On the number of solutions to semilinear boundary value problems

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Abstract

We consider semilinear elliptic problems of the form $\Delta u + g(u) = f(x)$ with Neumann boundary conditions or $\Delta u + \lambda_1 u + g(u) = f(x)$ with Dirichlet boundary conditions, and we derive conditions on g and f under which an upper bound on the number of solutions can be obtained.

1 Introduction

In this paper we consider semilinear elliptic problems of the form $\Delta u + g(u) = f(x)$ or $\Delta u + \lambda_1 u + g(u) = f(x)$ in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$, assuming Neumann or Dirichlet boundary conditions respectively on $\partial\Omega$. Our aim is to derive conditions that enforce upper bounds on the number of solutions to these equations. A key to such type of results is to verify that the associated linear equations $\Delta u + \alpha(x)u = 0$ or $\Delta u + \lambda_1 u + \alpha(x)u = 0$ do not have sign-changing solutions, since then (under certain assumptions on g and f) it follows that solutions of the nonlinear problems cannot cross. In this context one has to define $\alpha(x) = \frac{g(u_1(x)) - g(u_2(x))}{u_1(x) - u_2(x)}$ for two solutions u_1 and u_2 of the nonlinear equations to make the connection between the nonlinear and the linear problems. Therefore it is conceivable that conditions on the boundedness of g' will be helpful, and this corresponds to supposing that the $L^\infty(\Omega)$-norm of α is not too large. Hypotheses of this kind have been imposed in many papers, often in connection to Ambrosetti-Prodi type results; see [1] and the references therein.

The new feature of the present work is the observation that an assumption on an $L^\sigma(\Omega)$-norm of α (more precisely, on its positive part) for certain $\sigma < \infty$ will be sufficient to make the argument go through. Translated back to g and f (for instance in the Neumann case, where our results are more complete), this means that one can also allow for arbitrary g' and 'large' right-hand sides f, provided its average $\bar{f} = \frac{1}{|\Omega|}\int_\Omega f\,dx$ is small enough; see Corollary 4.3 below. Recently there have been some papers on periodic ODEs, where L^σ-bounds on the potential have been assumed in order to either study the stability of Hill's equation [13] or to investigate the bifurcation values for certain superlinear problems [8]. In this regard, our paper is a first attempt to transfer these results from ODEs to PDEs. It should also be mentioned that there is a connection to the classical Rozenblum-Lieb-Cwikel inequality (see e.g. [10]), since in essence everything comes down to showing that $N_0(-\alpha) \le 1$ for the number $N_0(-\alpha)$ of eigenvalues μ to $-\Delta - \alpha$ such that $\mu \le 0$, i.e., the number of non-negative eigenvalues to $L = \Delta + \alpha$ (for the Neumann boundary conditions). Hence the Rozenblum-Lieb-Cwikel inequality suggests that it is sufficient to bound an $L^\sigma(\Omega)$-norm of α_+ in order to obtain $N_0(-\alpha) \le 1$ as desired. However, in the particular case considered in this paper a direct approach to this question can be taken which does not rely on such very general arguments.

The paper is organized as follows. In Section 2 we consider the linearized problem corresponding to Neumann boundary conditions, whereas its Dirichlet counterpart is treated in Section 3. Applications to nonlinear problems are given in Section 4.

2 Neumann boundary conditions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. For $\alpha \in C(\overline{\Omega})$ we consider $Lu = \Delta u + \alpha(x)u$ together with the Neumann boundary condition $\frac{du}{dn} = 0$ on $\partial\Omega$. The eigenvalues of L are denoted $\mu_1 \ge \mu_2 \ge \mu_3 \ge \ldots$, and the corresponding eigenfunctions are $\phi_1, \phi_2, \phi_3, \ldots$ These eigenfunctions are understood as non-zero functions $\phi_i \in H^1(\Omega)$ such that

$$-\int_\Omega \nabla\phi_i \cdot \nabla u\, dx + \int_\Omega \alpha(x)\phi_i u\, dx = \mu_i \int_\Omega \phi_i u\, dx$$

for any test function $u \in H^1(\Omega)$. The standard regularity theory implies that ϕ_i belongs to $C(\overline{\Omega})$. We fix

$$\begin{cases} p \in [2,\infty] & \text{if} \quad n = 1 \\ p \in [2,\infty[& \text{if} \quad n = 2 \\ p \in [2, \frac{2n}{n-2}] & \text{if} \quad n \ge 3 \end{cases}, \tag{1}$$

and introduce

$$S_N(p;\Omega) = \inf\left\{ \|\nabla u\|^2_{L^2(\Omega)} : u \in H^1(\Omega),\ \|u\|_{L^p(\Omega)} = 1,\ \int_\Omega u\, dx = 0 \right\} > 0, \tag{2}$$

so that

$$S_N(p;\Omega)\|u\|^2_{L^p(\Omega)} \le \|\nabla u\|^2_{L^2(\Omega)} \quad \text{if} \quad u \in H^1(\Omega) \quad \text{and} \quad \int_\Omega u\, dx = 0. \tag{3}$$

The following lemma and its corollary play a key role for our results. Its proof is an adaptation of arguments we found in [3].

