INTEGRALITY PROPERTIES OF CLASS POLYNOMIALS FOR NON-HOLOMORPHIC MODULAR FUNCTIONS

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Abstract. In his paper *Traces of Singular Moduli* [14], Zagier studied values of certain modular functions at imaginary quadratic points known as singular moduli. He proved that "traces" of these algebraic integers are Fourier coefficients of certain half-integral weight modular forms. In this paper, he obtained similar results for certain non-holomorphic modular functions. However, he observed that these "singular moduli" are not necessarily algebraic integers. Based on numerical examples, the "class polynomials" whose roots are these singular moduli seem to have predictable denominators. Here we explain this phenomenon and provide a sharp bound on these denominators.

1. Introduction and Statement of Results

Classically, the term *singular modulus* refers to a value of the modular j-invariant at an imaginary quadratic point in the upper half plane. These are well-known to be algebraic integers, and they play a beautiful role in the theory of complex multiplication and explicit class field theory for imaginary quadratic fields. In [14], Zagier initiated the study of "traces" of singular moduli. He proved that the generating function associated to these numbers is a modular form of weight 3/2.

We say a function $f(z)$ is modular of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ if $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \varepsilon(\gamma)(cz+d)^k f(z)$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and z in the upper half plane $\mathbb{H}$. Here $\varepsilon(\gamma)$ is a certain multiplier depending on whether k is integral or not. In general we have $\varepsilon(\gamma)^4 = 1$ if $k \in \frac{1}{2}\mathbb{Z}$, and $\varepsilon(\gamma) = 1$ if $k \in \mathbb{Z}$ (for more on the theory of half-integral weight modular forms, see [12]). If $f(z)$ is modular of weight k, holomorphic on $\mathbb{H}$, and meromorphic at the cusps of Γ, we call $f(z)$ a *weakly holomorphic modular form* (or if $k = 0$, simply a *modular function*). For any k we denote the space of weakly holomorphic modular forms of weight k on $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ by $M_k^!$. If $f(z)$ is bounded (resp. vanishes) at all the cusps, we call f a holomorphic modular form (resp. a cusp form), and denote the space of all such forms by M_k (resp. S_k).

Zagier studied the special values of modular functions evaluated at CM points. Let $\mathcal{Q}_D$ be the set of binary positive definite integral quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ with discriminant $D = b^2 - 4ac$ and $a, b, c \in \mathbb{Z}$. Given such a

$Q(x,y)$, the associated CM point z_Q is given by

$$z_Q := \frac{-b+\sqrt{D}}{2a} \in \mathbb{H}.$$

Matrices in $\mathrm{SL}_2(\mathbb{Z})$ act on $\mathcal{Q}_D$ by $Q|\gamma(x,y) := Q((x,y)\gamma^T)$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. If $d \equiv 0,1 \pmod 4$, and D is any fundamental discriminant with $dD < 0$, then given a modular function $F \in M_0^!$, Zagier defined the twisted trace of singular moduli by

$$\mathrm{Tr}_{d,D}(F) := \sum_{Q \in \mathcal{Q}_{dD}/\Gamma} w_Q^{-1}\chi(Q)F(z_Q).$$

Here the factor $w_Q = 1$ unless $Q \sim a(x^2+y^2)$ or $Q \sim a(x^2+xy+y^2)$, in which case $w_Q = 2$ or 3 respectively, and the genus character $\chi(Q)$ is defined by

$$\chi(Q) := \chi(a,b,c) := \begin{cases} \chi_D(r) & \text{if } (a,b,c,D) = 1 \text{ and } Q \text{ represents } r, \text{ where } (r,D)=1; \\ 0 & \text{if } (a,b,c,D) > 1, \end{cases}$$

where χ_D is the Kronecker symbol $\left(\frac{D}{\cdot}\right)$.

To illustrate, let $j(z)$ be the usual modular j-invariant and consider the Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$,

$$J(z) := j(z) - 744 = q^{-1} + 196884q + 21493760q^2 + \dots$$

(where here and throughout this paper $q := e^{2\pi i z}$). Let $\eta(z)$ be the usual Dedekind eta-function and $E_k(z)$ the usual weight k Eisenstein series. Then we define the weight 3/2 modular form $g(z)$ by

$$g(z) := \theta_1(z) \cdot \frac{E_4(4z)}{\eta(4z)^6} = \sum_{n \geq -1} B(n)q^n,$$

where $\theta_1(z) := \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2}$.

Theorem (Zagier [14], Theorem 1)**.** *Let d be any positive integer such that $d \equiv 0,3 \pmod 4$. Then*

$$\mathrm{Tr}_{-d,1}(J(z)) = -B(d).$$

Zagier also considered examples of trace generating functions associated to modular forms of negative even weight k by taking the twisted trace of the non-holomorphic modular function

$$\partial f := R^{-k/2} f \tag{1.1}$$

where R^k is the *iterated Maass raising operator* defined in §2.1. Zagier also showed that these traces are the coefficients of certain half-integral weight modular forms.

