Maass Forms and Quantum Modular Forms

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Remarks

- *If $k = 0$, we call f a modular function.*
- *We can also define modular forms of half-integral weight.*

Congruence Subgroups

We are mainly interested in modular forms on groups like:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$$

Fourier Expansions

Any modular form of level N has a Fourier expansion

$$f(z) = \sum_{n \gg -\infty} a_n q^n,$$

where $q := e^{2\pi i z}$.

Examples

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2. The weight 12 modular discriminant function is the infinite product:

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3. The weight $\frac{1}{2}$ Jacobi theta function

$$\theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2}.$$

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- Zagier defined "traces of singular moduli", which he proved are often coefficients of modular forms.
- We consider integrality for the polynomials arising from non-holomorphic functions.

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Definition

Let Q_d be the set of positive definite binary quadratic forms of discriminant d. For a modular function F, define the trace:

$$\mathrm{Tr}_d(F) := \sum_{Q \in Q_d/\Gamma} w_Q^{-1} F(\tau_Q).$$

An Example of Zagier's Theory

Theorem (Zagier)

Let

$$J(z) := j(z) - 744$$

and

$$g(z) := \theta_1(z)\frac{E_4(4z)}{\eta(4z)^6} = \sum B(d)q^n$$

For any positive integer $d \equiv 0, 3 \pmod 4$, *we have*

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For any positive integer $d \equiv 0, 3 \pmod 4$, *we have*

$$\mathrm{Tr}_{\text{-d}}\left(\mathrm{J(z)}\right) = -B(d).$$

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Remark

It appears that the third symmetric function is always an integer.

Traces for Negative Weight Forms

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- For f of negative weight, ∂f is the iterated raising to weight 0.

Theorem 1

Theorem 1 (Griffin-R 2012)

Let $f(z) \in M_k^!$, $0 > k \in 2\mathbb{Z}$ have integral principal part. Denote the n^{th} symmetric function in the singular moduli of discriminant d for ∂f by $\mathcal{S}_f(n; d)$. Let

$$B(n,k) := \begin{cases} \frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\ \frac{1}{4}(-nk + 2k - 2) & \text{otherwise.} \end{cases}$$

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Then we have that

$$d^{B(n,k)} \cdot \mathcal{S}_f(n; d) \in \mathbb{Z}.$$

Special Cases

Corollary

For any $f(z) \in M^!_{-2}$ with integral principal part, we have that

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Remark

This theorem is sharp.

Sketch of Proof

- Use Newton's identities to reduce to sums of powers.

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- Unfortunately, powers of Maass forms are usually not finite sums of Maass forms.

The Spectral Decomposition

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Remark

The proof gives an explicit algorithm for computing the forms g_j.

Sketch of Proof (cont).

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- Bounding denominators on each piece gives a naïve bound.

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- Obstruction 1: Certain weights in the decomposition give the wrong denominators.
- We prove a vanishing condition on which forms in the decomposition actually appear.
- Obstruction 2: The coefficients $c_{i,j}$ in the previous theorem also introduce artificial denominators.
- We show that they cancel using the action of the Hecke algebra on Poincaré series.

Q.E.D.

Rankin-Cohen Brackets

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$$[f, g]_n^{(k,\ell)} := \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f^{(r)} \cdot g^{(s)}.$$

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- This gives a map

$$[\cdot,\cdot]_n^{(k),(\ell)} : \ M^!_k \otimes M^!_\ell \to M^!_{k+\ell+2n}.$$

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- In this case, we can expand in terms of Rankin-Cohen brackets.
- Using a calculation of Beyerl-James-Trentacoste-Xue, this reduces to a binomial sum identity, for j odd

$$\sum_{m=0}^{s}(-1)^{(j+m)}\cdot\frac{\binom{m+r}{j}\binom{s}{m}\binom{m-r-1}{r+m-j}}{\binom{-r-2s+m+j-1}{m+r-j}}=0.$$

Obstruction 2: Lining Up Principal Parts

- Raise the Zagier lifts of the pieces to the same weight and let:

$$Z(\tau) := \sum_{t=0}^{\left\lfloor \frac{E+1}{2} \right\rfloor} (-1)^{M+t} R^{M+t} \mathfrak{Z}_1(g_{2t-1}) + \sum_{t=0}^{M} (-1)^{M+t} R^{M-t} \mathfrak{Z}_1(g_{2t}).$$

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- By comparison with F, we observe that the holomorphic part Z^+ of Z has integral principal part.
- If all the coefficients of Z^+ are integral, then the $c_{i,j}$-denominators will cancel.

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- The Zagier lift is equivariant with the Hecke action:

$$\mathfrak{Z}_D(f|T(n)) = \mathfrak{Z}_D(f)|T(n^2).$$

Integrality of Coefficients

We construct a family of Hecke operators with “nice properties”.

