On the existence of a maximizer for the Strichartz inequality

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Abstract

It is shown that a maximizing function $u_* \in L^2$ does exist for the Strichartz inequality $\|e^{it\partial_x^2}u\|_{L_t^6(L_x^6)} \le S\|u\|_{L^2}$, with $S > 0$ being the sharp constant.

1 Introduction and main result

The $L_t^6(L_x^6)$-Strichartz inequality in one spatial dimension states that there is a constant $C > 0$ such that

$$\|e^{it\partial_x^2}u\|_{L_t^6(L_x^6)} \le C\|u\|_{L^2} \quad \text{for all} \quad u \in L^2 = L^2(\mathbb{R};\mathbb{C}),$$

see [12] for instance. The sharp (or best) constant for this estimate is

$$S := \sup\left\{\frac{\|e^{it\partial_x^2}u\|_{L_t^6(L_x^6)}}{\|u\|_{L^2}} : u \in L^2, u \neq 0\right\} = \sup\Big\{\|e^{it\partial_x^2}u\|_{L_t^6(L_x^6)} : u \in L^2, \|u\|_{L^2} = 1\Big\}.$$

Here $e^{it\partial_x^2}$ denotes the evolution operator of the free Schrödinger equation, so that $u(t,x) = (e^{it\partial_x^2}u_0)(x)$ by definition solves

$$iu_t + u_{xx} = 0, \quad u(0,x) = u_0(x). \tag{1.1}$$

The purpose of this paper is to verify that a maximizing function $u_* \in L^2$ does exist, i.e., u_* gives equality in the estimate $\|e^{it\partial_x^2}u\|_{L_t^6(L_x^6)} \le S\|u\|_{L^2}$. Stated differently, $\varphi(u_*) = S^6$ for some $u_* \in L^2$ with $\|u_*\|_{L^2} = 1$, where

$$\varphi(u) = \|e^{it\partial_x^2}u\|_{L_t^6(L_x^6)}^6 = \int_{\mathbb{R}}\int_{\mathbb{R}} |(e^{it\partial_x^2}u)(x)|^6\,dxdt. \tag{1.2}$$

The main difficulty of this problem results from the many invariances of φ: it is not hard to show that $\varphi(u(\cdot + x_0)) = \varphi(u)$ for $x_0 \in \mathbb{R}$, $\varphi(e^{ix\xi_0}u) = \varphi(u)$ for $\xi_0 \in \mathbb{R}$, and moreover $\varphi(u_\lambda) = \varphi(u)$ for $\lambda > 0$, where $u_\lambda(x) = \lambda^{1/2}u(\lambda x)$; see Corollary 2.3 below. Since all these invariances preserve the L^2-norm, it is in particular not true that every maximizing sequence (u_j) for φ under the constraint $\|u_j\|_{L^2} = 1$ converges strongly in L^2.

An outline of the proof in this paper that a maximizing function does exist is as follows. First the concentration compactness principle is applied to the sequence $(\hat{u}_j)$; the observation that it can be helpful to use this principle for the Fourier transforms rather than for the maximizing sequence itself seems to be new. It found a first application in [8], where a variational problem from nonlinear fiber optics was studied, although it turned out later that the proof could be simplified a lot and this idea in fact had not to be used. The concentration compactness principle asserts that basically there are three possible alternatives for a L^2-bounded sequence of functions: either it is tight (in the sense of measures), or it is "vanishing" (it tends to zero uniformly on every interval of fixed length), or it is "splitting" (into at least two parts with supports widely separated). The first issue will be to rule out vanishing and splitting. The former is achieved by suitably shifting the $\hat{u}_j$ (corresponding to multiplication of u_j with an appropriate $e^{ix\xi_j}$) and by rescaling the $\hat{u}_j$ (corresponding to a rescaling of the u_j) such that

$$\sup_{\xi_0\in\mathbb{R}} \int_{\xi_0-1}^{\xi_0+1} |\hat{u}_j|^2\, d\xi = \int_{-1}^{1} |\hat{u}_j|^2\, d\xi = \frac{1}{2}, \quad j \in \mathbb{N}, \tag{1.3}$$

is satisfied. Note that so far already two of the three invariances have been used. Next it has to be discussed why splitting of the sequence $(\hat{u}_j)$ cannot occur. For this we assume that $\hat{u}_j \sim \hat{v}_j + \hat{w}_j$, with $\|\hat{v}_j\|_{L^2} \sim \hat{\gamma} \in]0,1[$, $\|\hat{w}_j\|_{L^2} \sim (1-\hat{\gamma})$, and the supports of $\hat{v}_j$ and $\hat{w}_j$ widely separated, say $\operatorname{supp}(\hat{v}_j) \subset \{\xi : |\xi| \le a\}$ and $\operatorname{supp}(\hat{w}_j) \subset \{\xi : |\xi| \ge b\}$. From standard applications of the concentration compactness principle it is known that due to homogeneity properties of a functional a contradiction would be obtained if $\varphi(u_j) \sim \varphi(v_j) + \varphi(w_j)$ could be shown. Using $u_j \sim v_j + w_j$, it can be seen from the definition of φ that roughly

$$\begin{aligned} \varphi(u_j) - \varphi(v_j) - \varphi(w_j) \ &\sim\ \int_{\mathbb{R}}\int_{\mathbb{R}} |e^{it\partial_x^2} v_j|^3\, |e^{it\partial_x^2} w_j|^3\, dxdt \\ &\lesssim\ \|(e^{it\partial_x^2} v_j)^2 (e^{it\partial_x^2} w_j)\|_{L^2_{tx}} \|(e^{it\partial_x^2} v_j)(e^{it\partial_x^2} w_j)^2\|_{L^2_{tx}} \end{aligned} \tag{1.4}$$

holds. At this point it is helpful to recall that there are quite recent multilinear refinements of the Strichartz estimate, in particular of the kind which deal with functions whose Fourier supports are contained in different sets. The usefulness of such estimates has been recognized in [1] (in the case of two space dimensions), and later on a large number of variants and applications have been developed, also in different function spaces or for evolution equations different from the Schrödinger equation; see for instance [4, 7] and very many other papers. From [6, Lemma 3.1] we recall the particular multilinear estimate $\|(e^{it\partial_x^2}u)(e^{it\partial_x^2}v)(e^{it\partial_x^2}w)\|_{L^2_{tx}} \le C\|u\|_{H^{\frac14}}\|v\|_{H^{-\frac14}}\|w\|_{L^2}$ which is appropriate for our purposes. From (1.4) and $\|\hat{v}_j\|_{L^2}, \|\hat{w}_j\|_{L^2} \lesssim 1$ thus

$$\varphi(u_j) - \varphi(v_j) - \varphi(w_j) \lesssim \|v_j\|^2_{H^{\frac14}} \|w_j\|^2_{H^{-\frac14}} \lesssim a^{1/2} b^{-1/2}.$$

It appears to not have been noticed before that the concentration compactness principle can be (slightly) refined in such a way that in the case of a splitting sequence the two parts can be moved arbitrarily far apart; see Lemma 3.1 below. Hence $b \gg a$ can be achieved, and the multilinear estimate works together perfectly with the concentration compactness principle to imply that the sequence $(\hat{u}_j)$ cannot be splitting. Having now excluded two alternatives, it follows that the sequence of measures $\mu_j = |\hat{u}_j|^2\, d\xi$ is tight, i.e., roughly speaking localized by cutting off the high frequencies (which leads to an L^2-small remainder term). In particular, "almost" $(u_j) \subset H^1$ holds and if $\mu_j \rightharpoonup^* \mu$ as $j \to \infty$ (along a subsequence) in the sense of measures, then $\int_{\mathbb{R}} d\mu = 1$. However, this improvement is still not sufficient to ensure the needed strong convergence, since the shift invariance of φ has not been used yet. This observation leads to the idea to apply, in a next step, the concentration compactness principle also to (u_j). As soon as it were known that both vanishing and splitting is impossible for (u_j) the strong convergence would follow (if the u_j are shifted appropriately). Indeed, in this case the sequence is also localized in x-space to some interval $]-M,M[$, and it can be used that the embedding $H^1(]-M,M[)\subset L^2(]-M,M[)$ is compact. The fact that splitting cannot occur for (u_j) is proved by using the above-mentioned refinement of the splitting alternative: if $u_j\sim v_j+w_j$, $\|v_j\|_{L^2}\sim\gamma\in]0,1[$, and $\|w_j\|_{L^2}\sim(1-\gamma)$ are satisfied, and if $\mathrm{supp}(v_j)\subset\{x:|x|\le a\}$ and $\mathrm{supp}(w_j)\subset\{x:|x|\ge b\}$, then it can be shown that

$$\varphi(u_j)-\varphi(v_j)-\varphi(w_j)\lesssim\Big(\|v_j\|_{H^1}^{1/6}+\|w_j\|_{H^1}^{1/6}\Big)(1+a)^{1/2}(b-a)^{-1/12}\lesssim(1+a)^{1/2}(b-a)^{-1/12}.$$

The latter estimate holds, since we now have H^1-bounds also for v_j and w_j. By moving the two splitting components far enough apart (choosing $b\gg a+(1+a)^6$ for instance), the right-hand side can be made as small as necessary to verify that splitting of (u_j) is impossible. Therefore it remains to be seen that also vanishing of (u_j) cannot happen. It should be remarked that in general it is easy to construct (by shifting and scaling of any maximizing sequence) a maximizing sequence $(\tilde u_j)$ such that $(\hat{\tilde u}_j)$ is tight and $(\tilde u_j)$ is vanishing. Typically such a sequence will be concentrating at one or several points in ξ, i.e., $\mu=\sum a_l\delta_{\xi_l}$, but we additionally dispose of the normalization (1.3) which will show that concentration to a point is impossible for the special maximizing sequence (u_j). It is quite technical to make this argument rigorous in the case of the Strichartz estimate, since φ is a highly non-local functional. Stated differently, for test functions $\chi=\chi(x)$ there is no obvious relation between $\varphi(\chi u)$ and $\int_{\mathbb R}\int_{\mathbb R}\chi^6|e^{it\partial_x^2}u|^6\,dxdt$. To advance at this point it turns out to be helpful to consider φ as a functional of $\hat u$ rather than as a functional of u, i.e., we introduce $\psi(v)=\varphi(\check v)$. Using $(e^{it\partial_x^2}u_j)(x)=C\int_{\mathbb R}e^{i(x\xi-t\xi^2)}\hat u_j(\xi)\,d\xi$ and integrating out completely the functions $\delta_0(\xi_1-\xi_2+\xi_3-\xi_4+\xi_5-\xi_6)\delta_0(\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2+\xi_5^2-\xi_6^2)$ which then appear, it is possible to derive an explicit form $\psi(\hat u_j)=\int_{\mathbb R}f_j(\xi)\,d\xi$ of ψ, where in our case the f_j are non-negative functions which are related to u_j. From the tightness of $(\hat u_j)$ and the vanishing of (u_j) one can then deduce that $\psi(\chi u_j)\sim\int_{\mathbb R}\chi^6 f_j\,d\xi$ as $j\to\infty$ for test functions $\chi=\chi(\xi)$. To use this let $\nu_j=f_j\,d\xi$. First, tightness of $(\hat u_j)$ shows that the sequence of measures (ν_j) is tight as well, hence $\nu_j\rightharpoonup^*\nu$ as $j\to\infty$ (along a subsequence) in the sense of measures for a ν such that $\int_{\mathbb R}d\nu=\lim_{j\to\infty}\int_{\mathbb R}d\nu_j=\lim_{j\to\infty}\int_{\mathbb R}f_j\,d\xi=\lim_{j\to\infty}\psi(\hat u_j)=\lim_{j\to\infty}\varphi(u_j)=S^6=S^6\int_{\mathbb R}d\mu$. Second, for a test function $\chi=\chi(\xi)$ one gets

$$\int_{\mathbb R}\chi^6\,d\nu\sim\int_{\mathbb R}\chi^6\,d\nu_j\sim\psi(\chi\hat u_j)=\varphi(\check\chi*u_j)\le S^6\|\check\chi*u_j\|_{L^2}^6=S^6\Big(\int_{\mathbb R}\chi^2|\hat u_j|^2\,d\xi\Big)^3\sim S^6\Big(\int_{\mathbb R}\chi^2\,d\mu\Big)^3$$

as $j\to\infty$ by Strichartz' inequality. Therefore a reversed Hölder-type inequality relating $d\nu$ and $d\mu$ has been derived. In this situation a well-known result on the concentration of measures (see [11, Lemma I.2, p. 161]) can be applied to yield $\mu=\delta_{\xi_*}$ for some $\xi_*\in\mathbb R$, which gives a contradiction to (1.3).

Filling in the necessary details we obtain the following main result of this paper.

Theorem 1.1 *There exists a function $u_*\in L^2$ such that S is attained, i.e., $\|e^{it\partial_x^2}u_*\|_{L^6_t(L^6_x)}=S$ for some $u_*\in L^2$ with $\int_{\mathbb R}|u_*|^2\,dx=1$.*

It is to be expected that the technique of proof described above can also be applied successfully to other Strichartz-type inequalities to yield the existence of sharp constants, as soon as a multilinear refinement of the inequality in question is available. In connection with Theorem 1.1 an interesting open problem is to determine the numerical values of the sharp constants and to find (up to the invariances) the maximizing function(s); the associated Euler-Lagrange equation seems not to be very helpful in this respect. In the mathematical literature there are several examples of inequalities which are preserved by shift and scaling, for instance the Hardy-Littlewood-Sobolev inequality and the Sobolev inequality. The first general method to prove the existence of sharp constants for such problems has been developed by Lieb in [9] (see also the related [2]). In [9] it has also been possible in many cases to evaluate explicitly the sharp constants and the maximizing functions. The approach taken in [9] rests upon the fact that the left-hand side of the inequality does not decrease, whereas the right-hand side does not increase, if u is replaced by u^*, its symmetric decreasing rearrangement; this is the case for the Hardy-Littlewood-Sobolev inequality and for the Sobolev inequality. However, for the Strichartz inequality such rearrangement arguments do not appear to lead very far. A second general method which is closer to the approach used in the present paper can be found in [11], but in this way also no information on sharp constants or maximizing functions could be obtained.

The paper is organized as follows. In Section 2 some auxiliary results and estimates are collected. Section 3 discusses certain aspects related to concentration compactness, and the proof of Theorem 1.1 is developed in Section 4.