In general, it seems that these special values or "singular moduli" for a set of primitive quadratic forms of a fixed discriminant d form a single orbit of Galois conjugates, though this remains to be proven in general. These special values generate ring class fields over the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Analogously to the classical case of $j(z)$, we define the *generalized Hilbert class polynomial* of discriminant d for a holomorphic or non-holomorphic modular function $F(z)$, given by

$$H_d(F;x) := \prod_{Q\in\mathcal{Q}_d/\Gamma} (x - F(z_Q))^{\frac{1}{w_Q}}. \tag{1.2}$$

The degree of $H_d(F,x)$ is the Hurwitz-Kronecker class number $h(d)$. Note that $H_d(F;x)$ is not actually always a polynomial, but if d is a negative fundamental discriminant with $|d|>4$, then $H_d(F;x)$ is a polynomial, and we will denote its coefficients by

$$H_d(F;x) = \sum_{n=0}^{h(d)} \mathbf{e}_{n;F}(d)x^{h(d)-n}. \tag{1.3}$$

If $F(z)$ has integral coefficients, then $\mathbf{e}_{1;F}(d)$ will be integral; however the other coefficients may not be. For example, consider the weight -2 modular form

$$F_2(z) := \frac{E_4(z)E_6(z)}{\Delta} = q^{-1} - 240 - 141444q - 85292800q^2 - 238758390q^3 + \ldots,$$

and define the weight 0 non-holomorphic derivative

$$K(z) := \partial F_2 = R_{-2}F_2 = \frac{E_2^*(z)E_4(z)E_6(z) + 3E_4^3(z) + 2E_6(z)^2}{6\Delta(z)},$$

where $E_2^*(z) := E_2(z) - \frac{3}{\pi\Im z}$ is the non-holomorphic Eisenstein series of weight 2. The following table gives $H_d(K;x)$ for the first few negative fundamental discriminants of class number at least 3.

d	$H_d(K;x)$
-23	$x^3 - 23\cdot 141826x^2 - \frac{3945271661}{23}x - 7693330369871$
-31	$x^3 - 31\cdot 1201149x^2 - \frac{61346290410}{31}x + 1143159756791823$
-39	$x^4 - 39\cdot 8067588x^3 + \frac{8602826222178}{39}x^2 - 8402966980381003 5x + \frac{95749227855890319016073}{39^2}$

Note that in the examples above, both $\mathbf{e}_{1;F}(d)$, the third symmetric function $\mathbf{e}_{3;F}(d)$ also always appears to be integral. Extensive calculations along these lines suggest that these two are the only coefficients which are integral in general. We explain this phenomenon and give a bound on all other denominators of the coefficients of the class polynomials. This bound appears to be sharp in general.

Our result holds for a class of primes with possible exceptions dependent on the weight k of the original modular form and the specific coefficient $\mathbf{e}_{n;F}(d)$ under consideration. We call such primes *good for the pair* (k,n). The definition of a good prime relies on the Hecke algebra acting on modular forms of certain relevant weights. If k is a negative even integer and n a positive integer, then we say that an integer ℓ is a *relevant weight* for the pair (k,n) if $kn \leq \ell \leq 0$ and there is no integer i, $0 \leq i \leq -\frac{k}{2}-1$, such that $\ell = kn+4i+2$. If the Hecke operator T_p does not act nilpotently on the space of cusp forms $S_{2-\ell}$, then we say that p is good for the weight ℓ. If p is good for every ℓ relevant for the pair (k,n), we say that p is *good for the pair* (k,n).

Our main result is the following.

Theorem 1.1. *Let $f(z) \in M_k^!$ be a modular form of negative, even weight with integer coefficients, d be a negative fundamental discriminant, $d \neq -3$ or -4, and let $\mathbf{e}_{n;\partial f}(d)$ be defined as in (1.3). If $(p,d)=1$, then $\mathbf{e}_{n;\partial f}(d)$ is p-integral. Otherwise, let*

$$B(n,k) := \begin{cases} \frac{-nk}{4} & \text{if } 4 \mid nk \\ \frac{1}{4}(-nk+2k-2) & \text{otherwise.} \end{cases}$$

Then if p is good for the pair (k,n), we have that

$$d^{B(n,k)} \cdot \mathbf{e}_{n;f}(d)$$

is p-integral.

As there are no non-zero cusp forms of weight less than 12, we obtain the following corollary with no additional conditions to check.

Corollary 1.2. For any $f(z) \in M_{-2}^!$ with integral principal part, we have that

$$\mathbf{e}_{3;f}(d) \in \mathbb{Z}.$$

In particular, this explains the integrality pattern in the computed examples for $K(z)$.

Remark. Although the theorem is only stated for $D=1$ and d negative and primitive, it is clear from the proof that an analogous result for arbitrary "twisted class polynomials" holds in general, though care must be taken when d is -3 or -4 times a square.

1.1. **Outline of the proof and the paper.** When $4|nk$, or $p \nmid d$, the proof only requires the integrality results for singular values of $E_6(z)/\sqrt{\Delta(z)}$, $E_4(z)/\Delta(z)^{\frac{1}{3}}$, and $E_2^*(z)/\Delta(z)^{\frac{1}{6}}$ given, for instance, in [14]. The remaining case requires additional consideration.

