Estimates for the KdV-limit of the Camassa-Holm equation

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Abstract

In a number of scaling limits, we prove estimates relating the solutions of the Camassa-Holm equation to the solutions of the associated KdV equation. As a consequence, suitable solutions of the water wave problem and solutions of the Camassa-Holm equation stay close together for long times.

1 Introduction

The Camassa-Holm equation [11]

$$\begin{aligned} &\partial_t u + \partial_x u + \frac{3}{2}\varepsilon\, u\partial_x u + \delta^2\Big(\frac{1}{6} - \nu - \frac{\sigma}{2}\Big)\,\partial_x^3 u \\ &- \delta^2\nu\,\partial_t\partial_x^2 u - \frac{1}{2}\,\varepsilon\delta^2\nu(u\,\partial_x^3 u + 2\partial_x u\,\partial_x^2 u) + \,\delta^4\beta\,\partial_x^5 u = 0, \end{aligned} \tag{1}$$

with $u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, $t \in \mathbb{R}$, parameters $\nu, \sigma, \beta \in \mathbb{R}$, and small parameters $\varepsilon, \delta > 0$, has been derived [4, 12, 13] as an amplitude equation for the description of unidirectional surface water waves of an irrotational, inviscid fluid in an infinitely long canal of fixed constant depth, up to residual terms of order $\mathcal{O}(\delta^6 + \delta^4\varepsilon + \delta^2\varepsilon^2 + \varepsilon^3)$. It is obtained in the limit of small amplitude, i.e., the amplitude scales as ε, and in the limit of long waves, i.e., δ is the scaling of space and (as a consequence) also that of time. Often the parameter ν is fixed as

$$\nu = \frac{1}{60}\,\frac{19 - 30\sigma - 45\sigma^2}{1 - 3\sigma}, \quad (\sigma \neq 1/3), \tag{2}$$

where $\sigma \geq 0$ is the parameter for the surface tension in the linear dispersion relation

$$\omega^2 = (k + \sigma k^3)\tanh(k)$$

of the water wave problem. With this choice of ν, the coefficient

$$\beta = -\frac{1}{6}(1-3\sigma)\nu + \frac{1}{360}(19-30\sigma-45\sigma^2)$$

in front of the fifth-order term $\delta^4 \partial_x^5 u$ vanishes; see [11] for more details. However, since for technical reasons we will need $\nu > 0$, we just consider (1) with independent parameters $\nu > 0$ and $\sigma, \beta \in \mathbb{R}$.

The Camassa-Holm equation recently attracted a lot of interest, due to its complete integrability and the presence of so called peakon solutions; see for instance [4, 3, 1, 2] and the references therein. However, the question whether solutions of the water wave problem can really be approximated by solutions of the Camassa-Holm equation so far remained open. In [10, 11] it has been argued that the KdV 5-equation and the Camassa-Holm equation are formally related by a Kodama transformation. Moreover, special solutions and their occurrence in the water wave problem are discussed.

Our approach to the problem is as follows. In Section 2 we prove an approximation theorem, which says that in the limit $\delta^2 \leq \varepsilon \to 0$ solutions of the Camassa-Holm equation remain close to the solutions of the associated KdV equation

$$\partial_t v + \partial_x v + \frac{3}{2}\varepsilon\, v\partial_x v + \delta^2\Big(\frac{1}{6} - \frac{\sigma}{2}\Big)\partial_x^3 v = 0 \tag{3}$$

on an $\mathcal{O}(1/\varepsilon)$-time scale. For the particular choice $\varepsilon = \delta^2$, a number of approximation theorems have been established recently [8, 15, 16], showing that the water waves under consideration can be approximated on an $\mathcal{O}(1/\delta^2)$-time scale by the solutions of (3). Combining these two observations, we conclude that in this limit the solutions of the water wave problem can be approximated by the solutions of the Camassa-Holm equation.

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2 KdV dynamics in the Camassa-Holm equation

In this section we prove that in the limit $\delta^2 \leq \varepsilon \to 0$ the solutions of the Camassa-Holm equation (1) can be approximated through the solutions of the associated KdV equation (3) on an $\mathcal{O}(1/\varepsilon)$-time scale.

