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EXISTENCE OF A MAXIMAL SOLUTION FOR QUASIMONOTONE ELLIPTIC SYSTEMS

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Dedicated to the memory of Peter Hess

Abstract. Suppose the quasimonotone elliptic system has a supersolution above a subsolution. Then it is shown that there exists a maximal solution in between. Solutions and sub/supersolutions are defined in the spaces $C(\bar{\Omega})^n$ and $W_1^2(\bar{\Omega})^n$. The regularity assumptions for the equations are optimal: for fixed off diagonal terms the coupling functions are Carathéodory; for fixed diagonal and space variable these functions are increasing and not necessarily continuous. The basic ingredients are a version of the maximum principle, the Schauder fixed point theorem for the C-case and an existence theorem of J.L. Lions for the W-case.

1. Introduction. In this paper we consider systems of elliptic equations of the form

$$\begin{cases} -\Delta\vec{u} = F(\vec{u}) & \text{in } \Omega, \\ \vec{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^m$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a given function.

Problem (1) has been studied extensively in the literature. See e.g. [4], [10], [15], [30] or [18] and the references therein. For its connection with population dynamics and combustion theory, see [7], [9], [14], and [8].

Here we shall study system (1) by using a consequence of the maximum principle, namely the 'sub-supersolution method' (see [4]). It is well known ([27]) that the maximum principle in its various forms does not hold in the framework of vector valued functions (e.g. systems of equations) if one does not assume some structural condition on the coupling. For weakly coupled systems the structural condition is cooperativity (linear coupling) or quasimonotonicity (nonlinear coupling). If one allows coupling also in the derivatives then the maximum principle does not hold even for very simple systems ([27]). Hence in that case there is in general no hope to obtain existence theorems based on the maximum principle ([29]). The above reason

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motivates our assumption that the system under consideration is weakly coupled and of quasimonotonic type.

We will obtain existence of a maximal solution between a sub- and a supersolution for (1) under basically the following two conditions:

- F is a continuous function on the diagonal;
- F is increasing, not necessarily continuous, for off-diagonal terms; such F are called quasimonotone.

We will even allow functions F which on the diagonal are a combination of a continuous and an increasing function.

For the case that $n = 1$, the scalar function case, several authors have studied the existence of a solution between sub and supersolutions. Akô showed in [2] the existence of a strong solution between a supersolution above a subsolution for F Hölder continuous. For functions F, which are differentiable, Amann ([3]), respectively Amann and Crandall ([5]) show the existence of a maximal (strong) solution between a supersolution above a subsolution, which are both in $C^{2,\alpha}(\bar{\Omega})$, respectively $W^{2,q}(\Omega)$. Deuel and Hess ([17]) have shown that when there is a supersolution above a subsolution, both in $W^{1,q}(\Omega)$, there exists a $W_0^{1,q}(\Omega)$-solution in between. They did not assume that F is differentiable (or of bounded variation) and hence they could not use a monotone iteration procedure. The existence of a maximal weak solution is shown in [16].

For more general elliptic equations, results were obtained by Kura in [24]. For elliptic equations with discontinuous nonlinearities, see [11]. This last paper allows F to be a combination of a continuous and an increasing function. Using results in [6] we can make an implicit condition in [11] explicit.

A monotone iteration procedure can be used to show the existence of a (maximal) solution between classical sub- and a supersolution for systems like (1) when F is differentiable and $\frac{\partial}{\partial u_i} F_j \geq 0$ for $i \neq j$. Linear systems that satisfy the last property are called cooperative (see [20], [10], [28] or [15]. Existence between sub- and supersolutions for elliptic systems can be found in [30]. For systems with F Hölder-continuous McKenna and Walter in [26] show existence of a maximal and a minimal solution between sub/supersolutions in $C^2(\Omega) \cap C_0(\bar{\Omega})$. In the framework of viscosity solutions, existence of a maximal solution in between is obtained by Ishii and Koike in [21].

In this note we show that similar results as in [16] and [26] can be obtained for quasimonotone elliptic systems without F being differentiable or even continuous. We start with F that does not depend on ∇u and consider weak solutions in $C_0(\bar{\Omega})$ between a version of sup- and supersolutions in $C(\bar{\Omega})$. This case will not use Zorn's lemma. For F which depends on the diagonal of ∇u we will use (sup/super)solutions in $W_0^{1,2}(\Omega)$ (respectively $W^{1,2}(\Omega)$). Even without dependence of ∇u, neither version of sub- (super) solution is included in the other.

2. C-solutions. Let Ω be a bounded subset of $\mathbb{R}^m$, such that all points of $\partial\Omega$ are regular with respect to the Laplacian (see [19]). We assume that F satisfies the following condition (We will skip the arrow in our notation).

Condition A.

$$F(u)(x) = \begin{pmatrix} f_1(x, u(x), u_1(x)) \\ \vdots \\ f_n(x, u(x), u_n(x)) \end{pmatrix},$$

with $f_j : \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that for all $j = 1, \dots, n$:

1) $(x, u) \mapsto f_j(x, u, \tau)$ is measurable for all $\tau \in \mathbb{R}$,
2) $\tau \mapsto f_j(x, u, \tau)$ is continuous for all $u \in \mathbb{R}^n$ and almost all $x \in \overline{\Omega}$,
3) $u \mapsto f_j(x, u, \tau)$ is non decreasing for all $(x, \tau) \in \overline{\Omega} \times \mathbb{R}$,
4) $|f_j(x, u, \tau)| \le g(u, \tau)$ for some $g \in C(\mathbb{R}^n \times \mathbb{R})$.

Remarks. 1) Note that $f_j(x, u, \tau)$ is not uniquely defined by $F(u)$: the variable u_j appears both in u and τ. A similar construction appears in [11]. For later use we define $\tilde{F}$ on pairs of vector valued functions by

$$\tilde{F}_j(u, v)(x) = f_j(x, u(x), v_j(x)). \tag{2}$$

2) Condition A.3) shows that for all $i \neq j$ and $x \in \overline{\Omega}$, the functions

$$u_i \mapsto f_j\left(x, \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, , u_j\right)$$

are non decreasing, which generalizes the notion of cooperativity for differentiable F. Such F are called quasimonotone.