Thanks to Newton's identities, we may express the elementary symmetric functions in terms of power-sums. We recall that if

$$e_k(x_1,\ldots,x_n) := \sum_{1\leq J_1<j_2\ldots\leq j_k\leq n} x_{j_1}\ldots x_{j_k}$$

is the usual elementary symmetric polynomial of degree k in $x_1,\ldots,x_n$ and

$$p_k(x_1,\ldots,x_n) := \sum_{i=1}^{n} x_i^k$$

is the k^{th} power-sum, then Newton's identities state that

$$ke_k(x_1,\ldots,x_n)=\sum_{i=1}^{k}(-1)^{i-1}e_{k-i}(x_1,\ldots,x_n)p_i(x_1,\ldots,x_n). \tag{1.4}$$

Thus, our problem is reduced to the of study of traces of singular moduli for powers of $\partial f(z)$. We may decompose these powers of raises of modular forms further by using the following theorem due to Shimura. Theorem 1.3 allows to write such functions as sums of raises of weakly holomorphic modular forms. The statement of Theorem 1.3 requires the use of the iterated Maass lowering operator, L^n which is defined in (2.4).

Theorem 1.3 (Shimura [13])**.** *Suppose F is a smooth function on $\mathbb{H}$ which is modular of weight $k \leq 0$ and is in the kernel of L^{E+1}. Then there exist uniquely determined modular forms $g_j \in M^!_{k-2j}$ such that*

$$F=\sum_{j=0}^{E}R^j g_j.$$

We refer to the set of weights $\{k-2j \mid g_j \not\equiv 0\}$ in Theorem 1.3 as the *decomposition weights* of F, and the minimal E such that the theorem applies as the *depth* of F.

This decomposition allows us to apply the work of Duke and Jenkins [5] which generalizes Zagier's orignal paper and provides important integrality results. In particular, Duke and Jenkins define the lifting operations $\mathfrak{Z}_D(f)$ given in (3.1) which take a modular form f of negative even weight to forms of half-integral weight, whose coefficients are given in terms of the traces of values of $\partial f(z)$. Moreover they show that if f has integral coefficients, then so does $\mathfrak{Z}_D(f)$. Applying this to each of the pieces in Theorem 1.3 gives a bound on the denominators of $\mathbf{e}_{n;\partial f}(d)$, although this is far from optimal. It falls short for two reasons. Firstly, the use of the decomposition in Theorem 1.3 introduces artificial denominators which we need to show cancel out. Secondly, certain weights which can appear give larger denominators than in Theorem 1.1, so we need to show that they are irrelevant.

The proof of our full result requires a generalization of the Zagier lift, $\mathcal{Z}_D$, defined in (4.2), which we can apply to powers of $\partial f(z)$ and consider the Zagier lifts of every form in the decomposition at once. The analogous integrality result is as follows.

Theorem 1.4. *Let $F(z)$ be a non-holomorphic modular function, which may be decomposed as in Theorem 1.3, and let p be a prime which is good for each k in the set of decomposition weights of F. If $F(z) \in \mathbb{Z}_{(p)}\left[\frac{1}{4\pi y}\right]((q))$, then there is an explicit integer M dependent on the decomposition weights of F so that $R^M\mathcal{Z}_D(F(z)) \in \mathbb{Z}_{(p)}\left[\frac{1}{16\pi y}\right]((q))$. In particular, if p^ℓ exactly divides n, then the coefficients of q^n in $p^{\ell M}\mathcal{Z}_D(F(z))$ is p-integral.*

The formulae for the coefficients of $\mathcal{Z}_D(F(z))$ give bounds on the powers of primes which can divide the denominators of the traces of $F(z)$. These bounds are given in terms of the decomposition weights of F. *A priori*, the decomposition weights may sit in a broad range; however when $F(z) = (\partial f(z))^n$ with $f(z) \in M_k^!$, we find that only the weights which are relevant for the pair (k, n) appear, and thus can contribute to the bound.

Theorem 1.5. *Let $f \in M_k^!$ and consider the product $F = (\partial f)^n$. As in Theorem 1.3, write $F = \sum \partial(g_j)$, where each $g_j \in M_{-2j}^!$. If $-2m$ is not a relevant weight for (k, n), we have that $g_m \equiv 0$.*

Theorem 1.5 shows that certain weights which would weaken the bound cannot appear. Therefore these results, including the exact evaluation of the M given in theorem 1.4 in terms of the decomposition weights, together prove Theorem 1.1. The remainder of the paper is organized as follows. In §2 we define the Maass lowering and raising operators and use them to give a proof of the decomposition theorem. In §3, we recall the work of Duke-Jenkins which we will build on in §4 to prove Theorem 1.4. In §5, we introduce the Rankin-Cohen brackets and use them to prove Theorem 1.5.

2. Operators on Maass Forms and the Spectral Decomposition

In this section we recall some of the basic differential operators on Maass forms and prove Theorem 1.3.

2.1. **Maass forms and raising and lowering operators.** Maass forms are generalizations of the modular forms previously described. We say that a function $f \colon \mathbb{H} \to \mathbb{C}$ which is modular of weight k is a *weak Maass form* (or simply a Maass form) if it has at most linear exponential growth at the cusps and is an eigenfunction of the weight k hyperbolic Laplacian Δ_k, which is defined as

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

For more on the theory of such Maass forms, see e.g. [8].