Theorem 2.1 *Let $s \in \mathbb{N}$ and $s \geq 5$. For all $T_0 > 0$ and $C_1 > 0$ there exist $\varepsilon_0 > 0$ and $C_2 > 0$ such that for all $\delta^2 \leq \varepsilon \leq \varepsilon_0$ the following holds. Let $v \in C([0, T_0/\varepsilon], H^{s+5}(\mathbb{R}))$ be a solution of the KdV equation (3) satisfying*

$$\sup_{t\in[0,T_0/\varepsilon]} \|v(\cdot,t)\|_{H^{s+5}(\mathbb{R})} \leq C_1. \tag{4}$$

Then there exists a solution $u \in C([0,T_0/\varepsilon], H^s(\mathbb{R}))$ of the Camassa-Holm equation (1) so that $u(x,0) = v(x,0)$ for $x \in \mathbb{R}$ and

$$\sup_{t\in[0,T_0/\varepsilon]} \|u(\cdot,t) - v(\cdot,t)\|_{H^s(\mathbb{R})} \leq C_2\,\varepsilon.$$

Proof : The proof is based on suitable energy estimates. In order to find u, we make the ansatz $u = v + \varepsilon R$. A short calculation reveals that then we need solve the equation

$$\begin{aligned}
&\partial_t R + \partial_x R + \frac{3}{2}\varepsilon\,(v\partial_x R + R\partial_x v) + \delta^2(\frac{1}{6} - \nu - \frac{\sigma}{3})\,\partial_x^3 R - \delta^2\nu\partial_t\partial_x^2 R \\
&-\frac{1}{2}\,\varepsilon\delta^2\nu(v\partial_x^3 R + R\partial_x^3 v + 2\partial_x v\partial_x^2 R + 2\partial_x R\partial_x^2 v) + \delta^4\beta\partial_x^5 R + \frac{3}{2}\varepsilon^2\,R\partial_x R \\
&-\frac{1}{2}\,\varepsilon^2\delta^2\nu(R\partial_x^3 R + 2\partial_x R\partial_x^2 R) - \frac{1}{2}\,\delta^2\nu(v\partial_x^3 v + 2\partial_x v\partial_x^2 v) + \delta^4\varepsilon^{-1}\beta\,\partial_x^5 v \\
&-\delta^2\varepsilon^{-1}\nu\,(\partial_t\partial_x^2 v + \partial_x^3 v) = 0
\end{aligned}\tag{5}$$

for R, and in addition we have to verify that

$$\sup_{t\in[0,T_0/\varepsilon]} \|R(\cdot,t)\|_{H^s(\mathbb{R})} \leq C_2. \tag{6}$$

As initial data for R we choose $R(x,0) = 0$. Since the Camassa-Holm equation possesses local unique solutions in H^s, cf. [7, 14, 9], also (5) has local unique solutions in H^s. Furthermore, the solutions do not blow up, as long as the H^s-norm of $u = v + \varepsilon R$, resp. R, is bounded; here and in the sequel we write H^j in place of $H^j(\mathbb{R})$ and suppress the t-dependence of the functions. Hence it remains to show (6). For this, we let

$$E_j(R) = \int (\partial_x^j R)^2\,dx + \delta^2\nu \int (\partial_x^{j+1} R)^2\,dx, \quad j \in \{0,\ldots,s\},$$

and

$$\mathcal{E}(R) = \sum_{j=0}^{s} E_j(R);$$

it is for the definition of $E_j(R)$ that we had to assume $\nu > 0$. Note also that $\|R\|_{H^s}^2 + \delta^2\nu\|\partial_x^{s+1}R\|_{L^2}^2 \leq \mathcal{E}(R) \leq C(\|R\|_{H^s}^2 + \delta^2\|\partial_x^{s+1}R\|_{L^2}^2)$ for $\delta \leq 1$. In order to estimate $\mathcal{E}(R)$, we consider a smooth and compactly supported solution R of (5). We take ∂_x^j of (5), multiply by $\partial_x^j R$, and integrate over $x \in \mathbb{R}$. Then we obtain