Concerning notation, we write $L^p = L^p(\mathbb{R};\mathbb{C})$ and $H^s = H^s(\mathbb{R};\mathbb{C})$, with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$, respectively. The inner product on L^2 is $(u,v)_{L^2} = \int_{\mathbb{R}} u\bar{v}\,dx$, whereas the (spatial) Fourier transform of $u \in L^2$ is $\hat{u}(\xi) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-i\xi x}u(x)\,dx$ with inverse $\check{u}$. We mostly make explicit the (2π)-factors in our formulas, since some of them could perhaps be useful later to determine the sharp constant explicitly. For $s, b \in \mathbb{R}$ we denote $X^+_{s,b}$ the closure of $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|u\|^2_{X^+_{s,b}} = \int_{\mathbb{R}}\int_{\mathbb{R}} (1+\xi^2)^s(1+|\tau+\xi^2|)^b|\mathcal{F}u(\tau,\xi)|^2\,d\xi d\tau,$$

where

$$\mathcal{F}u(\tau,\xi) = (2\pi)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(\tau t+\xi x)}u(t,x)\,dxdt = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-it\tau}\hat{u}(t,\xi)\,dt$$

is the space-time Fourier transform of $u = u(t,x)$. In the particular case of $u(t,x) = (e^{it\partial_x^2}u)(x)$ it follows from $\hat{u}(t,\xi) = e^{-it\xi^2}\hat{u}(\xi)$ that $\mathcal{F}u(\tau,\xi) = (2\pi)^{1/2}\delta_0(\tau+\xi^2)\hat{u}(\xi)$, whence

$$\|e^{it\partial_x^2}u\|_{X^+_{s,b}} = (2\pi)^{1/2}\Big(\int_{\mathbb{R}}(1+\xi^2)^s|\hat{u}(\xi)|^2\,d\xi\Big)^{1/2} \sim \|u\|_{H^s}. \tag{1.5}$$

By C we denote unimportant positive numerical constants which may change from line to line.

2 Some preliminaries and technical lemmas

It will be useful to consider along with φ from (1.2) also its multilinear version Φ, which is defined as

$$\begin{aligned}&\Phi(u_1,u_2,u_3,u_4,u_5,u_6)\\ &\quad= \int_{\mathbb{R}}\int_{\mathbb{R}} (e^{it\partial_x^2}u_1)(x)\overline{(e^{it\partial_x^2}u_2)(x)}(e^{it\partial_x^2}u_3)(x)\overline{(e^{it\partial_x^2}u_4)(x)}(e^{it\partial_x^2}u_5)(x)\overline{(e^{it\partial_x^2}u_6)(x)}\,dxdt\end{aligned} \tag{2.1}$$

for $u_1,u_2,u_3,u_4,u_5,u_6 \in L^2$. Then

$$\varphi(u) = \Phi(u,u,u,u,u,u). \tag{2.2}$$

To derive certain estimates related to φ and Φ the special case $s = 1/4$ of the trilinear estimate from [6, Lemma 3.1] is recalled, which reads as

$$\|uvw\|_{L^2_{tx}} \leq C\|u\|_{X^+_{\frac{1}{4},b}} \|v\|_{X^+_{-\frac{1}{4},b}} \|w\|_{X^+_{0,b}},$$

where $b > 1/2$ and $C > 0$ depends only on b. In view of (1.5) this is specialized to

$$\|(e^{it\partial_x^2}u)(e^{it\partial_x^2}v)(e^{it\partial_x^2}w)\|_{L^2_{tx}} \leq C\|u\|_{H^{\frac{1}{4}}} \|v\|_{H^{-\frac{1}{4}}} \|w\|_{L^2}. \tag{2.3}$$

First we need to study the basic properties of φ and Φ in more detail.

Lemma 2.1 *The following assertions hold.*

(a) The estimate

$$|\Phi(u_1, u_2, u_3, u_4, u_5, u_6)| \leq S^6 \prod_{i=1}^{6} \|u_i\|_{L^2}$$

is satisfied for $u_1, u_2, u_3, u_4, u_5, u_6 \in L^2$.

(b) For $u_1, \ldots, u_6, v_1, \ldots, v_6 \in L^2$ *we have*

$$\begin{aligned} &|\Phi(u_1, u_2, u_3, u_4, u_5, u_6) - \Phi(v_1, v_2, v_3, v_4, v_5, v_6)| \\ &\quad \leq \; C\Big(\max_{1\leq i\leq 6}\{\|u_i\|_{L^2}, \|v_i\|_{L^2}\}\Big)^5 \Big(\max_{1\leq i\leq 6} \|u_i - v_i\|_{L^2}\Big), \end{aligned}$$

in particular, by (2.2),

$$|\varphi(u) - \varphi(v)| \leq C\Big(\max\{\|u\|_{L^2}, \|v\|_{L^2}\}\Big)^5 \|u - v\|_{L^2}, \quad u, v \in L^2. \tag{2.4}$$

(c) Assume $u = v + w$. *If* $\|v\|_{L^2}, \|w\|_{L^2} \leq 1$, *then*

$$|\varphi(u) - \varphi(v) - \varphi(w)| \leq C\Big(\|v\|_{H^{\frac{1}{4}}} \|w\|_{H^{-\frac{1}{4}}} + \|v\|^2_{H^{\frac{1}{4}}} \|w\|^2_{H^{-\frac{1}{4}}}\Big). \tag{2.5}$$

(d) Assume $u = v + w$, $a < b$, $\mathrm{supp}(v) \subset \{x \in \mathbb{R} : |x| \leq a\}$, *and* $\mathrm{supp}(w) \subset \{x \in \mathbb{R} : |x| \geq b\}$. *If* $\|v\|_{L^2}, \|w\|_{L^2} \leq 1$, *then*

$$|\varphi(u) - \varphi(v) - \varphi(w)| \leq C\Big(\|v\|^{1/6}_{H^1} + \|w\|^{1/6}_{H^1}\Big)(1 + a)^{1/2}(b - a)^{-1/12}. \tag{2.6}$$

Proof: (a) From Hölder's inequality in t, x with 6 factors and by definition of S we obtain

$$|\Phi(u_1, u_2, u_3, u_4, u_5, u_6)| \leq \prod_{i=1}^{6} \varphi(u_i)^{1/6} \leq S^6 \prod_{i=1}^{6} \|u_i\|_{L^2}.$$

(b) Due to the multilinearity of Φ we have

$$\begin{aligned} &|\Phi(u_1, u_2, u_3, u_4, u_5, u_6) - \Phi(v_1, v_2, v_3, v_4, v_5, v_6)| \\ &\quad = \Big|\Phi(u_1 - v_1, u_2, u_3, u_4, u_5, u_6) + \Phi(v_1, u_2 - v_2, u_3, u_4, u_5, u_6) + \Phi(v_1, v_2, u_3 - v_3, u_4, u_5, u_6) \\ &\qquad + \Phi(v_1, v_2, v_3, u_4 - v_4, u_5, u_6) + \Phi(v_1, v_2, v_3, v_4, u_5 - v_5, u_6) + \Phi(v_1, v_2, v_3, v_4, v_5, u_6 - v_6)\Big|, \end{aligned}$$

whence (a) applies. (c) Writing $v(t,x) = (e^{it\partial_x^2}v)(x)$ and $w(t,x) = (e^{it\partial_x^2}w)(x)$ we get

$$|\varphi(u)-\varphi(v)-\varphi(w)| \le C\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(|v|^5|w| + |v|^4|w|^2 + |v|^3|w|^3 + |v|^2|w|^4 + |v||w|^5\Big)\,dxdt \qquad (2.7)$$

from $u = v + w$. For the first term on the right-hand side we write $|v|^5|w| = |v|^3(|v|^2|w|)$ and apply Hölder's inequality in t,x with $p=2$, then Strichartz' inequality. This way the bound $(\dots) \le C\|v\|_{L^2}^3\|v^2w\|_{L^2_{tx}} \le C\|v^2w\|_{L^2_{tx}}$ is obtained. The fifth term is treated analogously, and for the third term we write $|v|^3|w|^3 = (|v|^2|w|)(|v||w|^2)$ and use again Hölder's inequality in t,x with $p=2$. From (2.3) it then follows that

$$\begin{aligned}|\varphi(u)-\varphi(v)-\varphi(w)| &\le C\Big(\|v^2w\|_{L^2_{tx}} + \|v^2w\|_{L^2_{tx}}^2 + \|v^2w\|_{L^2_{tx}}\|vw^2\|_{L^2_{tx}} + \|vw^2\|_{L^2_{tx}}^2 + \|vw^2\|_{L^2_{tx}}\Big)\\ &\le C\Big(\|v\|_{H^{\frac14}}\|w\|_{H^{-\frac14}} + \|v\|_{H^{\frac14}}^2\|w\|_{H^{-\frac14}}^2\Big).\end{aligned}$$

(d) We continue to use the notation from (c). As before, we start with (2.7). To estimate the first term on the right-hand side, we write now $|v|^5|w| = |v|^4(|v||w|)$ and apply Hölder's inequality in t,x with $p=3/2$ and $p'=3$. Strichartz' inequality and $\|v\|_{L^2}\le 1$ then yield the bound $C(\int_{\mathbb{R}}\int_{\mathbb{R}}|v|^3|w|^3\,dxdt)^{1/3}$. The second term can be bounded by the same expression, since $|v|^4|w|^2 = |v|^3|w|(|v||w|)$, and Hölder's inequality with $p_1=2$, $p_2=6$, and $p_3=3$ can be used. The other terms are handled similarly, resulting in

$$\begin{aligned}|\varphi(u)-\varphi(v)-\varphi(w)| &\le C\Big(\int_{\mathbb{R}}\int_{\mathbb{R}}|v|^3|w|^3\,dxdt\Big)^{1/3} + C\int_{\mathbb{R}}\int_{\mathbb{R}}|v|^3|w|^3\,dxdt\\ &\le C\Big(\int_{\mathbb{R}}\int_{\mathbb{R}}|v|^3|w|^3\,dxdt\Big)^{1/3}. \qquad (2.8)\end{aligned}$$

Next we fix $t_0>0$ and split $\int_{\mathbb{R}}\int_{\mathbb{R}}|v|^3|w|^3\,dxdt = \int_{|t|\le t_0}\int_{\mathbb{R}}|v|^3|w|^3\,dxdt + \int_{|t|>t_0}\int_{\mathbb{R}}|v|^3|w|^3\,dxdt =: (I)+(II)$. For (I) let $\Omega = \{x\in\mathbb{R} : |x|\le (a+b)/2\}$. Then

$$\begin{aligned}(I) &= \int_{|t|\le t_0}\int_{\Omega}|v|^3|w|^3\,dxdt + \int_{|t|\le t_0}\int_{\Omega^c}|v|^3|w|^3\,dxdt\\ &\le \int_{|t|\le t_0}dt\,\Big(\int_{\Omega}|v|^{10}\,dx\Big)^{3/10}\Big(\int_{\Omega}|w|^2\,dx\Big)^{1/2}\Big(\int_{\Omega}|w|^{10}\,dx\Big)^{1/5} + \int_{|t|\le t_0}\int_{\Omega^c}|v|^3|w|^3\,dxdt\\ &\le \Big(\sup_{|t|\le t_0}\int_{\Omega}|w(t)|^2\,dx\Big)^{1/2}\|e^{it\partial_x^2}v\|_{L_t^5(L_x^{10})}^3\|e^{it\partial_x^2}w\|_{L_t^5(L_x^{10})}^2 + \int_{|t|\le t_0}\int_{\Omega^c}|v|^3|w|^3\,dxdt\\ &\le C\Big(\sup_{|t|\le t_0}\int_{\Omega}|w(t)|^2\,dx\Big)^{1/2} + \int_{|t|\le t_0}\int_{\Omega^c}|v|^3|w|^3\,dxdt,\end{aligned}$$

by the general Strichartz inequality, cf. [3, Thm. 3.2.5(i)], since the pair $(q,r)=(5,10)$ is admissible in one spatial dimension. For the $\int_{\Omega^c}dx$-part one can argue in the same way, exchanging the roles of v and w. This yields

$$(I) \le C\Big(\sup_{|t|\le t_0}\int_{\Omega}|w(t)|^2\,dx\Big)^{1/2} + C\Big(\sup_{|t|\le t_0}\int_{\Omega^c}|v(t)|^2\,dx\Big)^{1/2}. \qquad (2.9)$$

In order to bound the right-hand side further, we use a well-known argument. We take a function $\beta\in C_0^\infty(\mathbb{R})$ with $\beta(x)\in[0,1]$ such that $\beta(x)=1$ for $|x|\le(a+b)/2$ and $\beta(x)=0$ for $|x|\ge$

$(a+3b)/4$. Then $\|\beta'\|_{L^\infty} \le C(b-a)^{-1}$. Defining $J(t) = \int_{\mathbb{R}} |w(t)|^2 \beta(x)\, dx$ we have $J(0) = 0$. From (1.1) we get $|\frac{d}{dt} J(t)| = |(-2)\,\mathrm{Im} \int_{\mathbb{R}} \bar{w}(t)\, \frac{dw}{dx}(t) \beta'\, dx| \le C(b-a)^{-1} \|w(t)\|_{L^2} \|\frac{dw}{dx}(t)\|_{L^2} = C(b-a)^{-1}\|w\|_{L^2}\|\frac{dw}{dx}\|_{L^2} \le C(b-a)^{-1}\|w\|_{H^1}$. Thus for $|t| \le t_0$,

$$\int_\Omega |w(t)|^2\, dx = \int_\Omega |w(t)|^2 \beta(x)\, dx \le J(t) \le C(b-a)^{-1}\|w\|_{H^1}|t| \le C(b-a)^{-1}\|w\|_{H^1} t_0.$$

In an analogous manner $\int_{\Omega^c} |v(t)|^2\, dx$ can be bounded, whence (2.9) gives

$$(I) \le C(b-a)^{-1/2}\Big(\|v\|_{H^1} + \|w\|_{H^1}\Big)\, t_0^{1/2}. \tag{2.10}$$

Concerning (II), we remind the pseudo-conformal estimate $\|v(t)\|_{L^6} \le C(\|v\|_{L^2} + \|xv\|_{L^2})|t|^{-1/3}$; see [3, Cor. 3.3.4(ii)] with $r = 6$. This yields

$$\begin{aligned} (II) &= \int_{|t|>t_0} \int_{\mathbb{R}} |v|^3|w|^3\, dxdt \le \Big(\int_{|t|>t_0}\int_{\mathbb{R}} |v|^6\, dxdt\Big)^{1/2} \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} |w|^6\, dxdt\Big)^{1/2} \\ &\le C\Big(\int_{|t|>t_0} \|v(t)\|_{L^6}^6\, dt\Big)^{1/2} \le C(1+\|xv\|_{L^2})^3 \Big(\int_{|t|>t_0} |t|^{-2}\, dt\Big)^{1/2} \le C(1+\|xv\|_{L^2})^3\, t_0^{-1/2}. \end{aligned}$$

Summarizing this bound and (2.10) it follows that for every $t_0 > 0$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |v|^3|w|^3\, dxdt \le C(b-a)^{-1/2}\Big(\|v\|_{H^1} + \|w\|_{H^1}\Big)\, t_0^{1/2} + C(1+\|xv\|_{L^2})^3\, t_0^{-1/2}.$$

One can then optimize this estimate with respect to t_0 to obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |v|^3|w|^3\, dxdt \le C(1+\|xv\|_{L^2})^{3/2}\Big(\|v\|_{H^1}^{1/2} + \|w\|_{H^1}^{1/2}\Big)(b-a)^{-1/4}.$$

Hence going back to (2.8) and noting $\|xv\|_{L^2} \le a\|v\|_{L^2} \le a$, (2.6) is seen to hold. □

The next lemma states a more explicit representation of Φ.