Lemma 2.1 *Suppose that*

$$\begin{cases} \sigma \in [1,\infty] & \text{if} \quad n = 1 \\ \sigma \in]1,\infty] & \text{if} \quad n = 2 \\ \sigma \in [\frac{n}{2},\infty] & \text{if} \quad n \ge 3 \end{cases}. \tag{4}$$

If

$$\|\alpha_+\|_{L^\sigma(\Omega)} < S_N(2\sigma^*;\Omega),$$

then $\mu_2 < 0$. *Here* $\alpha_+ = \max\{\alpha, 0\}$ *denotes the positive part of* α *and* $1/\sigma + 1/\sigma^* = 1$.

Proof: Assume on the contrary that $\mu_2 \geq 0$. For $c_1, c_2 \in \mathbb{R}$ we let $\psi = c_1\phi_1 + c_2\phi_2$ and apply the definition of eigenfunction with $u = \psi$. Since ϕ_1 and ϕ_2 are orthogonal in $L^2(\Omega)$ and in $H^1(\Omega)$, this yields

$$\begin{aligned} -\int_\Omega |\nabla\psi|^2\,dx + \int_\Omega \alpha(x)\psi^2\,dx &= \int_\Omega (c_1\mu_1\phi_1 + c_2\mu_2\phi_2)(c_1\phi_1 + c_2\phi_2)\,dx \\ &= c_1^2\mu_1 \int_\Omega \phi_1^2\,dx + c_2^2\mu_2 \int_\Omega \phi_2^2\,dx \geq 0, \end{aligned}$$

due to $\mu_1 \geq \mu_2 \geq 0$. Next we choose $c_1, c_2 \in \mathbb{R}$ such that $\psi \neq 0$, but $\int_\Omega \psi\,dx = 0$. Noting that (4) for σ is equivalent to (1) for $p = 2\sigma^*$, (3) implies

$$\begin{aligned} S_N(2\sigma^*;\Omega)\|\psi\|^2_{L^{2\sigma^*}(\Omega)} &\leq \|\nabla\psi\|^2_{L^2(\Omega)} \leq \int_\Omega \alpha(x)\psi^2\,dx \leq \int_\Omega \alpha_+(x)\psi^2\,dx \\ &\leq \Big(\int_\Omega |\alpha_+|^\sigma\,dx\Big)^{1/\sigma}\Big(\int_\Omega |\psi|^{2\sigma^*}\,dx\Big)^{1/\sigma^*} = \|\alpha_+\|_{L^\sigma(\Omega)}\|\psi\|^2_{L^{2\sigma^*}(\Omega)}, \end{aligned}$$

which leads to $S_N(2\sigma^*;\Omega) \leq \|\alpha_+\|_{L^\sigma(\Omega)}$ and contradicts our assumption. □

Corollary 2.2 *If*

$$\|\alpha_+\|_{L^\sigma(\Omega)} < S_N(2\sigma^*;\Omega) \tag{5}$$

and if there exists a nontrivial solution u of

$$\begin{cases} \Delta u + \alpha(x)u = 0 & \text{in } \Omega \\ \frac{du}{dn} = 0 & \text{on } \partial\Omega \end{cases},$$

then u does not change sign.

Proof: If $u \neq 0$, then u were a sign-changing eigenfunction of $L = \Delta + \alpha(x)$ corresponding to the eigenvalue $\mu = 0$. But the first eigenfunction ϕ_1 corresponding to μ_1 does not change sign (see [11, Thm. 11.10]), thus $0 \in \{\mu_2, \mu_3, \ldots\}$. Since $\mu_2 \geq \mu_3 \geq \ldots$, this yields $\mu_2 \geq 0$ in contradiction to Lemma 2.1. □

In the remaining part of this section we further discuss condition (5) and prove that it is optimal in one dimension if $\sigma \in [5/3, \infty[$.