Corollary

If $f_{k,1}|H$ has integer coefficients, p is ordinary for all eigenforms in a basis of S_k, and $f_{k,1}|H \equiv 0 + O(q) \pmod{p^n}$, then

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$$f_{k,1}|H \equiv 0 \pmod{p^n}.$$

A Tricky Question

Consider the integral

$$\int_{\alpha}^{i\infty} \frac{\eta(2z)^2/\eta(z)}{(z-\alpha)^{3/2}}\,\mathrm{d}z.$$

Question

How does one evaluate it?

What can we do?

What can we do?

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We give exact values for all of these integrals as algebraic multiples of π by specializing **one** *formula.*

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- Functions on $\mathbb{Q}$ which are modular up to a "nice function".
- They have connections to: unimodal sequences, ranks, cranks, Dedekind sums, Eichler integrals, mock theta functions ...

Defining Quantum Modular Forms

Definition

We say that a function $f : \mathbb{Q} \to \mathbb{C}$ is a quantum modular form if

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A "Strange" Quantum Modular Form

- A striking example of quantum modularity is given by the Kontsevich "strange function":

$$F(q) = \sum_{n=0}^{\infty} (1-q)(1-q^2)\cdots(1-q^n) = \sum_{n=0}^{\infty} (q;q)_n.$$

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Remark

This function is strange as it is not defined on any open subset of $\mathbb{C}$, but is well-defined at roots of unity.

Zagier's Result

Theorem (Zagier)

We have that $e^{\pi ix/12}F(e^{2\pi ix})$ *is a wt.* $3/2$ *quantum modular form.*

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- We study the vector-valued form:

$$H(q) = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} := \begin{pmatrix} \eta(z)^2/\eta(2z) \\ \eta(z)^2/\eta(z/2) \\ \eta(z)^2/\eta(\frac{z}{2}+\frac{1}{2}) \end{pmatrix}.$$

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- We then associate finite versions $\theta_{i,n}$ so that $\theta_{i,n} \to \theta_i$.
- The corresponding "strange" function is $\theta_i^S := \sum_{n=0}^{\infty} \theta_{i,n}$, which converges on some set of roots of unity.

A Vector-Valued Quantum Modular Form

Theorem 2 (R-Schneider 2012)

- *There are q-series* G_i *also defined for* $|q| < 1$ *with*

$$\theta_i^S(q^{-1}) = G_i(q).$$

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- *We find* $\left(\theta_1^S,\ \theta_2^S,\ \theta_3^S\right)^T$ *is a wt.* $3/2$ *quantum modular form.*

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k	$\pi i(i+1)\theta_1^S(\zeta_k)$	$\mathcal{I}(1/k, 10^9)$
3	$\pi i(i+1)(-2\zeta_3+3) \sim -7.1250 + 18.0078i$	$-7.1249 + 18.0078i$
5	$\pi i(i+1)(-2\zeta_5^3 - 2\zeta_5^2 - 8\zeta_5 + 3) \sim 12.078 + 35.7274i$	$12.078 + 35.7273i$
7	$\pi i(i+1)(6\zeta_7^4 - 2\zeta_7^2 - 10\zeta_7 + 7) \sim 52.0472 + 25.685i$	$52.0474 + 25.685i$
9	$\pi i(i+1)(8\zeta_9^4 - 16\zeta_9 + 3) \sim 76.4120 - 28.9837i$	$76.4116 - 28.9836i$

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$$= \eta(24z)D(q) + E(q)$$

where $E(q)$ is a "half-derivative" of $\eta(24z)$.

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  where $E(q)$ is a "half-derivative" of $\eta(24z)$.

- Thus, $F(q)$ equals a half-derivative of $\eta(24z)$ at roots of unity.
- Such a half-derivative is equal to an "Eichler integral", but now the integral lives in $\mathbb{H}^-$ and agrees at rationals.

Sketch of the Proof

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- The modularity of Eichler integrals comes from modularity of the original θ-functions.
- Our strategy is as follows:

$$\text{Strange function} \overset{\text{Sum of tails}}{\leftrightsquigarrow} \text{Half-Derivatives} \overset{\text{Reflection}}{\leftrightsquigarrow} \text{Eichler Integral}$$

Sums of Tails Identities

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Theorem (Andrews, Jimenez-Urroz, Ono)

As formal power series, we have

$$\sum_{n=0}^{\infty} (F_9(z) - F_{9,n}(z)) = 2F_9(z)E_1(z) + 2\sqrt{\theta}(F_9(z)),$$

$$\sum_{n=0}^{\infty} (F_{10}(z) - F_{10,n}(z)) = F_{10}(z)E_2(z) + \sqrt{\theta}(F_{10}(z)).$$

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Theorem (Andrews, Jimenez-Urroz, Ono)

As formal power series, we have

$$\sum_{n=0}^{\infty}(F_9(z) - F_{9,n}(z)) = 2F_9(z)E_1(z) + 2\sqrt{\theta}(F_9(z)),$$

$$\sum_{n=0}^{\infty}(F_{10}(z) - F_{10,n}(z)) = F_{10}(z)E_2(z) + \sqrt{\theta}(F_{10}(z)).$$

- Here $\sqrt{\theta}\sum a(n)q^n := \sum \sqrt{n}a(n)q^n$.