$$\begin{aligned}
&\frac{1}{2}\,\partial_t \int (\partial_x^j R)^2\,dx + \frac{3\varepsilon}{2}\int \partial_x^j(v\partial_x R + R\partial_x v)\,\partial_x^j R\,dx + \frac{1}{2}\,\delta^2\nu\partial_t \int (\partial_x^{j+1} R)^2\,dx \\
&-\frac{1}{2}\,\varepsilon\delta^2\nu \int \partial_x^j(v\partial_x^3 R + R\partial_x^3 v + 2\partial_x v\partial_x^2 R + 2\partial_x R\partial_x^2 v)\,\partial_x^j R\,dx \\
&+\frac{3}{2}\varepsilon^2 \int \partial_x^j(R\partial_x R)\,\partial_x^j R\,dx - \frac{1}{2}\,\varepsilon^2\delta^2\nu \int \partial_x^j(R\partial_x^3 R + 2\partial_x R\partial_x^2 R)\,\partial_x^j R\,dx \\
&-\frac{1}{2}\,\delta^2\nu \int \partial_x^j(v\partial_x^3 v + 2\partial_x v\partial_x^2 v)\,\partial_x^j R\,dx + \beta\,\delta^4\varepsilon^{-1}\int \partial_x^{j+5} v\,\partial_x^j R\,dx \\
&-\delta^2\varepsilon^{-1}\nu \int (\partial_t\partial_x^{j+2} v + \partial_x^{j+3} v)\,\partial_x^j R\,dx = 0,
\end{aligned}\tag{7}$$

where we used that

$$\int \partial_x^{j+1} R \, \partial_x^j R \, dx = 0, \quad \int \partial_x^{j+3} R \, \partial_x^j R \, dx = 0, \quad \int \partial_x^{j+5} R \, \partial_x^j R \, dx = 0;$$

this follows through an odd number of integrations by parts. The remaining contributions to (7) we have to consider term by term. To begin with, for coefficients c_k,

$$\begin{aligned} A_1 &= \int \partial_x^j (v \partial_x R) \, \partial_x^j R \, dx = \sum_{k=0}^{j-1} c_k \int \partial_x^{j-k} v \, \partial_x^{k+1} R \, \partial_x^j R \, dx + \int v \, \partial_x^{j+1} R \, \partial_x^j R \, dx \\ &=: a_{11} + a_{12}. \end{aligned}$$

Since $\sup_{t \in [0, T_0/\varepsilon]} \|v(t)\|_{C_b^{s+4}(\mathbb{R})} \le C$ by (4), we have $|a_{11}| \le C \|R\|_{H^j}^2$ by Hölder's inequality. For a_{12} we integrate by parts and get

$$a_{12} = \int v \, \partial_x^{j+1} R \, \partial_x^j R \, dx = -\int \partial_x (v \, \partial_x^j R) \, \partial_x^j R \, dx = -\frac{1}{2} \int \partial_x v \, (\partial_x^j R)^2 \, dx,$$

so that $|a_{12}| \le C \|R\|_{H^j}^2$ and hence $|A_1| \le C \|R\|_{H^j}^2 \le C \mathcal{E}(R)$. For the expression $A_2 = \int \partial_x^j (R \partial_x v) \, \partial_x^j R \, dx$ we directly obtain $|A_2| \le C \|R\|_{H^j}^2 \le C \mathcal{E}(R)$, as only derivatives of R occur up to the order j. Next we consider

$$\begin{aligned} A_3 &= \int \partial_x^j (v \partial_x^3 R) \, \partial_x^j R \, dx \\ &= \sum_{k=0}^{j-3} c_k \int \partial_x^{j-k} v \, \partial_x^{k+3} R \, \partial_x^j R \, dx + c_{j-2} \int \partial_x^2 v \, \partial_x^{j+1} R \, \partial_x^j R \, dx \\ &\quad + c_{j-1} \int \partial_x v \, \partial_x^{j+2} R \, \partial_x^j R \, dx + \int v \, \partial_x^{j+3} R \, \partial_x^j R \, dx \\ &=: a_{31} + a_{32} + a_{33} + a_{34}. \end{aligned}$$

