Optimal Control of Capital Injections by Reinsurance and Investments

University of Cologne

Julia Eisenberg

1 Introduction

1.1 The classical risk model

The **classical risk model** is a surplus process $X=\{X_t\}$ of the form

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i$$

where

- x: initial capital
- c: premium rate
- Z_i: iid claims
- N_t: Poisson process with intensity λ, independent of Z_i
- T_i: claims arrival times.

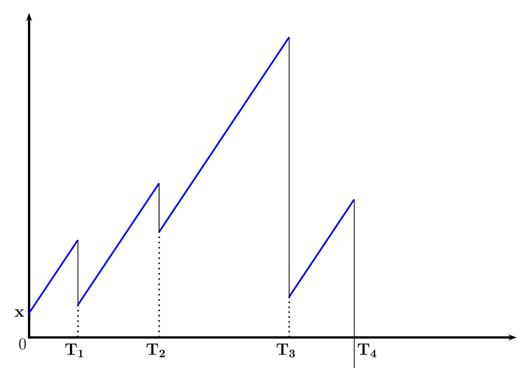

Figure 1: The process $\{X_t\}$.

1.2 Reinsurance

Reinsurer $\cong$ "an insurer assuming the risk of another under contract"

Problem of the first insurer: How much reinsurance should I buy?

The decision depends on

- the retention level $b \in [0,\tilde{b}]$;
- the self-insurance function $r(z,b)$. We assume, that r is continuous and increasing in both variables;
- the premium rate function $c(b)$. $c(b)$ denotes the premium remaining to the first insurer, if the retention level b was chosen.

1.3 Investments

In addition to the classical setup we allow the insurer to invest into a risky asset, modeled as a Black-Scholes model

$$\mathrm{d}Q_t = mQ_t\,\mathrm{d}t + \sigma Q_t\,\mathrm{d}W_t \quad\Leftrightarrow\quad Q_t = \exp\{(m-\tfrac{\sigma^2}{2})t + \sigma W_t\}$$

where $\{W_t\}$ is a standard Brownian motion and $m,\sigma>0$.
We are not interested in the asset price, but in the asset return!
The return of such a process is then the stochastic process $\{Q'_t\}$ given by the stochastic differential equation

$$\mathrm{d}Q'_t = \frac{\mathrm{d}Q_t}{Q_t} = m\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

1.4 Surplus process with reinsurance, investments and capital injections

The surplus process under reinsurance, investments and with capital injections fulfils

$$X_t^{A,B,Y} = x + \int_0^t c(b_s)\,\mathrm{d}s - \sum_{i=1}^{N_t} r(Z_i, b_{T_i-}) + m\int_0^t a_s\,\mathrm{d}s + \sigma\int_0^t a_s\,\mathrm{d}W_s + Y_t^{A,B}$$

where

- $A=\{a_t\}$ and $B=\{b_t\}$, $a_t\in\mathbb{R}$, $b_t\in[0,\tilde{b}]$ investment und reinsurance strategies respectively;
- $Y^{A,B}=\{Y_t^{A,B}\}$ accumulated capital injections, which are needed to prevent that the surplus process becomes negative.

We assume, that $\sum_{i=1}^{N_t} Z_i$ and W_t are independent and consider the filtration $\{\mathcal{F}_t\}$, generated by the two dimensional process $(\sum_{i=1}^{N_t} Z_i, W_t)$. We call a strategy (A,B) admissible, if A and B are cadlag and $\{F_t\}$ measurable.

2 Return and Value Functions

As the risk measure connected to some admissible strategy pair (A,B) we choose the value of expected discounted capital injections with some discounting factor $\delta\geq 0$.

$$\underbrace{V(x)}_{\text{value funktion}} = \inf_{(A,B)} \underbrace{V^{A,B}(x)}_{\text{return funktion}} = \inf_{(A,B)} \mathbb{E}_x\Big[\int_0^\infty e^{-\delta t}\,\mathrm{d}Y_t^{A,B}\Big]$$

It is clear, that it makes sense to inject capital only if the surplus becomes negative; and directly after the capital injection the surplus is equal to zero.