Given $k \in \frac{1}{2}\mathbb{Z}$, we define the *Maass raising operator* R_k by

$$R_k := \frac{1}{2\pi i} \frac{\partial}{\partial z} - \frac{k}{4\pi y}. \tag{2.1}$$

This operator preserves modularity, and sends Maass forms to Maass forms. If $f(z)$ is modular of weight k then $R_k f(z)$ is modular of weight $k+2$. If $f(z)$ has eigenvalue λ with respect to Δ_k, then $R_k f(z)$ has eigenvalue $\lambda + k$ with respect to Δ_{k+2} (see [3]). We also define the *iterated raising operator* R_k^d as the composition of d raising operators of the appropriate weights:

$$R_k^d := R_{k+2(d-1)} \circ \cdots \circ R_k.$$

Whenever the weight is clear from context, we suppress the dependence on the weight and simply write R^d. Standard formulas for the iterated raising operator (for example see [2, Ch. 1]) give us that

$$R_k^d f(z) = \sum_{j=0}^{d} \binom{d}{j} \frac{\Gamma(k+d)}{\Gamma(k+d-j)} \left(\frac{-1}{4\pi y}\right)^j \left(\frac{\partial}{2\pi i \partial z}\right) f(z). \tag{2.2}$$

As in (1.1), if $f \in M_k^!$ with k even and non-positive, we also define $\partial f := R^{-k/2} f$ so that ∂f is modular of weight 0.

In conjunction with the Maass raising operator, we will also need the *Maass lowering operator* L_k, defined as

$$L_k := -8\pi i y^2 \frac{\partial}{\partial \bar{z}}. \tag{2.3}$$

See [8] or [3] for more details (note that we normalize the factor of automorphy differently than in [3]). If f is a weight k Maass form with eigenvalue λ, then $L_k f$ is a Maass form of weight $k-2$ and eigenvalue $\lambda - k + 2$. We also define the iterated lowering operator

$$L_k^d = L^d := L_{k-2(d-1)} \circ \cdots \circ L_k. \tag{2.4}$$

We choose the specific normalizations for the raising and lowering operators above to preserve integrality in the following sense. Not all non-holomorphic modular forms are in $\mathbb{C}((q))\left[\frac{1}{4\pi y}\right]$; however for the purposes of this paper we restrict our attention to those which are. We refer to such a non-holomorphic modular form G with integral weight as having *integral coefficients* if $G \in \mathbb{Z}((q))\left[\frac{1}{4\pi y}\right]$. If G has half-integral weight, we will instead require that G be in $\mathbb{Z}((q))\left[\frac{1}{16\pi y}\right]$. Analogously, we say that G has rational or p-integral coefficients if G is in $\mathbb{Q}((q))\left[\frac{1}{4\pi y}\right]$ or $\mathbb{Z}_{(p)}((q))\left[\frac{1}{4\pi y}\right]$ respectively.

With these definitions, we have that if $G(z)$ has integral coefficients, then so does $R_k G(z)$, as does $L_k G(z)$ if G has integral weight or $4L_k G(z)$ if G has half-integral weight. We also have the following facts about the Maass raising and lowering operators.

Proposition 2.1. *For any complex functions f and g on $\mathbb{H}$ and any integer k, the following are true.*

(1) $R_{k-2}L_k = -\Delta_k$, *and* $L_{k+2}R_k = -\Delta_k - k$.
(2) R_k *and* L_k *both satisfy the Leibniz rule; that is*

$$R_{k+\ell}(fg) = (R_k f)\cdot g + f \cdot (R_\ell g) \quad \textit{and} \quad L_{k+\ell}(fg) = (L_k f)\cdot g + f\cdot (L_\ell g).$$

(3) *We have that* $L_k f = 0$ *if and only if* f *is holomorphic.*

These facts imply that sums and products of raises of weakly holomorphic modular forms must be in the kernel of some finite power of L. This allows a decomposition for such forms as in Theorem 1.3. This theorem is originally due to Shimura (see [13] Proposition 3.4, or [6] Section 10.1), however we give a short proof which gives explicit components which we will need later.

2.2. **Proof of Theorem 1.3.**

Proof. Since $L^{E+1}F = 0$, we have that $L^E F$ is weakly holomorphic. We define the g_i recursively beginning with g_E. Let

$$g_E = \frac{L^E F}{c_{E,E}},$$

and for each i with $0 \leq i < E$, let

$$g_i := \frac{1}{c_{i,i}} \left(L^i F - \sum_{j=i+1}^{E} c_{i,j} R^{j-i} g_j \right),$$

where

$$c_{i,j} := \frac{j!(-k+j+i)!}{(j-i)!(-k+j)!}.$$

By assumption, $k \leq 0$, so $c_{i,j}$ is defined for all $j \geq i$. Note that each g_i is modular of weight $k-2i$. By rearranging the definition of g_0, we see that $F = \sum_{i=0}^{E} R^i g_i$. Therefore we need only prove that each g_i is weakly holomorphic. We do so by inductively showing that $Lg_i = 0$ for each g_i.

