Class Polynomials for Non-holomorphic Modular Functions

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The Modular Invariant and its Special Values

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$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots \quad (q := e^{2\pi i\tau}).$$

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- Here are several examples:

$$j(i) = 1728, \qquad j\left(\frac{1+i\sqrt{7}}{2}\right) = -3375, \qquad j(i\sqrt{2}) = 8000.$$

Classical Theory of Complex Multiplication

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- Strange consequence:

$$e^{\pi\sqrt{163}} = 262537412640768743.99999999999925 \in \mathbb{Z} + \epsilon^2.$$

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Theorem

For all D, $H_D(x)$ is irreducible in $\mathbb{Z}[x]$ and its splitting field is a class field.

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- Weber defined several eponymous functions in terms of the η-function and used their properties to compute examples.
- Zagier's seminal paper Traces of Singular Moduli gives an automatic procedure for computing class polynomials.

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Remark

1. Zagier's theory provides a new proof of Borcherds' theorem and he shows that $A(1, d)$ is the trace of singular moduli.
2. Zagier's work applies to a much more general class of forms.

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Example

These appear in recent work of Bruinier-Ono on $p(n)$.

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Definition

Let Q_d be the set of positive definite binary quadratic forms of discriminant d. For a modular function F, define the trace:

$$\mathrm{Tr}_d(F) := \sum_{Q \in Q_d/\Gamma} w_Q^{-1} F(\tau_Q).$$

An Example of Zagier's Theory

Theorem (Zagier)

Let

$$J(z) := j(z) - 744$$

and

$$g(z) := \theta_1(z) \frac{E_4(4z)}{\eta(4z)^6} = \sum B(d) q^n$$

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$$\mathrm{Tr}_{\text{-d}}\left(\mathrm{J(z)}\right) = -B(d).$$

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Remark

It appears that the third symmetric function is always an integer.

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Answer

Our theorem predicts the correct/sharp denominators.

Traces for Negative Weight Forms

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- For f of negative weight, ∂f is the iterated raising to weight 0.

Our Main Result

Theorem (G-R)

Let $f(z) \in M_k^!$, $0 > k \in 2\mathbb{Z}$ have integral principal part. Denote the n^{th} symmetric function in the singular moduli of discriminant d for ∂f by $\mathcal{S}_f(n; d)$. Let

$$B(n,k) := \begin{cases} \frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\ \frac{1}{4}(-nk + 2k - 2) & \text{otherwise.} \end{cases}$$

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Then if $(p, d) = 1$, we have that $\mathcal{S}_f(n; d)$ is p-integral. If $p|d$ is good for (k, N), we have that

$$p^{B(n,k)} \cdot \mathcal{S}_f(n; d) \text{ is p-integral.}$$

Special Cases

Corollary

For any $f(z) \in M^{!}_{-2}$ with integral principal part, we have that

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Remark

This theorem is sharp.

Sketch of Proof

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- We prove the following fact.

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Let F be a product of "raises" of modular forms. Then there are modular forms $g_j \in M^!_{k-2j}$ such that

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Remark

The proof gives an explicit algorithm for computing the forms g_j.

Sketch of Proof (cont).

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- Bounding denominators on each piece gives a naïve bound.
- However, this falls far short of our theorem.

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- Obstruction 1: Certain weights in the decomposition give the wrong denominators.
- We prove a vanishing condition on which forms in the decomposition actually appear.
- Obstruction 2: The coefficients $c_{i,j}$ in the previous theorem also introduce artificial denominators.
- We show that they cancel using the action of the Hecke algebra on Poincaré series.

Q.E.D.

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- Here

$$c_{i,j} := \frac{j!(-k+j+i)!}{(j-i)!(-k+j)!}.$$

Work of Duke and Jenkins

- Let
$$\mathrm{Tr}^*_{d,D}(f) := (-1)^{\lfloor \frac{\hat{s}-1}{2} \rfloor} |d|^{\frac{-\hat{s}}{2}} |D|^{\frac{\hat{s}-1}{2}} \mathrm{Tr}_{d,D}(\partial f).$$

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- They define the D^{th} Zagier lift of f:

$$\mathfrak{Z}_D(f) := \sum_{m \geq 0} b(m) q^{-m} + \sum_{dD<0} \mathrm{Tr}^*_{d,D}(f) q^{|d|}.$$

Duke and Jenkins' Theorem

Theorem (Duke-Jenkins)

Suppose that $f \in M_k^!$, $k \leq 0$. If $f \in \mathbb{Z}[[q]]$, then $\mathfrak{Z}(f)$ is a half-integral weight modular form with integral coefficients.