Lemma 2.2 *For $\theta, \vartheta \in [0, 2\pi]$ we define*

$$\begin{aligned} a_1(\theta,\vartheta) &= -\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{6}}\sin\theta + \sqrt{\frac{2}{3}}\sin\vartheta, \quad a_2(\theta,\vartheta) = \frac{1}{\sqrt{2}}\cos\vartheta + \sqrt{\frac{3}{2}}\sin\vartheta, \\ a_3(\theta,\vartheta) &= \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{6}}\sin\theta + \sqrt{\frac{2}{3}}\sin\vartheta, \quad a_4(\theta,\vartheta) = -\frac{1}{\sqrt{2}}\cos\vartheta + \sqrt{\frac{3}{2}}\sin\vartheta, \\ a_5(\theta,\vartheta) &= \sqrt{\frac{2}{3}}\,(-\sin\theta + \sin\vartheta). \end{aligned}$$

Then

$$\begin{aligned} &\Phi(u_1,u_2,u_3,u_4,u_5,u_6) \\ &\quad= (4\pi)^{-1}\int_{\mathbb{R}} d\xi \int_0^\infty dr\, r \int_0^{2\pi} d\theta \int_0^{2\pi} d\vartheta\, \hat{u}_1(a_1(\theta,\vartheta)r+\xi)\, \bar{\hat{u}}_2(a_2(\theta,\vartheta)r+\xi)\, \hat{u}_3(a_3(\theta,\vartheta)r+\xi) \\ &\qquad\qquad \times \bar{\hat{u}}_4(a_4(\theta,\vartheta)r+\xi)\, \hat{u}_5(a_5(\theta,\vartheta)r+\xi)\, \bar{\hat{u}}_6(\xi) \qquad (2.11) \\ &\quad=: \Psi(\hat{u}_1,\hat{u}_2,\hat{u}_3,\hat{u}_4,\hat{u}_5,\hat{u}_6). \qquad (2.12) \end{aligned}$$

Proof: For $1 \le j \le 6$ we write

$$(e^{it\partial_x^2}u_j)(x) = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{i(x\xi - t\xi^2)}\hat{u}_j(\xi)\,d\xi \tag{2.13}$$

and insert these expressions with variable ξ_j for $\hat{u}_j$ into the definition of Φ, cf. (2.1). After integrating out $\int_{\mathbb{R}}\int_{\mathbb{R}} dxdt$, we arrive at

$$\begin{aligned}
&\Phi(u_1,u_2,u_3,u_4,u_5,u_6)\\
&= (2\pi)^{-1}\int_{\mathbb{R}}\dots\int_{\mathbb{R}} d\xi_1\dots d\xi_6\,\hat{u}_1(\xi_1)\bar{\hat{u}}_2(\xi_2)\hat{u}_3(\xi_3)\bar{\hat{u}}_4(\xi_4)\hat{u}_5(\xi_5)\bar{\hat{u}}_6(\xi_6)\\
&\qquad\times\delta_0(-\xi_1+\xi_2-\xi_3+\xi_4-\xi_5+\xi_6)\delta_0(\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2+\xi_5^2-\xi_6^2)\\
&= (2\pi)^{-1}\int_{\mathbb{R}}\dots\int_{\mathbb{R}} d\xi_1\dots d\xi_5\,\hat{u}_1(\xi_1)\bar{\hat{u}}_2(\xi_2)\hat{u}_3(\xi_3)\bar{\hat{u}}_4(\xi_4)\hat{u}_5(\xi_5)\bar{\hat{u}}_6(\xi_1-\xi_2+\xi_3-\xi_4+\xi_5)\\
&\qquad\times\delta_0\Big(\alpha(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\Big),
\end{aligned} \tag{2.14}$$

where

$$\alpha(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = (-2)\Big[(\xi_3+\xi_5-\xi_2-\xi_4)(\xi_1-\xi_2)+(\xi_5-\xi_4)(\xi_3-\xi_4)\Big]. \tag{2.15}$$

Next the transformation $\xi = Az$ is introduced, with the orthogonal matrix $A \in \mathbb{R}^{5\times 5}$ given by

$$A = \begin{pmatrix}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{5}} & 0 & -\frac{2}{3}\sqrt{\frac{3}{10}}\\
0 & 0 & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{10}}\\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{5}} & 0 & -\frac{2}{3}\sqrt{\frac{3}{10}}\\
0 & 0 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{2}} & \sqrt{\frac{3}{10}}\\
0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & 0 & -\frac{2}{3}\sqrt{\frac{3}{10}}
\end{pmatrix}. \tag{2.16}$$

Since $\alpha(Az) = z_1^2+z_2^2-z_4^2-5z_5^2$, this leads to

$$\begin{aligned}
&\Phi(u_1,u_2,u_3,u_4,u_5,u_6)\\
&= (2\pi)^{-1}\int_{\mathbb{R}}\dots\int_{\mathbb{R}} dz_1\dots dz_5\,\hat{u}_1\Big(-\frac{1}{\sqrt{2}}z_1+\frac{1}{\sqrt{6}}z_2+\frac{1}{\sqrt{5}}z_3-\frac{2}{3}\sqrt{\frac{3}{10}}z_5\Big)\\
&\qquad\times\bar{\hat{u}}_2\Big(\frac{1}{\sqrt{5}}z_3+\frac{1}{\sqrt{2}}z_4+\sqrt{\frac{3}{10}}z_5\Big)\\
&\qquad\times\hat{u}_3\Big(\frac{1}{\sqrt{2}}z_1+\frac{1}{\sqrt{6}}z_2+\frac{1}{\sqrt{5}}z_3-\frac{2}{3}\sqrt{\frac{3}{10}}z_5\Big)\\
&\qquad\times\bar{\hat{u}}_4\Big(\frac{1}{\sqrt{5}}z_3-\frac{1}{\sqrt{2}}z_4+\sqrt{\frac{3}{10}}z_5\Big)\\
&\qquad\times\hat{u}_5\Big(-\frac{2}{\sqrt{6}}z_2+\frac{1}{\sqrt{5}}z_3-\frac{2}{3}\sqrt{\frac{3}{10}}z_5\Big)\\
&\qquad\times\bar{\hat{u}}_6\Big(\frac{1}{\sqrt{5}}z_3-4\sqrt{\frac{3}{10}}z_5\Big)\delta_0(z_1^2+z_2^2-z_4^2-5z_5^2).
\end{aligned} \tag{2.17}$$

Then we pass to polar coordinates $(z_1, z_2) = r(\cos\theta, \sin\theta)$ and $(z_4, z_5) = s(\cos\vartheta, \frac{1}{\sqrt{5}}\sin\vartheta)$, and we set $\tilde{z}_3 = \frac{1}{\sqrt{5}}\, z_3$. If we also observe the relation $\int_0^\infty dr \int_0^\infty ds\, rsf(r,s)\delta_0(r^2 - s^2) = \frac{1}{2}\int_0^\infty dr\, rf(r,r)$, (2.17) may be rewritten as

$$
\begin{aligned}
&\Phi(u_1, u_2, u_3, u_4, u_5, u_6) \\
&= (4\pi)^{-1} \int_0^\infty dr\, r \int_0^{2\pi} d\theta \int_0^{2\pi} d\vartheta \int_{\mathbb{R}} d\tilde{z}_3\, \hat{u}_1\Big(\Big[-\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{6}}\sin\theta - \frac{2}{15}\sqrt{\frac{3}{2}}\sin\vartheta\Big]r + \tilde{z}_3\Big) \\
&\qquad \times \bar{\hat{u}}_2\Big(\Big[\frac{1}{\sqrt{2}}\cos\vartheta + \frac{1}{5}\sqrt{\frac{3}{2}}\sin\vartheta\Big]r + \tilde{z}_3\Big) \\
&\qquad \times \hat{u}_3\Big(\Big[\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{6}}\sin\theta - \frac{2}{15}\sqrt{\frac{3}{2}}\sin\vartheta\Big]r + \tilde{z}_3\Big) \\
&\qquad \times \bar{\hat{u}}_4\Big(\Big[-\frac{1}{\sqrt{2}}\cos\vartheta + \frac{1}{5}\sqrt{\frac{3}{2}}\sin\vartheta\Big]r + \tilde{z}_3\Big) \\
&\qquad \times \hat{u}_5\Big(\Big[-\frac{2}{\sqrt{6}}\sin\theta - \frac{2}{15}\sqrt{\frac{3}{2}}\sin\vartheta\Big]r + \tilde{z}_3\Big) \\
&\qquad \times \bar{\hat{u}}_6\Big(\Big[-\frac{4}{5}\sqrt{\frac{3}{2}}\sin\vartheta\Big]r + \tilde{z}_3\Big).
\end{aligned}
$$

Hence it remains to make the transformation $\xi = [-\frac{4}{5}\sqrt{\frac{3}{2}}\sin\vartheta]r + \tilde{z}_3$, $d\xi = d\tilde{z}_3$, to get (2.11). □

With Ψ from (2.12) let $\psi(v) := \Psi(v, v, v, v, v, v)$. Then

$$\psi(\hat{u}) = \Psi(\hat{u}, \hat{u}, \hat{u}, \hat{u}, \hat{u}, \hat{u}) = \Phi(u, u, u, u, u, u) = \varphi(u) \tag{2.18}$$

by Lemma 2.2 and (2.2). In particular, Strichartz' inequality yields

$$|\psi(v)| = \varphi(\check{v}) \le S^6 \|\check{v}\|_{L^2}^6 = S^6 \|v\|_{L^2}^6, \quad v \in L^2. \tag{2.19}$$

Due to (2.4) from Lemma 2.1(b), moreover

$$|\psi(v) - \psi(w)| \le C\Big(\max\{\|v\|_{L^2}, \|w\|_{L^2}\}\Big)^5 \|v - w\|_{L^2}, \quad v, w \in L^2. \tag{2.20}$$

We note some useful consequences.

Corollary 2.3 *The functional φ has the following invariances:*

(a) $\varphi(u(\cdot + x_0)) = \varphi(u)$ for $x_0 \in \mathbb{R}$;

(b) $\varphi(e^{ix\xi_0}u) = \varphi(u)$ for $\xi_0 \in \mathbb{R}$;

(c) $\varphi(u_\lambda) = \varphi(u)$ for $\lambda > 0$, where $u_\lambda(x) = \lambda^{1/2}u(\lambda x)$.

Proof: (a) Note that $(e^{it\partial_x^2}u(\cdot + x_0))(x) = (e^{it\partial_x^2}u)(x + x_0)$, since both sides have Fourier transform $e^{ix_0\xi}e^{-it\xi^2}\hat{u}(\xi)$. Hence (a) follows from the definition of φ, cf. (1.2). (b) From $\widehat{e^{ix\xi_0}u}(\xi) = \hat{u}(\xi - \xi_0)$ one gets $\varphi(e^{ix\xi_0}u) = \psi(\hat{u}(\cdot - \xi_0)) = \Psi(\hat{u}(\cdot - \xi_0), \ldots, \hat{u}(\cdot - \xi_0))$, see (2.18). But (2.11) implies that the latter equals $\Psi(\hat{u}, \ldots, \hat{u}) = \psi(\hat{u}) = \varphi(u)$, as the transformation $\tilde{\xi} = \xi - \xi_0$, $d\tilde{\xi} = d\xi$, in the $\int_{\mathbb{R}} d\xi$-integral can be made. (c) We have $(e^{it\partial_x^2}u_\lambda)(x) = \lambda^{1/2}(e^{i\lambda^2 t\partial_x^2}u)(\lambda x)$, whence (1.2) together with the substitution $(y, s) = (\lambda x, \lambda^2 t)$, $dyds = \lambda^3 dxdt$, show that $\varphi(u_\lambda) = \varphi(u)$. □

Corollary 2.4 *Let $u \in L^2$. Then $\varphi(u) \le \varphi(|\hat{u}|\check{\ })$.*

Proof: According to (2.18) the claim is equivalent to $\psi(\hat{u}) \le \psi(|\hat{u}|)$. But (2.11) yields $\psi(\hat{u}) = \Psi(\hat{u}, \dots, \hat{u}) \le \Psi(|\hat{u}|, \dots, |\hat{u}|) = \psi(|\hat{u}|)$. □

For $u \in L^2$ we moreover introduce

$$\begin{aligned} f_u(\xi) &= (4\pi)^{-1} \int_0^\infty dr\, r \int_0^{2\pi} d\theta \int_0^{2\pi} d\vartheta\, \hat{u}(a_1(\theta,\vartheta)r+\xi)\, \bar{\hat{u}}(a_2(\theta,\vartheta)r+\xi)\, \hat{u}(a_3(\theta,\vartheta)r+\xi) \\ &\qquad \times \bar{\hat{u}}(a_4(\theta,\vartheta)r+\xi)\, \hat{u}(a_5(\theta,\vartheta)r+\xi)\, \bar{\hat{u}}(\xi), \end{aligned} \tag{2.21}$$

so that

$$\int_{\mathbb{R}} f_u(\xi)\, d\xi = \psi(\hat{u}) = \varphi(u) \quad \text{and} \quad \int_{\mathbb{R}} \chi(\xi) f_u(\xi)\, d\xi = \Psi(\hat{u},\hat{u},\hat{u},\hat{u},\hat{u},\chi\hat{u}) = \Phi(u,u,u,u,u,\check{\chi} * u) \tag{2.22}$$

for χ real and bounded, by Lemma 2.2 and (2.18).

Lemma 2.5 *Let $\chi \in C_0^\infty(\mathbb{R})$, $\hat{u} \in L^1 \cap L^2$, and $\delta > 0$. Then*

$$\left|\psi(\chi\hat{u}) - \int_{\mathbb{R}} \chi(\xi)^6 f_u(\xi)\, d\xi\right| \le C\Big(\delta \|\chi\|_{L^\infty}^5 \|\chi'\|_{L^\infty} \|u\|_{L^2}^6 + \delta^{-1/3} \|\chi\|_{L^\infty}^6 \|u\|_{L^2}^{\frac{16}{3}} \|\hat{u}\|_{L^1}^{\frac{2}{3}}\Big),$$

with $C > 0$ independent of χ, u, and δ.