By hypothesis, $Lg_E = 0$. Suppose that $i < E$ is fixed, and that g_j is weakly holomorphic for each $i < j \leq E$. By construction, we have that

$$L^i F = \sum_{j=i}^{E} c_{i,j} R^{j-i} g_j. \tag{2.5}$$

Applying the lowering operator to g_i gives

$$Lg_i = \frac{1}{c_{i,i}} \left(L^{i+1} F - \sum_{j=i+1}^{E} c_{i,j} (LR) R^{j-i-1} g_j \right).$$

Since g_j is holomorphic, we have that $R^{j-i-1} g_j$ is an eigenfunction with respect to $LR = (\Delta_{k-2i-2} - (k-2i-2))$. A short calculation using Proposition 2.1 shows that the eigenvalue is
$(j-i)(-k+j+i+1)$. However, $c_{i,j}(j-i)(-k+j+i+1) = c_{i+1,j}$, so

$$c_{i,i} Lg_i = \left(L^{i+1} F - \sum_{j=i+1}^{E} c_{i+1,j} R^{j-i-1} g_j \right) = 0.$$

□

3. Integrality Results of Duke and Jenkins

In this section we describe the important work of Duke and Jenkins on integrality of traces of singular moduli in [5]. In particular, their results allow us to bound the denominators of the traces of singular moduli of each summand arising in the relevant case of Theorem 1.3. Following their paper, consider any $f = \sum_{n \gg -\infty} a(n)q^n \in M^!_{2-2s}$ with $s \in \mathbb{N}$. For convenience, set

$$\widehat{s} := \begin{cases} s & \text{if } (-1)^s D > 0 \\ 1-s & \text{otherwise.} \end{cases}$$

Also let

$$\mathrm{Tr}^*_{d,D}(f) := (-1)^{\lfloor \frac{\widehat{s}-1}{2} \rfloor} |d|^{\frac{-\widehat{s}}{2}} |D|^{\frac{\widehat{s}-1}{2}} \mathrm{Tr}_{d,D}((-1)^{s-1}\partial f),$$

where $\mathrm{Tr}_{d,D}$ is defined as in (1). For any fundamental discriminant D, they define the D^{th} *Zagier lift* of f to be:

(3.1)

$$\mathfrak{Z}_D(f) := \sum_{m>0} a(-m) m^{s-\widehat{s}} \sum_{n|m} \chi_D(n) n^{\widehat{s}-1} q^{-\frac{m^2|D|}{n^2}} + \frac{1}{2} L(1-s, \chi_D) a(0) + \sum_{dD<0} \mathrm{Tr}^*_{d,D}(f) q^{|d|}.$$

The main theorem of [5] states that $\mathfrak{Z}_D(\cdot)$ is a linear map between spaces of modular forms, which preserves integrality of Fourier coefficients. The image of $\mathfrak{Z}_D$ is in a distinguished subspace of the modular forms of half-integral weight on $\Gamma_0(4)$ which satisfy the Kohnen plus-space condition (see [7]). Specifically, if k is half-integral, we define $M^!_{\lambda+1/2}$ to be the subspace of weakly holomorphic modular forms $f(z)$ of weight $\lambda + 1/2$ on $\Gamma_0(4)$ which have a Fourier expansion

$$f(z) = \sum_{n \equiv 0 \text{ or } (-1)^\lambda \pmod 4} a(n) q^n.$$

Assuming this notation, their theorem is as follows.

Theorem 3.1 ([5, Theorem 1])**.** *Suppose that $f \in M^!_{2-2s}$ for an integer $s \geq 2$. If D is a fundamental discriminant and $\hat{s} = s$ or $1-s$ such that $(-1)^{\hat{s}} D > 0$, then we have that $\mathfrak{Z}_D(f) \in M^!_{3/2-\hat{s}}$. Furthermore, if f has integral Fourier coefficients, so does $\mathfrak{Z}_D(f)$.*

Although this theorem is only stated for D fundamental, it extends naturally to any discriminant D by way of the Hecke algebra. This theorem builds on Zagier's original work for $s = 1$. In that case Theorem 3.1 holds, as long as the constant term of f is 0. Note that $\mathrm{Tr}_{d,1}(1)$ is the Hurwitz-Kronecker class number for d, which is integral for d fundamental and not equal to -3 or -4. Therefore for the remainder of this paper we will assume that the constant term of each weight 0 modular function under consideration is zero. We also note that $\mathrm{Tr}_{d,D} = \mathrm{Tr}_{D,d}$, which implies a duality between coefficients of $\mathcal{Z}_D(f)$ for positive and negative D.

Duke and Jenkins also give a result which we will need later relating the twisted trace $\mathrm{Tr}^*_{m^2D',D}(f)$ to other twisted traces. In particular they show the following.

Proposition 3.2 ([5, Lemma 2])**.** *For D and D' fundamental discriminants with $DD' < 0$ and m a positive integer, we have*

$$\mathrm{Tr}^*_{m^2D',D}(f) = (-m)^{s-1} \sum_{a\cdot b|m} \mu(a)\chi_{D'}(a)\chi_D(b) \left(\frac{m}{ab}\right)^s \mathrm{Tr}^*_{\left(\frac{m}{ab}\right)^2 D,D'}(f). \tag{3.2}$$

4. Proof of Theorem 1.4

The Zagier lift $\mathfrak{Z}_D$ acts only on a single weakly holomorphic modular form. We wish to define a generalization which will give us information about the singular values of any non-holomorphic modular function $F(z)$ which can be decomposed by means of Theorem 1.3. The function F can be written as

$$F(z) = \sum_{j=0}^{e_1} \partial g_{2j}(z) + \sum_{j=1}^{e_2} \partial g_{2j-1}(z) \tag{4.1}$$

for some non-negative integers e_1 and e_2 chosen minimally, where $g_k(z) \in M^!_{-2k}$. Letting $D > 0$ and $d < 0$ be discriminants, we define