2.1 Properties of the value function

Lemma 1
The value function $V(x)$ has the following properties

1. $V(x)$ is decreasing with $\lim_{x\to\infty} V(x)=0$;
2. $V(x)$ is Lipschitz continuous on $[0,\infty)$ with $|V(x)-V(y)|\leq|x-y|$;
3. Let $c(b)$ be concave in b and $r(z,b)=zb$, $b\in[0,1]$. Then $V(x)$ is convex.

The **Hamilton–Jacobi–Bellman equation** of the considered problem is

$$0 = \inf_{\substack{a\in\mathbb{R}\\ b\in[0,\tilde{b}]}} \tfrac{\sigma^2a^2}{2}V''(x) + \lambda\int_0^\infty V(x-r(z,b))\,\mathrm{d}G(z) + (c(b)+am)V'(x) - (\delta+\lambda)V(x) \quad (1)$$

Assume that $V(x)$ is twice continuously differentiable. Minimimising with respect to a yields:

$$0 = \inf_{b\in[0,\tilde{b}]}\Big\{\lambda\int_0^\infty V(x-r(z,b))\,\mathrm{d}G(z) + c(b)V'(x)\Big\} - \tfrac{m^2V'(x)^2}{2\sigma^2V''(x)} - (\delta+\lambda)V(x)$$

In particular: If $V(x)$ is twice continuously differentiable, the optimal investment strategy is given by

$$a^*(x) = -\frac{mV'(x)}{\sigma^2V''(x)}\,.$$

Problem: We do not know, whether V is twice continuously differentiable!

Thus we have to use the concept of viscosity solutions.

2.2 Viscosity solutions

We say that a continuous function $\underline{u}:[0,\infty)\to\mathbb{R}_+$ is a **viscosity subsolution** to (1) at $x\in(0,\infty)$ if any twice continuously differentiable function $\psi:(0,\infty)\to\mathbb{R}$ with $\psi(x)=\underline{u}(x)$ such that $\underline{u}-\psi$ reaches the maximum at x satisfies

$$-\frac{m^2\psi'(x)^2}{2\sigma^2\psi''(x)} + \lambda\int_0^\infty \underline{u}(x-z)\,\mathrm{d}G(z) + c\psi'(x) - (\delta+\lambda)\underline{u}(x) \geq 0\,.$$

We say that a continuous function $\bar{u}:[0,\infty)\to\mathbb{R}_+$ is a **viscosity supersolution** to (1) at $x\in(0,\infty)$ if any twice continuously differentiable function $\phi:(0,\infty)\to\mathbb{R}$ with $\phi(x)=\bar{u}(x)$ such that $\bar{u}-\phi$ reaches the minimum at x satisfies

$$-\frac{m^2\phi'(x)^2}{2\sigma^2\phi''(x)} + \lambda\int_0^\infty \bar{u}(x-z)\,\mathrm{d}G(z) + c\phi'(x) - (\delta+\lambda)\bar{u}(x) \leq 0\,.$$

A **viscosity solution** to (1) is a continuous function $u:[0,\infty)\to\mathbb{R}_+$ if it is both a viscosity subsolution and a viscosity supersolution at any $x\in(0,\infty)$.

2.3 Main result

Theorem 1
$V(x)$ is a viscosity solution to (1).

Theorem 2: Comparison principle
Let $v(x)$ be a super- and $u(x)$ a subsolution to (1), fulfilling conditions 1 and 2 of **Lemma 1**. If it holds $u(0)\leq v(0)$, then $u(x)\leq v(x)$ on $[0,\infty)$.

Conclusion: The value function $V(x)$ is the unique viscosity solution to the HJB equation (1).

Similar results and proof techniques one can find for example in [1], [2] or [3].

3 Proportional Reinsurance and No-Discounting

In the special case of proportional reinsurance and $\delta=0$ we will be able to show th existence and uniqueness of the classical solution to the HJB equation. In the following we will refer to the HJB equation (1) with $\delta=0$.

3.1 The optimal strategy

Proportional reinsurance means $r(z,b)=zb$ for $b\in[0,1]$. Consider for the moment $c(b)$ calculated by the expected value principle:

$$c(b) = -\lambda\mu(\theta-\eta) + \lambda\mu(1+\theta)b\,,$$

where $\mu=\mathbb{E}[Z]$, η and θ are the safety coefficients of the insurer and reinsurer respectively.