3) If $f_j(x, u, \tau)$ does not depend on u then f_j is a Carathéodory function in (x, τ). For $f_j(x, u, \tau)$ that depends on u, condition A.1) and A.2) show that f_j is a Carathéodory function in $((x, u), \tau)$. What we need is for F to map measurable functions to measurable functions. For such F, the f_j are called superpositionally measurable (in short: supmeasurable). The measurability of f is, in general, not sufficient for its supmeasurability; see page 14 of [6]. It is sufficient that $(x, \tau) \mapsto f_j(x, u(x), \tau)$ is Carathéodory when $x \mapsto u(x)$ is measurable (or continuous). This is indeed true with the assumptions above. Since for all $x \in \overline{\Omega}$, $j \in \{1, \dots, n\}$, $u_k \in \mathbb{R}$ with $k \neq 1$ and $\tau \in \mathbb{R}$, the function

$$u_1 \mapsto f_j\left(x, \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \tau\right)$$

is non decreasing, Theorem 1.9 of [6] shows the following. If $u_1(\cdot)$ is measurable, then

$$(x, u_2, \dots, u_n) \mapsto f_j\left(x, \begin{pmatrix} u_1(x) \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \tau\right)$$

is measurable. Repeating the argument with the last function for u_2; etc., one shows the claim. Assuming that $u(\cdot)$ is continuous instead of measurable, does not improve the argument above.

Notations. In order to keep the notation simple we will use the following. The inequality $u \leq v$ for vector valued functions u and v on $\overline{\Omega}$ means $u_i(x) \leq v_i(x)$ for all $i \in \{1, 2, \dots, n\}$ and $x \in \overline{\Omega}$; $u < v$ means $u \leq v$ and $u \neq v$. Operators A on scalar valued functions will be extended to vector valued functions by

$$A \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} A(u_1) \\ \vdots \\ A(u_n) \end{pmatrix}.$$

The product of two vector valued functions φ and ψ will be defined by

$$\varphi\psi = \begin{pmatrix} \varphi_1\psi_1 \\ \vdots \\ \varphi_n\psi_n \end{pmatrix}.$$

Definition. (See [16] or [13]) The function u is called a C-subsolution if

1) $u \in (C(\bar{\Omega}))^n$,
2) $\int_\Omega (-u\Delta\varphi - F(u)\varphi)dx \leq 0$ for all $\varphi \in (\mathcal{D}^+(\Omega))^n$,
3) $u \leq 0$ on $\partial\Omega$,

where $\mathcal{D}^+(\Omega) = \{\varphi \in C_0^\infty(\Omega) : \varphi \geq 0\}$.

The function u is called a C-supersolution if the inequality signs are reversed. A function that is both a C-subsolution and a C-supersolution will be called a C-solution.

In this section subsolution will mean C-subsolution, etc. The maximum u_* of two vector functions u_α, u_β is defined by

$$u_{*,i}(x) = \max(u_{\alpha,i}(x), u_{\beta,i}(x))$$

with $i \in \{1, \dots, n\}$ and $x \in \overline{\Omega}$.

Theorem 1. *Let F satisfy condition A. Then the maximum of two C-subsolutions is again a C-subsolution.*

Proof. Let u_α and u_β be subsolutions and set $u_* = \max(u_\alpha, u_\beta)$. From the Kato inequality ([22]) it follows that for subsolutions $u_\alpha, u_\beta \in (W^{2,1}(\Omega))^n$ and $\varphi \in (\mathcal{D}^+(\Omega))^n$, we have

$$-\int_\Omega u_*\Delta\varphi\, dx == -\tfrac{1}{2}\int_\Omega (u_\alpha + u_\beta + |u_\alpha - u_\beta|)\Delta\varphi\, dx \tag{3}$$

$$\leq -\tfrac{1}{2}\int_\Omega (\Delta u_\alpha + \Delta u_\beta + \operatorname{sgn}(u_\alpha - u_\beta)(\Delta u_\alpha - \Delta u_\beta))\,\varphi\, dx$$

$$= \begin{pmatrix} -\int_{\Omega_1^>} \Delta u_{\alpha,1}\varphi_1 dx - \int_{\Omega_1^=} \frac{1}{2}(\Delta u_{\alpha,1} + \Delta u_{\beta,1})\varphi_1 dx - \int_{\Omega_1^<} \Delta u_{\beta,1}\varphi_1 dx \\ \vdots \\ -\int_{\Omega_n^>} \Delta u_{\alpha,n}\varphi_n dx - \int_{\Omega_n^=} \frac{1}{2}(\Delta u_{\alpha,n} + \Delta u_{\beta,n})\varphi_n dx - \int_{\Omega_n^<} \Delta u_{\beta,n}\varphi_n dx \end{pmatrix},$$

where $\Omega_i^\sigma = \{x \in \Omega; (u_{\alpha,i}(x))\sigma(u_{\beta,i}(x))\}$ with $i \in \{1, \dots, n\}$ and $\sigma \in \{<, =, >\}$. Let $\widetilde{F}$ be defined in (2). Using the facts that u_α and u_β are subsolutions and that F satisfies condition A.2) we obtain:

$$\begin{aligned}
&-\int_{\Omega_i^>} \Delta u_{\alpha,i}\varphi_i dx - \int_{\Omega_i^=} \tfrac{1}{2}(\Delta u_{\alpha,i} + \Delta u_{\beta,i})\varphi_i dx - \int_{\Omega_i^<} \Delta u_{\beta,i}\varphi_i dx \\
&\le \int_{\Omega_i^>} F_i(u_\alpha)\varphi_i dx + \int_{\Omega_i^=} \tfrac{1}{2}(F_i(u_\alpha) + F_i(u_\beta))\varphi_i dx + \int_{\Omega_i^<} F_i(u_\beta)\varphi_i dx \\
&= \int_{\Omega_i^>} \widetilde{F}_i(u_\alpha, u_*)\varphi_i dx + \int_{\Omega_i^=} \tfrac{1}{2}(\widetilde{F}_i(u_\alpha, u_*) + \widetilde{F}_i(u_\beta, u_*))\varphi_i dx \\
&+ \int_{\Omega_i^<} \widetilde{F}_i(u_\beta, u_*)\varphi_i \, dx \le \int_\Omega F_i(u_*)\varphi_i dx.
\end{aligned} \tag{4}$$