$$\mathcal{Z}_D(F) := \sum_{j=0}^{e_1} (-1)^{1+j} |D|^{e_2+j} R^{e_1-j} \mathfrak{Z}_D(g_{2j}) + \sum_{j=1}^{e_2} (-1)^{1-j} |D|^{e_2-j} R^{e_1+j} \mathfrak{Z}_D(g_{2j-1}), \tag{4.2}$$

which is a non-holomorphic modular form of weight $\frac{3}{2} + 2e_1$, and

$$\mathcal{Z}_d(F) := \sum_{j=0}^{e_1} (-1)^{-j} |d|^{e_1-j} R^{e_2+j} \mathfrak{Z}_d(g_{2j}) + \sum_{j=0}^{e_2} (-1)^{j} |d|^{e_1+j} R^{e_2-j} \mathfrak{Z}_d(g_{2j-1}),$$

which is a non-holomorphic modular form of weight $\frac{1}{2} + 2e_2$. Note that the coefficient of $q^{|d|}$ in $\mathcal{Z}_D(F)$ and the coefficient of $q^{|D|}$ in $\mathcal{Z}_d(F)$ are both equal to

$$|d|^{e_1} |D|^{e_2 - 1/2} \,\mathrm{Tr}_{d,D}(F). \tag{4.3}$$

Theorem 1.4 states that this must be p-integral for appropriate choices of p. Hence when $D = 1$, we have bounded the denominator by $|d|^{e_1}$. This bound determines the $B(n, k)$ of Theorem 1.1.

We will prove Theorem 1.4 by way of two propositions. In the following, we define the *principal part* of a function $F(z) = \sum_{m\gg-\infty} \sum_{n\geq 0} a(m,n) q^m \left(\frac{1}{4\pi y}\right)^n$ to be the polynomial in q^{-1} and $\frac{1}{4\pi y}$ given by $\sum_{m\leq 0} \sum_{n\geq 0} a(m,n) q^m \left(\frac{1}{4\pi y}\right)^n$.

Proposition 4.1. *Let $F(z)$ be a non-holomorphic modular function in $\mathbb{Q}((q))\left[\frac{1}{4\pi y}\right]$ with depth E and integral principal part, and let m be chosen so that then $R^M \mathcal{Z}_{\mathcal{D}}(\mathcal{F})$ has weight at least $3/2 + E$. Then $R^M \mathcal{Z}_{\mathcal{D}}(\mathcal{F})$ has integral principal part, and the coefficients* *of* $\left(\frac{1}{4\pi y}\right)^M q^n$ *can be written in terms of sums of traces of the modular functions* R^tL^tF *with* $0 \leq t \leq E$.

Proposition 4.2. *Let* $F(Z)$ *be a non-holomorphic modular function in* $\mathbb{Z}((q))\left[\frac{1}{4\pi y}\right]$. *If* M *is chosen as in Proposition 4.1, then* $R^M\mathcal{Z}_D(F)$ *has* p*-integral coefficients for any prime* p *which is good for* F.

Proposition 4.2 implies Theorem 1.4 since either $\mathcal{Z}_D(F)$ or $\mathcal{Z}_d(F)$ will satisfy the hypotheses of Proposition 4.1 with $M = 0$.

4.1. **Proof of Proposition 4.1.** The proof of Proposition 4.1 involves certain tedious combinatorial calculations. In the interest of brevity, we will omit some details. In particular, we only give the proof for for $D > 0$. Although the sign of D affects several details, a similar argument and calculations hold for $D < 0$, *mutatis mutandis*.

We wish to write $\mathcal{Z}_D(F)$ in terms of the traces of the non-holomorphic modular functions R^tL^tF. If G is modular with integral weight and rational coefficients, let $h_j(G, z)$ be the coefficient of $\left(\frac{1}{4\pi y}\right)^j$ in the expansion of G, so that

$$G = \sum_{j=0}^{E} h_j(G; z)\left(\frac{1}{4\pi y}\right)^j.$$

Similarly if G has half-integral weight, let $h_j^*(G, z)$ be the coefficient of $\left(\frac{1}{16\pi y}\right)^j$ in the expansion of G, so that

$$G = \sum_{j=0}^{E} h_j(G; z)\left(\frac{1}{16\pi y}\right)^j.$$

If t is a non-negative integer and F has integer coefficients, then $\frac{1}{t!}R^tL^tF$ has integer coefficients, and in particular we find that

$$h_0\left(\frac{1}{j!}R^jL^jF; z\right) = \left(\frac{1}{2\pi i}\cdot\frac{\partial}{\partial z}\right)^j h_j(F; z).$$

If $h_j(F; z) = \sum_{n \gg -\infty} a_j(n)q^n$, then let

$\widehat{h_j}(F; z) :=$

$$(-|D|)^{-j}\left(\sum_{m>0} a_\ell(-m)\sum_{n|m}\chi_D(n)n^j\left(\frac{m}{n}\right)^\delta q^{-\frac{m^2|D|}{n^2}} + \frac{1}{2}L(-j, \chi_D)a_j(0)\right) + \sum_{dD<0} B_{d,D}^{(j)}q^{|d|},$$

where d is a negative discriminant, $\delta = \begin{cases} 0 & \text{if } \ell \text{ is odd} \\ 1 & \text{if } \ell \text{ is even,} \end{cases}$ and

$$B_{d,D}^{(j)} = (-1)^{\left\lfloor\frac{j-1}{2}\right\rfloor}|d|^{\frac{-j-1+\delta}{2}}|D|^{\frac{-j-\delta}{2}}\operatorname{Tr}_{d,D}\left(\frac{1}{j!}R^jL^jF\right).$$

