Radiative friction for charges interacting with the radiation field: classical many-particle systems

SEBASTIAN BAUER & MARKUS KUNZE

Universität Duisburg-Essen, Fachbereich Mathematik,
D-45117 Essen, Germany

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Abstract

We consider an ensemble of classical particles modelled by means of a continuity equation for a distribution function which is coupled back to the self-induced fields. For such infinite dimensional systems simpler effective equations are derived in the limit $c \to \infty$. A main emphasis is on higher order approximations, in particular on the first post-Newtonian order where radiation starts to play a role.

1 Introduction

The most precise existing theory of gravitation, the theory of general relativity, predicts that certain astrophysical systems, such as colliding black holes or neutron stars, will give rise to gravitational radiation. There is a major international effort under way to detect these gravitational waves [Br04]. In order to relate the general theory to predictions of what the detectors will see it is necessary to use approximation methods since the exact theory is too complicated. The mathematical status of these approximations remains unclear and only very partial results exist. Therefore it is useful to start with model problems. One option is the relativistic Vlasov-Maxwell system which plays an important role in plasma physics. Although the field part of this system is electromagnetic and spacetime is flat, such a model is already fairly difficult. It is often used to gain a better understanding of the mathematical structures involved in more realistic gravity models. A further option for a simplified model is the scalar theory of gravitation, as described by the Vlasov-Nordström theory [C03]. It has already been considered as a model problem for numerical relativity in [ShT93].

Among the approximation methods used to study gravitational radiation those which are most accessible mathematically are the post-Newtonian approximations. Some information on these has been obtained in [R94, R92] but further rigorous progress seems difficult at this point. For that reason it seems to be useful to investigate (from the viewpoint of approximation methods) the two systems presented above, i.e., Vlasov matter coupled to the Maxwell fields and Vlasov matter coupled to a scalar gravitational field governed by the Nordström equation. We shall explain the post-Newtonian expansion of the Vlasov-Maxwell system in some detail in Section 2 and sketch our results concerning the Vlasov-Nordström system in Section 3.

Remark: Our contribution is one part of the research project jointly with G. Panati, H. Spohn, and S. Teufel within the Schwerpunkt. The second part will be covered in the separate contribution [PaSpT06].

2 The Vlasov-Maxwell system

The Vlasov-Maxwell system from kinetic theory models the evolution of a plasma or gas composed of many collisionless particles which move under the influence of their self-generated electromagnetic field. For the sake of simplicity we assume that there are only two different species of particles with mass normalized to unity and charge normalized to plus unity and minus unity, respectively. The particle distributions on phase space are modelled through the nonnegative distribution functions f^+ and f^-, $f^\pm = f^\pm(t,x,p)$, depending on time $t \in \mathbb{R}$, position $x \in \mathbb{R}^3$, and momentum $p \in \mathbb{R}^3$. It is assumed that collisions between single particles are sufficiently rare such that they can be neglected. Therefore all forces between the particles are mediated by the electromagnetic fields. The dynamics are governed by

$$\left.\begin{array}{c} \partial_t f^\pm + \hat{p}\cdot\nabla_x f^\pm \pm (E + c^{-1}\hat{p}\times B)\cdot\nabla_p f^\pm = 0, \\ \begin{array}{rclrcl} c\nabla\times E &=& -\partial_t B, & c\nabla\times B &=& \partial_t E + 4\pi j \\ \nabla\cdot E &=& 4\pi\rho, & \nabla\cdot B &=& 0, \\ \rho &:=& \displaystyle\int (f^+ - f^-)\,dp, & j &:=& \displaystyle\int \hat{p}\,(f^+ - f^-)\,dp \end{array}\end{array}\right\} \qquad (\mathrm{RVM}_c)$$

Here

$$\hat{p} = (1 + c^{-2}p^2)^{-1/2} p \in \mathbb{R}^3 \tag{2.1}$$

is the relativistic velocity associated to p. The Lorentz force $E + c^{-1}\hat{p}\times B$ realizes the coupling of the Maxwell fields $E = E(t,x) \in \mathbb{R}^3$ and $B = B(t,x) \in \mathbb{R}^3$ to the Vlasov equation, and conversely the density functions $f^\pm$ enter the field equations via the scalar charge density $\rho = \rho(t,x)$ and the current density $j = j(t,x) \in \mathbb{R}^3$, which act as source terms for the Maxwell equations. The parameter c denotes the speed of light for given units of time and space of the physical system. In order to state the Cauchy problem for (RVM_c) initial data for the densities and for the fields have to be prescribed,

$$f^\pm(0,x,p) = f^{\circ,\pm}(x,p), \quad E(0,x) = E^\circ(x), \quad B(0,x) = B^\circ(x). \tag{2.2}$$

