Optimal Control of Reinvestments by Reinsurance

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11th March 2008

Diffusion Approximations

We consider a diffusion process on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ of the form

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By $\mathscr{S}_x$ we denote the set of admissible strategies for an initial value x.

We are interested in the value function $V(x) = \inf\limits_{Y\in\mathscr{S}_x} V^Y(x)$ with $\lim\limits_{x\to\infty} V(x) = 0$ and in, if it exists, the optimal strategy.

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And S. Watanabe showed that the strategy Y^* is unique.

Solution for Brownian Motions

Shreve, Lechoczky and Gaver found the explicit solution for Brownian motions $X_t = x + mt + \sigma W_t$.
The value function has in this case the form

$$V(x) = \begin{cases} \frac{\sigma^2}{m+\sqrt{m^2+2\delta\sigma^2}} \exp\left(-\frac{m+\sqrt{m^2+2\delta\sigma^2}}{\sigma^2}x\right) & : \quad x \geq 0 \\ \frac{\sigma^2}{m+\sqrt{m^2+2\delta\sigma^2}} - x & : \quad x < 0. \end{cases}$$

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$$X_t^{(n)} = x + \left[(1+\eta/\sqrt{n})\lambda\mu_1\sqrt{n}\right]t - \sum_{i=1}^{N_t^{(n)}} Z_i/\sqrt{n}.$$

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$$X_t^{(n)} = x + \left[(1 + \eta/\sqrt{n})\lambda\mu_1\sqrt{n}\right]t - \sum_{i=1}^{N_t^{(n)}} Z_i/\sqrt{n}.$$

As weak limit we obtain:

$$X_t = x + \lambda\mu_1\eta t + \sqrt{\lambda\mu_2}W_t,$$

where W_t is a standard Brownian motion.

Diffusion Approximations with Reinsurance

We consider now diffusion approximations to a classical risk process where the claims are reinsured by some reinsurance:

$$dX_t^B = \lambda\theta\Big[\big(\tfrac{\eta}{\theta} - 1\big)\mu + \mathbb{E}(r(Z, b_t))\Big]dt + \sqrt{\lambda\mathbb{E}(r(Z, b_t)^2)}dW_t.$$

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$b \in [0, \tilde{b}]$ with $b = 0$ meaning "full reinsurance" and $b = \tilde{b}$ meaning "no reinsurance".

Let $\mathscr{U}$ be the set of adapted strategies, i.e. strategies $B = (b_t)_{t\geq 0}$ with $b_t \in [0, \tilde{b}]$.
We denote the value connected to the strategy $Y_t^B = \sup\limits_{0\leq s\leq t} (-X_s^B) \vee 0$ by $V^B(x)$ and define the value function again as $V(x) = \inf\limits_{B\in\mathscr{U}} V^B(x)$.

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Let $\tau_x^B = \inf\{t \geq 0 : X_t^B < 0,\ X_0^B = x\}$ be the ruin time.

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In order to minimize $V^B(x)$, we have to minimize $L^B(x) := \mathbb{E}(e^{-\delta\tau_x^B})$ and find $L(x) := \inf\limits_{B\in\mathscr{U}} L^B(x)$.

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Consequently $L(x) = L^B(x)$ is an exponential function in x.

Because $V^B(x)$ is decreasing in x and $\frac{dV^B}{dx}(0) = -1$, there is $\beta(B) > 0$ independent of x with $L^B(x) = \exp(-\beta(B)x)$.

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Because $V^B(x)$ is decreasing in x and $\frac{dV^B}{dx}(0) = -1$, there is $\beta(B) > 0$ independent of x with $L^B(x) = \exp(-\beta(B)x)$.

On the other hand $\beta(B)$ independent of x means B is independent of x, i.e. $B = b$ is a constant strategy.