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- Recall that $\mathcal{E}_f$ is modular up to a period polynomial:
$$g(x) := c_k \int_0^{\infty} f(z)(z-x)^{k-2} \, \mathrm{d}z.$$

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- This can be fixed by defining an integral in the lower half plane which agrees with $\sqrt{\theta}(f)$ at rationals.
- The obstruction to modularity is not a polynomial, but it is still a $\mathcal{C}^\infty$-function on $\mathbb{R}$.

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$$H(-1/z) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} H(z).$$

Proof of the Theorem (cont.)

- Extension of the strange functions to the upper half plane (after reflection) follows from power series manipulations, e.g.

$$\theta_1^S(q^{-1}) = 2\sum_{n=0}^{\infty} \frac{q^{2n+1}(q;q)_{2n}}{(1+q^{2n+1})(-q;q)_{2n}}.$$

The great anticipator of mathematics

Srinivasa Ramanujan (1887-1920)

"Death bed letter"

"Dear Hardy, I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (partially studied by Rogers), they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples."

Ramanujan, January 12, 1920.

The first example

$$f(q) = 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots$$

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"Theorem"

Ramanujan's mock theta functions are holomorphic parts of weight 1/2 harmonic Maass forms.

Defining Maass forms

Notation. Throughout, let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$.

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Hyperbolic Laplacian.

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

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2. *We have that* $\Delta_k f = 0$.

HMFs have two parts

"Fundamental Lemma"

If $f \in H_{2-k}$ and $\Gamma(a,x)$ is the incomplete Γ-function, then

$$f(z) = \sum_{n \gg -\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|y)q^n.$$

$\updownarrow$ **Holomorphic part f^+**

$\updownarrow$ **Nonholomorphic part f^-**

Remark

The mock theta functions are examples of f^+.

So many recent applications

- q-series and partitions
- Modular L-functions (e.g. BSD numbers)
- Eichler-Shimura theory
- Probability models
- Generalized Borcherds products
- Moonshine for affine Lie superalgebras and M_{24}
- Donaldson invariants
- Black holes
- ...

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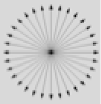
Ramanujan's last letter.

- Asymptotics, near roots of unity, of "Eulerian modular forms".



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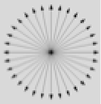

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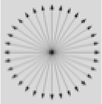

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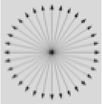


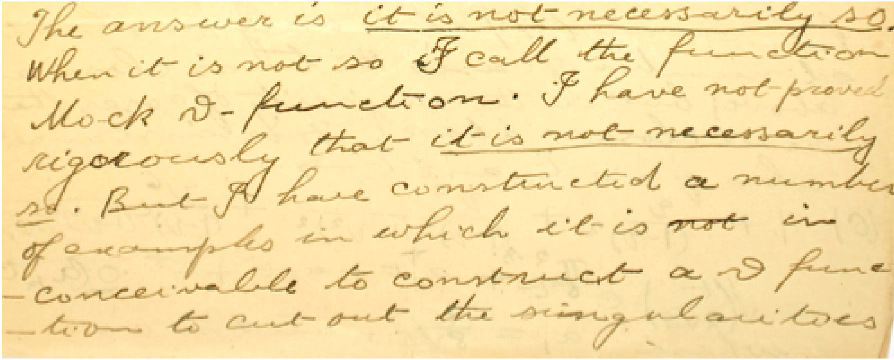
- Raises **one** question and conjectures the answer.
- Gives **one example** supporting his conjectured answer.
- Concludes with a list of his mock theta functions.

Ramanujan's question

Question (Ramanujan)

Must Eulerian series with "similar asymptotics" be the sum of a modular form and a function which is $O(1)$ at all roots of unity?

Ramanujan's answer



The answer is it is not necessarily so. When it is not so I call the function Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is ~~not~~ inconceivable to construct a ϑ function to cut out the singularities

Ramanujan's last words

"it is inconceivable to construct a ϑ function to cut out the singularities of a mock theta function..."

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Theorem 3 (Griffin-Ono-R 2013)

Ramanujan's examples satisfy his own definition.

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Theorem 3 (Griffin-Ono-R 2013)

Ramanujan's examples satisfy his own definition. More precisely, a mock theta function and a modular form never cut out exactly the same singularities.

Sketch of proof: parallel weight

- Suppose a mock theta function f of weight k is cut out by a modular form g of weight k'.

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- By the Bruinier-Funke pairing, any HMF has a nonzero principal part at some cusp.

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- If f cut out g, then $\tilde{f}$ cuts out $\tilde{g}$ where $\tilde{g}$ is the result of twisting g.
- We ruled out the case $k = k'$. If $k \neq k'$, it is easy to show this cannot happen for two modular forms.

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- Symmetric functions in singular moduli for nonholomorphic modular functions.
- A new example of a quantum modular form.
- Ramanujan's original definition of a mock modular form.

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Thank you!