From (3) and (4) it follows that

$$-\int_\Omega u_* \, (\Delta\varphi) dx \le \int_\Omega F(u_*)\varphi \, dx.$$

For subsolutions $u_\alpha, u_\beta \in (C(\bar{\Omega}))^n$ we have to use the mollifier J_ϵ as in [16]. One shows that for $\epsilon < \delta$

$$-\Delta(J_\epsilon * u_\alpha) \le J_\epsilon * F(u_\alpha) \quad \text{in } \Omega_\delta, \tag{5}$$

where $\Omega_\delta = \{x \in \Omega; d(x, \partial\Omega) > \delta\}$. Indeed, let $\varphi \in (\mathcal{D}^+(\Omega_\delta))^n$. It follows that $J_\epsilon * \varphi \in (\mathcal{D}^+(\Omega))^n$ for $\epsilon < \delta$ and

$$\begin{aligned}
0 &\ge \int_\Omega (-u_\alpha \Delta(J_\epsilon * \varphi) - F(u_\alpha)(J_\epsilon * \varphi)) \, dx \\
&= \int_\Omega (u_\alpha(J_\epsilon * -\Delta\varphi) - F(u_\alpha)(J_\epsilon * \varphi)) \, dx \\
&= \int_{\mathbb{R}^n} ((J_\epsilon * u_\alpha)(-\Delta\varphi) - (J_\epsilon * F(u_\alpha))\varphi) dx \\
&= \int_{\Omega_\delta} (-\Delta(J_\epsilon * u_\alpha) - J_\epsilon * F(u_\alpha))\varphi \, dx.
\end{aligned} \tag{6}$$

Since $(-\Delta(J_\epsilon * u_\alpha) - J_\epsilon * F(u_\alpha))$ is continuous and since (6) holds for all $\varphi \in (\mathcal{D}^+(\Omega_\delta))^n$, we get (5). The mollifier J_ϵ is nonnegative and has support $\{x; |x| \le \epsilon\}$. Set $v^\epsilon = J_\epsilon * v$ and define F^ϵ componentwise by

$$F_i^\epsilon(x) = \begin{cases} J_\epsilon * F_i(u_\alpha(x)) & \text{if } u_{\alpha,i}^\epsilon(x) > u_{\beta,i}^\epsilon(x), \\ \tfrac{1}{2} J_\epsilon * (F_i(u_\alpha(x)) + F_i(u_\beta(x))) & \text{if } u_{\alpha,i}^\epsilon(x) = u_{\beta,i}^\epsilon(x), \\ J_\epsilon * F_i(u_\beta(x)) & \text{if } u_{\alpha,i}^\epsilon(x) < u_{\beta,i}^\epsilon(x). \end{cases}$$

One shows as in (3) that for $\varphi \in (\mathcal{D}^+(\Omega_\delta))^n$ and $\epsilon < \delta$:

$$-\int_\Omega \max(u_\alpha^\epsilon, u_\beta^\epsilon)\ \Delta\varphi\, dx \le \int_\Omega F^\epsilon\ \varphi\, dx.$$

Since u_α and u_β are continuous, one has $\int_\Omega \max(u_\alpha^\epsilon, u_\beta^\epsilon)\ \Delta\varphi\, dx \to \int_\Omega u_*\Delta\varphi\, dx$ when $\epsilon \downarrow 0$. It remains to show that

$$\overline{\lim_{\epsilon\downarrow 0}} \int_\Omega F^\epsilon\ \varphi\, dx \le \int_\Omega F(u_*)\, \varphi\, dx.$$

Note that the condition A.2) shows that $F_i(u_\alpha) \le \widetilde{F}_i(u_*, u_\alpha)$ and hence $J_\epsilon * F_i(u_\alpha) \le J_\epsilon * \widetilde{F}_i(u_*, u_\alpha)$. For the i^{th} component one shows that

$$\begin{aligned}
&\int_\Omega F_i^\epsilon\ \varphi_i\, dx - \int_\Omega F_i(u_*)\ \varphi_i\, dx \\
&\le \int_{[u_{\alpha,i}^\epsilon > u_{\beta,i}^\epsilon]} (J_\epsilon * \widetilde{F}_i(u_*, u_\alpha) - F_i(u_*))\varphi_i\, dx \\
&+ \int_{[u_{\alpha,i}^\epsilon < u_{\beta,i}^\epsilon]} (J_\epsilon * \widetilde{F}_i(u_*, u_\beta) - F_i(u_*))\varphi_i\, dx \\
&+ \int_{[u_{\alpha,i}^\epsilon = u_{\beta,i}^\epsilon]} (\tfrac{1}{2}J_\epsilon * \widetilde{F}_i(u_*, u_\alpha) + \tfrac{1}{2}J_\epsilon * \widetilde{F}_i(u_*, u_\beta) - F_i(u_*))\varphi_i\, dx.
\end{aligned} \tag{7}$$