A Useful Vanishing Criterion

Definition

Let $0 > k \in 2\mathbb{Z}$ *and* $n \in \mathbb{N}$*. We say* m *is a* ***bad weight*** *for* (k, n) *if* m *is of the form* $kn + 4i + 2$ *for* $0 \leq i \leq -\frac{k}{2} - 1$*.*

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Theorem (G-R)

Let $f \in M_k^!$ and consider the product $F = (\partial f)^n$. Decompose $F = \sum \partial(g_i)$. Then if g_i has bad weight for (k, n), $g_i \equiv 0$.

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- This gives an (essentially unique) map

$$[\cdot,\cdot]_n^{(k),(\ell)} : \; M_k^! \otimes M_\ell^! \to M_{k+\ell+2n}^!.$$

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- We can expand in terms of Rankin-Cohen brackets.
- Using a calculation of Beyerl-James-Trentacoste-Xue, this reduces to a binomial sum identity, for j odd

$$\sum_{m=0}^{s} (-1)^{(j+m)} \cdot \frac{\binom{m+r}{j}\binom{s}{m}\binom{m-r-1}{r+m-j}}{\binom{-r-2s+m+j-1}{m+r-j}} = 0.$$

Obstruction 2: Lining Up Principal Parts

- Raise the Zagier lifts of the pieces to the same weight and let:

$$Z(\tau) := \sum_{t=0}^{\lfloor \frac{E+1}{2} \rfloor} (-1)^{M+t} R^{M+t} \mathfrak{z}_1(g_{2t-1}) + \sum_{t=0}^{M} (-1)^{M+t} R^{M-t} \mathfrak{z}_1(g_{2t}).$$

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- If all the coefficients of Z^+ are integral, then the $c_{i,j}$-denominators will cancel.

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- The Zagier lift is equivariant with the Hecke action:

$$\mathfrak{Z}_D(f|T(n)) = \mathfrak{Z}_D(f)|T(n^2).$$

Hypotheses

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- We assume p is ordinary for all eigenforms in a basis of S_k.

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2. *If* $\mathfrak{H}_n$ *and* $\mathfrak{H}'_n$ *are two such operators, then*

$$f_{k,1}|\mathfrak{H}_n \equiv f_{k,1}|\mathfrak{H}'_n \pmod{p^n}.$$

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Then is a Hecke operator $\mathfrak{H}_n$ *such that*

$$f_{2-k,1}|\mathfrak{H}_n \in M^!_{2-k}, \qquad f_{k,1}|\mathfrak{H}_n \in \mathbb{Z}((q)),$$

and $f_{k,1}|\mathfrak{H}_n \equiv q^{-1} + O(q) \pmod{p^n}$. *Any such* $\mathfrak{H}_n$ *satisfies:*

1. *If* $f_{2-k,1}|H$ *is weakly holomorphic and* $f_{k,1}|H$ *has integer coefficients, then* $(f_{k,1}|\mathfrak{H}_n)\,|H \equiv f_{k,1}|H \pmod{p^n}$.
2. *If* $\mathfrak{H}_n$ *and* $\mathfrak{H}'_n$ *are two such operators, then*

$$f_{k,1}|\mathfrak{H}_n \equiv f_{k,1}|\mathfrak{H}'_n \pmod{p^n}.$$

3. *If* $(f_{k,1}|\mathfrak{H}_n)\,|H \equiv 0 + O(q) \pmod{p^m}$ *for some* $m \leq n$*, then* $(f_{k,1}|\mathfrak{H}_n)\,|H \equiv 0 \pmod{p^m}$.

Integrality of Coefficients

Corollary

If $f_{k,1}|H$ *has integer coefficients,* p *is ordinary for all eigenforms in a basis of* S_k*, and* $f_{k,1}|H \equiv 0 + O(q) \pmod{p^n}$*, then*

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- The holomorphic part of $\mathcal{Z}_D(f)$ has integral principal part.

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- The holomorphic part of $\mathcal{Z}_D(f)$ has integral principal part.
- Use induction to extend the corollary to linear combinations.

Our Main Theorem

Theorem (G-R)

Let $f(z) \in M_k^!$, $0 > k \in 2\mathbb{Z}$ have integral principal part. Denote the n^{th} symmetric function in the singular moduli of discriminant d for ∂f by $\mathcal{S}_f(n; d)$. Let

$$B(n,k) := \begin{cases} \frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\ \frac{1}{4}(-nk+2k-2) & \text{otherwise.} \end{cases}$$

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Then if $(p, d) = 1$, we have that $\mathcal{S}_f(n; d)$ is p-integral. If $p|d$ is good for (k, N), we have that

$$p^{B(n,k)} \cdot \mathcal{S}_f(n; d) \text{ is } p\text{-integral.}$$