

# INTEGRALITY PROPERTIES OF THE CM-VALUES OF CERTAIN WEAK MAASS FORMS

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ABSTRACT. In a recent paper, Bruinier and Ono prove that the coefficients of certain weight  $-1/2$  harmonic Maass forms are traces of singular moduli for weak Maass forms. In particular, for the partition function  $p(n)$ , they prove that

$$p(n) = \frac{1}{24n-1} \cdot \sum P(\alpha_Q),$$

where  $P$  is a weak Maass form and  $\alpha_Q$  ranges over a finite set of discriminant  $-24n+1$  CM points. Moreover, they show that  $6 \cdot (24n-1) \cdot P(\alpha_Q)$  is always an algebraic integer, and they conjecture that  $(24n-1) \cdot P(\alpha_Q)$  is always an algebraic integer. Here we prove a general theorem which implies this conjecture as a corollary.

## 1. INTRODUCTION AND STATEMENT OF RESULTS

A *partition* of a positive integer  $n$  is any nonincreasing sequence of positive integers which sum to  $n$ . The partition function  $p(n)$ , which counts the number of partitions of  $n$ , is an important function in number theory whose study has a long history. One of the celebrated results of Hardy and Ramanujan on this function, giving rise to the “circle” method, quantifies the growth rate:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}}.$$

This asymptotic and its method of proof were later refined by Rademacher, yielding an “exact” formula in terms of a modified Bessel function of the first kind  $I_{\frac{3}{2}}(\cdot)$  and a Kloosterman sum  $A_k(n)$ :

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

One can compute values of  $p(n)$  from this formula by using sufficiently accurate truncations. Bounding the resulting error is a well-known difficult problem; the best-known bounds are due to Folsom and Masri [4].

In recent work [2], Bruinier and Ono prove a new formula for  $p(n)$  as a finite sum of algebraic numbers. These numbers are *singular moduli* for a *weak Maass form* which

they describe in terms of Dedekind's eta function and the quasimodular Eisenstein series  $E_2$ , which are defined in terms of  $q := e^{2\pi iz}$  as

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{and} \quad E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} dq^n.$$

They then define the  $\Gamma_0(6)$  weight  $-2$  meromorphic modular form:

$$(1) \quad F_p(z) := \frac{1}{2} \cdot \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{\eta(z)^2 \eta(2z)^2 \eta(3z)^2 \eta(6z)^2} = q^{-1} - 10 - 29q - \dots$$

Using the convention that  $z := x + iy$ , with  $x, y \in \mathbb{R}$ , they define the weak Maass form:

$$(2) \quad P_p(z) := - \left( \frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y} \right) F_p(z) = \left( 1 - \frac{1}{2\pi y} \right) q^{-1} + \frac{5}{\pi y} + \left( 29 + \frac{29}{2\pi y} q \right) + \dots$$

Bruinier and Ono give a formula for  $p(n)$  in terms of discriminant  $-24n+1 = b^2 - 4ac$  positive definite integral binary quadratic forms  $Q(x, y) = ax^2 + bxy + cy^2$  satisfying the condition  $6 \mid a$ . The group  $\Gamma_0(6)$  acts on such forms, and we let  $\mathcal{Q}_n$  be any set of representatives of those equivalence classes with  $a > 0$  and  $b \equiv 1 \pmod{12}$ . To each such  $Q$ , we associate the CM point  $\alpha_Q$  defined to be the root of  $Q(x, 1) = 0$  lying in the upper half of the complex plane. Then the formula of Bruinier and Ono states:

$$(3) \quad p(n) = \frac{1}{24n-1} \cdot \sum_{Q \in \mathcal{Q}_n} P(\alpha_Q).$$

They further prove that each  $6 \cdot (24n-1) \cdot P_p(\alpha_Q)$  is an algebraic integer. They also show that the numbers  $P(\alpha_Q)$ , as  $Q$  varies over  $\mathcal{Q}_n$ , form a multiset which is a union of Galois orbits for the discriminant  $-24n+1$  ring class field. Based on numerics, they made the following conjecture:

**Conjecture** (Bruinier and Ono [2]). *For the Maass form  $P(z)$  above and for the  $\alpha_Q$  in the formula for  $p(n)$ , we have that  $(24n-1) \cdot P_p(\alpha_Q)$  is an algebraic integer.*

We prove that this is indeed the case. In fact, we prove that this is true for all the CM points of discriminant  $-24n+1$  of a wider class of Maass forms. Namely, we have the following:

**Theorem 1.1.** *Suppose  $F$  is a weakly holomorphic, weight  $-2$  modular form on a congruence subgroup such that the Fourier expansions of*

$$F \quad \text{and} \quad q \frac{dF}{dq} + F \cdot \frac{E_2 E_4 - E_6}{6E_4}$$

at all cusps have coefficients that are algebraic integers. Let  $\alpha_Q$  be the CM point in  $\mathbb{H}$  corresponding to a quadratic form  $Q(x, y)$  of discriminant  $-24n + 1$ , and let  $P(z)$  be the weak Maass form

$$P(z) = - \left( \frac{1}{2\pi i} \cdot \frac{d}{dz} + \frac{1}{2\pi y} \right) F(z).$$

Then  $(24n - 1) \cdot P(\alpha_Q)$  is an algebraic integer.

*Remark.* We recall that a meromorphic modular form is said to be *weakly holomorphic* if its poles are supported on the cusps.

The form  $F_p(z)$  studied by Bruinier and Ono satisfies these conditions. One can see this because  $F_p(z)$  has level 6, so the group of Atkin-Lehner involutions acts transitively on the cusps. Since  $F_p(z)$  is an eigenform for all of the Atkin-Lehner involutions and has an integral Fourier expansion at infinity, it follows that the Fourier expansions of  $F_p$  at all cusps is integral. Moreover, since the Atkin-Lehner involutions commute with the Maass raising operator

$$R_{-2} = -4\pi q \frac{d}{dq} - \frac{2}{y},$$

the Fourier expansion of

$$q \frac{dF}{dq} + F \cdot \frac{E_2 E_4 - E_6}{6E_4} = F \cdot \frac{(E_2 - \frac{3}{\pi \operatorname{Im} z}) E_4 - E_6}{6E_4} - \frac{1}{4\pi} R_{-2} F$$

at all cusps is integral as well. Therefore, Theorem 1.1 implies the following:

**Corollary 1.2.** *The conjecture of Bruinier and Ono is true.*

*Remark.* Corollary 1.2 is sharp for small (and possibly all)  $n$ . For example, we have

$$\prod_{m=1}^3 (x - P_p(\alpha_{Q_m})) = x^3 - 23x^2 + \frac{3592}{23}x - 419,$$

where  $Q_m$  ranges over any choice of representatives of  $\mathcal{Q}_1$ .

Returning to the general case of  $P(z)$  as in Theorem 1.1, the work of Bruinier and Ono (Theorem 4.5 of [2]) implies  $6 \cdot (24n - 1) \cdot P(\alpha_Q)$  is an algebraic integer. Although this theorem is stated for squarefree level and when  $F$  is an eigenfunction of the Atkin-Lehner involutions, an inspection of the proof shows that the assumptions in the statement of Theorem 1.1 are also sufficient. Thus it suffices to show that  $P(\alpha_Q)$  is integral at primes lying over 6. We will henceforth refer to this property as *6-integrality*. For this purpose, it is convenient to decompose  $P$  as

$$(4) \quad P = A + B \cdot C,$$

where

$$(5) \quad A = -q \frac{dF}{dq} - \frac{1}{6} F E_2 + \frac{F E_6 (7j - 6912)}{6 E_4 (j - 1728)},$$

$$(6) \quad B = \frac{F E_6 j}{E_4},$$

$$(7) \quad C = \frac{E_4}{6 E_6 j} \left( E_2 - \frac{3}{\pi \operatorname{Im} z} \right) - \frac{7j - 6912}{6j(j - 1728)}.$$

To establish the 6-integrality of  $P(\alpha_Q)$ , it suffices to establish the 6-integrality of each of  $A(\alpha_Q)$ ,  $B(\alpha_Q)$  and  $C(\alpha_Q)$ . In Section 2, we will use methods similar to those of [2] to show that  $A(\alpha_Q)$  and  $B(\alpha_Q)$  are 6-integral. Then in Section 3, we show that  $C(\alpha_Q)$  is 6-integral using a description of  $C$  in terms of classical modular polynomials due to Masser.