We obtain from the HJB equation (1)

1. $V''(x)>0$: $V''(x)\geq 0$ follows from the convexity; $V''(x)\neq 0$ follows from
$$\frac{\sigma^2a^2}{2}V''(x) + \big(c(0)+am\big)V'(x) < 0 \ \text{ für } \ a=\frac{1-c(0)}{m}\,.$$
2. The optimal strategy at $x=0$ is given by
$$(a^*,b^*) = \begin{cases} (\frac{2\lambda\mu(\theta-\eta)}{m},0) & : V'(0)\in(-\frac{1}{1+\theta},0)\,,\\ (a^*,1) & : V'(0)\in(-1,-\frac{1}{1+\theta})\,,\\ (\frac{2\lambda\mu(\theta-\eta)}{m},b) & : V'(0)=-\frac{1}{1+\theta}\ ,b\in[0,1]\,.\end{cases}$$
3. If $V'(0)\in(-1,-\frac{1}{1+\theta})$, then $b^*(x)=1$ für $x\in[0,\epsilon)$, $\epsilon>0$.

3.2 Existence and uniqueness of the value function

Theorem 3
Let $f(x)$ be a decreasing, twice continuously differentiable solution to (1) with $\lim_{x\to\infty} f(\infty)=0$. Then $f(x)=V(x)$ and the optimal strategy is the strategy of the feedback form $(A^*(X_t),B^*(X_t))$.

The uniqueness of the classical solution allows us to show the following result:

Lemma 2 Assume the value function $V(x)$ is the unique, twice continuously differentiable, vanishing at infinity solution to the HJB equation (1); and the net profit condition $c>\lambda\mu$ is fulfilled, then the optimal investment strategy at $x=0$ is $a^*=0$.

Theorem 4 There is a unique decreasing, twice continuously differentiable solution to (1), if the claims distribution function $G(x)$ has a bounded density and $\lim_{b\to 1}\frac{c-c(b)}{1-b}>0$.

The proofs of the Theorems 3 and 4 are similar to the proofs in [4].

4 Numerical Results

Assume now $Z\sim\text{Exp}(1/\mu)$ and $\sigma^2=0.01$, $m=0.03$, $\delta=0.04$, $\mu=\lambda=1$, $\eta=0.3$ und $\theta=0.5$.

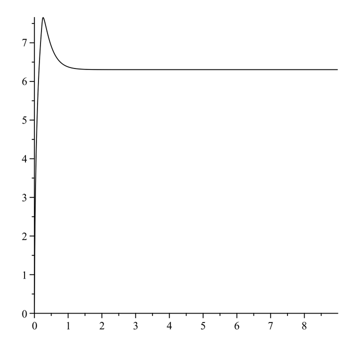

Figure 2: Optimal investment strategy

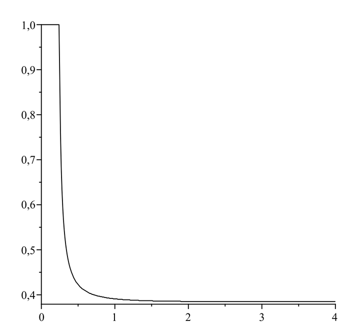

Figure 3: Optimal reinsurance strategy

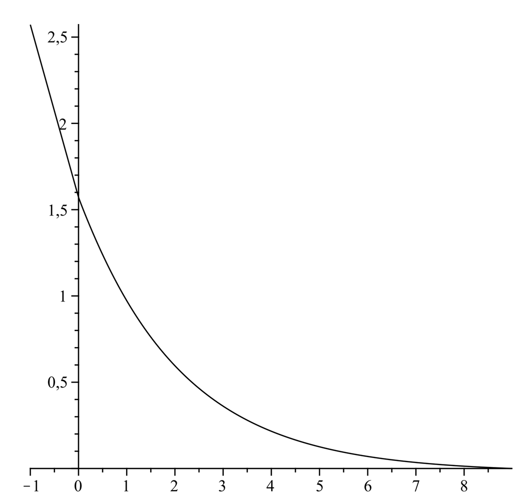

Figure 4: Value function

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