 0\}\end{aligned}$$

and similarly $\mathcal{Z}_V$.

ii) Between two consecutive zeros of U, the function V has opposite sign somewhere. Let $\alpha_1, \alpha_2 \in [-1,1]$ with $\alpha_1 < \alpha_2$ denote two consecutive zeros of U. Without loss of generality we may assume that $U = u_1 - u_2 > 0$ on (α_1, α_2). We find that

$$-\phi_p(u_1')' + \phi_p(u_2')' = -\lambda\left(\phi_p(v_1) - \phi_p(v_2)\right).$$

Multiplying both sides by $u_1 - u_2$ we obtain, after an integration by parts and using the fact that ϕ_p is strictly increasing, that

$$0 < \int_{\alpha_1}^{\alpha_2} \left(\phi_p(u_1') - \phi_p(u_2')\right)(u_1' - u_2')\, dx =$$

$$= \int_{\alpha_1}^{\alpha_2} \left(-\phi_p(u_1')' + \phi_p(u_2')'\right)(u_1 - u_2)\, dx =$$

$$= -\lambda \int_{\alpha_1}^{\alpha_2} \left(\phi_p(v_1) - \phi_p(v_2)\right)(u_1 - u_2)\, dx.$$

Since $u_1 - u_2 > 0$ on (α_1, α_2) we find that there is some $x \in (\alpha_1, \alpha_2)$ with $\phi_p(v_1(x)) - \phi_p(v_2(x)) < 0$ and hence $V(x) = v_1(x) - v_2(x) < 0$.

iii) Between two consecutive zeros of V, the function U has the same sign somewhere. The argument is similar as above. Let $\beta_1 < \beta_2$ be two consecutive zeros of V. We use

$$-\phi_p(v_1')' + \phi_p(v_2')' = \phi_p(u_1) - \phi_p(u_2),$$

multiply by $v_1 - v_2$ and integrate by parts to find

$$\int_{\beta_1}^{\beta_2} \left(\phi_p(u_1) - \phi_p(u_2)\right)(v_1 - v_2)\, dx > 0.$$

iv) Elements of $\mathcal{Z}_U$ and $\mathcal{Z}_V$ keep alternating. Set

$$\begin{aligned}\mathcal{Z}_U &= \{\alpha_i; i = 0, 1, 2, \ldots\}, \\ \mathcal{Z}_V &= \{\beta_i; i = 0, 1, 2, \ldots\},\end{aligned}$$

with

$$\begin{aligned}-1 &= \alpha_0 < \alpha_1 < \alpha_2 < \ldots \\ -1 &= \beta_0 < \beta_1 < \beta_2 < \ldots\end{aligned}$$

and suppose without loss of generality that

$$\operatorname{sign} U = (-1)^i \text{ on } (\alpha_i, \alpha_{i+1}). \tag{36}$$

Note that by *i)* both $\mathcal{Z}_U$ and $\mathcal{Z}_V$ contain at least three elements.

First assume that

$$\operatorname{sign} V = (-1)^i \text{ on } (\beta_i, \beta_{i+1}) \tag{37}$$

which implies with *ii)* that

$$-1 = \beta_0 = \alpha_0 < \beta_1 < \alpha_1. \tag{38}$$

We will prove that

$$\beta_{i-1} \leq \alpha_{i-1} < \beta_i < \alpha_i < 1 \tag{39}$$

implies that there exist β_{i+1} and α_{i+1} with

$$\beta_i \leq \alpha_i < \beta_{i+1} < \alpha_{i+1} < 1. \tag{40}$$

For $i = 1$ (39) is satisfied by *i)* and (38).

Since $\beta_i, \alpha_i < 1$ there exist $\beta_{i+1}, \alpha_{i+1} \leq 1$. Now suppose that (40) does not hold. Then either $\alpha_i \geq \beta_{i+1}$, or $\beta_{i+1} \geq \alpha_{i+1}$, or $\alpha_i < \beta_{i+1} < \alpha_{i+1} = 1$. Note that we will have $\beta_i \leq \alpha_i$.

If $\alpha_i \geq \beta_{i+1}$ we have $\alpha_{i-1} < \beta_i < \beta_{i+1} \leq \alpha_i$. It follows that $\operatorname{sign} U = (-1)^{i-1} = -\operatorname{sign} V$ on (β_i, β_{i+1}) which contradicts *iii)*.

If $\beta_{i+1} \geq \alpha_{i+1}$ then we have $\beta_i \leq \alpha_i < \alpha_{i+1} \leq \beta_{i+1}$ implying that $\operatorname{sign} U = (-1)^i = \operatorname{sign} V$ on (α_i, α_{i+1}) which contradicts *ii)*.

If $\alpha_i < \beta_{i+1} < \alpha_{i+1} = 1$ then there exists β_{i+2} and $\alpha_i < \beta_{i+1} < \beta_{i+2} \leq \alpha_{i+1} = 1$, and again a contradiction by *iii)*.