Lemma 2.3 *If $I =]a, b[\subset \mathbb{R}$ and $\sigma \in [5/3, \infty[$, then there exists $\alpha \in C(\overline{I})$ such that*

(i) $\|\alpha_+\|_{L^\sigma(I)} = S_N(2\sigma^*; I)$, *and*

(ii) there is a nontrivial and sign-changing solution u of

$$\begin{cases} u'' + \alpha(x)u = 0 & \text{in } I \\ u'(a) = u'(b) = 0 \end{cases}. \tag{6}$$

Proof: For bounded intervals $]a,b[\subset\mathbb{R}$ we define

$$S_D(p;a,b)=\inf\Big\{\|u'\|^2_{L^2(]a,b[)}:u\in H^1_0(]a,b[),\ \|u\|_{L^p(]a,b[)}=1\Big\},$$

and we also let $S_N(p;a,b):=S_N(p;]a,b[)$; recall (2). By using a shift, it suffices for the proof to consider the case where $I=]0,l[$ for some $l>0$. For $p=2\sigma^*\in]2,5]$ define $\gamma=S_D(p;0,l)>0$. Let ϕ denote the unique solution of $\phi''+\gamma|\phi|^{p-2}\phi=0$ with minimal period $2l$ such that $A=\phi(0)>0$ and $\phi'(0)=0$. This solution does exist, since the orbits of the ODE cover the (ϕ,ϕ')-phase plane, with a continuous and monotone minimal period function that approaches infinity at the origin and zero at infinity. Then $\phi(-x)=\phi(x)$ holds, since $x\mapsto\phi(-x)$ satisfies the same initial value problem as ϕ. Let $x_0>0$ be the first zero of ϕ', so that $\phi'(x)>0$ for $x\in]-x_0,0[$, $\phi'(x)<0$ for $x\in]0,x_0[$, and $\phi'(\pm x_0)=0$. As $E=E(x)=\frac{1}{2}\phi'(x)^2+\frac{\gamma}{p}|\phi(x)|^p=\frac{\gamma}{p}A^p$ is constant along the orbit, it follows that $|\phi(\pm x_0)|=A$. From the definition of x_0 we deduce that $\phi(\pm x_0)<A$ and so $\phi(\pm x_0)=-A$. The solutions $\phi(x)$ and $-\phi(x+x_0)$ satisfy the same initial conditions at $x=0$ and therefore they coincide. The identity $\phi(x)=-\phi(x+x_0)$ implies that $2x_0$ is the minimal period of ϕ and this leads to the identity $\phi(x)=-\phi(x+l)$. In particular, $\phi(l/2)=-\phi(-l/2)=-\phi(l/2)$ gives $\phi(-l/2)=\phi(l/2)=0$. Next we define

$$u_N(x)=\phi(x),\quad \alpha(x)=\gamma|u_N(x)|^{p-2}=\gamma|u_N(x)|^{2(\sigma^*-1)},\quad\text{and}\quad u_D(x)=\phi(x-l/2),\quad x\in[0,l].$$

Then u_N solves (6) on $[0,l]$ and changes sign. Finally to verify (i), we observe that also $u_D''+\gamma|u_D|^{p-2}u_D=0$, $u>0$ on $]0,l[$, and $u_D(0)=u_D(l)=0$. Next let $u\in H^1_0(]0,l[)$ be the positive minimizer for $S_D(p;0,l)=\gamma$. Then $u''+\lambda|u|^{p-2}u=0$ for some Lagrange multiplier $\lambda\in\mathbb{R}$, and $u(0)=u(l)=0$. Upon multiplication of the equation by u and integration over $[0,l]$, we get $\gamma=\int_0^l(u')^2\,dx=\lambda\int_0^l u^p\,dx=\lambda\|u\|^p_{L^p(]0,l[)}=\lambda$, and consequently $u_D=u$ by the uniqueness for positive solutions of the Dirichlet problem. Therefore $\phi(x)=-\phi(x-l)$ results in

$$\begin{aligned}\|\alpha_+\|_{L^\sigma(]0,l[)}&=\gamma\Big(\int_0^l|u_N(x)|^{2\sigma(\sigma^*-1)}\,dx\Big)^{1/\sigma}=\gamma\|u_N\|^{p/\sigma}_{L^p(]0,l[)}=\gamma\|u_D\|^{p/\sigma}_{L^p(]0,l[)}\\&=\gamma\|u\|^{p/\sigma}_{L^p(]0,l[)}=\gamma=S_D(p;0,l)=S_N(p;0,l)=S_N(2\sigma^*;]0,l[),\end{aligned}$$

where we also used the following Lemma 2.4. This completes the proof. □

Lemma 2.4 *For bounded intervals* $]a,b[\subset\mathbb{R}$ *we consider*

$$S_D(p;a,b)=\inf\Big\{\|u'\|^2_{L^2(]a,b[)}:u\in H^1_0(]a,b[),\ \|u\|_{L^p(]a,b[)}=1\Big\}$$

and

$$S_N(p;a,b)=\inf\Big\{\|u'\|^2_{L^2(]a,b[)}:u\in H^1(]a,b[),\ \|u\|_{L^p(]a,b[)}=1,\ \int_a^b u\,dx=0\Big\},$$

see (2). Then

$$S_D(p;a,b)=S_N(p;a,b),\quad p\in]1,5].$$

Proof: Passing from a function $u(x)$ for $x\in]a,b[$ to $\tilde u(x)=u((b-a)x/2+(a+b)/2)$ for $x\in]-1,1[$, it is found that S_D and S_N scale as