Proof: By definition we have

$$\begin{aligned} &\psi(\chi\hat{u}) - \int_{\mathbb{R}} \chi^6(\xi) f_u(\xi)\, d\xi \\ &= (4\pi)^{-1} \int_0^\infty dr\, r \int_0^{2\pi} d\theta \int_0^{2\pi} d\vartheta \int_{\mathbb{R}} d\xi\, \Big(\chi(a_1 r+\xi)\hat{u}(a_1 r+\xi)\, \chi(a_2 r+\xi)\bar{\hat{u}}(a_2 r+\xi) \\ &\qquad \times \chi(a_3 r+\xi)\hat{u}(a_3 r+\xi)\, \chi(a_4 r+\xi)\bar{\hat{u}}(a_4 r+\xi) \\ &\qquad \times [\chi(a_5 r+\xi) - \chi(\xi)]\hat{u}(a_5 r+\xi)\, \chi(\xi)\bar{\hat{u}}(\xi) \\ &\quad + \chi(a_1 r+\xi)\hat{u}(a_1 r+\xi)\, \chi(a_2 r+\xi)\bar{\hat{u}}(a_2 r+\xi)\, \chi(a_3 r+\xi)\hat{u}(a_3 r+\xi) \\ &\qquad \times [\chi(a_4 r+\xi) - \chi(\xi)]\bar{\hat{u}}(a_4 r+\xi)\, \chi(\xi)\hat{u}(a_5 r+\xi)\, \chi(\xi)\bar{\hat{u}}(\xi) \\ &\quad + \chi(a_1 r+\xi)\hat{u}(a_1 r+\xi)\, \chi(a_2 r+\xi)\bar{\hat{u}}(a_2 r+\xi)\, [\chi(a_3 r+\xi) - \chi(\xi)]\hat{u}(a_3 r+\xi) \\ &\qquad \times \chi(\xi)\bar{\hat{u}}(a_4 r+\xi)\, \chi(\xi)\hat{u}(a_5 r+\xi)\, \chi(\xi)\bar{\hat{u}}(\xi) \\ &\quad + \chi(a_1 r+\xi)\hat{u}(a_1 r+\xi)\, [\chi(a_2 r+\xi) - \chi(\xi)]\bar{\hat{u}}(a_2 r+\xi)\, \chi(\xi)\hat{u}(a_3 r+\xi) \\ &\qquad \times \chi(\xi)\bar{\hat{u}}(a_4 r+\xi)\, \chi(\xi)\hat{u}(a_5 r+\xi)\, \chi(\xi)\bar{\hat{u}}(\xi) \\ &\quad + [\chi(a_1 r+\xi) - \chi(\xi)]\hat{u}(a_1 r+\xi)\, \chi(\xi)\bar{\hat{u}}(a_2 r+\xi)\, \chi(\xi)\hat{u}(a_3 r+\xi) \\ &\qquad \times \chi(\xi)\bar{\hat{u}}(a_4 r+\xi)\, \chi(\xi)\hat{u}(a_5 r+\xi)\, \chi(\xi)\bar{\hat{u}}(\xi)\Big). \end{aligned}$$

Now we split $\int_0^\infty dr = \int_0^\delta dr + \int_\delta^\infty dr$. Noting $|a_i(\theta,\vartheta)| \le 3$ for $1 \le i \le 5$ and all $\theta, \vartheta \in [0, 2\pi]$, we can estimate $|\chi(a_i r+\xi) - \chi(\xi)| \le 3\|\chi'\|_{L^\infty} r \le 3\|\chi'\|_{L^\infty}\delta$ in the first part (I) of the above integral where $r \in [0, \delta]$. Hence

$$\begin{aligned} (I) &\le C\delta \|\chi\|_{L^\infty}^5 \|\chi'\|_{L^\infty} \int_0^\infty dr\, r \int_0^{2\pi} d\theta \int_0^{2\pi} d\vartheta \int_{\mathbb{R}} d\xi\, |\hat{u}(a_1 r+\xi)|\, |\hat{u}(a_2 r+\xi)|\, |\hat{u}(a_3 r+\xi)| \\ &\qquad \times |\hat{u}(a_4 r+\xi)|\, |\hat{u}(a_5 r+\xi)|\, |\hat{u}(\xi)| \\ &\le C\delta \|\chi\|_{L^\infty}^5 \|\chi'\|_{L^\infty}\, \psi(|\hat{u}|) \le C\delta \|\chi\|_{L^\infty}^5 \|\chi'\|_{L^\infty} \|u\|_{L^2}^6, \end{aligned} \tag{2.23}$$

cf. (2.19). The second part (II) where $r \in]\delta, \infty[$ is bounded by

$$(II) \;\leq\; C\|\chi\|_{L^\infty}^6 \int_\delta^\infty dr\, r \int_0^{2\pi} d\theta \int_0^{2\pi} d\vartheta \int_{\mathbb{R}} d\xi\, |\hat{u}(a_1 r+\xi)|\,|\hat{u}(a_2 r+\xi)|\,|\hat{u}(a_3 r+\xi)| \times |\hat{u}(a_4 r+\xi)|\,|\hat{u}(a_5 r+\xi)|\,|\hat{u}(\xi)|.$$

To take advantage of the fact that $r > \delta$, we will undo the transformations from Lemma 2.2 which led to (2.11). Going back the steps to (2.17), it hence follows with $v := |\hat{u}|$ that

$$\begin{aligned}(II) \;\leq\; C\|\chi\|_{L^\infty}^6 \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} dz_1 \ldots dz_5 \mathbf{1}_M(z_1,\ldots,z_5)\, v\Big(-\frac{1}{\sqrt{2}}\,z_1+\frac{1}{\sqrt{6}}\,z_2+\frac{1}{\sqrt{5}}\,z_3-\frac{2}{3}\sqrt{\frac{3}{10}}\,z_5\Big) \\ \times v\Big(\frac{1}{\sqrt{5}}\,z_3+\frac{1}{\sqrt{2}}\,z_4+\sqrt{\frac{3}{10}}\,z_5\Big) \\ \times v\Big(\frac{1}{\sqrt{2}}\,z_1+\frac{1}{\sqrt{6}}\,z_2+\frac{1}{\sqrt{5}}\,z_3-\frac{2}{3}\sqrt{\frac{3}{10}}\,z_5\Big) \\ \times v\Big(\frac{1}{\sqrt{5}}\,z_3-\frac{1}{\sqrt{2}}\,z_4+\sqrt{\frac{3}{10}}\,z_5\Big) \\ \times v\Big(-\frac{2}{\sqrt{6}}\,z_2+\frac{1}{\sqrt{5}}\,z_3-\frac{2}{3}\sqrt{\frac{3}{10}}\,z_5\Big) \\ \times v\Big(\frac{1}{\sqrt{5}}\,z_3-4\sqrt{\frac{3}{10}}\,z_5\Big)\delta_0(z_1^2+z_2^2-z_4^2-5z_5^2), \qquad (2.24)\end{aligned}$$

where $M = \{(z_1,\ldots,z_5) : z_1^2+z_2^2 \geq \delta^2, z_4^2+5z_5^2 \geq \delta^2\}$; observe that $z_1^2+z_2^2 = r^2$ and $z_4^2+5z_5^2 = s^2$. Next we set $z = A^{-1}\xi = A^t\xi \in \mathbb{R}^5$, with A from (2.16). Since

$$z_1 = \frac{1}{\sqrt{2}}\,(\xi_3-\xi_1), \quad z_2 = \frac{1}{\sqrt{6}}\,(\xi_1+\xi_3-2\xi_5),$$

it follows that $A(M) \subset N^{(1)} \cup N^{(2)}$, where

$$N^{(1)} = \{(\xi_1,\ldots,\xi_5) : |\xi_3-\xi_1| \geq \delta/2\}, \quad N^{(2)} = \{(\xi_1,\ldots,\xi_5) : |\xi_5-\xi_1| \geq \delta/2\}.$$

Indeed, otherwise for some $\xi = Az \in A(M)$ we would have $|\xi_3-\xi_1| < \delta/2$ as well as $|\xi_5-\xi_1| < \delta/2$, whence $z_1^2+z_2^2 = \frac{1}{2}\,(\xi_3-\xi_1)^2+\frac{1}{6}\,([\xi_3-\xi_1]+2[\xi_1-\xi_5])^2 \leq \frac{1}{2}(\delta^2/4)+\frac{1}{6}((\delta/2)+2(\delta/2))^2 = \delta^2/2$, in contradiction to $z \in M$. Thus if the transformation $\xi = Az$ in (2.24) is introduced, we get

$$\begin{aligned}(II) \;&\leq\; C\|\chi\|_{L^\infty}^6 \sum_{i=1}^2 \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} d\xi_1 \ldots d\xi_5\, \mathbf{1}_{N^{(i)}}(\xi_1,\ldots,\xi_5)\, v(\xi_1)v(\xi_2)v(\xi_3)v(\xi_4) \\ &\qquad \times v(\xi_5)v(\xi_1-\xi_2+\xi_3-\xi_4+\xi_5)\delta_0\Big(\alpha(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)\Big) \\ &=\; C\|\chi\|_{L^\infty}^6 \sum_{i=1}^2 \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} d\xi_1 \ldots d\xi_6\, \mathbf{1}_{N^{(i)}}(\xi_1,\ldots,\xi_5)\, v(\xi_1)v(\xi_2)v(\xi_3)v(\xi_4)v(\xi_5)v(\xi_6) \\ &\qquad \times \delta_0(-\xi_1+\xi_2-\xi_3+\xi_4-\xi_5+\xi_6)\delta_0(\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2+\xi_5^2-\xi_6^2),\end{aligned}$$

cf. (2.14) and (2.15). Since the term with $\mathbf{1}_{N^{(2)}}$ is transformed into the term with $\mathbf{1}_{N^{(1)}}$ when ξ_3 and ξ_5 are exchanged, it suffices to consider the contribution with $\mathbf{1}_{N^{(1)}}$. Hence,

$$\begin{aligned}(II) &\leq C\|\chi\|_{L^\infty}^6 \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} d\xi_1 \ldots d\xi_6 \, \mathbf{1}_{N^{(1)}}(\xi_1, \ldots, \xi_5)\, v(\xi_1)v(\xi_2)v(\xi_3)v(\xi_4)v(\xi_5)v(\xi_6) \\ &\qquad \times \delta_0(-\xi_1+\xi_2-\xi_3+\xi_4-\xi_5+\xi_6)\delta_0(\xi_1^2-\xi_2^2+\xi_3^2-\xi_4^2+\xi_5^2-\xi_6^2) \\ &= C\|\chi\|_{L^\infty}^6 \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{|\xi_3-\xi_1|\geq\delta/2\}} v(\xi_1)v(\xi_3)\, G(-\xi_1^2-\xi_3^2, \xi_1+\xi_3)\, d\xi_1 d\xi_3, \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} G(\tau,\xi) &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} d\xi_2\, d\xi_4\, d\xi_5\, d\xi_6\, v(\xi_2)v(\xi_4)v(\xi_5)v(\xi_6) \\ &\qquad \times \delta_0(-\xi+\xi_2+\xi_4-\xi_5+\xi_6)\delta_0(-\tau-\xi_2^2-\xi_4^2+\xi_5^2-\xi_6^2). \end{aligned}$$

Basically we can now follow a standard proof of the Strichartz inequality to obtain the desired estimate. Using (2.13) it is straightforward to verify that $G(\tau,\xi) = 2\pi\mathcal{F}\Big(|e^{it\partial_x^2}\check{v}|^2(\overline{e^{it\partial_x^2}\check{v}})^2\Big)(-\tau,-\xi)$. Thus the Hausdorff-Young inequality in conjunction with Strichartz' inequality imply

$$\|G\|_{L^3_{\tau\xi}} \leq C\Big\||e^{it\partial_x^2}\check{v}|^2(\overline{e^{it\partial_x^2}\check{v}})^2\Big\|_{L^{3/2}_{tx}} \leq C\|\check{v}\|_{L^2}^4 = C\|u\|_{L^2}^4, \tag{2.26}$$

recall $v = |\hat{u}|$. Therefore we can continue in (2.25),

$$\begin{aligned}(II) &\leq C\|\chi\|_{L^\infty}^6 \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{|\xi_3-\xi_1|\geq\delta/2\}} \frac{v(\xi_1)v(\xi_3)}{|\xi_3-\xi_1|^{1/3}}\, |\xi_3-\xi_1|^{1/3}\, G(-\xi_1^2-\xi_3^2,\xi_1+\xi_3)\, d\xi_1 d\xi_3 \\ &\leq C\|\chi\|_{L^\infty}^6 \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{|\xi_3-\xi_1|\geq\delta/2\}} \frac{v(\xi_1)^{3/2}v(\xi_3)^{3/2}}{|\xi_3-\xi_1|^{1/2}}\, d\xi_1 d\xi_3\Big)^{2/3} \\ &\qquad \times \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} |\xi_3-\xi_1|\, G(-\xi_1^2-\xi_3^2,\xi_1+\xi_3)^3\, d\xi_1 d\xi_3\Big)^{1/3} \\ &\leq C\delta^{-1/3}\|\chi\|_{L^\infty}^6 \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} v(\xi_1)^{3/2}v(\xi_3)^{3/2}\, d\xi_1 d\xi_3\Big)^{2/3} \|G\|_{L^3_{\tau\xi}}, \end{aligned} \tag{2.27}$$

the latter through the substitution $\tau = -\xi_1^2-\xi_3^2$, $\xi = \xi_1+\xi_3$, which has $d\tau d\xi = 2|\xi_3-\xi_1| d\xi_1 d\xi_3$. Hence it follows from (2.26) that

$$(II) \leq C\delta^{-1/3}\|\chi\|_{L^\infty}^6 \|u\|_{L^2}^4 \|v\|_{L^{\frac{3}{2}}}^2 \leq C\delta^{-1/3}\|\chi\|_{L^\infty}^6 \|u\|_{L^2}^4 \|\hat{u}\|_{L^1}^{\frac{2}{3}} \|\hat{u}\|_{L^2}^{\frac{4}{3}}.$$

Summarizing this estimate and (2.23), the proof of the lemma is complete. □

We remark that using $|\xi_3-\xi_1|^{1/2} \geq C|\xi_3-\xi_1|^{1/2-\kappa}\delta^\kappa$ and the Hardy-Littlewood-Sobolev inequality in (2.27), the bound on (II) can be improved to $C\delta^{-2\kappa/3}\|\chi\|_{L^\infty}^6\|u\|_{L^2}^{6-\frac{4\kappa}{3}}\|\hat{u}\|_{L^1}^{\frac{4\kappa}{3}}$ for $\kappa \in]0,\frac{1}{2}]$, but this fact will be of no relevance to us here.

For a proof of the following lemma see [8, Lemma 2.6].

Lemma 2.6 *Let $(u_j) \subset L^2$ be bounded and such that for any $A > 0$*

$$\lim_{j\to\infty} \sup_{x_0\in\mathbb{R}} \int_{x_0-A}^{x_0+A} |u_j|^2\, dx = 0,$$

i.e., (u_j) is "vanishing". With $\phi \in \mathcal{S}(\mathbb{R})$ (Schwartz functions) we define $u_j^{(l)} = (\phi\hat{u}_j)\check{} = \check{\phi} * u_j$. Then $(u_j^{(l)})$ is also vanishing.

3 Concentration compactness

This terminology is used for the fact that basically there are three possibilities for a (L^2 or H^1-) bounded sequence of functions: either it is tight (in the sense of measures), or it is "vanishing" (it tends to zero uniformly on every interval of fixed length), or it is "splitting" (into at least two parts with supports widely separated). This principle, see [10], has turned out to be very helpful in a number of variational problems. For our purposes here we need a small refinement which relies on the observation that in the case of a splitting sequence the two parts can be moved arbitrarily far apart, see (3.1) below. Since we will need a very explicit form of alternative (3), we include some details.