It suffices to prove the proposition with M chosen so that $R^M\mathcal{Z}_{\mathcal{D}}(\mathcal{F})$ has weight $3/2+E$. In this case, we will show that $R^M\mathcal{Z}_{\mathcal{D}}(\mathcal{F})$ is given by

$$\sum_{0\le\ell\le E/2}(-1)^{1+\ell}|D|^{e_2+\ell}R_{3/2+2\ell}^{\lfloor E/2\rfloor-\ell}\left[\widehat{h_{2\ell}}(F;z)\left(\frac{1}{16\pi y}\right)^{2\ell}+\widehat{h_{2\ell+1}}(F;z)\left(\frac{1}{16\pi y}\right)^{2\ell+1}\right].$$

By Theorem 1.3, $F(z)$ has a decomposition $\sum_{s=1}^{E+1}R^s g_s(z)$, where $g_s\in M^!_{-2s}$, and so we may consider the contributions of the $g_s(z)$ independently. If s is *odd*, $s=2t+1$, then Theorem 3.1 gives us that $\mathfrak{Z}_D(g_s(z))$ has weight $-1/2-2t$. We wish to show that $R^s\mathfrak{Z}_D(g_s(z))$ is equal to

$$\sum_{0\le\ell\le t}(-|D|)^{t+1+\ell}R_{3/2+2\ell}^{t-\ell}\left[\widehat{h_{2\ell}}(\partial g_s;z)\left(\frac{1}{16\pi y}\right)^{2\ell}+\widehat{h_{2\ell+1}}(\partial g_s;q)\left(\frac{1}{16\pi y}\right)^{2\ell+1}\right].\tag{4.4}$$

Here we have factored out some of the raising operators since $h_j(\partial g_s;z)=0$ if $j>s$. Using (2.2) to calculate $\partial g_s(z)$, we find that

$$h_j(\partial g_s,z)=\frac{(s+j)!}{(j)!(s-j)!}\left(\frac{\partial}{\partial z}\right)g_s(z)$$

Similarly, if we use (2.2) to calculate $R^{t+1+\ell}\mathfrak{Z}_D(g_s(z))$, we find the coefficient of $\left(\frac{1}{16\pi y}\right)^{2\ell+1}$ reduces to

$$\frac{(s+2\ell+1)!}{(2\ell+1)!(s-2\ell-1)!}\left(\frac{1}{2\pi i}\cdot\frac{\partial}{\partial z}\right)^{t-\ell}\mathfrak{Z}_D(G_s(z))=(-|D|)^{t+\ell+1}\widehat{h_{2\ell+1}}(\partial g_s,z).$$

The coefficient of $\left(\frac{1}{16\pi y}\right)^{2\ell}$ is not itself $(-|D|)^{t+\ell+1}\widehat{h_{2\ell}}(\partial g_s,z)$, however if we take the partial derivative $\frac{1}{2\pi i}\cdot\frac{\partial}{\partial z}$ of this coefficient, the result is. We now have that $R^s\mathfrak{Z}_D(g_s(z))$ and (4.4) must agree for powers of $\left(\frac{1}{16\pi y}\right)$ up to $2t+1$, and therefore these are equal.

For even s, say $s=2t$,, we have that $\mathfrak{Z}_D(g_s(z))$ has weight $3/2+2t$. After factoring out powers of the raising operator, we wish to show that $\mathfrak{Z}_D(g_s(z))$ is equal to

$$\sum_{0\le\ell\le t}(-|D|)^{\ell-t}R_{3/2+2\ell}^{t-\ell}\left[\widehat{h_{2\ell}}(\partial g_s,z)\left(\frac{1}{16\pi y}\right)^{2\ell}+\widehat{h_{2\ell+1}}(\partial g_s,z)\left(\frac{1}{16\pi y}\right)^{2\ell+1}\right].\tag{4.5}$$

Although (4.5) is not obviously weakly holomorphic as is $\mathfrak{Z}_D(g_s(z))$, we find that cancellation occurs for all positive powers of $\frac{1}{16\pi y}$. Using (2.2), we can expand (4.5) as a sum over powers of $\frac{1}{16\pi y}$. This process yields

$$\sum_{\substack{i,j,\ell,m\ge0\\ i+j+\ell+m=t}}(-1)^j(4)^{i+j}\binom{t-\ell}{i,j,m}\frac{\Gamma(3/2+t+\ell)}{\Gamma(3/2+t+\ell-j)}\left(\alpha(2\ell)+\alpha(2\ell+1)\right),$$

where

$$\alpha(k) = \frac{(s+k)!(k+i-1)!}{(k)!(s-k)!(k-1)!}\left(\frac{1}{16\pi y}\right)^{k+i+j} \mathfrak{Z}_D^{(-k-i-j)}(g_s),$$

and by $\mathfrak{Z}_D^{(-r)}(g_s)$ we mean a formal anti-derivative, so that if $\mathfrak{Z}_D(g_s) = \sum_{n\neq 0} a(n)q^n$ then $\mathfrak{Z}_D^{(-r)}(g_s) = \sum_{n\neq 0} a(n)n^{-r}q^n$. This expression may be rearranged to find the coefficients of $\left(\frac{1}{16\pi y}\right)^n$, however the resulting expression seems to have few obvious simplifications. Using an implementation of the Wilf-Zeilberger method [9] for Mathematica [11], we find that this new expression is identically zero for positive values of n. For $n = 0$, a quick calculation shows that we have recovered $\mathfrak{Z}_D(g_s)$.