Notice that

$$\begin{aligned}
&\int_{[u_{\alpha,i}^\epsilon > u_{\beta,i}^\epsilon]} (J_\epsilon * \widetilde{F}_i(u_*, u_\alpha) - F_i(u_*))\varphi_i\, dx \\
&= \int_{[u_{\alpha,i}^\epsilon > u_{\beta,i}^\epsilon]} (\widetilde{F}_i(u_*, u_\alpha) - F_i(u_*))\varphi_i\, dx - \int_{[u_{\alpha,i}^\epsilon > u_{\beta,i}^\epsilon]} (\mathbf{1} - J_\epsilon *)\widetilde{F}_i(u_*, u_\alpha)\varphi_i\, dx \\
&\le \int_\Omega \mathbf{1}_{[u_{\alpha,i}^\epsilon > u_{\beta,i}^\epsilon]}\mathbf{1}_{[u_{\alpha,i} < u_{\beta,i}]}(\widetilde{F}_i(u_*, u_\alpha) - F_i(u_*))\varphi_i dx + \int_\Omega \left|(\mathbf{1} - J_\epsilon *)\widetilde{F}_i(u_*, u_\alpha)\right|\varphi_i dx
\end{aligned}$$

has zero limit. The first term in the last expression goes to zero by the Lebesgue dominated convergence theorem. The second term needs more care. Note that $\widetilde{F}_i(u_*, u_\alpha) \in L^\infty(\Omega)$. For $g \in C(\bar{\Omega})$ it follows that $(\mathbf{1} - J_\epsilon *)g \to 0$ uniformly as $\epsilon \downarrow 0$ and hence

$$\lim_{\epsilon\downarrow 0} \int_\Omega |(\mathbf{1} - J_\epsilon *)g|\, \varphi_i\, dx = 0. \tag{8}$$

Taking into account that $C(\bar{\Omega})$ is dense in $L^1(\Omega)$ we obtain (8) for $h \in L^p(\Omega) \subset L^1(\Omega)$. Indeed we have that

$$\begin{aligned}
&\int_\Omega |(\mathbf{1} - J_\epsilon *)h|\, \varphi_i\, dx \\
&\le \int_\Omega |h - g|\, \varphi_i dx + \int_\Omega |(\mathbf{1} - J_\epsilon *)g|\, \varphi_i dx + \int_\Omega |J_\epsilon * (g - h)|\, \varphi_i dx
\end{aligned}$$

$$\leq \|h-g\|_1\,\|\varphi_i\|_\infty + \int_\Omega |(1-J_\epsilon*)g|\,\varphi_i dx + \|J_\epsilon\|_1\,\|h-g\|_1\,\|\varphi_i\|_\infty$$
$$=2\,\|h-g\|_1\,\|\varphi_i\|_\infty + \int_\Omega |(1-J_\epsilon*)g|\,\varphi_i dx.$$

Similar estimates hold for the other terms in the right hand side of (7). Hence

$$-\int_\Omega u_*\,\Delta\varphi dx \leq \overline{\lim_{\epsilon\downarrow 0}} \int_\Omega F_\epsilon\,\varphi\,dx \leq \int_\Omega F(u_*)\,\varphi\,dx.$$

Proposition 2. *Suppose $n=1$ and $F(u)(x)=f(x,u)$ such that f is Carathéodory and satisfies condition A.4. Let u_α be a C-subsolution and let u_β be a C-supersolution with $u_\alpha \leq u_\beta$. Then there exists a C-solution in between.*

Proof. The proof is almost standard. For the sake of completeness we will include it here.

1) Define the function $\Phi : C(\bar\Omega) \to C(\bar\Omega)$ by

$$\Phi(u)(x) = \begin{cases} u_\beta(x) & \text{if } u_\beta(x) < u(x), \\ u(x) & \text{if } u_\alpha(x) \leq u(x) \leq u_\beta(x), \\ u_\alpha(x) & \text{if } u(x) < u_\alpha(x). \end{cases}$$

Hence Φ is a bounded and continuous function on $C(\bar\Omega)$. Since $F \circ \Phi$ is uniformly bounded and Carathéodory, it is continuous from $C(\bar\Omega)$ to $L^p(\Omega)$ for every $p \in [1,\infty)$ (see Theorem 3.7 of [6]).

2) Let B denote the inverse of $-\Delta$ with zero Dirichlet boundary condition. Then $B : L^p(\Omega) \to C(\bar\Omega)$ can be written as follows: $B(f) = w - h$, where $w = \Gamma * f$ with Γ the newtonian potential and h is the harmonic function on Ω that satisfies $h = w$ on $\partial\Omega$. Since $\Gamma* : L^p(\Omega) \to W^{2,p}(\Omega) \hookrightarrow C(\bar\Omega)$ is compact for p large enough and since $w \mapsto h$ from $C(\bar\Omega)$ in $C(\bar\Omega)$ is continuous, one finds that B is compact. Hence $B \circ F \circ \Phi : C(\bar\Omega) \to C(\bar\Omega)$ is compact and bounded. The Schauder fixed point theorem gives a solution of $u = BF(\Phi(u))$ in $C(\bar\Omega)$, and hence a solution of

$$-\Delta u = F \circ \Phi(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{9}$$

3) We have to show that if u solves (9), then u solves (1). Let us denote with u a solution of (9). Suppose that the set $\Omega^\star = \{x \in \Omega : u_\beta(x) < u(x)\}$ is nonempty. Since u (u_β) is a (super) solution, it follows that

$$\int_{\Omega^\star} (u-u_\beta)(-\Delta\varphi)dx \leq \int_{\Omega^\star} (F(\Phi(u)) - F(u_\beta))\varphi\,dx \leq 0 \quad \text{for any } \varphi \in \mathcal{D}^+(\Omega^\star).$$

Hence $u - u_\beta$ is subharmonic in $\Omega^\star$ and $u - u_\beta \leq 0$ on $\partial\Omega^\star$. An application of the maximum principle gives $u - u_\beta \leq 0$ in $\Omega^\star$, which is a contradiction. □

Let $\mathcal{S} \in C(\bar\Omega)^n$ denote the set of C-subsolutions $\underline{u}$ with $u_\alpha \leq \underline{u} \leq u_\beta$. The supremum $u_\star$ of the functions in $\mathcal{S}$ is defined as follows:

$$(u_\star)_j(x) = \sup\left\{\underline{u}_j(x) : \underline{u} \in \mathcal{S}\right\}. \tag{10}$$

Theorem 3. *Suppose condition* A *holds. Let u_α be a C-subsolution and let u_β be a C-supersolution with $u_\alpha \le u_\beta$. Then the supremum $u_\star$ of all C-subsolutions between u_α and u_β is a C-solution.*

As an immediate consequence we get:

Corollary 4. *Suppose condition* A *holds and that u_α is a C-subsolution, u_β is a C-supersolution with $u_\alpha \le u_\beta$. Then every C-solution u with $u_\alpha \le u \le u_\beta$ satisfies $u \le u_\star$.*

Proof. **1)** Since $u_\alpha \in \mathcal{S}$, the set $\mathcal{S}$ is nonempty. By its definition, $\mathcal{S}$ is bounded from above. Hence $u_\star$ is well defined.