*Remark.* For the remainder of this paper, we fix  $D \equiv 1 \pmod{24}$  with  $D < 0$ , and we let  $\alpha_Q$  denote any CM point of discriminant  $D$ .

## 2. PROOF OF 6-INTEGRALITY OF $A$ AND $B$

In this section, we prove the 6-integrality of  $A$  and  $B$  at the CM-points  $\alpha_Q$ . We begin by showing that  $j(\alpha_Q)$  is a unit at 2 and 3.

**Lemma 2.1.** *Let  $p \in \{2, 3\}$  and  $E$  be an elliptic curve defined over a number field  $K$  having complex multiplication by an order in a quadratic field  $F$ . If  $E$  has good ordinary reduction at all primes lying over  $p$ , then  $j(E)$  is coprime to  $p$ .*

*Proof.* Assume to the contrary that  $j(E)$  was not coprime to  $p$ ; write  $\mathfrak{p}$  for a prime ideal lying over  $p$  containing  $j(E)$ , and write  $k$  for the residue field  $\mathcal{O}_K/\mathfrak{p}$ .

When  $p = 2$ , the elliptic curve  $E^2 = \mathbb{C}/\mathbb{Z}[\omega]$  (where  $\omega$  is a primitive cube root of unity) has good supersingular reduction at  $\mathfrak{p}$ . But  $j(\omega) = 0$ , so  $E_{/k}^2 \simeq E_{/k}$ , so  $E_{/k}$  is supersingular, which is a contradiction.

Similarly, when  $p = 3$ , the elliptic curve  $E^3 = \mathbb{C}/\mathbb{Z}[i]$  has good supersingular reduction at  $\mathfrak{p}$ . But  $j(\omega) = 1728$ , so  $E_{/k}^3 \simeq E_{/k}$ , so  $E_{/k}$  is supersingular, which is a contradiction.  $\square$

By this lemma, it suffices to show that both  $B$  and  $A' := A \cdot j \cdot (j - 1728)$  assume integral values at all CM-points.

**Lemma 2.2.** *The modular functions  $A'$  and  $B$  are weakly holomorphic and have integral Fourier expansions at all cusps.*

*Proof.* By definition, we have

$$B = F \cdot E_6 \cdot \frac{j}{E_4},$$

and by direct examination, all three of the above terms are weakly holomorphic and have integral Fourier expansions at all cusps. Similarly, by definition, we have

$$A' = F \cdot E_6(j - 864) \cdot \frac{j}{E_4} - (j - 1728) \cdot \left[ j \cdot \left( q \frac{dF}{dq} + F \cdot \frac{E_2 E_4 - E_6}{6E_4} \right) \right]$$

and all of the above terms are weakly holomorphic and have integral Fourier expansions at all cusps.  $\square$

**Lemma 2.3.** *A weakly holomorphic modular function  $g$  for a congruence subgroup  $\Gamma_g$  that has integral Fourier expansions at all cusps is integral at any CM-point.*

*Proof.* (The following argument is due to Bruinier and Ono; see Lemma 4.3 of [2].) We consider the polynomial

$$\Psi_g(X, z) = \prod_{\gamma \in \Gamma_g \backslash \Gamma(1)} (X - g(\gamma z)).$$

This is a monic polynomial in  $X$  of degree  $[\Gamma(1) : \Gamma_g]$  whose coefficients are weakly holomorphic modular functions in  $z$  for the group  $\Gamma(1)$ , so  $\Psi_g(X, z) \in \mathbb{C}[j(z), X]$ .