$$S_D(p;a,b)=S_D(p)\Big(\frac{b-a}{2}\Big)^{-(1+2/p)}\quad\text{and}\quad S_N(p;a,b)=S_N(p)\Big(\frac{b-a}{2}\Big)^{-(1+2/p)},$$

where $S_D(p) := S_D(p;-1,1)$ and $S_N(p) := S_N(p;-1,1)$, respectively. Therefore it suffices to verify that

$$S_D(p) = S_N(p), \quad p \in]1,5]. \tag{7}$$

1.) Let

$$\tilde{S}_N(p) = \inf\left\{ \frac{\|u'\|_{L^2(]-1,1[)}}{\|u\|_{L^p(]-1,1[)}} : u \in H^1(]-1,1[),\ \int_{-1}^{1} u\,dx = 0 \right\} > 0.$$

Due to [6, Thm. 1a)] there exists a minimizer $\tilde{u}$ such that u is odd. Then $u(x) = \|\tilde{u}\|^{-1}_{L^p(]-1,1[)}\tilde{u}(x)$ yields an odd minimizer for $S_N(p)$. Next we introduce

$$v(x) = \begin{cases} u(x+1) & : \ x \in [-1,0] \\ -u(x-1) & : \ x \in [0,1] \end{cases},$$

and note that $v(-1) = u(0) = 0 = -u(0) = v(1)$, whence $v \in H^1_0(]-1,1[)$. Also $\|v\|_{L^p(]-1,1[)} = \|u\|_{L^p(]-1,1[)} = 1$ is found, and consequently $S_D(p) \le \|v'\|^2_{L^2(]-1,1[)} = \|u'\|^2_{L^2(]-1,1[)} = S_N(p)$. In other words, $S_D(p) \le S_N(p)$ holds for $p \in]1,5]$.

2.) Conversely, by rearrangement there exists an even minimizer u for $S_D(p)$ such that u is radially decreasing and $u(-1) = u(1) = 0 = u'(0)$; see [6, Thm. 3a)] for a related result. Letting

$$v(x) = \begin{cases} u(x+1) & : \ x \in [-1,0] \\ -u(x-1) & : \ x \in [0,1] \end{cases},$$

we get $\int_{-1}^1 v\,dx = \int_0^1 u\,dx - \int_{-1}^0 u\,dx = 0$ due to $u(-x) = u(x)$, and moreover $\|v\|_{L^p(]-1,1[)} = \|u\|_{L^p(]-1,1[)} = 1$. Hence $S_N(p) \le \|v'\|^2_{L^2(]-1,1[)} = \|u'\|^2_{L^2(]-1,1[)} = S_D(p)$. Accordingly, $S_N(p) \le S_D(p)$ is satisfied for $p \in]1,\infty[$. □

Remark 2.5 (a) From [6, Thm. 1b)] it is known that the Neumann minimizers for $\Omega =]-1,1[$ are not odd for $p > 6$, so that (7) will not hold in this regime. The range $p \in]5,6]$ seems to be open.

(b) In view of [2, (5.40)] we have $\|u\|_{L^p(]0,1[)} \le c(p)\|u'\|_{L^2(]0,1[)}$ for $u \in H^1_0(]0,1[)$, where

$$c(p) = \frac{1}{2}\frac{p(1+2/p)^{1/2}}{(1+p/2)^{1/p}\beta(1/p,1/2)}$$

is the best constant, with $\beta(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ the beta function. Therefore $S_D(p;0,1) = c(p)^{-2}$, and thus using Lemma 2.4 and provided that $\sigma \in [5/3,\infty[$, we obtain the explicit expression

$$\begin{aligned} S_N(2\sigma^*; I) &= S_D(2\sigma^*; I) = S_D(2\sigma^*; 0,1)|I|^{-(1+1/\sigma^*)} = c(2\sigma^*)^{-2}|I|^{-(1+1/\sigma^*)} \\ &= \frac{\beta(1/2\sigma^*, 1/2)^2}{\sigma^*(1+\sigma^*)^{1/\sigma}}\,|I|^{-(1+1/\sigma^*)} \end{aligned}$$

for the relevant constant from (5) in the one-dimensional case.

(c) The paper [8] deals with periodic solutions, but some of its results are easily adapted to the Neumann problem. In particular one can apply Proposition 2.1 in [8] to deduce that, for $\Omega =]a,b[$, the conclusion of Corollary 2.2 still holds if (5) is replaced by

$$\|\alpha_+\|_{L^\sigma(]a,b[)} < S_D(2\sigma^*; a, b).$$

Hence (5) is not optimal for Ω an interval and $2\sigma^* > 6$, which is equivalent to $\sigma < 3/2$. ♢

It is an open problem to determine for which $\Omega \subset \mathbb{R}^n$ condition (5) is optimal in dimensions $n \geq 2$. Certainly it seems reasonable to expect that the optimal values could only be determined for special domains like balls or rectangles.