Lemma 3.1 *Let $(f_j) \subset L^2$ be a sequence such that $\|f_j\|_{L^2} = 1$ for $j \in \mathbb{N}$. Then there is a subsequence (not relabelled) such that exactly one of the following three possibilities occurs.*

(1) There exists a sequence $(z_j) \subset \mathbb{R}$ such that for every $\varepsilon > 0$ there is $R = R_\varepsilon > 0$ with the property that

$$\int_{z_j-R}^{z_j+R} |f_j|^2\,dz \geq 1-\varepsilon, \quad j \in \mathbb{N}.$$

(2) For every $A > 0$ we have

$$\lim_{j\to\infty} \sup_{z_0\in\mathbb{R}} \int_{z_0-A}^{z_0+A} |f_j|^2\,dz = 0.$$

(3) There is $\gamma \in]0,1[$ with the following property. For every $\delta \in]0,\gamma[$ there exist $j_0 = j_0(\delta) \in \mathbb{N}$ and $z_1^*, z_2^* > 0$ with

$$z_2^* \geq z_1^* + 4\delta^{-1} + \delta^{-6}(z_1^* + 2\delta^{-1})^6 \tag{3.1}$$

such that

$$\gamma - \delta < \sup_{z_0\in\mathbb{R}} \int_{z_0-z_2^*}^{z_0+z_2^*} |f_j|^2\,dz < \gamma + \delta, \quad j \geq j_0, \tag{3.2}$$

and for every $j \geq j_0$ we may select $z_j \in \mathbb{R}$ satisfying

$$\gamma - \delta < \int_{z_j-z_1^*}^{z_j+z_1^*} |f_j|^2\,dz < \gamma + \delta. \tag{3.3}$$

In particular, if we fix functions $\rho, \eta \in C_0^\infty(\mathbb{R})$ with values in $[0,1]$ which satisfy $\rho(z) = 1$ for $|z| \leq z_1^*$, $\rho(z) = 0$ for $|z| \geq z_1^* + 2\delta^{-1}$, $\eta(z) = 0$ for $|z| \leq z_2^* - 2\delta^{-1}$, and $\eta(z) = 1$ for $|z| \geq z_2^*$, then defining $v_j(z) = \rho(z-z_j)f_j(z)$ and $w_j(z) = \eta(z-z_j)f_j(z)$ one obtains for $j \geq j_0$ the estimates

$$\|f_j - (v_j + w_j)\|_{L^2}^2 \leq 2\delta, \quad \Big|\|v_j\|_{L^2}^2 - \gamma\Big| \leq 3\delta, \quad \text{and} \quad \Big|\|w_j\|_{L^2}^2 - (1-\gamma)\Big| \leq 9\delta.$$

Proof: The argument relies on the Lévy concentration functions $\Gamma_j(z) = \sup_{z_0\in\mathbb{R}} \int_{z_0-z}^{z_0+z} |f_j(y)|^2\,dy$, $z \geq 0$. Then $0 \leq \Gamma_j(z) \leq 1$ and Γ_j is non-decreasing. Hence there exists a subsequence of (f_j), a countable set $E \subset [0,\infty[$, and a non-negative and non-decreasing function Γ such that $\Gamma_j(z) \to \Gamma(z)$ as $j \to \infty$ for every $z \in [0,\infty[\setminus E$. With $\gamma := \lim_{z\to\infty} \Gamma(z) \in [0,1]$, there are three possibilities: the cases $\gamma = 1$ or $\gamma = 0$ lead to alternative (1) or (2), respectively, cf. [3, Lemma 8.3.8] (here it is not needed that (f_j) is bounded in H^1). So it remains to show that $\gamma \in]0,1[$ implies (3). To see this, we fix $\delta \in]0, \gamma[$ and choose $z^* > 0$ such that $\gamma - \delta < \Gamma(z) \le \gamma$ for $z \ge z^*$. Then we take two widely separated points where Γ_j converges to Γ, i.e., we fix some $z_1^* \in ([0,\infty[\setminus E) \cap [z^*,\infty[$ and then choose $z_2^* \in ([0,\infty[\setminus E) \cap [z^*,\infty[$ satisfying $z_2^* \ge z_1^* + 4\delta^{-1} + \delta^{-6}(z_1^* + 2\delta^{-1})^6$. Note that then in particular $(z_2^* - 2\delta^{-1}) - (z_1^* + 2\delta^{-1}) > 0$, whence the supports of ρ and η are separated. Since z_1^* and z_2^* are convergence points, we also find $j_0 \in \mathbb{N}$ with $\gamma - \delta < \Gamma_j(z_1^*) \le \Gamma_j(z_2^*) < \gamma + \delta$ for $j \ge j_0$. By definition of Γ_j, this yields (3.2), and moreover for every $j \ge j_0$ we find $z_j \in \mathbb{R}$ such that (3.3) holds. With ρ and η as in (3), we then define v_j and w_j. In view of (3.2) and (3.3) we have

$$\int_{z_1^* \le |z-z_j| \le z_2^*} |f_j(z)|^2\,dz = \int_{z_j - z_2^*}^{z_j + z_2^*} |f_j|^2\,dz - \int_{z_j - z_1^*}^{z_j + z_1^*} |f_j|^2\,dz \le \gamma + \delta - (\gamma - \delta) = 2\delta. \tag{3.4}$$

Due to the support properties of v_j and w_j therefore

$$\begin{aligned} \|f_j - (v_j + w_j)\|_{L^2}^2 &= \int_{z_1^* \le |z-z_j| \le z_2^*} (1 - \rho(z - z_j) - \eta(z - z_j))^2 |f_j(z)|^2\,dz \\ &\le \int_{z_1^* \le |z-z_j| \le z_2^*} |f_j(z)|^2\,dz \le 2\delta. \end{aligned}$$

In addition, (3.3), $z_1^* + 2\delta^{-1} \le z_2^*$, and (3.4) imply

$$\begin{aligned} \Big| \|v_j\|_{L^2}^2 - \gamma \Big| &= \Big| \int_{z_j - z_1^*}^{z_j + z_1^*} |f_j(z)|^2\,dz - \gamma + \int_{z_1^* \le |z-z_j| \le z_1^* + 2\delta^{-1}} \rho(z - z_j)^2 |f_j(z)|^2\,dz \Big| \\ &\le \delta + \int_{z_1^* \le |z-z_j| \le z_2^*} |f_j(z)|^2\,dz \le 3\delta. \end{aligned}$$

Finally, from this and $\|f_j\|_{L^2} = 1$ we get

$$\begin{aligned} \Big| \|w_j\|_{L^2}^2 - (1-\gamma) \Big| &\le 3\delta + \Big| \|w_j\|_{L^2}^2 + \|v_j\|_{L^2}^2 - \|f_j\|_{L^2}^2 \Big| \\ &= 3\delta + \Big| \int_{|z-z_j| > z_2^*} |f_j(z)|^2\,dz + \int_{z_2^* - 2\delta^{-1} \le |z-z_j| \le z_2^*} \eta(z - z_j)^2 |f_j(z)|^2\,dz \\ &\qquad + \int_{|z-z_j| < z_1^*} |f_j(z)|^2\,dz + \int_{z_1^* \le |z-z_j| \le z_1^* + 2\delta^{-1}} \rho(z - z_j)^2 |f_j(z)|^2\,dz \\ &\qquad - \int_{\mathbb{R}} |f_j(z)|^2\,dz \Big| \\ &\le 3\delta + \int_{z_1^* \le |z-z_j| \le z_2^*} |f_j(z)|^2\,dz + \int_{z_2^* - 2\delta^{-1} \le |z-z_j| \le z_2^*} |f_j(z)|^2\,dz \\ &\qquad + \int_{z_1^* \le |z-z_j| \le z_1^* + 2\delta^{-1}} |f_j(z)|^2\,dz \le 9\delta, \end{aligned}$$

where once more $z_1^* + 2\delta^{-1} \le z_2^*$ and (3.4) have been used. □

The following result examines what can be said in the case that a reversed Hölder-type estimate holds for measures. It is a special case of [11, Lemma I.2, p. 161], see also [5, p. 13].

Lemma 3.2 *Let ν, μ be finite non-negative measures on $\mathbb{R}$ such that for some $1 < p < q < \infty$ and $S > 0$ the estimate*

$$\Big(\int_{\mathbb{R}} |\chi|^q\,d\nu \Big)^{1/q} \le S \Big(\int_{\mathbb{R}} |\chi|^p\,d\mu \Big)^{1/p}, \quad \chi \in C_0^\infty(\mathbb{R}),$$

holds. If $\nu(\mathbb{R})^{1/q} \ge S\mu(\mathbb{R})^{1/p}$, then $\nu = \gamma\delta_{\xi_*}$ and $\mu = \gamma^{p/q} S^{-p} \delta_{\xi_*}$ for some $\gamma \ge 0$ and $\xi_* \in \mathbb{R}$.

4 Proof of Theorem 1.1

We consider a maximizing sequence, i.e., $(U_j) \subset L^2$ is such that $\|U_j\|_{L^2} = 1$ for $j \in \mathbb{N}$ and $\varphi(U_j) \to S^6$ as $j \to \infty$. Then we introduce $V_j = |\hat{U}_j|\check{}$ and note that $\hat{V}_j = |\hat{U}_j| \geq 0$ as well as $\hat{V}_j \in L^2$. Hence we can select $w_j \in \mathcal{S}(\mathbb{R})$ satisfying $w_j \geq 0$ and $\|w_j - \hat{V}_j\|_{L^2} \leq 1/j$. With these definitions we let $u_j = \|w_j\|_{L^2}^{-1} \check{w}_j$. Then $u_j \in \mathcal{S}(\mathbb{R})$, $\|u_j\|_{L^2} = 1$, and $\hat{u}_j \geq 0$. Therefore Strichartz' inequality yields $\varphi(u_j) \leq S^6$. Since $\|\hat{V}_j\|_{L^2} = \|U_j\|_{L^2} = 1$, we also have $|\|w_j\|_{L^2} - 1| \leq \|w_j - \hat{V}_j\|_{L^2} \leq 1/j$. Hence e.g. $\|w_j\|_{L^2} \leq 2$, and it follows from Corollary 2.4 and with (2.4) from Lemma 2.1(b) that

$$\begin{aligned} 0 &\leq S^6 - \varphi(u_j) \leq |S^6 - \varphi(U_j)| + \varphi(U_j) - \varphi(u_j) \leq |S^6 - \varphi(U_j)| + \varphi(|\hat{U}_j|\check{}\,) - \varphi(u_j) \\ &= |S^6 - \varphi(U_j)| + \varphi(V_j) - \|w_j\|_{L^2}^{-6}\, \varphi(\check{w}_j) \\ &\leq |S^6 - \varphi(U_j)| + |\varphi(V_j) - \varphi(\check{w}_j)| + \Big| \|w_j\|_{L^2}^{-6} - 1 \Big|\, \varphi(\check{w}_j) \\ &\leq |S^6 - \varphi(U_j)| + C\|V_j - \check{w}_j\|_{L^2} + C\Big| \|w_j\|_{L^2}^{-6} - 1 \Big| \to 0, \quad j \to \infty. \end{aligned}$$

Thus $\varphi(u_j) \to S^6$, i.e., (u_j) is a maximizing sequence with the additional properties that $u_j \in \mathcal{S}(\mathbb{R})$ and $\hat{u}_j \geq 0$. We need to modify the u_j in other respects, too. For this we fix $j \in \mathbb{N}$ and examine the concentration function $\hat{\Gamma}_j(z) = \sup_{\xi_0 \in \mathbb{R}} \int_{\xi_0 - z}^{\xi_0 + z} |\hat{u}_j|^2\, d\xi$ of $\hat{u}_j$. Since $\lim_{z \to \infty} \hat{\Gamma}_j(z) = 1$ we may select $\lambda_j > 0$ such that $\hat{\Gamma}_j(\lambda_j^{-1}) = 1/2$. In addition, the function $\xi_0 \mapsto \int_{\xi_0 - \lambda_j^{-1}}^{\xi_0 + \lambda_j^{-1}} |\hat{u}_j|^2\, d\xi$ is continuous and tending to zero as $\xi_0 \to \pm\infty$. Hence we also find $\xi_{0,j} \in \mathbb{R}$ with $\int_{\xi_{0,j} - \lambda_j^{-1}}^{\xi_{0,j} + \lambda_j^{-1}} |\hat{u}_j|^2\, d\xi = \sup_{\xi_0 \in \mathbb{R}}(\ldots) = \hat{\Gamma}_j(\lambda_j^{-1}) = 1/2$. We then define $\xi_j = -\lambda_j \xi_{0,j}$ and $\tilde{u}_j(x) = e^{ix\xi_j} \lambda_j^{1/2} u_j(\lambda_j x)$. If follows that $\|\tilde{u}_j\|_{L^2} = 1$, $\tilde{u}_j \in \mathcal{S}(\mathbb{R})$, and $\hat{\tilde{u}}_j(\xi) = \lambda_j^{-1/2} \hat{u}_j(\lambda_j^{-1}(\xi - \xi_j)) \geq 0$. From Corollary 2.3(b) and (c) we deduce that $\varphi(\tilde{u}_j) = \varphi(u_j)$, and we calculate

$$\begin{aligned} \sup_{\xi_0 \in \mathbb{R}} \int_{\xi_0 - 1}^{\xi_0 + 1} |\hat{\tilde{u}}_j(\xi)|^2\, d\xi &= \sup_{\xi_0 \in \mathbb{R}} \int_{\lambda_j^{-1}(\xi_0 - 1 - \xi_j)}^{\lambda_j^{-1}(\xi_0 + 1 - \xi_j)} |\hat{u}_j(\xi)|^2\, d\xi = \sup_{\xi_0 \in \mathbb{R}} \int_{\xi_0 - \lambda_j^{-1}}^{\xi_0 + \lambda_j^{-1}} |\hat{u}_j(\xi)|^2\, d\xi \\ &= \hat{\Gamma}_j(\lambda_j^{-1}) = 1/2. \end{aligned}$$

Moreover,

$$\int_{-1}^{1} |\hat{\tilde{u}}_j(\xi)|^2\, d\xi = \int_{\lambda_j^{-1}(-1 - \xi_j)}^{\lambda_j^{-1}(1 - \xi_j)} |\hat{u}_j(\xi)|^2\, d\xi = \int_{\xi_{0,j} - \lambda_j^{-1}}^{\xi_{0,j} + \lambda_j^{-1}} |\hat{u}_j(\xi)|^2\, d\xi = \frac{1}{2}\,.$$

Renaming $\tilde{u}_j$ to u_j, we can summarize the foregoing modifications as follows. There exists $(u_j) \subset \mathcal{S}(\mathbb{R})$ such that $\|u_j\|_{L^2} = 1$, $\hat{u}_j \geq 0$, and $\varphi(u_j) \to S^6$ as $j \to \infty$, and also

$$\sup_{\xi_0 \in \mathbb{R}} \int_{\xi_0 - 1}^{\xi_0 + 1} |\hat{u}_j(\xi)|^2\, d\xi = \int_{-1}^{1} |\hat{u}_j(\xi)|^2\, d\xi = \frac{1}{2}\,, \quad j \in \mathbb{N}, \tag{4.1}$$

is satisfied. With this special improved maximizing sequence we are going to work in the sequel.