2) We define $\mathcal{S}_0 = \mathcal{S} \cap C_0(\bar{\Omega})^n$. Then we have

$$(u_\star)_j(x) = \sup\left\{\underline{u}_j(x) : \underline{u} \in \mathcal{S}_0\right\}.$$

Indeed, let $\underline{u} \in \mathcal{S}$, then $\underline{u}$ is a subsolution of

$$-\Delta u = \widetilde{F}(\underline{u}, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \tag{11}$$

Since condition A.3) shows that $\widetilde{F}(\underline{u}, u_\beta) \le F(u_\beta)$, one finds that u_β is a supersolution of (11). System (11) is not coupled and every equation satisfies the conditions of Theorem 2. Hence there exists a solution $u_o \in C_0(\bar{\Omega})^n$ with $\underline{u} \le u_o \le u_\beta$ of (11). Since $\widetilde{F}(\underline{u}, u_o) \le F(u_o)$, the function u_o is a subsolution of (1).

3) We may assume that

$$(u_\star)_j(x) = \sup\left\{\underline{u}_j(x) : \underline{u} \in \mathcal{S}_1\right\},$$

where $\mathcal{S}_1 = \left\{u \in \mathcal{S}_0 : u \text{ solves (11) for some } \underline{u} \in \mathcal{S}\right\}$. Since

$$\widetilde{F}(u_\alpha, u) \le \widetilde{F}(\underline{u}, u) \le \widetilde{F}(u_\beta, u)$$

and

$$|\widetilde{F}_j(u_\sigma, u)| \le g(u_\sigma, u_j) \le M \quad \text{for } \sigma \in \{\alpha, \beta\},\ 1 \le j \le n,$$

$$\left(\text{here } M = \max_{\substack{j=1,\dots,n \\ \sigma=\alpha,\beta\ x\in\overline{\Omega}}} \{g(u_\sigma, \tau);\ \min u_{\alpha,j}(x) \le \tau \le \max u_{\beta,j}(x)\}\right)$$

hold, one finds that for all $p \in (1, \infty)$ and any compact $\Omega' \subset \Omega$ we have

$$\|u\|_{W^{2,p}(\Omega')} \le c_p(\Omega')M \quad \text{for all } u \in \mathcal{S}_1. \tag{12}$$

Moreover, if v is the solution of

$$-\Delta v = 1 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega, \tag{13}$$

then by the maximum principle

$$|u_j| \le Mv \quad \text{for all } u \in \mathcal{S}_1. \tag{14}$$

The estimates (12) and (14) show that $\mathcal{S}_1$ is an equicontinuous family in $C_0(\bar{\Omega})^n$ and hence that $u_\star \in C_0(\bar{\Omega})^n$ holds.

4) Let $\Omega_q = \{x^k : k \in \mathbb{N}\}$ be a countable dense subset of Ω, and let $\{u^{j,k,l}\}_{l=1}^{\infty} \subset \mathcal{S}_1$ be such that $u_j^{j,k,l}(x^k) \to u_{\star,j}(x^k)$ as $l \to \infty$. It follows from Theorem 1 that $\underline{u}^{k,l}$, defined by

$$\underline{u}^{k,l} = \max u^{j,k,\lambda}, \quad 1 \le j \le n,\ 0 \le \lambda \le l$$

is a subsolution. Moreover we have $\underline{u}^{k,1} \le \underline{u}^{k,2} \le \dots \le u_\beta$. By the previous steps we know that there exist

$$\begin{aligned} &u^{k,1} \in \mathcal{S}_1 \quad \text{with } \underline{u}^{k,1} \le u^{k,1} \le u_\beta, \\ &u^{k,2} \in \mathcal{S}_1 \quad \text{with } \max(u^{k,1}, \underline{u}^{k,2}) \le u^{k,2} \le u_\beta, \\ &u^{k,3} \in \mathcal{S}_1 \quad \text{with } \max(u^{k,2}, \underline{u}^{k,3}) \le u^{k,3} \le u_\beta; \quad \text{etc.} \end{aligned}$$

Hence for every k there exists a sequence $\{u^{k,l}\}_{l=1}^{\infty} \subset \mathcal{S}_1$ with $u^{k,1} \le u^{k,2} \le \dots$ and such that $\lim_{l\to\infty} u^{k,l}(x^k) = u_\star(x^k)$. Again it follows from Theorem 1 that $\underline{u}^l$, defined by

$$\underline{u}^l = \max_{0\le k\le l} u^{k,l},$$

is a subsolution and $\underline{u}^{l_1} = \max_{0\le k\le l_1} u^{k,l_1} \le \max_{0\le k\le l_1} u^{k,l_2} \le \underline{u}^{l_2}$ for $l_1 \le l_2$. Again, by the previous steps there exist

$$\begin{aligned} &u^1 \in \mathcal{S}_1 \quad \text{with } \underline{u}^1 \le u^1 \le u_\beta, \\ &u^2 \in \mathcal{S}_1 \quad \text{with } \max(u^1, \underline{u}^2) \le u^2 \le u_\beta, \\ &u^3 \in \mathcal{S}_1 \quad \text{with } \max(u^2, \underline{u}^3) \le u^3 \le u_\beta; \quad \text{etc.} \end{aligned}$$