Clearly $|a_{31}| + |a_{32}| \le C \|R\|_{H^j}^2$ as before, cf. a_{11} and a_{12}. Now we observe the useful general formulas

$$\int f \partial_x g \, \partial_x^2 g \, dx = -\frac{1}{2} \int \partial_x f \, (\partial_x g)^2 \, dx, \tag{8}$$

$$\int f g \, \partial_x^3 g \, dx = -\frac{1}{2} \int \partial_x^3 f \, g^2 \, dx + \frac{3}{2} \int \partial_x f \, (\partial_x g)^2 \, dx, \tag{9}$$

for smooth and compactly supported functions f and g. Thus $|a_{33}| \le C \|\partial_x^{j+1} R\|_{L^2}^2$ as well as $|a_{34}| \le C(\|R\|_{H^j}^2 + \|\partial_x^{j+1} R\|_{L^2}^2)$ is found, and consequently $|A_3| \le C(\|R\|_{H^j}^2 + \|\partial_x^{j+1} R\|_{L^2}^2) \le C(\mathcal{E}(R) + \|\partial_x^{j+1} R\|_{L^2}^2)$. For the term

$$A_4 = \int \partial_x^j (R \partial_x^3 v + 2 \partial_x v \partial_x^2 R + 2 \partial_x R \partial_x^2 v) \, \partial_x^j R \, dx$$

we can argue analogously using (8) and (9), so that also $|A_4| \le C(\mathcal{E}(R) + \|\partial_x^{j+1} R\|_{L^2}^2)$. The next contribution to (7) is

$$A_5 = \int \partial_x^j (R \partial_x R)\, \partial_x^j R\, dx.$$

If $j = 0$, then $A_5 = \frac{1}{3} \int \partial_x (R^3)\, dx = 0$. If $j = 1$ we get $A_5 = \frac{1}{2} \int (\partial_x R)^3\, dx$ through integration by parts, whence $|A_5| \le C\|R\|_{H^2} \|R\|_{H^1}^2 \le C\mathcal{E}(R)^{3/2}$ for $j = 1$. If $j \ge 2$, then we split

$$\begin{aligned} A_5 &= \sum_{k=1}^{j-1} c_k \int \partial_x^{j-k} R\, \partial_x^{k+1} R\, \partial_x^j R\, dx + \int \partial_x R\, (\partial_x^j R)^2\, dx + \int R\, \partial_x^{j+1} R\, \partial_x^j R\, dx \\ &=: a_{51} + a_{52} + a_{53}. \end{aligned}$$

It follows that $|a_{51}| \le C\|R\|_{H^j}^3$, since $\|R\|_{C_b^{l-1}} \le C\|R\|_{H^l}$ for $l \ge 1$. For the same reason we have $|a_{52}| \le C\|R\|_{H^j}^3$, and (8) shows that $a_{53} = -\frac{1}{2} a_{52}$ obeys the same estimate $|a_{53}| \le C\|R\|_{H^j}^3$. Consequently, we see that $|A_5| \le C\mathcal{E}(R)^{3/2}$ for all j. Further, let

$$A_6 = \int \partial_x^j (R \partial_x^3 R + 2\partial_x R\, \partial_x^2 R)\, \partial_x^j R\, dx.$$

If $j = 0$, then $A_6 = \int \partial_x (R^2 \partial_x^2 R)\, dx = 0$. If $j = 1$, then

$$A_6 = -\int (R\partial_x^3 R + 2\partial_x R\, \partial_x^2 R)\, \partial_x^2 R\, dx = -\frac{3}{2} \int \partial_x R\, (\partial_x^2 R)^2\, dx$$

by (8), so that $|A_6| \le C\|R\|_{H^2}^3 \le C\mathcal{E}(R)^{3/2}$ for $j = 1$. If $j = 2$, then

$$\begin{aligned} A_6 &= -\int \partial_x (R\partial_x^3 R + 2\partial_x R\, \partial_x^2 R)\, \partial_x^3 R\, dx = -\frac{5}{2} \int \partial_x R\, (\partial_x^3 R)^2\, dx + \frac{2}{3} \int \partial_x (\partial_x^2 R)^3\, dx \\ &= -\frac{5}{2} \int \partial_x R\, (\partial_x^3 R)^2\, dx, \end{aligned}$$

once again by (8). Thus $|A_6| \le C\|R\|_{H^2} \|R\|_{H^3}^2$ for $j = 2$. If $j = 3$, then a similar calculation leads to

$$A_6 = -\int \partial_x^2 (R\partial_x^3 R + 2\partial_x R\, \partial_x^2 R)\, \partial_x^4 R\, dx = -\frac{7}{2} \int \partial_x R\, (\partial_x^4 R)^2\, dx - 7 \int \partial_x^2 R\, \partial_x^3 R\, \partial_x^4 R\, dx,$$