3 Dirichlet boundary conditions

In this section we consider the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Let $\lambda_1 > 0$ be the corresponding first eigenvalue of $-\Delta$ with associated eigenfunction $\phi > 0$; the other eigenvalues are denoted $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$. We recall the Hardy inequality

$$S_H(p, \tau; \Omega) \left\| \frac{u}{\phi^\tau} \right\|^2_{L^p(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)}, \quad u \in H^1_0(\Omega), \quad \tau \in [0,1], \quad \frac{1}{p} \geq \frac{1}{2} - \frac{1-\tau}{n}; \tag{8}$$

see [5, p. 75]. This reference gives (8) for $\frac{1}{p} = \frac{1}{2} - \frac{1-\tau}{n}$ only, but since $\|\frac{u}{\phi^\tau}\|^2_{L^q(\Omega)} \leq |\Omega|^{\frac{2(p-q)}{pq}} \|\frac{u}{\phi^\tau}\|^2_{L^p(\Omega)}$ for $q \leq p$, (8) also holds in the generality stated for a suitable constant $S_H(p, \tau; \Omega) > 0$. Furthermore, we will need the constant

$$\tilde{S}_D(p; \Omega) = \inf \left\{ \|\nabla u\|^2_{L^2(\Omega)} : u \in H^1_0(\Omega),\ \|u\|_{L^p(\Omega)} = 1,\ \int_\Omega u\phi \, dx = 0 \right\} > 0$$

for $p = 2$. In this case an expansion of $u \in H^1_0(\Omega)$ in terms of the eigenfunctions of $-\Delta$ shows that $\tilde{S}_D(2; \Omega) = \lambda_2$. Consequently,

$$\lambda_2 \|u\|^2_{L^2(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)} \quad \text{if} \quad u \in H^1_0(\Omega) \quad \text{and} \quad \int_\Omega u\phi \, dx = 0. \tag{9}$$

The following lemma parallels Lemma 2.1. Here we write $\mu_1 \geq \mu_2 \geq \ldots$ for the eigenvalues of $Lu = \Delta u + \lambda_1 u + \alpha(x)u$ and $\phi_1, \phi_2, \ldots$ for the associated eigenfunctions.

Lemma 3.1 *Suppose that* $\sigma \in [\frac{n+1}{2}, \infty]$. *If*

$$\|\alpha_+\|_{L^\sigma(\Omega,\, \phi dx)} < (1 - \lambda_1 \lambda_2^{-1}) S_H(2\sigma^*, 1/2\sigma; \Omega),$$

then $\mu_2 < 0$.

Notice that the first eigenfunction ϕ of $-\Delta$ is acting as a weight in the L^σ-norm.

Proof: Let us assume that on the contrary $\mu_2 \geq 0$. If we set $\tau = \frac{1}{2\sigma}$ and $p = 2\sigma^*$, then $\tau \in [0,1]$ and $\frac{1}{p} \geq \frac{1}{2} - \frac{1-\tau}{n}$ is found. Next we consider $\psi = c_1\phi_1 + c_2\phi_2$ for $c_1, c_2 \in \mathbb{R}$ to obtain $L\psi = c_1\mu_1\phi_1 + c_2\mu_2\phi_2$. Since $\psi \in H^1_0(\Omega)$, this yields

$$-\int_\Omega |\nabla\psi|^2 \, dx + \lambda_1 \int_\Omega \psi^2 \, dx + \int_\Omega \alpha(x)\psi^2 \, dx \geq c_1^2\mu_1 \int_\Omega \phi_1^2 \, dx + c_2^2\mu_2 \int_\Omega \phi_2^2 \, dx \geq 0,$$

in view of $\mu_1 \geq \mu_2 \geq 0$. If $c_1, c_2 \in \mathbb{R}$ are chosen such that $\psi \neq 0$ and $\int_\Omega \psi\phi \, dx = 0$, then (9) implies

$$\int_\Omega |\nabla\psi|^2 \, dx \leq \lambda_1 \int_\Omega \psi^2 \, dx + \int_\Omega \alpha(x)\psi^2 \, dx \leq \lambda_1\lambda_2^{-1} \int_\Omega |\nabla\psi|^2 \, dx + \int_\Omega \alpha(x)\psi^2 \, dx.$$