Since also $\|\hat{u}_j\|_{L^2} = 1$ for $j \in \mathbb{N}$, Lemma 3.1 can be applied to the sequence $(f_j) = (\hat{u}_j)$. The following two subsections 4.1 and 4.2 deal with the two possibilities which then may occur (for a subsequence which is not relabelled) according to Lemma 3.1; note that the Fourier transforms cannot vanish in the sense of alternative (2) in Lemma 3.1, as $\lim_{j \to \infty} \sup_{\xi_0 \in \mathbb{R}} \int_{\xi_0 - 1}^{\xi_0 + 1} |\hat{u}_j|^2\, d\xi = 1/2 \neq 0$ by (4.1).

4.1 The Fourier transforms cannot be splitting

In this section we suppose that alternative (3) from Lemma 3.1 is satisfied for $(\hat{u}_j)$. Then we have $\hat{\gamma} \in]0,1[$ for $\hat{\gamma} = \lim_{\xi\to\infty} \hat{\Gamma}(\xi)$, the function $\hat{\Gamma}$ being the pointwise limit (outside a countable set) of the concentration functions $\hat{\Gamma}_j(\xi) := \sup_{\xi_0\in\mathbb{R}} \int_{\xi_0-\xi}^{\xi_0+\xi} |\hat{u}_j|^2\,d\zeta$ of the $\hat{u}_j$. We fix $\delta \in]0,\hat{\gamma}[$ and select $j_0 \in \mathbb{N}$, $\xi_1^* = z_1^* > 0$, $\xi_2^* = z_2^* > 0$, $\xi_j = z_j$ for $j \geq j_0$, and moreover the functions ρ and η as stated in (3) of Lemma 3.1; all these quantities are depending on δ. With $a_j(\xi) = \rho(\xi - \xi_j)\hat{u}_j(\xi)$ and $b_j(\xi) = \eta(\xi - \xi_j)\hat{u}_j(\xi)$, we then have $\|a_j\|_{L^2} \leq 1$, $\|b_j\|_{L^2} \leq 1$, and in addition for $j \geq j_0$ the estimates $\|\hat{u}_j - (a_j + b_j)\|_{L^2}^2 \leq 2\delta$, $|\|a_j\|_{L^2}^2 - \hat{\gamma}| \leq 3\delta$, as well as $|\|b_j\|_{L^2}^2 - (1-\hat{\gamma})| \leq 9\delta$. Setting $v_j = \check{a}_j$ and $w_j = \check{b}_j$, this leads to $\|v_j\|_{L^2} \leq 1$, $\|w_j\|_{L^2} \leq 1$, and also

$$\|u_j - (v_j + w_j)\|_{L^2}^2 \leq 2\delta, \quad \Big|\|v_j\|_{L^2}^2 - \hat{\gamma}\Big| \leq 3\delta, \quad \text{and} \quad \Big|\|w_j\|_{L^2}^2 - (1-\hat{\gamma})\Big| \leq 9\delta \tag{4.2}$$

for $j \geq j_0$. Then (2.4) from Lemma 2.1(b), (4.2), and Corollary 2.3(b) imply the estimate

$$\begin{aligned} |\varphi(u_j) - \varphi(v_j) - \varphi(w_j)| &\leq |\varphi(u_j) - \varphi(v_j + w_j)| + |\varphi(v_j + w_j) - \varphi(v_j) - \varphi(w_j)| \\ &\leq C\Big(\max\{\|u_j\|_{L^2}, \|v_j + w_j\|_{L^2}\}\Big)^5 \|u_j - (v_j + w_j)\|_{L^2} \\ &\qquad + |\varphi(v_j + w_j) - \varphi(v_j) - \varphi(w_j)| \\ &\leq C\delta^{1/2} + |\varphi(\tilde{v}_j + \tilde{w}_j) - \varphi(\tilde{v}_j) - \varphi(\tilde{w}_j)|, \end{aligned}$$

where $\tilde{v}_j(x) := e^{-ix\xi_j} v_j(x)$ and $\tilde{w}_j(x) := e^{-ix\xi_j} w_j(x)$. Since $\|\tilde{v}_j\|_{L^2} = \|v_j\|_{L^2} \leq 1$ and $\|\tilde{w}_j\|_{L^2} = \|w_j\|_{L^2} \leq 1$ we can then apply (2.5) from Lemma 2.1 to get

$$|\varphi(u_j) - \varphi(v_j) - \varphi(w_j)| \leq C\delta^{1/2} + C\Big(\|\tilde{v}_j\|_{H^{\frac14}} \|\tilde{w}_j\|_{H^{-\frac14}} + \|\tilde{v}_j\|_{H^{\frac14}}^2 \|\tilde{w}_j\|_{H^{-\frac14}}^2\Big). \tag{4.3}$$

The second term on the right-hand side can be handled by means of the following observation, where the notation from Lemma 3.1 is used.

Lemma 4.1 *We have the bound*

$$\|\tilde{v}_j\|_{H^{\frac14}} \|\tilde{w}_j\|_{H^{-\frac14}} \leq 2^{1/8}\delta^{1/4}.$$

Proof: Due to $\hat{\tilde{v}}_j(\xi) = \hat{v}_j(\xi + \xi_j) = a_j(\xi + \xi_j) = \rho(\xi)\hat{u}_j(\xi + \xi_j)$ we find

$$\|\tilde{v}_j\|_{H^{\frac14}}^2 = \int_{\mathbb{R}} (1+\xi^2)^{1/4} |\hat{\tilde{v}}_j(\xi)|^2\,d\xi = \int_{\mathbb{R}} (1+\xi^2)^{1/4} \rho(\xi)^2 |\hat{u}_j(\xi+\xi_j)|^2\,d\xi.$$

By definition of ρ we have $\rho(\xi) = 0$ for $|\xi| \geq \xi_1^* + 2\delta^{-1} = z_1^* + 2\delta^{-1}$. Therefore $\rho(\xi) \in [0,1]$ yields

$$\|\tilde{v}_j\|_{H^{\frac14}}^2 \leq (1 + [z_1^* + 2\delta^{-1}]^2)^{1/4} \int_{\mathbb{R}} |\hat{u}_j(\xi+\xi_j)|^2\,d\xi = (1 + [z_1^* + 2\delta^{-1}]^2)^{1/4}.$$

Similarly, we have $\hat{\tilde{w}}_j(\xi) = \eta(\xi)\hat{u}_j(\xi+\xi_j)$, thus

$$\|\tilde{w}_j\|_{H^{-\frac14}}^2 = \int_{\mathbb{R}} (1+\xi^2)^{-1/4} \eta(\xi)^2 |\hat{u}_j(\xi+\xi_j)|^2\,d\xi.$$

Recalling that $\eta(\xi) = 0$ for $|\xi| \le \xi_2^* - 2\delta^{-1} = z_2^* - 2\delta^{-1}$ and $\eta(\xi) \in [0,1]$, we obtain

$$\|\tilde{w}_j\|^2_{H^{-\frac{1}{4}}} \le (1 + [z_2^* - 2\delta^{-1}]^2)^{-1/4} \int_{\mathbb{R}} |\hat{u}_j(\xi + \xi_j)|^2 \, d\xi = (1 + [z_2^* - 2\delta^{-1}]^2)^{-1/4}.$$

Therefore we arrive at

$$\|\tilde{v}_j\|_{H^{\frac{1}{4}}} \|\tilde{w}_j\|_{H^{-\frac{1}{4}}} \le (1 + [z_1^* + 2\delta^{-1}]^2)^{1/8} (1 + [z_2^* - 2\delta^{-1}]^2)^{-1/8} \le 2^{1/8} (z_1^* + 2\delta^{-1})^{1/4} (z_2^* - 2\delta^{-1})^{-1/4},$$

where $\delta < \hat{\gamma} < 1 \le 2$ has been used, hence $1 \le z_1^* + 2\delta^{-1}$. Due to (3.1) we have $z_2^* \ge z_1^* + 4\delta^{-1} + \delta^{-6}(z_1^* + 2\delta^{-1})^6 \ge 2\delta^{-1} + \delta^{-1}(z_1^* + 2\delta^{-1})$. This yields $(z_1^* + 2\delta^{-1})(z_2^* - 2\delta^{-1})^{-1} \le \delta$, hence the claimed estimate holds. □

By Lemma 4.1 we can continue from (4.3) as

$$|\varphi(u_j) - \varphi(v_j) - \varphi(w_j)| \le C\delta^{1/2} + C\delta^{1/4} \le C\delta^{1/4}, \quad j \ge j_0. \tag{4.4}$$

Using Strichartz' inequality, this yields

$$\begin{aligned} \varphi(u_j) &\le C\delta^{1/4} + \varphi(v_j) + \varphi(w_j) \le C\delta^{1/4} + S^6 \|v_j\|_{L^2}^6 + S^6 \|w_j\|_{L^2}^6 \\ &\le C\delta^{1/4} + S^6 (3\delta + \hat{\gamma})^3 + S^6 (9\delta + (1 - \hat{\gamma}))^3. \end{aligned}$$

At the beginning of the argument $\delta \in]0, \hat{\gamma}[$ has been fixed, and we have found that the latter estimate holds for all $j \ge j_0 = j_0(\delta)$. Since (u_j) is a maximizing sequence, as $j \to \infty$ this yields

$$S^6 \le C\delta^{1/4} + S^6 (3\delta + \hat{\gamma})^3 + S^6 (9\delta + (1 - \hat{\gamma}))^3.$$

Taking the limit $\delta \to 0$ we finally arrive at $1 \le \hat{\gamma}^3 + (1 + \hat{\gamma})^3$, contradicting $\hat{\gamma} \in]0,1[$. Hence it is not possible that the Fourier transforms are splitting.

4.2 The Fourier transforms are tight

So far we have shown that the alternatives (2) and (3) from Lemma 3.1 cannot hold for the sequence $(\hat{u}_j)$. Therefore (1) has to be satisfied, i.e., there exists a sequence $(\xi_j) \subset \mathbb{R}$ such that for every $\varepsilon > 0$ there is $R = R_\varepsilon > 0$ with the property that $\int_{\xi_j - R}^{\xi_j + R} |\hat{u}_j|^2 \, d\xi \ge 1 - \varepsilon$ for $j \in \mathbb{N}$. It is well-known that in this kind of argument, in view of (4.1), one can assume $\xi_j = 0$ for every $j \in \mathbb{N}$ by replacing R_ε with $2R_\varepsilon + 1$; see e.g. [13, p. 48]. Indeed, if $\varepsilon < 1/2$, then we choose a corresponding R_ε and note that $I_{j,\varepsilon} = [\xi_j - R_\varepsilon, \xi_j + R_\varepsilon] \cap [-1,1] \ne \emptyset$, since otherwise by (4.1) the contradiction $1 = \int_{\mathbb{R}} |\hat{u}_j|^2 \, d\xi \ge \int_{\xi_j - R_\varepsilon}^{\xi_j + R_\varepsilon} |\hat{u}_j|^2 \, d\xi + \int_{-1}^{1} |\hat{u}_j|^2 \, d\xi \ge 1 - \varepsilon + 1/2 = 3/2 - \varepsilon$ would be obtained. But $I_{j,\varepsilon} \ne \emptyset$ implies $[\xi_j - R_\varepsilon, \xi_j + R_\varepsilon] \subset [-(2R_\varepsilon + 1), 2R_\varepsilon + 1]$, whence $\int_{-(2R_\varepsilon+1)}^{2R_\varepsilon+1} |\hat{u}_j|^2 \, d\xi \ge \int_{\xi_j - R_\varepsilon}^{\xi_j + R_\varepsilon} |\hat{u}_j|^2 \, d\xi \ge 1 - \varepsilon$. Hence we can assume that

$$\forall\, \varepsilon \in]0, 1/2[\quad \exists\, R = R_\varepsilon > 0 : \quad \int_{-R}^{R} |\hat{u}_j|^2 \, d\xi \ge 1 - \varepsilon, \quad j \in \mathbb{N}, \tag{4.5}$$

is satisfied.

Lemma 4.2 *For every $j \in \mathbb{N}$ let the measures ν_j and μ_j on $\mathbb{R}$ be defined as $\nu_j = f_{u_j}\, d\xi$ and $\mu_j = |\hat{u}_j|^2\, d\xi$, where f_{u_j} is given by (2.21). Then ν_j and μ_j are non-negative measures, and the sequences (ν_j) and (μ_j) are tight. There exist non-negative measures ν and μ on $\mathbb{R}$ such that, possibly after selecting subsequences, $\nu_j \rightharpoonup^* \nu$ as well as $\mu_j \rightharpoonup^* \mu$ as $j \to \infty$ in the sense of measures. In addition,*

$$\int_{\mathbb{R}} d\nu = S^6 \quad \textit{and} \quad \int_{\mathbb{R}} d\mu = 1. \tag{4.6}$$

Proof: By construction we have $\hat{u}_j \geq 0$, whence (2.21) shows that $f_{u_j} \geq 0$. Moreover, (2.22) implies $\int_{\mathbb{R}} d\nu_j = \int_{\mathbb{R}} f_{u_j}\, d\xi = \varphi(u_j) \to S^6$ as $j \to \infty$. Using the fact that $\|\hat{u}_j\|_{L^2} = 1$, we also get $\int_{\mathbb{R}} d\mu_j = \int_{\mathbb{R}} |\hat{u}_j|^2\, d\xi = 1$ for $j \in \mathbb{N}$. From (4.5) we deduce that (μ_j) is tight. Hence, by passing to a subsequence if necessary, $\mu_j \rightharpoonup^* \mu$ as $j \to \infty$ in the sense of measures (i.e., $\int_{\mathbb{R}} \chi\, d\mu_j \to \int_{\mathbb{R}} \chi\, d\mu$ for all bounded $\chi \in C(\mathbb{R})$), by Prochorov's compactness theorem. In view of the Portmanteau theorem this in turn implies $\int_{\mathbb{R}} d\mu = 1$. Therefore it remains to be verified that (ν_j) is tight, too. For this we note that, due to the second relation from (2.22) and Lemma 2.1(a),

$$\begin{aligned} \int_{|\xi|\geq R} d\nu_j &= \int_{\mathbb{R}} \mathbf{1}_{|\xi|\geq R} f_{u_j}\, d\xi = \Phi(u_j, u_j, u_j, u_j, u_j, (\mathbf{1}_{|\xi|\geq R})\check{} * u_j) \leq S^6 \|(\mathbf{1}_{|\xi|\geq R})\check{} * u_j\|_{L^2} \\ &= S^6 \Big(1 - \int_{|\xi|<R} |\hat{u}_j|^2\, d\xi\Big)^{1/2} \leq S^6 \varepsilon^{1/2}, \quad j \in \mathbb{N}, \end{aligned}$$

the latter provided that $\varepsilon \in]0, 1/2[$ is prescribed and $R = R_\varepsilon$ is chosen according to (4.5). Hence (ν_j) is tight and we may argue as before to obtain $\nu_j \rightharpoonup^* \nu$ along a subsequence and $\int_{\mathbb{R}} d\nu = \lim_{j\to\infty} \int_{\mathbb{R}} d\nu_j = S^6$. □

The next step consists in the application of Lemma 3.1 to $(f_j) = (u_j)$, yielding a further subsequence of (u_j), which however will be not relabelled for notational convenience. The argument is then divided once more according to which one of the possibilities (1), (2), or (3) from Lemma 3.1 arises.