Our assumption that  $g$  has integral Fourier expansion at all cusps implies that for any  $\gamma \in \Gamma(1)$ , the modular function  $g \mid \gamma$  has a Fourier expansion at infinity whose coefficients are algebraic integers. Thus, the coefficients of  $\Psi_g(X, z)$  are polynomials in  $j(z)$  whose coefficients are algebraic integers.

Since  $j$  is integral at any CM-point  $\alpha$ , the value  $g(\alpha)$  satisfies a monic polynomial whose coefficients are algebraic integers, and is therefore an algebraic integer.  $\square$

*Remark.* In Appendix A, we give values of the polynomials  $\Psi_{A'}$  and  $\Psi_B$  for the form  $F_p$  considered by Bruinier and Ono, thus providing a direct proof of the integrality of  $A'(\alpha_Q)$  and  $B(\alpha_Q)$  in this case.

**Lemma 2.4.** *If  $\alpha_Q$  is an CM-point with discriminant  $D \equiv 1 \pmod{24}$ , then  $A(\alpha_Q)$  and  $B(\alpha_Q)$  are 6-integral.*

*Proof.* This follows from combining Lemmas 2.1, 2.2, and 2.3.  $\square$

### 3. PROOF OF 6-INTEGRALITY FOR $C(\alpha_Q)$

In this section, we finish the proof of Theorem 1.1 by showing that  $C(\alpha_Q)$  is 6-integral at the required CM points  $\alpha_Q$ . To do this, we study the classical modular polynomials  $\Phi_{-D}$ , in a fashion similar to Appendix 1 of [3]. We begin by reviewing the definition of  $\Phi_{-D}$ .

**Definition 1.** We say that two matrices  $B_1$  and  $B_2$  are *equivalent* if  $B_1 = X \cdot B_2$  for some  $X \in \mathrm{SL}_2(\mathbb{Z})$ .

It is well-known that there are only finitely many equivalence classes of primitive integer matrices of determinant  $-D$ . Write  $M_1, M_2, \dots, M_n$  for these equivalence classes and suppose  $M_1$  is such that  $\alpha_Q = M_1\alpha_Q$ .

**Definition 2.** We write  $\Phi_{-D}(X, Y)$  for the *classical modular polynomial*, i.e. the polynomial such that

$$\Phi_{-D}(j(z), Y) = \prod_{i=1}^n (Y - j(M_i z)).$$

By [1], Theorem 1 of Section 3.4, the polynomial  $\Phi_{-D}(X, Y)$  is symmetric in  $X$  and  $Y$  and has coefficients that are rational integers. In particular, we can expand  $\Phi_{-D}(X, Y)$  in a power series about  $X = Y = j(\alpha_Q)$  as

$$\Phi(X, Y) = \sum_{\mu, \nu} \beta_{\mu, \nu} (X - j(\alpha_Q))^\mu (Y - j(\alpha_Q))^\nu,$$

where  $\beta_{\mu, \nu} = \beta_{\nu, \mu}$ . We write  $\beta = \beta_{0,1} = \beta_{1,0}$ .

We define  $Q$  to be *special* if there is more than one equivalence class of matrices  $M$  such that  $M\alpha_Q = \alpha_Q$ . This can only happen if  $D = 3d^2$  for some integer  $d$  (see [3], Appendix 1), so in particular forms of discriminant  $-24n + 1$  are not special.

**Lemma 3.1** (Masser). *If  $Q$  is not special, we have  $\beta \neq 0$  and*

$$C(\alpha_Q) = \frac{\beta_{0,2} - \beta_{1,1} + \beta_{2,0}}{\beta}.$$

*Proof.* See [3], Appendix 1 (in particular, equations (100) and (106), and the definition of  $\gamma$  on page 118).  $\square$

By definition, the  $\beta_{\mu, \nu}$  are algebraic integers. Thus, to prove that  $C(\alpha_Q)$  is integral at primes lying over 6, it suffices to show that  $\beta$  is a unit at primes lying over 6. From the definition of  $\beta$ , we have

$$\beta = \prod_{i=2}^n (j(\alpha_Q) - j(M_i\alpha_Q)).$$

Thus, it suffices to show that for any prime  $\mathfrak{p}$  lying over 6, we have  $j(\alpha_Q) \not\equiv j(M_i\alpha_Q) \pmod{\mathfrak{p}}$ . To show this, it is enough to establish the following lemma:

