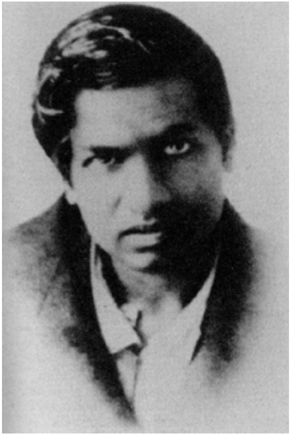
Ramanujan's Deathbed Letter

Larry Rolen

Emory University

The great anticipator of mathematics



Srinivasa Ramanujan (1887-1920)

"Death bed letter"

Dear Hardy,

"I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (partially studied by Rogers), they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."

Ramanujan, January 12, 1920.

The first example

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$

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"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered....This remains a challenge for the future..."

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"Theorem"

Ramanujan's mock theta functions are holomorphic parts of weight 1/2 harmonic Maass forms.

Defining Maass forms

Notation. Throughout, let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$.

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Hyperbolic Laplacian.

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

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Remark

Modular forms *are holomorphic functions which satisfy (1).*

HMFs have two parts ($q := e^{2\pi i z}$)

Fundamental Lemma

If $f \in H_{2-k}$ and $\Gamma(a, x)$ is the incomplete Γ-function, then

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n.$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

Holomorphic part f^+ **Nonholomorphic part** f^-

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Holomorphic part f^+ **Nonholomorphic part** f^-

Remark

The mock theta functions are examples of f^+.

So many recent applications

- q-series and partitions
- Modular L-functions (e.g. BSD numbers)
- Eichler-Shimura Theory
- Probability models
- Generalized Borcherds Products
- *Moonshine* for affine Lie superalgebras and M_{24}
- Donaldson invariants
- Black holes
- ...

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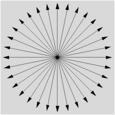
Ramanujan's last letter.

- Asymptotics, near roots of unity, of "Eulerian" modular forms.



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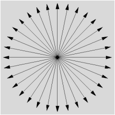

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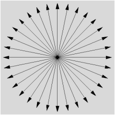

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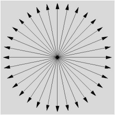


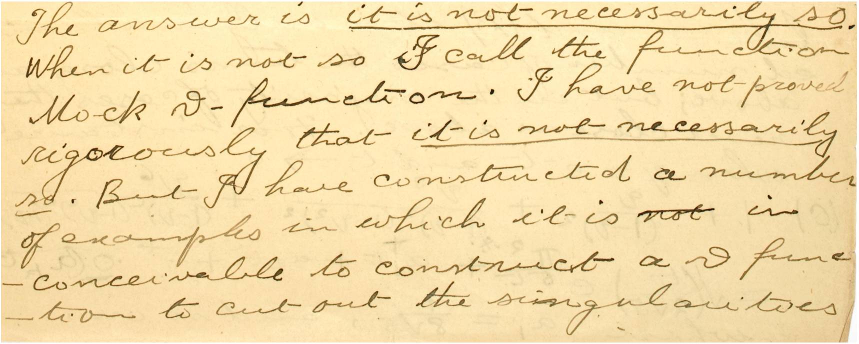
- Raises **one** question and conjectures the answer.
- Gives **one example** supporting his conjectured answer.
- Concludes with a list of his mock theta functions.