4.2.1 The case that the sequence (u_j) is vanishing

Throughout this subsection we assume that (2) in Lemma 3.1 occurs for (u_j), i.e.,

$$\lim_{j\to\infty} \sup_{x_0\in\mathbb{R}} \int_{x_0-A}^{x_0+A} |u_j|^2\, dx = 0 \tag{4.7}$$

is satisfied for every $A > 0$. Our aim is to derive a contradiction from this, whence in fact (u_j) cannot be vanishing.

Lemma 4.3 *With ν and μ from Lemma 4.2 the estimate*

$$\Big(\int_{\mathbb{R}} \chi^6\, d\nu\Big)^{1/6} \leq S\Big(\int_{\mathbb{R}} \chi^2\, d\mu\Big)^{1/2}, \quad \chi \in C_0^\infty(\mathbb{R}),$$

holds.

Before we are going on to the proof of Lemma 4.3, let us first argue why this leads to a contradiction. From (4.6) in Lemma 4.2 we know that $\nu(\mathbb{R})^{1/6} = S = S\mu(\mathbb{R})^{1/2}$. Hence we can apply Lemma 3.2 with $q = 6$ and $p = 2$ to find that $\nu = \gamma\delta_{\xi_*}$ and $\mu = \gamma^{1/3}S^{-2}\delta_{\xi_*}$ for some $\gamma \geq 0$ and $\xi_* \in \mathbb{R}$. Then $\nu(\mathbb{R}) = S^6$ yields $\gamma = S^6$ and $\mu = \delta_{\xi_*}$. Therefore $1 = \mu(]\xi_* - 1, \xi_* + 1[) \leq \liminf_{j\to\infty} \mu_j(]\xi_* - 1, \xi_* + 1[) = \liminf_{j\to\infty} \int_{\xi_*-1}^{\xi_*+1} |\hat{u}_j|^2 \, d\xi \leq \liminf_{j\to\infty} \sup_{\xi_0\in\mathbb{R}} \int_{\xi_0-1}^{\xi_0+1} |\hat{u}_j|^2 \, d\xi \leq 1/2$ by (4.1), which is a contradiction.

Proof of Lemma 4.3: Using (4.5) we will first split every u_j into a regular low-frequency part and a small high-frequency part as follows. We choose $\phi \in C_0^\infty(\mathbb{R})$ with values in $[0, 1]$ such that $\phi(\xi) = 1$ for $|\xi| \leq 1$ and $\phi(\xi) = 0$ for $|\xi| \geq 2$, and for $R > 0$ we define $\phi_R(\xi) = \phi(\xi/R)$. With a fixed sequence $(\varepsilon_k) \subset]0, 1/2[$, $\varepsilon_k \to 0$ as $k \to \infty$, we select $R_k := R_{\varepsilon_k}$ corresponding to ε_k via (4.5) for $k \in \mathbb{N}$. Setting $\phi_k = \phi_{R_k}$, we decompose every u_j as

$$u_j = u_{j,k}^{(l)} + u_{j,k}^{(h)}, \quad j, k \in \mathbb{N}, \tag{4.8}$$

where

$$u_{j,k}^{(l)} = (\phi_k \hat{u}_j)\check{} = \check{\phi}_k * u_j \quad \text{and} \quad u_{j,k}^{(h)} = ((1 - \phi_k)\hat{u}_j)\check{} = (1 - \phi_k)\check{} * u_j. \tag{4.9}$$

Since $u_j \in \mathcal{S}(\mathbb{R})$ and $\hat{u}_j \geq 0$ by construction, we get $u_{j,k}^{(l)} \in \mathcal{S}(\mathbb{R})$ and $\hat{u}_{j,k}^{(l)} = \phi_k \hat{u}_j \geq 0$. For all $j, k \in \mathbb{N}$,

$$\Big\| \frac{du_{j,k}^{(l)}}{dx} \Big\|_{L^2}^2 = \|u_{j,k}^{(l)}\|_{\dot{H}^1}^2 = \int_{\mathbb{R}} |\xi|^2 |\phi_k \hat{u}_j|^2 \, d\xi = \int_{|\xi|\leq 2R_k} |\xi|^2 |\phi_k \hat{u}_j|^2 \, d\xi \leq CR_k^2 \|\hat{u}_j\|_{L^2}^2 = CR_k^2, \tag{4.10}$$

and also, by (4.8) and (4.5),

$$\|u_j - u_{j,k}^{(l)}\|_{L^2}^2 = \|u_{j,k}^{(h)}\|_{L^2}^2 = \int_{|\xi|\geq R_k} |(1 - \phi_k)\hat{u}_j|^2 \, d\xi \leq \int_{|\xi|\geq R_k} |\hat{u}_j|^2 \, d\xi = 1 - \int_{|\xi|<R_k} |\hat{u}_j|^2 \, d\xi \leq \varepsilon_k \, . \tag{4.11}$$

In addition,

$$\|u_{j,k}^{(l)}\|_{L^2} \leq \|\hat{u}_j\|_{L^2} = 1. \tag{4.12}$$

In the following we fix $\chi \in C_0^\infty(\mathbb{R})$. Then by definition of ν_j, the second relation in (2.22), Lemma 2.1(b), (4.12), and (4.11),

$$\begin{aligned}
\int_{\mathbb{R}} \chi^6 \, d\nu_j &= \int_{\mathbb{R}} \chi^6 f_{u_j} \, d\xi = \int_{\mathbb{R}} \chi^6 f_{u_{j,k}^{(l)}} \, d\xi + \int_{\mathbb{R}} \chi^6 f_{u_j} \, d\xi - \int_{\mathbb{R}} \chi^6 f_{u_{j,k}^{(l)}} \, d\xi \\
&= \int_{\mathbb{R}} \chi^6 f_{u_{j,k}^{(l)}} \, d\xi + \Phi\Big(u_j, u_j, u_j, u_j, u_j, (\chi^6)\check{} * u_j\Big) \\
&\quad - \Phi\Big(u_{j,k}^{(l)}, u_{j,k}^{(l)}, u_{j,k}^{(l)}, u_{j,k}^{(l)}, u_{j,k}^{(l)}, (\chi^6)\check{} * u_{j,k}^{(l)}\Big) \\
&\leq \int_{\mathbb{R}} \chi^6 f_{u_{j,k}^{(l)}} \, d\xi + C\Big(\max\Big\{ \|u_j\|_{L^2}, \|(\chi^6)\check{} * u_j\|_{L^2}, \|u_{j,k}^{(l)}\|_{L^2}, \|(\chi^6)\check{} * u_{j,k}^{(l)}\|_{L^2} \Big\} \Big)^5 \\
&\qquad\qquad \times \max\Big\{ \|u_j - u_{j,k}^{(l)}\|_{L^2}, \Big\|(\chi^6)\check{} * u_j - (\chi^6)\check{} * u_{j,k}^{(l)}\Big\|_{L^2} \Big\} \\
&\leq \psi\Big(\chi \hat{u}_{j,k}^{(l)}\Big) + \Big| \int_{\mathbb{R}} \chi^6 f_{u_{j,k}^{(l)}} \, d\xi - \psi\Big(\chi \hat{u}_{j,k}^{(l)}\Big) \Big| + C\varepsilon_k^{1/2}, \quad j, k \in \mathbb{N},
\end{aligned}$$

where here and in the sequel $C > 0$ may depend on χ, but not on k or j. Next we can invoke Lemma 2.5 with $u = u_{j,k}^{(l)}$ satisfying $\hat{u} = \hat{u}_{j,k}^{(l)} \in \mathcal{S}(\mathbb{R}) \subset L^1 \cap L^2$ and $\delta = \varepsilon_k^{1/2}$ to obtain from (2.20), Strichartz' inequality, and (4.11),

$$
\begin{aligned}
&\int_{\mathbb{R}} \chi^6 \, d\nu_j \\
\leq\;& \psi\Big(\chi \hat{u}^{(l)}_{j,k}\Big) + C\Big(\varepsilon_k^{1/2} \|\chi\|^5_{L^\infty} \|\chi'\|_{L^\infty} \, \|u^{(l)}_{j,k}\|^6_{L^2} + \varepsilon_k^{-1/6} \|\chi\|^6_{L^\infty} \|u^{(l)}_{j,k}\|^{\frac{16}{3}}_{L^2} \|\hat{u}^{(l)}_{j,k}\|^{\frac{2}{3}}_{L^1}\Big) + C\varepsilon_k^{1/2} \\
\leq\;& \Big|\psi\Big(\chi \hat{u}^{(l)}_{j,k}\Big) - \psi(\chi \hat{u}_j)\Big| + \varphi\Big(\check{\chi} * u_j\Big) + C\varepsilon_k^{1/2} + C\varepsilon_k^{-1/6} \|\hat{u}^{(l)}_{j,k}\|^{\frac{2}{3}}_{L^1} \\
\leq\;& C\Big(\max\Big\{\|\chi \hat{u}^{(l)}_{j,k}\|_{L^2}, \|\chi \hat{u}_j\|_{L^2}\Big\}\Big)^5 \Big\|\chi \hat{u}^{(l)}_{j,k} - \chi \hat{u}_j\Big\|_{L^2} + S^6 \, \|\check{\chi} * u_j\|^6_{L^2} \\
&+ C\varepsilon_k^{1/2} + C\varepsilon_k^{-1/6} \|\hat{u}^{(l)}_{j,k}\|^{\frac{2}{3}}_{L^1} \\
\leq\;& S^6 \, \Big(\int_{\mathbb{R}} \chi^2 |\hat{u}_j|^2 \, d\xi\Big)^3 + C\varepsilon_k^{1/2} + C\varepsilon_k^{-1/6} \|\hat{u}^{(l)}_{j,k}\|^{\frac{2}{3}}_{L^1} \\
=\;& S^6 \, \Big(\int_{\mathbb{R}} \chi^2 \, d\mu_j\Big)^3 + C\varepsilon_k^{1/2} + C\varepsilon_k^{-1/6} \|\hat{u}^{(l)}_{j,k}\|^{\frac{2}{3}}_{L^1} . \qquad (4.13)
\end{aligned}
$$

Here $C = C(\chi)$, and this estimate holds for all $j, k \in \mathbb{N}$. We claim that for fixed $k \in \mathbb{N}$

$$\|\hat{u}^{(l)}_{j,k}\|_{L^1} \to 0 \quad \text{as} \quad j \to \infty. \qquad (4.14)$$

We remark that in order to verify (4.14) it is sufficient to prove that there exists a subsequence $(j') \subset (j)$ such that $\lim_{j' \to \infty} \|\hat{u}^{(l)}_{j',k}\|_{L^1} = 0$, since the following argument can also be used if we start with a subsequence of the original sequence $(\hat{u}^{(l)}_{j,k})_{j \in \mathbb{N}}$. To finally invoke (4.7), we apply Lemma 2.6 to $\phi = \phi_k \in C_0^\infty(\mathbb{R})$. This yields, cf. (4.9), $\lim_{j\to\infty} \sup_{x_0 \in \mathbb{R}} \int_{x_0 - A}^{x_0 + A} |u^{(l)}_{j,k}|^2 \, dx = 0$ for every $A > 0$. Hence in particular $u^{(l)}_{j,k} \to 0$ as $j \to \infty$ in $L^2_{\rm loc}$. Then a diagonal argument implies that there is a subsequence $(j') \subset (j)$ such that $u^{(l)}_{j',k}(x) \to 0$ as $j' \to \infty$ for a.e. $x \in \mathbb{R}$. Recalling $u^{(l)}_{j',k} \in \mathcal{S}(\mathbb{R})$ and (4.10), it follows that $|u^{(l)}_{j',k}(x) - u^{(l)}_{j',k}(y)| \leq C R_k |x - y|^{1/2}$. Thus we must in fact have $u^{(l)}_{j',k}(x) \to 0$ as $j' \to \infty$ for every $x \in \mathbb{R}$. Therefore $\hat{u}^{(l)}_{j',k} \geq 0$ leads to

$$\|\hat{u}^{(l)}_{j',k}\|_{L^1} = \int_{\mathbb{R}} \hat{u}^{(l)}_{j',k}(\xi) \, d\xi = (2\pi)^{1/2} u^{(l)}_{j',k}(0) \to 0 \quad \text{as} \quad j' \to \infty.$$

This completes the proof of the claim (4.14). Then going back to (4.13) we fix $k \in \mathbb{N}$ and take the limit $j \to \infty$. Since $\nu_j \rightharpoonup^* \nu$ and $\mu_j \rightharpoonup^* \mu$ as $j \to \infty$ in the sense of measures, cf. Lemma 4.2, it follows from (4.14) that

$$\int_{\mathbb{R}} |\chi|^6 \, d\nu \leq S^6 \, \Big(\int_{\mathbb{R}} |\chi|^2 \, d\mu\Big)^3 + C\varepsilon_k^{1/2}.$$

As this estimate holds for every $k \in \mathbb{N}$, we can pass to the limit $k \to \infty$ to finish the proof of Lemma 4.3. □

4.2.2 The case that the sequence (u_j) is splitting

In this subsection we suppose that (3) in Lemma 3.1 is satisfied for (u_j), and again our goal is to show that this is impossible. Since (4.5) holds, we can once more use the decomposition of the u_j in low and high frequencies, recall (4.8)-(4.12). Furthermore, we have $\gamma \in]0, 1[$, where

$\gamma = \lim_{x\to\infty}\Gamma(x)$. Here $\Gamma(x) = \lim_{j\to\infty}\Gamma_j(x) = \lim_{j\to\infty}\sup_{x_0\in\mathbb{R}}\int_{x_0-x}^{x_0+x}|u_j|^2\,dy$ is the pointwise (outside a countable set) limit of the concentration functions corresponding to (u_j). We now fix $\delta\in]0,\gamma[$ and choose $j_0\in\mathbb{N}$, $x_1^*=z_1^*>0$, $x_2^*=z_2^*>0$, $x_j=z_j$ for $j\ge j_0$, and moreover the functions ρ and η as described in (3) of Lemma 3.1; once again, all these quantities are depending on δ. Defining $v_j(x)=\rho(x-x_j)u_j(x)$ and $w_j(x)=\eta(x-x_j)u_j(x)$, we remind that then $\|v_j\|_{L^2}\le 1$, $\|w_j\|_{L^2}\le 1$, and also

$$\|u_j-(v_j+w_j)\|_{L^2}^2\le 2\delta,\qquad \Big|\|v_j\|_{L^2}^2-\gamma\Big|\le 3\delta,\quad\text{and}\quad \Big|\|w_j\|_{L^2}^2-(1-\gamma)\Big|\le 9\delta \tag{4.15}$$

holds for $j\ge j_0$. Next we are going to transfer these estimates for every $k\in\mathbb{N}$ to the functions obtained in an analogous way from the low-frequency parts $u_{j,k}^{(l)}$ of u_j, cf. (4.9). To this end, we introduce

$$v_{j,k}(x)=\rho(x-x_j)u_{j,k}^{(l)}(x)\quad\text{and}\quad w_{j,k}(x)=\eta(x-x_j)u_{j,k}^{(l)}(x). \tag{4.16}$$