Hence from (8) we get

$$\begin{aligned}
&(1-\lambda_1\lambda_2^{-1})S_H(2\sigma^*,1/2\sigma;\Omega)\left\|\frac{\psi}{\phi^{1/2\sigma}}\right\|^2_{L^{2\sigma^*}(\Omega)}\\
\le\;& (1-\lambda_1\lambda_2^{-1})\int_\Omega|\nabla\psi|^2\,dx \le \int_\Omega \alpha(x)\psi^2\,dx \le \int_\Omega(\alpha_+(x)\phi^{1/\sigma})(\psi^2\phi^{-1/\sigma})\,dx\\
\le\;& \Big(\int_\Omega|\alpha_+|^\sigma\phi\,dx\Big)^{1/\sigma}\Big(\int_\Omega|\psi|^{2\sigma^*}\phi^{-\sigma^*/\sigma}\,dx\Big)^{1/\sigma^*} = \|\alpha_+\|_{L^\sigma(\Omega,\,\phi dx)}\left\|\frac{\psi}{\phi^{1/2\sigma}}\right\|^2_{L^{2\sigma^*}(\Omega)}.
\end{aligned}$$

Therefore the contradiction $(1-\lambda_1\lambda_2^{-1})S_H(2\sigma^*,1/2\sigma;\Omega)\le\|\alpha_+\|_{L^\sigma(\Omega,\,\phi dx)}$ is obtained. □

Remark 3.2 A variant of Lemma 3.1 is as follows. If

$$\|(\lambda_1+\alpha)_+\|_{L^\sigma(\Omega,\,\phi dx)} < S_H(2\sigma^*,1/2\sigma;\Omega),$$

then $\mu_2<0$. The proof is analogous to that of Lemma 3.1. ♢

Corollary 3.3 *If*

$$\|\alpha_+\|_{L^\sigma(\Omega,\,\phi dx)} < (1-\lambda_1\lambda_2^{-1})S_H(2\sigma^*,1/2\sigma;\Omega) \quad \textit{or} \quad \|(\lambda_1+\alpha)_+\|_{L^\sigma(\Omega,\,\phi dx)} < S_H(2\sigma^*,1/2\sigma;\Omega),$$

and if there exists a nontrivial solution u of

$$\left\{\begin{array}{rcl}\Delta u+\lambda_1 u+\alpha(x)u &=& 0 \quad \text{in } \Omega\\ u &=& 0 \quad \text{on } \partial\Omega\end{array}\right.,$$

then u does not change sign.

Proof: Any sign-changing solution $u\ne 0$ is an eigenfunction of $L=\Delta+\lambda_1+\alpha(x)$ corresponding to the eigenvalue $\mu=0$. Since the first eigenfunction ϕ_1 of L does not change sign, we get $0\in\{\mu_2,\mu_3,\dots\}$ and therefore $\mu_2\ge 0$. But this is impossible due to Lemma 3.1 or Remark 3.2. □

4 Some applications to nonlinear problems

4.1 Semilinear Neumann problems

For certain nonlinearities g we consider the semilinear Neumann problems

$$\left\{\begin{array}{rcl}\Delta u+g(u) &=& f(x) \quad \text{in } \Omega\\ \frac{du}{dn} &=& 0 \quad \text{on } \partial\Omega\end{array}\right., \tag{10}$$

with $f\in L^\infty(\Omega)$ and we will decompose $f(x)=\tilde f(x)+\bar f$, where $\bar f=\frac{1}{|\Omega|}\int_\Omega f\,dx$ denotes the average of f over Ω and $\int_\Omega\tilde f(x)\,dx=0$. The following result asserts that the solutions of (10) are ordered, if we assume (11) and (12) below for g and f. Here we say that u is a solution of (10), provided that $u\in C(\overline{\Omega})\cap H^1(\Omega)$ and u solves the equation in the variational sense.

Theorem 4.1 *Suppose that $\sigma < \infty$ satisfies (4) and $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous such that*

$$\Big(\frac{g(s_1) - g(s_2)}{s_1 - s_2}\Big)_+^\sigma \le A\Big(\frac{g(s_1) + g(s_2)}{2}\Big) + B, \quad s_1, s_2 \in \mathbb{R}, \tag{11}$$

for some constants $A, B \ge 0$. If we denote $\bar f = \frac{1}{|\Omega|}\int_\Omega f\,dx$ and if

$$A\bar f + B < \frac{1}{|\Omega|}\, S_N(2\sigma^*;\Omega)^\sigma, \tag{12}$$

then different solutions of (10) cannot cross.