Hence there exists $\{u^l\}_{l=1}^{\infty} \subset \mathcal{S}_1$ with $u^1 \le u^2 \le \dots$ and $\lim_{l\to\infty} u^l(x) = u_\star(x)$ for all $x \in \Omega_q$. Since $\mathcal{S}_1$ is equicontinuous on $\overline{\Omega}$, we find

$$\lim_{l\to\infty} u^l(x) = u_\star(x) \quad \text{for all } x \in \overline{\Omega}.$$

5) It remains to show that $u_\star$ satisfies the equality related to 2) in the definition of subsolution. Since $u_\star$ and all u^l are equicontinuous,

$$\lim_{l\mapsto\infty} \int_\Omega u^l \Delta\varphi\, dx = \int_\Omega u_\star \Delta\varphi\, dx$$

for all $\varphi \in \mathcal{D}^+(\Omega)^n$. Now from $F(u^l) \le \widetilde{F}(u_\star, u^l)$ and $\lim_{l\to\infty} \widetilde{F}(u_\star, u^l) = \widetilde{F}(u_\star, u_\star) = F(u_\star)$ it follows that

$$\int_\Omega (-u_\star \Delta\varphi - F(u_\star)\varphi)\, dx \le \overline{\lim_{l\to\infty}} \int_\Omega (-u^l \Delta\varphi - F(u^l)\varphi)\, dx \le 0$$

for all $\varphi \in \mathcal{D}^+(\Omega)^n$. If the last inequality is strict for some $\varphi \in \mathcal{D}^+(\Omega)^n$ then $u_\star$ is a subsolution and not a solution. By the second step there exists a solution u of (11), with $\underline{u}$ replaced by $u_\star$, and $u_\star < u \leq u_\beta$. Moreover, u is a subsolution of (1) with $u_\star < u$, a contradiction.

3. W-solutions. Let Ω be a general bounded domain in $\mathbb{R}^m$. Assume that F satisfies the following condition.

Condition B.

$$F(u)(x) = \begin{pmatrix} f_1(x, u(x), u_1(x), \nabla u_1(x)) \\ \vdots \\ f_n(x, u(x), u_n(x), \nabla u_n(x)) \end{pmatrix},$$

with $f_j : \overline{\Omega} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ such that for all $j = 1, \dots, n$:

1) $(x, u) \mapsto f_j(x, u, \tau, \sigma)$ is measurable for all $(\tau, \sigma) \in \mathbb{R} \times \mathbb{R}^m$,
2) $(\tau, \sigma) \mapsto f_j(x, u, \tau, \sigma)$ is continuous for all $u \in \mathbb{R}^n$ and almost all $x \in \overline{\Omega}$,
3) $u \mapsto f_j(x, u, \tau, \sigma)$ is non decreasing for all $(x, \tau, \sigma) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^m$.

We will use sub- and supersolutions as in the second part of [16].

Definition (See [16]). The function u is called a W-subsolution if

1) $u \in (W^{1,2}(\Omega))^n$,
2) $\int_\Omega (-u\Delta\varphi - F(u)\varphi)dx \leq 0$ for all $\varphi \in (\mathcal{D}^+(\Omega))^n$,
3) $u \leq 0$ on $\partial\Omega$ in the sense of Kinderlehrer and Stampacchia.

Remark. A function $u \in W^{1,2}(\Omega)$ satisfies $u \geq 0$ on $E \subset \Omega$ in the sense of Kinderlehrer and Stampacchia, if there is a sequence $u_n \in W^{1,\infty}(\Omega)$ $(\subset C(\bar{\Omega}))$ with $u_n(x) \geq 0$ for $x \in E$ and $\|u - u_n\|_{W^{1,2}(\Omega)} \to 0$ for $n \to \infty$. See [23] for a discussion of this notion of positivity.

As usual, a supersolution is defined by reversing the inequalities. A function that is both a W-sub- and a W-supersolution will be a W-solution. In this section a subsolution will be a W-subsolution.

Theorem 5. *Suppose condition B holds and let u_α, u_β respectively be a W-sub- and a W-supersolution with $u_\alpha \leq u_\beta$. Moreover, suppose that there are $K \in \mathbb{R}$ and $p > \frac{2n}{n+2}$ ($p > 1$ for $n = 1$) with $k \in L^p(\Omega)$, such that for all $j \in \{1, \dots, n\}$ and $(u, \tau) \in \mathbb{R}^n \times \mathbb{R}$, with $(u_\alpha(x), u_{\alpha,j}(x)) \leq (u, \tau) \leq (u_\beta(x), u_{\beta,j}(x))$ we have:*

$$\left| f_j(x, u, \tau, \sigma) \right| \leq k(x) + K\,|\sigma| \quad \textit{for all } x \in \overline{\Omega} \textit{ and } \sigma \in \mathbb{R}^m. \tag{15}$$

Then there exists a W-solution u_*, such that

$$u_\alpha \leq u \leq u_* \leq u_\beta$$

for every W-solution u with $u_\alpha \leq u \leq u_\beta$.

Remark. With additional regularity assumptions for the sub- and supersolution and for $\partial\Omega$ we may improve the growth of the gradient depending part. Instead of the linear growth in (15) it will be sufficient to assume quadratic growth:

$$\left|f_j(x, u, \tau, \sigma)\right| \leq k(x) + K\,|\sigma|^2 \quad \text{for all } x \in \overline{\Omega} \text{ and } \sigma \in \mathbb{R}^m.$$

We refer to [16, Remark 1, page 539].

The proof will start with similar steps as in the C-solution case. In order to obtain a maximal solution we will use Zorn's lemma.