Since ρ and η attain their values in $[0,1]$, it follows from (4.12) that

$$\|v_{j,k}\|_{L^2}\le\|u_{j,k}^{(l)}\|_{L^2}\le 1\quad\text{and}\quad\|w_{j,k}\|_{L^2}\le\|u_{j,k}^{(l)}\|_{L^2}\le 1. \tag{4.17}$$

Moreover, $v_{j,k}\in H^1$ and $w_{j,k}\in H^1$. Due to $\|\rho'\|_{L^\infty}\sim\delta\le 1$ and $\|\eta'\|_{L^\infty}\sim\delta\le 1$, we obtain, using (4.10), the bounds

$$\|v_{j,k}\|_{H^1}+\|w_{j,k}\|_{H^1}\le CR_k. \tag{4.18}$$

The estimates from (4.15) are modified to

$$\|u_j-(v_{j,k}+w_{j,k})\|_{L^2}\le 2(\delta^{1/2}+\varepsilon_k^{1/2}),\qquad\Big|\|v_{j,k}\|_{L^2}^2-\gamma\Big|\le 3(\delta+\varepsilon_k^{1/2}), \tag{4.19}$$

$$\text{and}\quad\Big|\|w_{j,k}\|_{L^2}^2-(1-\gamma)\Big|\le 9(\delta+\varepsilon_k^{1/2}) \tag{4.20}$$

for $j\ge j_0$ and $k\in\mathbb{N}$. Indeed, since $\rho(x)\in[0,1]$, (4.8) and (4.11) imply

$$\|v_j-v_{j,k}\|_{L^2}^2=\int_{\mathbb{R}}\rho(x-x_j)^2|u_j(x)-u_{j,k}^{(l)}(x)|^2\,dx\le\left\|u_j-u_{j,k}^{(l)}\right\|_{L^2}^2=\left\|u_{j,k}^{(h)}\right\|_{L^2}^2\le\varepsilon_k,$$

and in the same way $\|w_j-w_{j,k}\|_{L^2}^2\le\varepsilon_k$ follows, whence (4.19) and (4.20) are obtained. The following argument is similar to that given in Section 4.1. By (2.4) in Lemma 2.1(b), (4.17), the first estimate from (4.19), and Corollary 2.3(a),

$$\begin{aligned}|\varphi(u_j)-\varphi(v_{j,k})-\varphi(w_{j,k})| &\le |\varphi(u_j)-\varphi(v_{j,k}+w_{j,k})|+|\varphi(v_{j,k}+w_{j,k})-\varphi(v_{j,k})-\varphi(w_{j,k})|\\ &\le C\Big(\max\{\|u_j\|_{L^2},\|v_{j,k}+w_{j,k}\|_{L^2}\}\Big)^5\|u_j-(v_{j,k}+w_{j,k})\|_{L^2}\\ &\quad +|\varphi(v_{j,k}+w_{j,k})-\varphi(v_{j,k})-\varphi(w_{j,k})|\\ &\le C(\delta^{1/2}+\varepsilon_k^{1/2})+|\varphi(\tilde v_{j,k}+\tilde w_{j,k})-\varphi(\tilde v_{j,k})-\varphi(\tilde w_{j,k})|,\end{aligned} \tag{4.21}$$

where $\tilde v_{j,k}(x):=v_{j,k}(x+x_j)=\rho(x)u_{j,k}^{(l)}(x+x_j)$ and $\tilde w_{j,k}(x):=w_{j,k}(x+x_j)=\eta(x)u_{j,k}^{(l)}(x+x_j)$, cf. (4.16). We recall from Lemma 3.1 that $\mathrm{supp}(\rho)\subset\{x\in\mathbb{R}:|x|\le x_1^*+2\delta^{-1}=z_1^*+2\delta^{-1}\}$ and $\mathrm{supp}(\eta)\subset\{x\in\mathbb{R}:|x|\ge x_2^*-2\delta^{-1}=z_2^*-2\delta^{-1}\}$. Since $\|\tilde v_{j,k}\|_{L^2}=\|v_{j,k}\|_{L^2}\le 1$ and $\|\tilde w_{j,k}\|_{L^2}=\|w_{j,k}\|_{L^2}\le 1$ by (4.17), (2.6) from Lemma 2.1(d) can thus be applied with $a=z_1^*+2\delta^{-1}$ and $b=z_2^*-2\delta^{-1}$. Using (4.18) we obtain

$$\begin{aligned}|\varphi(\tilde v_{j,k}+\tilde w_{j,k})-\varphi(\tilde v_{j,k})-\varphi(\tilde w_{j,k})| &\le C\Big(\|\tilde v_{j,k}\|_{H^1}^{1/6}+\|\tilde w_{j,k}\|_{H^1}^{1/6}\Big)(1+a)^{1/2}(b-a)^{-1/12}\\ &= C\Big(\|v_{j,k}\|_{H^1}^{1/6}+\|w_{j,k}\|_{H^1}^{1/6}\Big)(1+a)^{1/2}(b-a)^{-1/12}\\ &\le CR_k^{1/6}(1+a)^{1/2}(b-a)^{-1/12}.\end{aligned} \tag{4.22}$$

Lemma 4.4 *With $a = z_1^* + 2\delta^{-1}$ and $b = z_2^* - 2\delta^{-1}$ the estimate $(1+a)^{1/2}(b-a)^{-1/12} \leq 2^{1/2}\delta^{1/2}$ holds.*

Proof: By choice of z_2^*, see (3.1), $z_2^* \geq z_1^* + 4\delta^{-1} + \delta^{-6}(z_1^* + 2\delta^{-1})^6$ which is equivalent to $(z_1^* + 2\delta^{-1})(z_2^* - z_1^* - 4\delta^{-1})^{-1/6} \leq \delta$. From $\delta < \gamma < 1 \leq 2$ and $z_1^* > 0$ we get $1 \leq z_1^* + 2\delta^{-1}$, hence $(1+a)^{1/2}(b-a)^{-1/12} \leq 2^{1/2}(z_1^* + 2\delta^{-1})^{1/2}(z_2^* - z_1^* - 4\delta^{-1})^{-1/12} \leq 2^{1/2}\delta^{1/2}$. □

From (4.21), (4.22), and Lemma 4.4,

$$|\varphi(u_j) - \varphi(v_{j,k}) - \varphi(w_{j,k})| \leq C(\delta^{1/2} + \varepsilon_k^{1/2}) + CR_k^{1/6}\delta^{1/2}.$$

This estimate holds for all $k \in \mathbb{N}$, $\delta \in]0, \gamma[$, and $j \geq j_0(\delta)$, with however $v_{j,k}$ and $w_{j,k}$ depending on δ, j, and k. Thus Strichartz' inequality, the second bound from (4.19), and (4.20) yield

$$\begin{aligned} \varphi(u_j) &\leq C(\delta^{1/2} + \varepsilon_k^{1/2}) + CR_k^{1/6}\delta^{1/2} + \varphi(v_{j,k}) + \varphi(w_{j,k}) \\ &\leq C(\delta^{1/2} + \varepsilon_k^{1/2}) + CR_k^{1/6}\delta^{1/2} + S^6\|v_{j,k}\|_{L^2}^6 + S^6\|w_{j,k}\|_{L^2}^6 \\ &\leq C(\delta^{1/2} + \varepsilon_k^{1/2}) + CR_k^{1/6}\delta^{1/2} + S^6\Big(3(\delta + \varepsilon_k^{1/2}) + \gamma\Big)^3 + S^6\Big(9(\delta + \varepsilon_k^{1/2}) + (1-\gamma)\Big)^3. \end{aligned}$$

Since we have got rid of the functions $v_{j,k}$ and $w_{j,k}$, we may take the limit $j \to \infty$ to find

$$S^6 \leq C(\delta^{1/2} + \varepsilon_k^{1/2}) + CR_k^{1/6}\delta^{1/2} + S^6\Big(3(\delta + \varepsilon_k^{1/2}) + \gamma\Big)^3 + S^6\Big(9(\delta + \varepsilon_k^{1/2}) + (1-\gamma)\Big)^3.$$

This estimate is satisfied for all $\delta \in]0, \gamma[$ and all $k \in \mathbb{N}$. Hence we can pass successively to the limits first $\delta \to 0$ and then $k \to \infty$ to arrive at $S^6 \leq S^6\gamma^3 + S^6(1-\gamma)^3$, which is a contradiction to $\gamma \in]0, 1[$.

4.2.3 The case that the sequence (u_j) is tight

Summarizing the results of the preceding sections, we have constructed a maximizing sequence (u_j) which satisfies (4.5) and then applied the concentration compactness principle to (u_j). Since the alternatives (2) and (3) do not occur by Sections 4.2.1 and 4.2.2, it follows that (1) holds, i.e., there exists a sequence $(x_j) \subset \mathbb{R}$ such that for every $\delta > 0$ there is $M = M_\delta > 0$ with $\int_{x_j-M}^{x_j+M} |u_j|^2\,dx \geq 1 - \delta$ for all $j \in \mathbb{N}$. Finally we let $\tilde{u}_j(x) = u_j(x + x_j)$ and claim that $(\tilde{u}_j)$ is a maximizing sequence which (along a subsequence) strongly converges in L^2. Indeed, since $\|\tilde{u}_j\|_{L^2} = \|u_j\|_{L^2} = 1$ and, by Corollary 2.3(a), $\varphi(\tilde{u}_j) = \varphi(u_j) \to S^6$ as $j \to \infty$, $(\tilde{u}_j)$ is a maximizing sequence. It follows from $\hat{\tilde{u}}_j(\xi) = e^{ix_j\xi}\hat{u}_j(\xi)$ that (4.5) is satisfied for $(\tilde{u}_j)$, too. In addition,

$$\forall\,\delta > 0 \quad \exists\,M = M_\delta > 0: \quad \int_{-M}^{M} |\tilde{u}_j|^2\,dx \geq 1 - \delta, \quad j \in \mathbb{N}. \tag{4.23}$$

Passing to a subsequence (which is not relabelled) we can suppose that $\tilde{u}_j \rightharpoonup u_*$ in L^2 as $j \to \infty$ for some $u_* \in L^2$ with $\|u_*\|_{L^2} \leq 1$. Again we fix a sequence $\varepsilon_k \to 0$, choose R_k according to (4.5), and define $\tilde{u}_{j,k}^{(l)} = (\phi_k\hat{\tilde{u}}_j)\check{}$ and $\tilde{u}_{j,k}^{(h)} = ((1-\phi_k)\hat{\tilde{u}}_j)\check{}$. Here $\phi_k(\xi) = \phi(\xi/R_k)$, with $\phi \in C_0^\infty(\mathbb{R})$ taking values in $[0, 1]$ such that $\phi(\xi) = 1$ for $|\xi| \leq 1$ and $\phi(\xi) = 0$ for $|\xi| \geq 2$. Then again $\|\frac{d\tilde{u}_{j,k}^{(l)}}{dx}\|_{L^2} = \|\tilde{u}_{j,k}^{(l)}\|_{\dot{H}^1} \leq CR_k$, $\|\tilde{u}_j - \tilde{u}_{j,k}^{(l)}\|_{L^2} = \|\tilde{u}_{j,k}^{(h)}\|_{L^2} \leq \varepsilon_k^{1/2}$, and $\|\tilde{u}_{j,k}^{(l)}\|_{L^2} \leq 1$ are satisfied, see

(4.10), (4.11), and (4.12). Now we fix $k \in \mathbb{N}$. Then there exists a subsequence $(j') \subset (j)$, $\tilde{v}_k \in H^1$, and $\tilde{w}_k \in L^2$ such that $\tilde{u}^{(l)}_{j',k} \rightharpoonup \tilde{v}_k$ in H^1 and $\tilde{u}^{(h)}_{j',k} \rightharpoonup \tilde{w}_k$ in L^2 as $j' \to \infty$. In particular,

$$\|\tilde{w}_k\|_{L^2} \leq \liminf_{j'\to\infty} \|\tilde{u}^{(h)}_{j',k}\|_{L^2} \leq \varepsilon_k^{1/2}, \tag{4.24}$$

and also $u_* = \tilde{v}_k + \tilde{w}_k$ due to $\tilde{u}_j = \tilde{u}^{(l)}_{j,k} + \tilde{u}^{(h)}_{j,k}$. Next we fix $\delta > 0$ and choose $M = M_\delta$ according to (4.23). By compactness of the embedding $H^1 \subset L^2(]-M,M[)$, $\tilde{u}^{(l)}_{j',k} \to \tilde{v}_k$ strongly in $L^2(]-M,M[)$ as $j' \to \infty$. Therefore, using (4.24), the above bounds, and (4.23),

$$\begin{aligned}
\|u_*\|_{L^2} &\geq \|u_*\|_{L^2(]-M,M[)} = \|\tilde{v}_k + \tilde{w}_k\|_{L^2(]-M,M[)} \geq \|\tilde{v}_k\|_{L^2(]-M,M[)} - \|\tilde{w}_k\|_{L^2} \\
&\geq \lim_{j'\to\infty} \|\tilde{u}^{(l)}_{j',k}\|_{L^2(]-M,M[)} - \varepsilon_k^{1/2} = \lim_{j'\to\infty} \|\tilde{u}_{j'} - \tilde{u}^{(h)}_{j',k}\|_{L^2(]-M,M[)} - \varepsilon_k^{1/2} \\
&\geq \limsup_{j'\to\infty} \left(\|\tilde{u}_{j'}\|_{L^2(]-M,M[)} - \|\tilde{u}^{(h)}_{j',k}\|_{L^2} \right) - \varepsilon_k^{1/2} \\
&\geq \limsup_{j'\to\infty} \|\tilde{u}_{j'}\|_{L^2(]-M,M[)} - 2\varepsilon_k^{1/2} \geq (1-\delta)^{1/2} - 2\varepsilon_k^{1/2}.
\end{aligned}$$

This estimate holds for every $\delta > 0$, whence $\|u_*\|_{L^2} \geq 1 - 2\varepsilon_k^{1/2}$ for every $k \in \mathbb{N}$. Passing to the limit $k \to \infty$ thus $\|u_*\|_{L^2} = 1 = \lim_{j\to\infty} \|\tilde{u}_j\|_{L^2}$. Since also $\tilde{u}_j \rightharpoonup u_*$ in L^2, we obtain $\tilde{u}_j \to u_*$ strongly in L^2. By continuity of $\varphi : L^2 \to \mathbb{R}$, cf. (2.4), this finally yields $\varphi(u_*) = \lim_{j\to\infty} \varphi(\tilde{u}_j) = S^6$, i.e., u_* is a maximizing function and the proof of Theorem 1.1 is completed. □

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