Proof: Let u_1, u_2 be solutions of (10) with $u_1 \neq u_2$. We have to show that $u(x) = u_1(x) - u_2(x) \neq 0$ for $x \in \Omega$. First we note that u is a solution of

$$\Delta u + \alpha(x) u = 0 \text{ in } \Omega, \quad \frac{du}{dn} = 0 \text{ on } \partial\Omega,$$

where

$$\alpha(x) = \begin{cases} \frac{g(u_1(x)) - g(u_2(x))}{u_1(x) - u_2(x)} & : \ u_1(x) \neq u_2(x) \\ 0 & : \ u_1(x) = u_2(x) \end{cases}.$$

Since g is locally Lipschitz continuous, α is bounded. In view of (11) we have

$$\|\alpha_+\|^\sigma_{L^\sigma(\Omega)} = \int_\Omega \Big(\frac{g(u_1(x)) - g(u_2(x))}{u_1(x) - u_2(x)}\Big)_+^\sigma dx \le A\int_\Omega \Big(\frac{g(u_1(x)) + g(u_2(x))}{2}\Big)\, dx + B|\Omega|. \tag{13}$$

On the other hand, if we employ the test function $u = 1$ in the definition of solution of the Neumann problems associated to the equations $\Delta u_j + g(u_j) = f = \tilde f + \bar f$ over Ω, it follows that $\int_\Omega g(u_j)\,dx = |\Omega|\bar f$. Hence (13) yields the bound

$$\|\alpha_+\|^\sigma_{L^\sigma(\Omega)} \le (A\bar f + B)|\Omega|.$$

Therefore (12) in conjunction with Corollary 2.2 completes the proof of the theorem. □

Note that (12) allows for right-hand sides f of 'arbitrary size', provided that B and the average $\bar f$ of f are small enough. Some examples of nonlinearities that are admissible in the sense of (11) are given below. Here we say that a locally Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfies the condition $\mathcal{C}(\sigma; A, B)$, if (11) holds.

Example 4.2 (a) Let $g(s) = |s|^p$ or $g(s) = (s_+)^p$ for $s \in \mathbb{R}$ and $p \in]1,\infty[$. Then g satisfies $\mathcal{C}(p^*; p^{p^*}, 0)$. Hence Theorem 4.1 applies, provided that

$$\begin{cases} p \in [1,\infty] & \text{if} \quad n = 1 \\ p \in [1,\infty[& \text{if} \quad n = 2 \\ p \in [1, \frac{n}{n-2}] & \text{if} \quad n \ge 3 \end{cases}, \quad \text{and} \quad \bar f < \frac{S_N(2p;\Omega)^{p^*}}{p^{p^*}\,|\Omega|}.$$

The existence of solution for this type of problems has been discussed by several authors. Some references are [12, 7, 9].

(b) Let $g(s) = e^s$ for $s \in \mathbb{R}$. Then g satisfies $\mathcal{C}(1; 1, 0)$. Therefore in order to apply Theorem 4.1 we can take $n = 1$ (where $\sigma = 1$ is admissible) and require $\bar f < \frac{1}{|\Omega|}\, S_N(\infty;\Omega)$.

(c) Let $g(s) = e^s + \sin s$ for $s \in \mathbb{R}$. Then g satisfies $\mathcal{C}(1; 1, 2)$. Here Theorem 4.1 can be used, if $n = 1$ and $\bar f + 2 < \frac{1}{|\Omega|}\, S_N(\infty;\Omega)$. ◇

See [8, Lemma 3.1]. The proof of (a) relies on the (sharp) inequality

$$\left|\frac{|s_1|^p - |s_2|^p}{s_1 - s_2}\right|^{p^*} \le p^{p^*}\Big(\frac{|s_1|^p + |s_2|^p}{2}\Big), \quad s_1, s_2 \in \mathbb{R}, \quad s_1 \ne s_2,$$

for $p \in]1, \infty[$ and $1/p + 1/p^* = 1$.

Theorem 4.1 has some direct consequences for the number of solutions to (10).

Corollary 4.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous such that $\mathcal{C}(\sigma; A, B)$ holds. We moreover assume that $f \in L^\infty(\Omega)$ satisfies (12).*

(a) If g is strictly increasing, then (10) can have at most one solution.

(b) If g is strictly convex, then (10) can have at most two solutions.

Proof: (a) If there were two solutions u_1, u_2 of (10), then by Theorem 4.1 we may suppose that $u_1 > u_2$ in Ω. Integrating the equations over Ω, this yields the contradiction $|\Omega|\bar f = \int_\Omega g(u_1)\,dx > \int_\Omega g(u_2)\,dx = |\Omega|\bar f$; cf. the proof of Theorem 4.1. (b) If there were three solutions u_1, u_2, and u_3 of (10), then due to Theorem 4.1 we can assume that $u_1 > u_2 > u_3$ in Ω. Thus if we set

$$\alpha_1(x) = \frac{g(u_1(x)) - g(u_2(x))}{u_1(x) - u_2(x)}, \quad \alpha_2(x) = \frac{g(u_2(x)) - g(u_3(x))}{u_2(x) - u_3(x)}, \quad x \in \Omega,$$

then $\alpha_1(x) > \alpha_2(x)$ a.e. by the strict convexity of g. In addition, from (11) and (12) we obtain

$$\|\alpha_{1,+}\|_{L^\sigma(\Omega)} < S_N(2\sigma^*; \Omega) \quad \text{and} \quad \|\alpha_{2,+}\|_{L^\sigma(\Omega)} < S_N(2\sigma^*; \Omega)$$

as in the proof of Theorem 4.1. Denoting $v_1 = u_1 - u_2$ and $v_2 = u_2 - u_3$, then v_i is a nontrivial positive solution to

$$\Delta v + \alpha_i(x) v = 0 \text{ in } \Omega, \quad \frac{dv}{dn} = 0 \text{ on } \partial\Omega. \tag{14}$$