HJB Equation

In order to find the optimal constant strategy b^* we consider the Hamilton-Jacobi-Bellman equation for the function $L(x)$:

$$\inf_{b\in[0,\tilde{b}]} \frac{\lambda\mathbb{E}(r(Z,b)^2)}{2} L''(x) + \lambda\theta\left[\left(\frac{\eta}{\theta}-1\right)\mu + \mathbb{E}(r(Z,b))\right] L'(x) - \delta L(x) = 0.$$

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If there is an optimal strategy b^*, i.e. $L(x)=L^{b^*}(x)$, then the value function is also an exponential function and the HJB equation has the form:

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$$\frac{\lambda\mathbb{E}(r(Z,b^*)^2)}{2}\beta(b^*)^2+\lambda\theta\Big[(\tfrac{\eta}{\theta}-1)\mu+\mathbb{E}(r(Z,b^*))\Big]\beta(b^*)-\delta=0,$$

i.e. a quadratic equation in $\beta(b^*)$.

Existence of a Solution

With notation $\alpha(b) = \lambda\theta\Big[(\frac{\eta}{\theta} - 1)\mu + \mathbb{E}(r(Z,b))\Big]$, $b \neq 0$ we obtain

$$\beta(b) = [\lambda\mathbb{E}(r(Z,b)^2)]^{-1}\Big[\alpha(b) + \sqrt{\alpha(b)^2 + 2\delta\lambda\mathbb{E}(r(Z,b)^2)}\Big] > 0$$

as a positive solution to the HJB equation.

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If we set $\beta(0) = 0$, then $\beta(b)$ is continuous on the set $[0, \tilde{b}]$ and has therefore a maximum in $(0, \tilde{b}]$.

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Because $V^b(x) = \frac{1}{\beta(b)}\exp(-\beta(b)x)$ the minimum of $V^b(x)$ in b is at the same time the maximum of $\beta(b)$.

Verification Theorem

Verification theorem:

There is a $b^* \in (0, \tilde{b}]$ with $L(x) = \exp(-\beta(b^*)x)$. The constant strategy $B = b^*$ is an optimal reinsurance strategy, whereas b^* is a maximum point of $\beta(b)$.
The corresponding value function has the form:

$$V^{b^*}(x) = \begin{cases} \frac{1}{\beta(b^*)} e^{-\beta(b^*)x} & : \quad x \geq 0 \\ \frac{1}{\beta(b^*)} - x & : \quad x < 0. \end{cases}$$

The uniqueness of the optimal strategy, as the case may be, can be shown for a concrete function $r(Z, b)$.

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We set $\lambda\mu=1$, $\sqrt{\lambda\mu_2}=\sigma$ and $b_0=1-\frac{\eta}{\theta}$ and consider the process

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$$X_t^b = x + (b\theta - (\theta - \eta))t + b\sigma W_t.$$

It is easy to verify, that the optimal constant strategy b^* is given through:

$$b^* = \begin{cases} 1 & : \quad \theta \ge \eta + \sqrt{\eta^2 + 2\delta\sigma^2} \\ \frac{2b_0}{1+2\delta\sigma^2/\theta^2} & : \quad \eta < \theta < \eta + \sqrt{\eta^2 + 2\delta\sigma^2}. \end{cases}$$

Choose for example $\theta = 0.5$, $\eta = 0.3$, $\lambda = 0.05$, $\delta = 0.04$ and $\mu_2 = 80$. Then we have for the optimal deductible $b^* = 0.351 < 1$
In this case it holds:

$$V^{0.351}(x) = \begin{cases} 0.235^{-1} e^{-0.235x} & : \quad x \geq 0 \\ 0.235^{-1} - x & : \quad x < 0. \end{cases}$$