**Lemma 3.2.** *Suppose  $\mathfrak{p}$  is a prime ideal of a number field  $K$ . Suppose  $E$  and  $E'$  are two elliptic curves over  $K$  with complex multiplication by the same order  $R$  in a quadratic field  $F$ . Suppose the index  $[\mathcal{O}_F : R]$  is coprime to the residue characteristic of  $\mathfrak{p}$ . If both curves have good ordinary reduction at  $\mathfrak{p}$  and the reduced curves are isomorphic, then  $E$  and  $E'$  are also isomorphic.*

*Proof.* Write  $k$  for the residue field  $\mathcal{O}_K/\mathfrak{p}$  and  $p = \text{char}(k)$ . As the index of  $R$  in  $\mathcal{O}_F$  is coprime to  $p$ , there is an isogeny  $f: E \rightarrow E'$  whose degree is coprime to  $p$ . Since  $E$  has ordinary reduction at  $\mathfrak{p}$ , its endomorphisms over  $k$  are a rank-2 submodule  $S$  of  $\mathcal{O}_F$  which contains  $R$ . As the index of  $R$  in  $\mathcal{O}_F$  is coprime to  $p$ , the index  $d$  of  $R$  in  $S$  is also coprime to  $p$ . Choose an isomorphism between the reductions  $E_{/k}$  and  $E'_{/k}$ . Composing this with the isogeny  $f$  gives an endomorphism of  $E_{/k}$ , and multiplying this endomorphism by  $d$  gives an endomorphism which lifts to an endomorphism  $g$  of  $E$  whose degree is coprime to  $p$ . Now the specializations of the kernels of  $f \circ d$  and  $g$  coincide by construction, and both kernels are subgroups whose order is coprime to  $p$ . Thus,  $\ker f \circ d = \ker g$ , and therefore  $E \cong E'$ .  $\square$

This completes the proof of the 6-integrality of  $C(\alpha_Q)$ , as the assumption  $D \equiv 1 \pmod{24}$  shows that the conditions of the above lemma are satisfied. By the discussion in Section 1, this establishes Theorem 1.1.

#### APPENDIX A. THE POLYNOMIALS $\Psi_{A'}$ AND $\Psi_B$ FOR $F = F_p$

Here, we give the explicit values of the polynomials  $\Psi_{A'}$  and  $\Psi_B$  when  $F = F_p$  is the form considered by Bruinier and Ono in [2]. Namely, we have

$$\Psi_{A'} = X^{12} + \sum_{i=0}^{11} a_i X^i \quad \text{and} \quad \Psi_B = X^{12} + \sum_{i=0}^{11} b_i X^i,$$

where the  $a_i$  and  $b_i$  are the polynomials in  $j$  with integer coefficients given below.

$$\begin{aligned} a_{11} &= -2 \cdot (j - 2^6 \cdot 3^3) \cdot (j - 2^5 \cdot 3^3) \cdot j \\ a_{10} &= -(j - 2^6 \cdot 3^3) \cdot j^2 \cdot (7 \cdot 67 \cdot j^2 - 2^6 \cdot 3^2 \cdot 2053 \cdot j + 2^{11} \cdot 3^5 \cdot 31 \cdot 53) \\ a_9 &= 2 \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^2 \cdot (3^2 \cdot j^4 - 2^3 \cdot 6379 \cdot j^3 + 2^6 \cdot 3^2 \cdot 162713 \cdot j^2 \\ &\quad - 2^{12} \cdot 3^5 \cdot 72797 \cdot j + 2^{25} \cdot 3^{12}) \\ a_8 &= 2 \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^3 \cdot (2 \cdot 7 \cdot 13^2 \cdot j^5 - 3^2 \cdot 409 \cdot 3373 \cdot j^4 \\ &\quad + 2^7 \cdot 3^4 \cdot 1237 \cdot 1973 \cdot j^3 - 2^{14} \cdot 3^7 \cdot 5 \cdot 311 \cdot 443 \cdot j^2 \\ &\quad + 2^{21} \cdot 3^{10} \cdot 31 \cdot 2897 \cdot j - 2^{31} \cdot 3^{14} \cdot 163) \\ a_7 &= 2^2 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (11 \cdot 61 \cdot 193 \cdot j^5 - 2^3 \cdot 3 \cdot 27510443 \cdot j^4 \\ &\quad + 2^9 \cdot 3^3 \cdot 97550587 \cdot j^3 - 2^{16} \cdot 3^6 \cdot 11 \cdot 2599451 \cdot j^2 \\ &\quad + 2^{23} \cdot 3^9 \cdot 5 \cdot 739 \cdot 1109 \cdot j - 2^{34} \cdot 3^{13} \cdot 4691) \end{aligned}$$