Henceforth we treat the speed of light c as a parameter and study the behavior of the system as $c \to \infty$. It will be explained below that after a suitable rescaling the fields are slowly varying in their space and time variables. Thus the limit $c \to \infty$ corresponds to an adiabatic limit and to slowly moving particles. Our general goal is to establish conditions under which the solutions of (RVM_c) converge to a solution of an effective system. For the Vlasov-Maxwell system the first result in this direction was obtained in [S86], where it has been shown that as $c \to \infty$ the solutions of (RVM_c) approach a solution of the Vlasov-Poisson system at the rate $\mathcal{O}(c^{-1})$; see [AU86, D86] for similar results and [Le04] for the case of two spatial dimensions. The respective Newtonian limits of other related systems are derived in [R94, CLe04].

It was one aim of this project to replace the Vlasov-Poisson system by other effective equations to achieve higher order convergence and more precise approximations. In [BK05] this led to an effective system whose solutions stay as close as $\mathcal{O}(c^{-3})$ to a solution of the full Vlasov-Maxwell system. In the context of individual particles, this post-Newtonian (PN) order of approximation is usually called the Darwin order; see [KSp00, Sp04] and the references therein. We also mention that weak convergence properties of other kinds of Darwin approximations for the Vlasov-Maxwell system have been studied in [DRa92, BeFLaSo03].

In the next order, and in analogy to the case of individual particles [KSp01], radiation effects play a role for the first time. Therefore the well-known problems related to the existence of unphysical solutions, usually called "run-away solutions", are expected to turn up [J99]. For individual particles these problems can be resolved rigorously by restricting the dynamics to a suitable center-like manifold in the phase space; see [KSp01, Sp04]. Since the phase space of the Vlasov-Maxwell system is infinite dimensional (as densities are considered), it is clear that several new mathematical difficulties have to be surrounded in this step. In [B06b] we determined effective equations for the Vlasov-Maxwell system on the center manifold, which led to a slightly dissipative Vlasov-like effective equation, free of "run-away" solutions, and we proved that solutions of these equations stay as close as $\mathcal{O}(c^{-4})$ to a solution of the full Vlasov-Maxwell system.

Compared to systems of coupled individual particles, for the Vlasov-Maxwell system one immediately is faced with the fact that so far only the existence of local solutions is known in general. These solutions are global under additional conditions, for instance if a suitable a priori bound on the velocities is available; see [GSt86]. This means that from the onset we will have to restrict ourselves to solutions of (RVM$_c$) which are only defined on some time interval $[0, T]$ that may be very small. On the other hand, in [S86] it has been shown that such a time interval can be found which is uniform in $c \geq 1$, so it seems reasonable to accept this limit.

In Section 2.1 we first carry out a formal expansion of (RVM$_c$) in c^{-1} as $c \to \infty$. It turns out that up to the Darwin order this approximation yields the correct effective system. However, for the next order this naive procedure does not yield the optimal result. To get a clue on how an improved approximation may look like we derive the leading order dipole radiation term in Section 2.2, and thereafter we establish an improved system (called the radiation approximation) which gives a better approximation to the full Vlasov-Maxwell system. More details on this are elaborated in Section 2.3. Finally a comparison of the continuous models to the individual particle models is done in Section 2.4.

2.1 The naive post-Newtonian expansion

We adopt the definition of a post-Newtonian approximation from [KR01b]; see also [R92] for the Einstein case. Thus the matter and the fields are described by a one-parameter family $(f^{\pm}(c), E(c), B(c))$ of solutions to (RVM$_c$), depending on the parameter $c \in [c_0, \infty[$. This means that $(f^{\pm}(c), E(c), B(c))$ describes a family of solutions of physical systems which are represented in parameter-dependent units, where the numerical value of the speed of light is given by c. A more conventional physical description of the post-Newtonian expansion would say that in a fixed system of units the occurring velocities are small compared to the speed of light. Taking this viewpoint means that we consider (RVM$_c$) at a fixed c (say $c = 1$) by rescaling the prescribed nonnegative initial densities $f^{\circ,\pm}$, for which we suppose that $f^{\circ,\pm} \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ have compact support. To be more precise, denote $\bar{v} = \int\int \hat{p} f^{\circ,\pm}(x,p)\,dx\,dp$, where $\hat{p}$ is taken for $c = \varepsilon^{-1/2}$; cf. (2.1). Then $\bar{v}$ is viewed as an average velocity of the system. Then we introduce $f^{\circ,\pm,\varepsilon}(x,p) = \varepsilon^{3/2} f^{\circ,\pm}(\varepsilon x, \varepsilon^{-1/2} p)$ and consider $f^{\circ,\pm,\varepsilon}$ for $c = 1$. It follows that