4.2. **Proof of Proposition 4.2.** By Proposition 4.1, we have that the principal part of $R^M\mathcal{Z}_D(F)$ is integral. If $R^M\mathcal{Z}_D(F)$ is not entirely p-integral, take ℓ to be the largest integer such that $h_\ell^*(R^M\mathcal{Z}_D(F);z)$ is not p-integral, and let p^n exactly divide the denominator. Then $p^n\frac{4^\ell}{\ell!}L^\ell R^M\mathcal{Z}_D(F) \equiv p^n h_\ell^*(R^M\mathcal{Z}_D(F);z) \pmod{p}$. Since the principal part of $h_\ell^*(R^M\mathcal{Z}_D(F);z)$ is integral, we have that $p^n h_\ell^*(R^M\mathcal{Z}_D(F);z)$ is congruent to a cusp form of the appropriate weight, say k. We note that k must be a decomposition weight for $Z_D(F)$, and therefore the corresponding weight $3-2k$ must be a decomposition weight for F. Moreover, since the coefficients of $h_\ell^*(R^M\mathcal{Z}_D(F);z)$ are given in terms of Zagier lifts of weakly holomorphic modular forms, (3.2) shows that this cusp form is eventually annihilated $\pmod{p}$ by the Hecke operator T_{p^2}. Since T_{p^2} acts nilpotently on the space of cusp forms S_k, it must also act nilpotently on S_{2k-1}, since these spaces are isomorphic by way of the Shimura correspondence [7]. Thus p is not good for $3-2k$.

5. A Useful Vanishing Condition

In this section, we prove Theorem 1.5, which is essentially a combinatorial fact. In particular, we use the structure of *Rankin-Cohen brackets* to provide a convenient basis for expressing the combinatorics of the spectral decomposition of the product of two forms.

5.1. **Rankin-Cohen Brackets.** In [4] and [10], Rankin and Cohen utilized certain polynomials in derivatives of modular forms which are again modular, called the *Rankin-Cohen Brackets*. Let f be a modular form of weight k, g a modular form of weight ℓ, and n a non-negative integer. Then the n^{th} Rankin-Cohen bracket is defined as:

$$[f,g]_n^{(k,\ell)} := \sum_{r+s=n} (-1)^r \binom{n+k-1}{s}\binom{n+\ell-1}{r} f^{(r)}\cdot g^{(s)}.$$

Here $f^{(n)} := \left(\frac{1}{2\pi i}\frac{d}{dz}\right)^n f$. We will suppress the dependence on the weights k and ℓ and write simply $[f,g]_n$ when the dependence is clear from context. The key fact is that:

$$[\cdot,\cdot]_n^{(k,\ell)} :\ M_k^! \otimes M_\ell^! \to M_{k+\ell+2n}^!. \tag{5.1}$$

This can be seen, for example, using the Cohen-Kuznetsov lifting to Jacobi-like forms.

5.2. **The Vanishing Lemma.** Using the Rankin-Cohen brackets, we are now in position to prove Theorem 1.5. Using an inductive argument and the spectral decomposition of Theorem 1.3, it suffices to prove the following lemma for the product of raisings of just two forms of possibly different weights.

Lemma 5.1. *Let $f \in M_k^!$ and $g \in M_\ell^!$ have negative even weight. Set $F := \partial f \cdot \partial g$ and write $F = \sum \partial(g_i)$ for the modular forms g_i defined in Theorem* 1.3*. Suppose $m = k + \ell + 4i + 2$ where $0 \leq i \leq -\frac{\min\{k,\ell\}}{2} - 1$. Then if $g_{m/2}$ has weight m, we have that $g_{m/2} \equiv 0$.*

Proof. In Proposition 2.3 of [1], the authors consider a similar combinatorial expansion which is given in terms of Rankin-Cohen brackets. Using their proposition, it suffices to prove the following (setting $k = -2r$, $\ell = -2s$) whenever $j < r$ and j is odd:

$$S(j) := \sum_{m=0}^{s} (-1)^{(j+m)} \cdot \frac{\binom{m+r}{j}\binom{s}{m}\binom{m-r-1}{r+m-j}}{\binom{-r-2s+m+j-1}{m+r-j}} = 0.$$

Using the Wilf-Zeilberger method [9], one finds that the function $S(j)$ satisfies the following recursion in the range $j < r$:

$$(2+j)(1+j-2r)(1+j-2s)(j-2r-2s)\cdot S(j+2)$$
$$-4(1+2j-2r-2s)(3+2j-2r-2s)(j-r-s)(1+j-r-s)\cdot S(j) = 0.$$

For the base case, $j = 1$, we must show that g_{E-1} vanishes in the notation of Theorem 1.3. A calculation shows that $L^{E-1}[(\partial f)\cdot(\partial g)]$ is some nonzero multiple of $R(f \cdot g)$, so that by Theorem 1.3, we have that g_{E-1} is a multiple of Rg_E. However g_{E-1} is holomorphic, whereas Rg_E is not, which implies that $g_{E-1} \equiv 0$. □

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