$$\begin{aligned}
a_6 &= 2^3 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (2^4 \cdot 3^2 \cdot j^8 + 7 \cdot 199 \cdot 1373 \cdot j^7 \\
&\quad - 2^2 \cdot 29 \cdot 37 \cdot 281 \cdot 13913 \cdot j^6 + 2^{13} \cdot 3^3 \cdot 7 \cdot 233 \cdot 143281 \cdot j^5 \\
&\quad - 2^{15} \cdot 3^7 \cdot 5 \cdot 11 \cdot 21117827 \cdot j^4 + 2^{23} \cdot 3^9 \cdot 3943 \cdot 117577 \cdot j^3 \\
&\quad - 2^{31} \cdot 3^{12} \cdot 769 \cdot 45317 \cdot j^2 + 2^{41} \cdot 3^{16} \cdot 7 \cdot 15923 \cdot j - 2^{50} \cdot 3^{20} \cdot 269) \\
a_5 &= 2^4 \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^5 \cdot (2^6 \cdot 3^4 \cdot 5 \cdot j^8 - 7 \cdot 5051 \cdot 5939 \cdot j^7 \\
&\quad + 2^3 \cdot 3^2 \cdot 5 \cdot 61 \cdot 101 \cdot 330037 \cdot j^6 - 2^9 \cdot 3^5 \cdot 96289 \cdot 119173 \cdot j^5 \\
&\quad + 2^{16} \cdot 3^9 \cdot 17 \cdot 77252741 \cdot j^4 - 2^{22} \cdot 3^{11} \cdot 11 \cdot 71 \cdot 523 \cdot 4091 \cdot j^3 \\
&\quad + 2^{35} \cdot 3^{14} \cdot 5 \cdot 673 \cdot 977 \cdot j^2 - 2^{41} \cdot 3^{18} \cdot 79 \cdot 1831 \cdot j + 2^{55} \cdot 3^{24}) \\
a_4 &= (j - 2^6 \cdot 3^3)^4 \cdot j^6 \cdot (2^8 \cdot 3^3 \cdot 5 \cdot 2003 \cdot j^9 - 409 \cdot 39157 \cdot 44483 \cdot j^8 \\
&\quad + 2^9 \cdot 3 \cdot 2092618568983 \cdot j^7 - 2^{20} \cdot 3^4 \cdot 98512996093 \cdot j^6 \\
&\quad + 2^{20} \cdot 3^7 \cdot 41 \cdot 242261 \cdot 608831 \cdot j^5 - 2^{28} \cdot 3^{10} \cdot 5 \cdot 1231 \cdot 155631757 \cdot j^4 \\
&\quad + 2^{32} \cdot 3^{13} \cdot 521 \cdot 3077579657 \cdot j^3 - 2^{42} \cdot 3^{16} \cdot 997 \cdot 1607 \cdot 16657 \cdot j^2 \\