and accordingly $|A_6| \le C\|R\|_{H^3} \|R\|_{H^4}^2$ for $j = 3$. If $j \ge 4$, we write

$$\begin{aligned} A_6 &= \Big(\sum_{k=0}^{j-2} c_k \int (\partial_x^{j-k} R\, \partial_x^{k+3} R + 2\partial_x^{j-k+1} R\, \partial_x^{k+2} R)\, \partial_x^j R\, dx \\ &\qquad + 2c_{j-1} \int \partial_x^2 R\, \partial_x^{j+1} R\, \partial_x^j R\, dx \Big) + (c_{j-1} + 2) \int \partial_x R\, \partial_x^{j+2} R\, \partial_x^j R\, dx \\ &\quad + \int R\, \partial_x^{j+3} R\, \partial_x^j R\, dx \\ &=: a_{61} + a_{62} + a_{63}. \end{aligned}$$

Then $|a_{61}| \leq C\|R\|_{H^j}\|R\|^2_{H^{j+1}}$, since in each term of a_{61} one order of derivative is $\leq (j-1)$. Also by (8), $|a_{62}| \leq C|\int \partial_x^2 R\,(\partial_x^{j+1}R)^2\,dx| \leq C\|R\|_{H^j}\|R\|^2_{H^{j+1}}$, and (9) yields

$$|a_{63}| \leq C\left| -\frac{1}{2}\int \partial_x^3 R\,(\partial_x^j R)^2\,dx + \frac{3}{2}\int \partial_x R\,(\partial_x^{j+1}R)^2\,dx\right| \leq C\|R\|_{H^j}\|R\|^2_{H^{j+1}}.$$

In summary, we obtain $|A_6| \leq C\|R\|_{H^j}\|R\|^2_{H^{j+1}} \leq C\mathcal{E}(R)^{1/2}\|R\|^2_{H^{j+1}}$ for $j \neq 1$, whereas $|A_6| \leq C\mathcal{E}(R)^{3/2}$ for $j = 1$. Next, for the expression $A_7 = \int \partial_x^j(v\partial_x^3 v + 2\partial_x v\partial_x^2 v)\,\partial_x^j R\,dx$ we directly get $|A_7| \leq C\|R\|_{H^j} \leq C\mathcal{E}(R)^{1/2}$ from the bound on v. Then $A_8 = \int \partial_x^{j+5}v\,\partial_x^j R\,dx$ is estimated as $|A_8| \leq C\|R\|_{H^j} \leq C\mathcal{E}(R)^{1/2}$, and finally we have to bound the residual terms

$$A_9 = \int (\partial_t\partial_x^{j+2}v + \partial_x^{j+3}v)\,\partial_x^j R\,dx.$$

Using the KdV equation (3), A_9 can be rewritten as

$$A_9 = -\frac{3}{2}\varepsilon\int \partial_x^{j+2}(v\partial_x v)\,\partial_x^j R\,dx - \delta^2\Big(\frac{1}{6} - \frac{\sigma}{2}\Big)\int \partial_x^{j+5}v\,\partial_x^j R\,dx,$$

from where we get $|A_9| \leq C\max\{\varepsilon,\delta^2\}\|R\|_{H^j} \leq C\max\{\varepsilon,\delta^2\}\mathcal{E}(R)^{1/2}$. Summarizing the foregoing estimates on $A_1,\ldots,A_9$, we conclude from (7) that for $\delta^2 \leq \varepsilon \leq 1$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(R) &\leq C\sum_{j=0}^{s}\Big(\varepsilon\mathcal{E}(R) + \varepsilon\delta^2(\mathcal{E}(R) + \|\partial_x^{j+1}R\|^2_{L^2}) + \varepsilon^2\mathcal{E}(R)^{3/2} \\
&\qquad + \varepsilon^2\delta^2(\mathcal{E}(R)^{1/2}\|R\|^2_{H^{j+1}} + \mathcal{E}(R)^{3/2}) + \delta^2\mathcal{E}(R)^{1/2} + \delta^4\varepsilon^{-1}\mathcal{E}(R)^{1/2} \\
&\qquad + \delta^2\varepsilon^{-1}\max\{\varepsilon,\delta^2\}\mathcal{E}(R)^{1/2}\Big) \\
&\leq C\Big(\varepsilon\mathcal{E}(R) + \varepsilon^2\mathcal{E}(R)^{3/2} + \varepsilon\mathcal{E}(R)^{1/2}\Big) \\
&\leq C^*\Big(\varepsilon\mathcal{E}(R) + \varepsilon^2\mathcal{E}(R)^{3/2} + \varepsilon\Big), \qquad (10)
\end{aligned}$$