1) *A smoother subsolution above a subsolution.*

Lemma 6. *Let the assumptions be as in the previous theorem. Then there exists a bound M such that for every W-subsolution u_γ, with $u_\alpha \leq u_\gamma \leq u_\beta$, there exists a W-subsolution u_γ^* such that*

i) $u_\alpha \leq u_\gamma \leq u_\gamma^* \leq u_\beta$,

ii) $u_\gamma^* \in W_0^{1,2}(\Omega) \cap W_{\text{loc}}^{2,p}(\Omega)$,

iii) $\left\|u_\gamma^*\right\|_{W^{1,2}(\Omega)} \leq M$.

Proof. We will fix u_γ and solve

$$\begin{cases} -\Delta u = \widetilde{F}(u_\gamma, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where $\widetilde{F}$ is defined like in (2):

$$\widetilde{F}_j(v, u)(x) = f_j(x, v(x), u_j(x), \nabla u_j(x)).$$

Notice that (16) is uncoupled. Hence we may consider the equation case and use a modification of the proof of Deuel and Hess that is also used in [16].

Similar as in [17] we use the truncation operator $T : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$ defined by

$$T(u) = \begin{cases} u_{\gamma,j} & \text{for } u < u_{\gamma,j}, \\ u & \text{for } u_{\gamma,j} \leq u \leq u_{\beta,j}, \\ u_{\beta,j} & \text{for } u_{\beta,j} < u. \end{cases}$$

A lemma in [17] shows that T is bounded and continuous. From (15) and a Sobolev imbedding it follows that

$$\begin{aligned} \left|\int_\Omega \widetilde{F}_j(u_\gamma, T(u))\varphi\, dx\right| &\leq \|k\|_{0,p}\, \|\varphi\|_{0,q} + Kc_1\, \|u\|_{1,2}\, \|\varphi\|_{0,2} \\ &\leq c_2\, \|\varphi\|_{1,2} + c_3\|u\|_{1,2}\, \|\varphi\|_{0,2} \leq c_2\, \|\varphi\|_{1,2} + \epsilon\|u\|_{1,2}^2 + \tfrac{1}{4}\epsilon^{-1}(c_3)^2\, \|\varphi\|_{0,2}^2 \end{aligned}$$

for all $u, \varphi \in W^{1,2}(\Omega)$. The operator $u \mapsto \widetilde{F}_j(u_\gamma, T(u))$ from $W^{1,2}(\Omega)$ to $L^p(\Omega)$ is bounded and continuous. We denote $\|u\|_{1,2} = \|u\|_{W^{1,2}(\Omega)}$ and so on.

Similarly, we may construct a penalty term π defined by

$$\pi(x,u) = \begin{cases} u - u_{\gamma,j}(x) & \text{for } u < u_{\gamma,j}(x), \\ 0 & \text{for } u_{\gamma,j}(x) \le u \le u_{\beta,j}(x), \\ u_{\beta,j}(x) - u & \text{for } u_{\beta,j} < u. \end{cases}$$

Then π is Carathéodory and for some c_4, which we may take to be depending on u_α and u_β (and not on u_γ), one finds

$$\int_\Omega \pi(\cdot, u) u \, dx \ge \tfrac{1}{2} \|u\|_{0,2}^2 - c_4 \tag{18}$$

for all $u \in W^{1,2}(\Omega)$.

By the Poincaré-Friedrichs inequality there exists c_5 such that

$$\|u\|_{1,2}^2 \le c_5 \int_\Omega |\nabla u|^2 \, dx \quad \text{for } u \in W_0^{1,2}(\Omega).$$

Now define the form $B(\cdot,\cdot)$ on $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, where $W_0^{1,2}(\Omega)$ is equipped with the norm $\|\cdot\|_{1,2}$, by

$$B(u,\varphi) = \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega \widetilde{F}_j(u_\gamma, T(u))\varphi \, dx + C \int_\Omega \pi(\cdot, u)\varphi \, dx,$$

with $C = \frac{1}{2} c_5 \, c_3^2$. Using the second estimate of (17) and (18), it follows

$$B(u,\cdot) : W_0^{1,2}(\Omega) \to (W^{1,2}(\Omega))^*$$

is bounded and coercive. Indeed we have

$$|B(u,\varphi)| \le (c_2 + (1 + c_3 + C)\, \|u\|_{1,2})\, \|\varphi\|_{1,2}, \tag{19}$$

and moreover

$$B(u,u) \ge \tfrac{1}{2} c_5 \, \|u\|_{1,2}^2 - c_2 \, \|u\|_{1,2} - C c_4, \tag{20}$$

which implies for $u \in W_0^{1,2}(\Omega)$, that

$$\frac{B(u,u)}{\|u\|_{1,2}} \to \infty \quad \text{as} \quad \|u\|_{1,2} \to \infty. \tag{21}$$

Moreover, if $u^k \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$, then

$$\begin{aligned} \liminf_{k\to\infty} B(u^k, u^k - v) &= \liminf_{k\to\infty} B(u^k, u^k) - \lim_{k\to\infty} B(u^k, v) \\ &\ge B(u,u) - \lim_{k\to\infty} B(u^k, v) = B(u, u - v). \end{aligned} \tag{22}$$

Because of (19), (21) and (22), one may use Theorem 2.7 of [25]. This theorem shows that there is a $u_o \in W_0^{1,2}(\Omega)$ with

$$B(u_o, \varphi) = 0 \quad \text{for all } \varphi \in W_0^{1,2}(\Omega), \tag{23}$$

which shows that there exists a solution for the truncated problem with the penalty term. Substituting φ with u_o in (23) and using (20), one finds that $\|u_o\|_{1,2} \le M$ for some constant $M = M(k(\cdot), K, u_\alpha, u_\beta, \Omega)$. In order to show that u_o satisfies $u_o \le u_{\beta,j}$, one can use the argument of [16] page 538. By this argument we know that $(u_o - u_{\beta,j})^+ \in W_0^{1,2}(\Omega)$ and hence we may apply the method of [17, p. 53]. Similarly we can prove that $u_o \ge u_{\alpha,j}$ and hence u_o is a solution of the j-th component of (16).