&\quad + 2^{52} \cdot 3^{20} \cdot 23 \cdot 541 \cdot 6863 \cdot j - 2^{63} \cdot 3^{24} \cdot 5 \cdot 11987) \\
a_3 &= 2 \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^6 \cdot (3^2 \cdot 377732207 \cdot j^{10} - 2^6 \cdot 5^2 \cdot 7 \cdot 101 \cdot 28520381 \cdot j^9 \\
&\quad + 2^{11} \cdot 11 \cdot 337 \cdot 17990477821 \cdot j^8 - 2^{20} \cdot 3^3 \cdot 179 \cdot 389 \cdot 171956657 \cdot j^7 \\
&\quad + 2^{23} \cdot 3^6 \cdot 5 \cdot 479 \cdot 37193046587 \cdot j^6 - 2^{30} \cdot 3^9 \cdot 1283 \cdot 28703 \cdot 758137 \cdot j^5 \\
&\quad + 2^{36} \cdot 3^{12} \cdot 7 \cdot 31 \cdot 54791988203 \cdot j^4 - 2^{45} \cdot 3^{15} \cdot 19^2 \cdot 151 \cdot 7738067 \cdot j^3 \\
&\quad + 2^{55} \cdot 3^{20} \cdot 41 \cdot 12810583 \cdot j^2 - 2^{65} \cdot 3^{24} \cdot 1103107 \cdot j + 2^{76} \cdot 3^{27} \cdot 1447) \\
a_2 &= 2^2 \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^7 \cdot (42967 \cdot 2406947 \cdot j^{11} - 2^3 \cdot 557 \cdot 1783 \cdot 140768209 \cdot j^{10} \\
&\quad + 2^9 \cdot 3^4 \cdot 6205891 \cdot 21226039 \cdot j^9 - 2^{19} \cdot 3^7 \cdot 5 \cdot 11 \cdot 251872948013 \cdot j^8 \\
&\quad + 2^{24} \cdot 3^9 \cdot 5 \cdot 13 \cdot 23 \cdot 37 \cdot 521 \cdot 3203149 \cdot j^7 - 2^{29} \cdot 3^{13} \cdot 47242981376477 \cdot j^6 \\
&\quad + 2^{35} \cdot 3^{16} \cdot 227 \cdot 112292655271 \cdot j^5 - 2^{41} \cdot 3^{18} \cdot 107 \cdot 269749728667 \cdot j^4 \\
&\quad + 2^{54} \cdot 3^{22} \cdot 43 \cdot 449215127 \cdot j^3 - 2^{61} \cdot 3^{27} \cdot 5 \cdot 653 \cdot 54193 \cdot j^2 \\
&\quad + 2^{72} \cdot 3^{30} \cdot 139 \cdot 3719 \cdot j - 2^{82} \cdot 3^{35} \cdot 139)
\end{aligned}$$