with the constants C and C^* depending only on C_1 from (4); we also used the general inequality $|x|^{1/2} \leq 1 + |x|$. Let $C_2 = C^*T_0\exp(2C^*T_0)$ and consider the longest time interval $[0,\tau_1]$ so that $\mathcal{E}(R(t)) \leq C_2$ on this time interval. Then (10) in conjunction with $R(x,0) = 0$ and Gronwall's inequality yields for $t \in [0, T_0/\varepsilon]$ the estimate

$$\mathcal{E}(R(t)) \leq C^*\varepsilon t\exp(C^*[1 + C_2^{1/2}\varepsilon]\varepsilon t) \leq C^*T_0\exp(C^*[1 + C_2^{1/2}\varepsilon]T_0) < C_2,$$

provided that $\varepsilon \leq \varepsilon_0$ and $\varepsilon_0 > 0$ is chosen sufficiently small. Therefore $[0,\tau_1] \supset [0, T_0/\varepsilon]$, and the bound $\|R(\cdot,t)\|^2_{H^s} \leq \mathcal{E}(R(t)) \leq C_2$ is verified for all $t \in [0, T_0/\varepsilon]$. Writing C_2 in the place of $C_2^{1/2}$, this shows (6) and hence completes the proof of Theorem 2.1. ■

3 Camassa-Holm dynamics in the water wave problem

It is the purpose of this section to provide error estimates relating the solutions of the Camassa-Holm equation (1) to the solutions of the associated water wave problem in the case where $\varepsilon = \delta^2$. We refer to the water wave problem without and with surface tension as stated in [15, 16]. There are used the spaces $H^s(2)$, equipped with the norm $\|u\|_{H^s(2)} = \|u\rho\|_{H^s}$, where $\rho(x) = 1 + x^2$. Moreover, we have to make the following remark.

Remark 3.1 The KdV equation (3) still contains the small parameter δ. However, it can be made independent of δ by changing variables $v(x,t) = w(x-t, \delta^2 t)$ for $w = w(\xi, \tau)$. Then

$$\partial_\tau w + (\frac{1}{6} - \frac{\sigma}{2})\, \partial_\xi^3 w + \frac{3}{2}\, w \partial_\xi w = 0, \tag{11}$$

where δ dropped out. In this sense the KdV equation is unique amongst all long wave models.

The solutions to the water wave problem are determined by the evolution of the free top surface. For the solutions under consideration, the elevation of this free top surface is a graph $\eta = \eta(x,t)$ over the flat bottom which is parameterized by $x \in \mathbb{R}$. In [15, 16] the following has been shown.

(KdV App) *Fix $\sigma \geq 0$ with $\sigma \neq 1/3$ and take $s \in \mathbb{N}$ with $s \geq 6$. Then for every $A_1 > 0$ there exists $\delta_0 > 0$ and $A_2 > 0$ such that the following holds. If $\delta \in (0, \delta_0)$ and if $w \in C([0, T_0], H^s(2))$ is a solution of the KdV equation (11) so that $\|w(\cdot, 0)\|_{H^s(2)} \leq A_1$ for the initial data, then there is a solution η of the water wave problem satisfying*

$$\sup_{t \in [0, T_0/\delta^2]} \sup_{x \in \mathbb{R}} |\eta(x,t) - w(x-t, \delta^2 t)| \leq A_2\, \delta^{1/2},$$

or equivalently,

$$\sup_{t \in [0, T_0/\delta^2]} \sup_{x \in \mathbb{R}} |\eta(x,t) - v(x,t)| \leq A_2\, \delta^{1/2}.$$

Remark 3.2 In comparing [15, 16] with (KdV App), note that x, t, η in (KdV App) are already scaled variables which correspond to $\delta x, \delta t, \delta^2 \eta$ in [15, 16].