Since the above proof holds for every component, we obtain a W-solution of (16). Denote this solution by u_γ^*. Since $u_\gamma \le u_\gamma^*$, one finds $\widetilde{F}(u_\gamma, u_\gamma^*) \le \widetilde{F}(u_\gamma^*, u_\gamma^*) = F(u_\gamma^*)$ and hence, that u_γ^* is a W-subsolution of (1). □

Set

$$\mathcal{S}_W = \Big\{ u^* \in (W_0^{1,2}(\Omega))^n : \text{there is } W\text{-subsolution } u \text{ of (1) such that}$$
$$\text{i) } u_\alpha \le u \le u^* \le u_\beta,$$
$$\text{ii) } u^* \text{ is a } W\text{-solution of } \begin{pmatrix} -\Delta u^* = \widetilde{F}(u, u^*) \text{ in } \Omega \\ u^* = 0 \quad \text{on } \partial\Omega \end{pmatrix} \Big\}.$$

By the previous lemma, $\mathcal{S}_W$ is nonempty and every u^* satisfies $\|u^*\|_{W^{1,2}(\Omega)} \le M$. Moreover, standard regularity arguments (see Theorem 9.13 of [19]) show that $u^* \in (W^{2,p}(\Omega'))^n$ for any compact $\Omega' \subset \Omega$ and even

$$\left\| u^* \right\|_{W^{2,p}(\Omega')} \le C(M, \Omega'). \tag{24}$$

2) *Zorn's lemma; $\mathcal{S}_W$ has a maximal element.*

Let $\{u^i\}_{i \in I}$ be a completely ordered family of subsolutions in $\mathcal{S}_W$. Let $\{u^{i_k}\}_{k \in \mathbb{N}}$ be a sequence that is cofinal in this family with respect to the ordering $\le$. Since the unit ball in $W_0^{1,2}(\Omega)$, and hence in $(W_0^{1,2}(\Omega))^n$, is weakly sequentially compact, there is an ordered subsequence $\{u^{i_{k_\kappa}}\}_{\kappa \in \mathbb{N}}$ in $\{u^{i_k}\}_{k \in \mathbb{N}}$ that converges weakly to some $u \in (W_0^{1,2}(\Omega))^n$. Indeed, since $W_0^{1,2}(\Omega)$ is compactly imbedded in $L^2(\Omega)$ and since every u^i satisfies (24), we may take subsequences denoted again by u^k such that:

i) $u^k \to u$ strongly in $(L^2(\Omega))^n$,

ii) $u^k \to u$ strongly in $(W_{\text{loc}}^{1,2}(\Omega))^n$,

iii) $u^k \to u$ pointwise almost everywhere in Ω.

Since $u_j^k(x)$ is increasing for every component j and because of Condition B.3, it follows that for $\varphi_j \in \mathcal{D}^+(\Omega)$, we obtain

$$\int_\Omega f_j(x, u^k(x), u_j^k(x), \nabla u_j^k(x))\varphi_j(x)\,dx \le \int_\Omega f_j(x, u(x), u_j^k(x), \nabla u_j^k(x))\varphi_j(x)\,dx.$$

Since $(x, (\tau, \sigma)) \to f_j(x, u(x), \tau, \sigma)$ is Carathéodory and φ_j has compact support in Ω, it follows that $u_j^k \to u_j$, $\nabla u_j^k \to \nabla u_j$ strongly in $W^{1,2}$ (support φ_j) as $k \to \infty$ and

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega & f_j(x, u(x), u_j^k(x), \nabla u_j^k(x))\varphi_j(x)\,dx \\ &= \int_\Omega f_j(x, u(x), u_j(x), \nabla u_j(x))\varphi_j(x)\,dx.\end{aligned}$$

Since

$$\begin{aligned}\int_\Omega u_j^k(x)\ \Delta\varphi_j(x)\,dx = -\int_\Omega \nabla u_j^k(x)\ \nabla\varphi_j(x)\,dx &\to -\int_\Omega \nabla u_j(x)\ \nabla\varphi_j(x)\,dx \\ &= \int_\Omega u_j(x)\ \Delta\varphi_j(x)\,dx \quad \text{for } \varphi_j \in \mathcal{D}^+(\Omega),\end{aligned}$$

we obtain that the limit u is a subsolution. By the previous lemma there exists $u^\star \in \mathcal{S}_W$ with $u_\alpha \le u \le u^\star \le u_\beta$. Hence the completely ordered subset has an upper bound in $\mathcal{S}_W$. By Zorn's lemma, $\mathcal{S}_W$ has at least one maximal element in the sense of partial ordering.

3) *The maximum of two subsolutions in $\mathcal{S}_W$ is a W-subsolution.*

This result follows straightforwardly from (3) and (4). Since elements of $\mathcal{S}_W$ are in $(W^{2,p}_{\text{loc}}(\Omega))^n$, we do not need a mollifier.

As a consequence, we find that $\mathcal{S}_W$ has exactly one maximal element. If there were two maximal elements, the pointwise maximum would be a subsolution and Lemma 6 would yield an element above, that is, there exists $u^\star \in \mathcal{S}_W$ such that every $u \in \mathcal{S}_W$ satisfies $u \le u^\star$.

4) *A solution for the original problem.*

If $u^\star$ is not a solution of (1), then by the lemma there is a solution $u_\gamma^\star \in \mathcal{S}_W$ of (16), with $u_\gamma = u^\star$. Then, either $u^\star = u_\gamma^\star$ and $u^\star$ is a solution, or $u^\star < u_\gamma^\star$ and $u^\star$ is not maximal. Hence $u^\star$ is a solution.

If u is a solution of (1) with $u_\alpha \le u \le u_\beta$, then u is a solution of (23) with $u_\gamma = u$, and hence $u \in \mathcal{S}_W$. In other words, every solution between u_α and u_β lies in $\mathcal{S}_W$. Hence $u^\star$ is the maximal solution.

Note added in proof. We recently learned that there are related results by Carl, Heikkilä and Kumpulainen in [12].

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