$$\bar{v}^\varepsilon = \int\int \hat{p} f^{\circ,\pm,\varepsilon}(x,p)\,dx\,dp = \sqrt{\varepsilon}\int\int \hat{p} f^{\circ}(x,p)\,dx\,dp = \sqrt{\varepsilon}\,\bar{v},$$

i.e., the systems with initial distribution functions $f^{\circ,\pm,\varepsilon}$ have small velocities compared to the systems associated to $f^{\circ,\pm}$. Under this scaling the masses remain unchanged, as $\int\int f^{\circ,\pm,\varepsilon}(x,p)\,dx\,dp = \int\int f^{\circ,\pm}(x,p)\,dx\,dp$. Next observe that $(f^{\pm}, E, B)$ is a solution of (RVM$_c$) with $c = \varepsilon^{-1/2}$ if and only if

$$\begin{aligned} f^{\pm,\varepsilon}(t,x,p) &= \varepsilon^{3/2} f^{\pm}(\varepsilon^{3/2}t, \varepsilon x, \varepsilon^{-1/2}p), \\ E^{\varepsilon}(t,x) &= \varepsilon^{2} E(\varepsilon^{3/2}t, \varepsilon x), \\ B^{\varepsilon}(t,x) &= \varepsilon^{2} B(\varepsilon^{3/2}t, \varepsilon x), \end{aligned} \tag{2.3}$$

is a solution of (RVM$_c$) with $c = 1$. By definition of the rescaled fields these fields are slowly varying in their space and time variables. Thus the limit $c \to \infty$ corresponds to an adiabatic limit. Henceforth we will return to the original formulation and consider the system as $c \to \infty$, treating the speed of light c as a parameter. However, due to the rescaling outlined above all theorems can also be formulated in a parameter independent fashion. In that case the value of c is fixed, say $c = 1$, and the initial data have to be modified according to (2.3); see [BK05] for details.

We start with a formal expansion of all quantities occurring in (RVM$_c$) in powers of c^{-1},

$$\begin{aligned} f^{\pm} &= f_0^{\pm} + c^{-1} f_1^{\pm} + c^{-2} f_2^{\pm} + c^{-3} f_3^{\pm} + \dots, \\ E &= E_0 + c^{-1} E_1 + c^{-2} E_2 + c^{-3} E_3 + \dots, \\ B &= B_0 + c^{-1} B_1 + c^{-2} B_2 + c^{-3} B_3 + \dots, \\ \rho &= \rho_0 + c^{-1} \rho_1 + c^{-2} \rho_2 + c^{-3} \rho_3 + \dots, \\ j &= j_0 + c^{-1} j_1 + c^{-2} j_2 + c^{-3} j_3 + \dots. \end{aligned} \tag{2.4}$$

In addition, also the initial densities are assumed to allow an expansion as $f^{\circ,\pm} = f_0^{\circ,\pm} + c^{-1} f_1^{\circ,\pm} + \dots$. Finally $\hat{p} = p - (c^{-2}/2)p^2 p + \dots$ by (2.1), where $p^2 = |p|^2$. These expansions can be substituted into (RVM$_c$). Comparing coefficients at every order gives a hierarchy of equations for the coefficients. The equations at order k will be addressed as the $k/2$ PN equations, and

$$f^{\pm,k/2\,\mathrm{PN}} = \sum_{j=0}^{k} c^{-j} f_j^{\pm}, \quad E^{k/2\,\mathrm{PN}} = \sum_{j=0}^{k} c^{-j} E_j, \quad B^{k/2\,\mathrm{PN}} = \sum_{j=0}^{k} c^{-j} B_j,$$

is the $k/2$ PN approximation. This notation is used due to the fact that in the context of general relativity post-Newtonian approximations are usually counted in orders of c^{-2}.