Ramanujan's question

Question (Ramanujan)

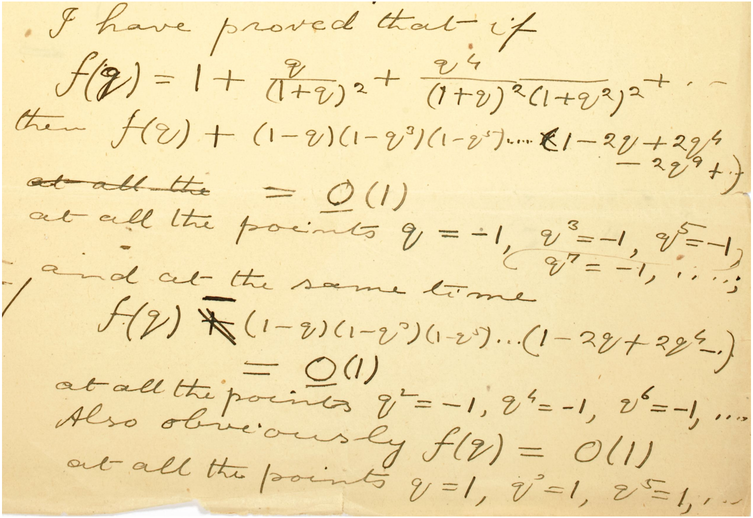
Must Eulerian series with "similar asymptotics" be the sum of a modular form and a function which is $O(1)$ at all roots of unity?

Ramanujan's Answer



The answer is it is not necessarily so. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is ~~not~~ in-conceivable to construct a ϑ function to cut out the singularities

Ramanujan's "Example"



I have proved that if

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \cdots$$

then $f(q) + (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)$

~~at all the~~ $= O(1)$

at all the points $q=-1$, $q^3=-1$, $q^5=-1$, $q^7=-1$, ...;

and at the same time

$$f(q) - (1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-\cdots) = O(1)$$

at all the points $q^2=-1$, $q^4=-1$, $q^6=-1$, ...

Also obviously $f(q) = O(1)$

at all the points $q=1$, $q^3=1$, $q^5=1$, ...

Ramanujan's "Near Miss Example"

Define the mock theta $f(q)$ and the modular form $b(q)$ by

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

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Conjecture (Ramanujan)

If q approaches an even order $2k$ root of unity, then

$$f(q) - (-1)^k b(q) = O(1).$$

Numerics

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As $q \to -1$, we have

$$f(-0.994) \sim -1{\cdot}10^{31},\; f(-0.996) \sim -1{\cdot}10^{46},\; f(-0.998) \sim -6{\cdot}10^{90},$$

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$$f(-0.998185) \sim -\text{Googol}$$

Numerics continued...

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Amazingly, Ramanujan's guess gives:

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\ldots$	$3.969\ldots$	$3.976\ldots$	$3.984\ldots$	$3.992\ldots$

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This suggests that

$$\lim_{q\to -1}(f(q)+b(q)) = 4.$$

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q	$0.992i$	$0.994i$	$0.996i$
$f(q)$	$2 \cdot 10^6 - 4.6 \cdot 10^6 i$	$2 \cdot 10^8 - 4 \cdot 10^8 i$	$1.0 \cdot 10^{12} - 2 \cdot 10^{12} i$
$f(q) - b(q)$	$\sim 0.05 + 3.85i$	$\sim 0.04 + 3.89i$	$\sim 0.03 + 3.92i$

As $q \longrightarrow i$

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This suggests that

$$\lim_{q \to i} (f(q) - b(q)) = 4i.$$

Natural Questions

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Question

If he is right, then what are the **mysterious** $O(1)$ *numbers in*

$$\lim_{q \to \zeta} (f(q) - (-1)^k b(q)) = O(1)?$$

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Theorem (Folsom, Ono, Rhoades)

If ζ is an even $2k$ order root of unity, then

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

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Remark

This Theorem follows from "quantum" modularity.

Ramanujan's last words

"it is inconceivable to construct a ϑ *function to cut out the singularities of a mock theta function..."*

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"...it has not been **proved** *that* **any** *of Ramanujan's mock theta functions really are mock theta functions according to his definition."*

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Theorem (G-Ono-Rolen (2013))

Ramanujan's examples satisfy his own definition.

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Theorem (G-Ono-Rolen (2013))

Ramanujan's examples satisfy his own definition. More precisely, a mock theta function and a modular form never cut out exactly the same singularities.