We have proven that (39) implies (40). A similar argument, with α_i and β_i interchanged, holds when we assume that

$$\operatorname{sign} V = (-1)^{i+1} \text{ on } (\beta_i, \beta_{i+1}).$$

iv) Finitely many zeroes isn't possible. The previous argument shows that $\mathcal{Z}_U$ or $\mathcal{Z}_V$ cannot have finitely many elements. Hence U and V have at least infinitely many zeros, a contradiction. $\square$

Remark 9: We cannot prove uniqueness when for two solutions either $u_1 - u_2$ or $v_1 - v_2$ has an infinite number of zeros. Hence there is at least one accumulation point $\tilde{\alpha}$ of zeros, both of $u_1 - u_2$ and $v_1 - v_2$. As a result (u_1, v_1) and (u_2, v_2) satisfy the same initial value problem at $\tilde{\alpha}$. We may rephrase the system of differential equations to

$$\left\{ \begin{array}{rcl} z' & = & \phi_p(v) - f, \\ u' & = & \phi_p^{inv}(z), \\ w' & = & -\phi_p(u), \\ v' & = & \phi_p^{inv}(w). \end{array} \right. \tag{41}$$

If the right hand side of (41) is Lipschitz at $\tilde{\alpha}$, there is a unique solution to the initial value problem. Since we have proven that u' and v' are nonzero at the boundary the right hand side of (41) satisfies the Lipschitz-condition at the boundary for $p \geq 2$. Hence in that case no accumulation of zeros of $u_1 - u_2$ (or $v_1 - v_2$) can occur at -1 and 1. Similarly, for $p < 2$ such an accumulation of zeros can only occur at the boundary since the Lipschitz-condition for (41) only fails when $u = 0$ or $v = 0$.

A Appendix: the strong maximum principle

The strong maximum principle in one dimension is a straightforward exercise. For the sake of easy reference we state and prove it.

Proposition A *Let $p \in (1, \infty)$. If $f, g \in C[-1, 1]$ with $f \leq g$ and $f \neq g$, then there is $c > 0$ such that for all $x \in [-1, 1]$ we have*

$$G_p(g)(x) - G_p(f)(x) \geq c(1 - |x|). \tag{42}$$

Proof. Set $u = G_p(f)$ and $v = G_p(g)$ and note that $u, v \in C^1[-1, 1]$. We also define

$$h(x) = \phi_p(v'(x)) - \phi_p(u'(x)).$$

Since $h' = f - g \leq 0$ it follows that h is decreasing. Since $u \neq v$ and $u(x) = v(x) = 0$ for $|x| = 1$ there exists an extremum point for $u - v$, say ξ. Since $v'(\xi) = u'(\xi)$ implies $h(\xi) = 0$ one finds that $h(-1) > 0$ and $h(1) < 0$. These inequalities are strict: $h(-1) = 0$ would imply $h \equiv 0$ on $[-1, \xi]$ and hence $u(\xi) = v(\xi)$, a contradiction. One finishes by observing that $\operatorname{sign} h = \operatorname{sign}(v' - u')$. □

B Appendix: some inequalities

We have used the following elementary inequalities.

Lemma B *For all $a, b \geq 0$ we have:*

i. if $p \geq 2$

$$(a+b)^{\frac{1}{p-1}} \leq a^{\frac{1}{p-1}} + b^{\frac{1}{p-1}}; \tag{43}$$

ii. if $1 < p < 2$:

$$\phi_p^{inv}(a+b) - \phi_p^{inv}(a) \leq 2^{\frac{1}{p-1}} \left(b \ a^{\frac{2-p}{p-1}} + b^{\frac{1}{p-1}} \right), \tag{44}$$

$$\phi_p^{inv}(a) - \phi_p^{inv}(a-b) \leq \frac{1}{p-1} \left(b \ a^{\frac{2-p}{p-1}} + b^{\frac{1}{p-1}} \right), \tag{45}$$

$$a^{\frac{2-p}{p-1}} \leq b^{\frac{2-p}{p-1}} + b^{-1} \ a^{\frac{1}{p-1}}. \tag{46}$$

Proof. We will prove the last three. Hence we have that $\frac{1}{p-1} > 1$. First we assume that $b \leq a$ and denote $x = \frac{b}{a}$. Since $x \in [0,1]$ we find that (44), (45) follow from

$$(1+x)^{\frac{1}{p-1}} - 1 \leq 2^{\frac{1}{p-1}} x \leq 2^{\frac{1}{p-1}} \left(x + x^{\frac{1}{p-1}} \right) \text{ for } 0 \leq x \leq 1$$

respectively

$$(1-x)^{\frac{1}{p-1}} - 1 \geq -\frac{1}{p-1} x \geq -\frac{1}{p-1} \left(x + x^{\frac{1}{p-1}} \right) \text{ for } 0 \leq x.$$

If $b > a$ then

$$\phi_p^{inv}(a+b) - \phi_p^{inv}(a) \leq \phi_p^{inv}(a+b) \leq 2^{\frac{1}{p-1}} b^{\frac{1}{p-1}}$$

and

$$\phi_p^{inv}(a-b) - \phi_p^{inv}(a) = -(b-a)^{\frac{1}{p-1}} - a^{\frac{1}{p-1}} \geq -b^{\frac{1}{p-1}}.$$

Inequality (46) is immediate by distinguishing $a \geq b$ and $a \leq b$. $\square$

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