$$\begin{aligned}
a_1 &= 2^3 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \cdot (1847032397279 \cdot j^{11} - 2^6 \cdot 47 \cdot 157 \cdot 3691 \cdot 11660843 \cdot j^{10} \\
&\quad + 2^{14} \cdot 3^4 \cdot 383 \cdot 25679 \cdot 7797631 \cdot j^9 - 2^{20} \cdot 3^6 \cdot 400129001343469 \cdot j^8 \\
&\quad + 2^{24} \cdot 3^9 \cdot 5 \cdot 41 \cdot 503 \cdot 67307 \cdot 267271 \cdot j^7 \\
&\quad - 2^{30} \cdot 3^{12} \cdot 19 \cdot 509 \cdot 13597 \cdot 11431571 \cdot j^6 \\
&\quad + 2^{37} \cdot 3^{15} \cdot 31 \cdot 3038701 \cdot 4610147 \cdot j^5 - 2^{43} \cdot 3^{20} \cdot 7^2 \cdot 41 \cdot 73 \cdot 2381 \cdot 56891 \cdot j^4 \\
&\quad + 2^{52} \cdot 3^{21} \cdot 5 \cdot 139 \cdot 9239401667 \cdot j^3 - 2^{62} \cdot 3^{25} \cdot 5 \cdot 1381 \cdot 3698087 \cdot j^2 \\
&\quad + 2^{73} \cdot 3^{29} \cdot 11 \cdot 47 \cdot 58693 \cdot j - 2^{85} \cdot 3^{33} \cdot 8161) \\
a_0 &= -2^4 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \cdot (2^3 \cdot 3^2 \cdot 7^6 \cdot j^{14} - 5 \cdot 13 \cdot 3109 \cdot 76441597 \cdot j^{13} \\
&\quad + 2^4 \cdot 3449 \cdot 4363 \cdot 873750089 \cdot j^{12} - 2^{11} \cdot 3^4 \cdot 7 \cdot 2087 \cdot 57859 \cdot 9420337 \cdot j^{11} \\
&\quad + 2^{16} \cdot 3^8 \cdot 11^2 \cdot 73 \cdot 125183 \cdot 10636957 \cdot j^{10} - 2^{26} \cdot 3^9 \cdot 691 \cdot 14434308694753 \cdot j^9 \\
&\quad + 2^{31} \cdot 3^{13} \cdot 101 \cdot 283 \cdot 252059913139 \cdot j^8 \\
&\quad - 2^{37} \cdot 3^{16} \cdot 11 \cdot 13 \cdot 17 \cdot 647 \cdot 863 \cdot 4253233 \cdot j^7 \\
&\quad + 2^{43} \cdot 3^{18} \cdot 631819 \cdot 16451871913 \cdot j^6 - 2^{48} \cdot 3^{23} \cdot 149 \cdot 233 \cdot 90533 \cdot 330413 \cdot j^5 \\
&\quad + 2^{59} \cdot 3^{25} \cdot 23 \cdot 1408302006413 \cdot j^4 - 2^{70} \cdot 3^{27} \cdot 726838208711 \cdot j^3 \\
&\quad + 2^{80} \cdot 3^{32} \cdot 7 \cdot 263 \cdot 337 \cdot 1327 \cdot j^2 - 2^{90} \cdot 3^{37} \cdot 569731 \cdot j + 2^{100} \cdot 3^{39} \cdot 17^3) \\
b_{11} &= -(j - 2^6 \cdot 3^3) \cdot j \\
b_{10} &= -2 \cdot 13 \cdot 3^2 \cdot (j - 2^6 \cdot 3^3) \cdot j^2 \\
b_9 &= 2^2 \cdot (j - 2^3 \cdot 3^6) \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^2 \\
b_8 &= 3^4 \cdot (13 \cdot j - 2^5 \cdot 3 \cdot 163) \cdot (j - 2^6 \cdot 3^3)^2 \cdot j^3 \\
b_7 &= 5 \cdot 2^5 \cdot 3^6 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \\
b_6 &= 2^2 \cdot 3^3 \cdot (j - 2^6 \cdot 3^3)^3 \cdot j^4 \cdot (j^2 + 2^4 \cdot 3^5 \cdot 13 \cdot j - 2^9 \cdot 3^5 \cdot 269) \\
b_5 &= 2^5 \cdot 3^5 \cdot (5 \cdot j - 2^6 \cdot 3^4) \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^5 \\
b_4 &= 2^5 \cdot 3^8 \cdot (31 \cdot j - 2^3 \cdot 3^2 \cdot 1471) \cdot (j - 2^6 \cdot 3^3)^4 \cdot j^6 \\
b_3 &= 2^8 \cdot 3^8 \cdot (383 \cdot j - 2^6 \cdot 3 \cdot 1447) \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^6 \\
b_2 &= 2^9 \cdot 3^9 \cdot (3923 \cdot j - 2^6 \cdot 3^5 \cdot 139) \cdot (j - 2^6 \cdot 3^3)^5 \cdot j^7 \\
b_1 &= 13 \cdot 19 \cdot 3^{11} \cdot 2^{15} \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \\
b_0 &= -2^8 \cdot 3^9 \cdot (j - 2^6 \cdot 3^3)^6 \cdot j^8 \cdot (j^2 - 2^7 \cdot 3^3 \cdot 1399 \cdot j + 2^{12} \cdot 3^6 \cdot 17^3)
\end{aligned}$$

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