Thus $\mu = 0$ is an eigenvalue of $L_1 = \Delta + \alpha_1(x)$ with Neumann boundary conditions, and $v = v_1$ is a corresponding eigenfunction. Therefore, by the Fredholm alternative, the inhomogeneous problem $L_1 v = h$ can have a solution v only if $\int_\Omega h v_1\,dx = 0$. However, from (14) we obtain $L_1 v_2 = (\alpha_2(x) - \alpha_1(x)) v_2$, hence $\int_\Omega (\alpha_2(x) - \alpha_1(x)) v_2 v_1\,dx < 0$ leads to a contradiction. □

4.2 Semilinear Dirichlet problems

In analogy to Section 4.1 we consider the semilinear Dirichlet problems

$$\left\{\begin{array}{rcll} \Delta u + \lambda_1 u + g(u) & = & f(x) & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega \end{array}\right., \tag{15}$$

and we retain the notation introduced previously. The only difference is in the decomposition of f: here we use the splitting $f = \overline{f}\phi + \tilde f$ with $\overline{f} \in \mathbb{R}$ and $\int_\Omega \tilde f \phi\,dx = 0$. The eigenfunction ϕ has been normalized so that $\|\phi\|_{L^2(\Omega)} = 1$. The counterpart of Theorem 4.1 is

Theorem 4.4 *Suppose* $\sigma \in [\frac{n+1}{2}, \infty[$ *and* $g : \mathbb{R} \to \mathbb{R}$ *is locally Lipschitz continuous such that*

$$\Big(\frac{g(s_1) - g(s_2)}{s_1 - s_2}\Big)_+^\sigma \le A\Big(\frac{g(s_1) + g(s_2)}{2}\Big) + B, \quad s_1, s_2 \in \mathbb{R}, \tag{16}$$

for some constants $A, B \ge 0$*. If*

$$A\overline{f} + B \int_\Omega \phi \, dx < (1 - \lambda_1 \lambda_2^{-1})^\sigma S_H(2\sigma^*, 1/2\sigma; \Omega)^\sigma, \tag{17}$$

then different weak solutions of (15) cannot cross.

Proof: Let u_1, u_2 be solutions of (15) with $u_1 \ne u_2$. Then $u = u_1 - u_2$ is a solution to

$$\Delta u + \lambda_1 u + \alpha(x) u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

with

$$\alpha(x) = \begin{cases} \frac{g(u_1(x)) - g(u_2(x))}{u_1(x) - u_2(x)} & : \ u_1(x) \ne u_2(x) \\ 0 & : \ u_1(x) = u_2(x) \end{cases}.$$

From (16) we deduce

$$\begin{aligned} \|\alpha_+\|^\sigma_{L^\sigma(\Omega, \phi dx)} &= \int_\Omega \Big(\frac{g(u_1(x)) - g(u_2(x))}{u_1(x) - u_2(x)}\Big)_+^\sigma \phi(x) \, dx \\ &\le A \int_\Omega \Big(\frac{g(u_1(x)) + g(u_2(x))}{2}\Big) \phi(x) \, dx + B \int_\Omega \phi \, dx \\ &= \int_\Omega (Af + B)\phi \, dx = A\overline{f} + B \int_\Omega \phi \, dx, \end{aligned}$$

where we used that $\int_\Omega g(u_j)\phi \, dx = \int_\Omega f \phi \, dx$ as a consequence of $-\Delta\phi = \lambda_1 \phi$. Hence (17) in conjunction with Corollary 3.3 shows that u is of one sign in Ω. □

Example 4.5 We consider $g(s) = |s|^p$ or $g(s) = (s_+)^p$ for $s \in \mathbb{R}$ and $p \in [1, \frac{n+1}{n-1}]$. Then we may choose $\sigma = p^*$, $A = p^{p^*}$, and $B = 0$ in (16); see Example 4.2(a). Thus Theorem 4.4 can be used, if

$$A\overline{f} + B \int_\Omega \phi \, dx < p^{-p^*} (1 - \lambda_1 \lambda_2^{-1})^{p^*} S_H(2p, 1/2p^*; \Omega)^{p^*}.$$

However, this condition may be not easy to verify for concrete problems. A recent result about the existence of solution for $g(s) = (s_+)^p$ can be found in [4]. ♢

Finally we mention that due to Theorem 4.4 upper bounds on the number of solutions to (15) can be derived in a fashion similar to Corollary 4.3.

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