Combining Theorem 2.1 with (KdV App) gives that solutions of the water wave problem and solutions of the Camassa-Holm equation stay close together over a time interval of length $\mathcal{O}(1/\delta^2)$.

Theorem 3.3 *Fix $\sigma \geq 0$ with $\sigma \neq 1/3$, let $\nu > 0$, assume $s \in \mathbb{N}$ and $s \geq 6$, and take $\varepsilon = \delta^2$. Then for all $C_3 > 0$ there exist $\delta_0 > 0$ and $C_4, C_5 > 0$ such that for all $\delta \in (0, \delta_0)$ the following holds. Let $u_0 \in H^{s+5}(2)$ be such that $\|u_0\|_{H^{s+5}(2)} \leq C_3$. Then the associated solution $u \in C([0, T_0/\delta^2], H^{s+5}(\mathbb{R}))$ of the Camassa-Holm equation (1) with initial data $u(x,0) = u_0(x)$ satisfies*

$$\sup_{t \in [0, T_0/\delta^2]} \|u(\cdot, t)\|_{H^s(\mathbb{R})} \leq C_4, \tag{12}$$

and there is a solution η of the water wave problem so that

$$\sup_{t\in[0,T_0/\delta^2]} \sup_{x\in\mathbb{R}} |\eta(x,t) - u(x,t)| \leq C_5\, \delta^{1/2}.$$

Proof : Solve the KdV equation (11) with initial data $w(x,0) = u_0(x) \in H^{s+5}(2)$. The solutions of the KdV equation (11) are known to exist in $H^{s+5}(2)$ for all $t \in \mathbb{R}$, see [15]. By the continuity of the map which sends initial data to the solutions, there exists $C_1 > 0$ such that for the solutions w of (11) with initial data bounded by C_3,

$$\sup_{\tau\in[0,T_0]} \|w(\cdot,\tau)\|_{H^{s+5}(2)} \leq C_1.$$

Since (11) is independent of δ, C_1 is independent of δ as well. Therefore

$$\sup_{t\in[0,T_0/\delta^2]} \|w(\cdot,\delta^2 t)\|_{H^{s+5}} \leq C_1,$$

or equivalently for the solutions $v(x,t) = w(x-t,\delta^2 t)$ of (3),

$$\sup_{t\in[0,T_0/\delta^2]} \|v(\cdot,t)\|_{H^{s+5}} \leq C_1.$$

Hence the assumptions of Theorem 2.1 and (KdV App) are satisfied, taking $A_1 = C_1$ for the latter. Consequently there is a solution $u \in C([0,T_0/\delta^2], H^s(\mathbb{R}))$ of the Camassa-Holm equation (1) such that $u(x,0) = v(x,0) = w(x,0) = u_0(x)$ and

$$\sup_{t\in[0,T_0/\delta^2]} \|u(\cdot,t) - v(\cdot,t)\|_{H^s} \leq C_2\delta^2.$$

This yields (12), and using (KdV App) we can moreover estimate

$$|\eta(x,t) - u(x,t)| \leq |\eta(x,t) - v(x,t)| + \|v(\cdot,t) - u(\cdot,t)\|_{L^\infty} \leq A_2\,\delta^{1/2} + C\delta^2$$

uniformly in $t \in [0,T_0/\delta^2]$ and $x \in \mathbb{R}$, which gives the desired result. ■

Remark 3.4 If we retain the condition $\beta = 0$, as a consequence of (2), then the Camassa-Holm equation (1) can be rewritten in terms of $m = u - 10g\nu\delta^2\partial_x^2 u$ as

$$\partial_t m + \partial_x m + \frac{\varepsilon}{2}(u\partial_x m + 2m\partial_x u) + \delta^2\Big(\frac{1}{6} - \frac{\sigma}{2}\Big)\partial_x^3 u = 0.$$

In this formulation, the peakon equation (the equation which possesses the solitary waves with discontinuous derivative at the peaks) can be obtained in the case where $\sigma = 1/3$. However, in the original Camassa-Holm equation, the limit $\sigma \to 1/3$ is singular by the choice of ν in (2), and as already stated in [11], the peakon equation cannot strictly be derived from the Euler equation and hence it is at most a phenomenological model. Moreover, the limit equation for $\sigma = 1/3$ is no longer the KdV equation, but the Kawahara equation; see [16].

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