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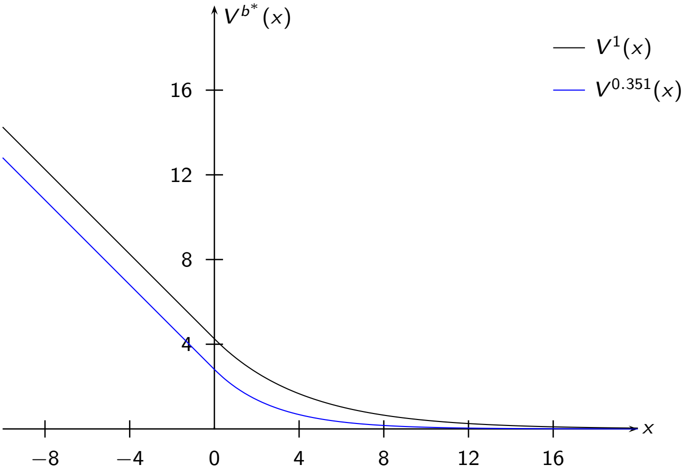
The parameters $\theta = 0.8$, $\eta = 0.2$, $\lambda = 0.05$, $\delta = 0.04$ and $\mu_2 = 80$ yield $b^* = 1$, i.e. the value function:

$$V^{1}(x) = \begin{cases} 0.356^{-1} e^{-0.356x} & : \quad x \geq 0 \\ 0.356^{-1} - x & : \quad x < 0. \end{cases}$$

Value functions for optimal strategies $b^* = 1$ and $b^* = 0.351$.



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$$X_t^b = x + \lambda\theta\Big[\big(\tfrac{\eta}{\theta} - 1\big)\mu + \mathbb{E}(\min(Z, b))\Big]t + \sqrt{\lambda\mathbb{E}(\min(Z, b)^2)}W_t.$$

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We assume $\mathbb{E}(Z^2) < \infty$.

Value function for $Z \sim \mathrm{Exp}(\frac{1}{\mu})$

We consider at first the function

$$\beta(b) = [\lambda\mathbb{E}(\min(Z,b)^2)]^{-1}\left[\alpha(b) + \sqrt{\alpha(b)^2 + 2\delta\lambda\mathbb{E}(\min(Z,b)^2)}\right]$$

and show, that there is a unique $b^* \in \left(\frac{2\mu\theta^2\lambda(1-\eta/\theta)}{2\delta+\theta^2\lambda}, \frac{\mu\theta^2\lambda(1-\eta/\theta)}{\delta}\right)$.

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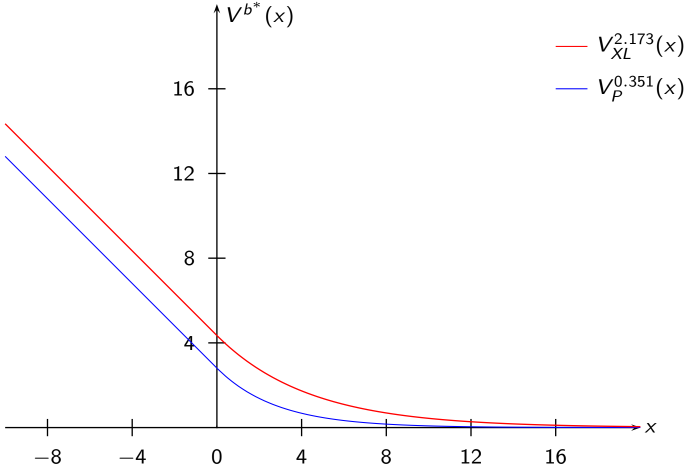
We assume $Z \sim \mathrm{Exp}(\frac{1}{\mu})$ and obtain for the parameters $\theta = 0.5$, $\eta = 0.3$, $\lambda = 0.05$, $\delta = 0.04$ and $\mu_2 = 80$ the value function

$$V^{2.173}(x) = \begin{cases} 0.23^{-1}e^{-0.23x} & : \quad x \geq 0 \\ 0.23^{-1} - x & : \quad x < 0. \end{cases}$$

Value functions for the XL (V_{XL}) and proportional (V_P) reinsurances with the same parameters $\theta = 0.5$, $\eta = 0.3$, $\lambda = 0.05$, $\delta = 0.04$ and $\mu_2 = 80$.



References

Shreve, S.E., Lehocsky, J.P. and Gaver, D.P. (1984). Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM J. Control Optim.* **22**, 55–75.

Watanabe, S. (1971). On stochastic differential equations for multidimensional diffusion processes with boundary conditions. *J. Math. Kyoto Univ.* **11**, 169–180.

Thank you for your attention