At order zero the well known Vlasov-Poisson system of plasma physics is obtained,

$$\left.\begin{aligned} &\partial_t f_0^{\pm} + p \cdot \nabla_x f_0^{\pm} \pm E_0 \cdot \nabla_p f_0^{\pm} = 0, \\ E_0(t,x) &= -\int |z|^{-2} \bar{z}\, \rho_0(t, x+z)\, dz, \\ \rho_0 &= \int (f_0^{+} - f_0^{-})\, dp, \\ f_0^{\pm}(0,x,p) &= f_0^{\circ,\pm}(x,p), \end{aligned}\right\} \tag{VP$_{\mathrm{plasma}}$}$$

where $\bar{z} = |z|^{-1} z$. Note that the degrees of freedom of the electromagnetic fields up to this order are lost, reflecting that the limit $c \to \infty$ is singular and the hyperbolic field equations become elliptic. As mentioned above, this 0 PN approximation is made rigorous in [S86].

Concerning a general k we assume that the lower order coefficients have already been computed. Then the fields at order k have to solve

$$\begin{array}{rclcrcl}\nabla \times E_k &=& -\partial_t B_{k-1}, & \qquad & \nabla \cdot E_k &=& 4\pi\rho_k,\\ \nabla \times B_k &=& \partial_t E_{k-1} + 4\pi j_{k-1}, & \qquad & \nabla \cdot B_k &=& 0.\end{array}$$

The Vlasov equation to that order is

$$\partial_t f_k^\pm + p\cdot\nabla_x f_k^\pm \pm E_0\cdot\nabla_p f_k^\pm = \mp E_k\cdot\nabla_p f_0^\pm + R_k^\pm,$$

where the $R_k^\pm$ can be calculated from the known quantities $f_j^\pm$, $\nabla_x f_j^\pm$, $\nabla_p f_j^\pm$, E_j, and B_j for $j = 0,\ldots,k-1$. A special feature of this hierarchy is as follows. If we assume for the initial data that $f_k^{\circ,\pm} = 0$ for all odd k, then using the explicit form of R_k it can be shown that we can set

$$f_{2l+1}^\pm = 0, \quad E_{2l+1} = 0, \quad B_{2l} = 0 \tag{2.5}$$

consistently, for $l = 0, 1, 2, \ldots$. This simplification will be employed throughout. To solve the equations for (f_k, E_k, B_k), we observe that once E_k is known, then $f_k^\pm$ can be calculated using characteristics. Note that for all orders k the characteristic flow is determined by the vector field $(p, \pm E_0)$. On the other hand, if the $f_k^\pm$ are known, then ρ_k and j_k are fixed. Using the vector identity $-\nabla\times\nabla\times + \nabla\nabla\cdot = \Delta$, we can rewrite the field equations as

$$\begin{aligned} E_{2k} &= 4\pi\Delta^{-1}(\nabla\rho_{2k} + \partial_t j_{2k-2}) + \Delta^{-1}(\partial_t^2 E_{2k-2}),\\ B_{2k+1} &= \Delta^{-1}(\partial_t^2 B_{2k-1}) - 4\pi\Delta^{-1}(\nabla\times j_{2k}), \end{aligned} \tag{2.6}$$

where quantities carrying a negative index are understood to be zero. Assuming that all densities are compactly supported we can solve these field equations. Of course without boundary conditions the solutions are not unique, and at least for higher orders they will not vanish at infinity. Nevertheless, if we take those fields, then the coupled equations can be solved by a fixed-point iteration for E_k. Thus on a formal level the (naive) PN approximation scheme is well defined.

According to this scheme, B_1 is given by

$$B_1(t,x) = \int |z|^{-2}\bar z \times j_0(t, x+z)\,dz, \tag{2.7}$$

where $j_0 = \int p\,(f_0^+ - f_0^-)\,dp$. The couple $(f_2^\pm, E_2)$ is the solution to

$$\left.\begin{aligned} &\partial_t f_2^\pm + p\cdot\nabla_x f_2^\pm \pm E_0\cdot\nabla_p f_2^\pm\\ &\qquad = \tfrac12 p^2\, p\cdot\nabla_x f_0^\pm \mp (E_2 + p\times B_1)\cdot\nabla_p f_0^\pm,\\ &E_2(t,x) = \frac12\int \bar z\,\partial_t^2\rho_0(t,x+z)\,dz - \int |z|^{-1}\partial_t j_0(t,x+z)\,dz\\ &\qquad - \int |z|^{-2}\bar z\,\rho_2(t,x+z)\,dz,\\ &\rho_2 = \int (f_2^+ - f_2^-)\,dp,\\ &f_2(0,x,p) = f_2^{\circ,\pm}(x,p). \end{aligned}\right\} \tag{$\mathrm{LVP_{plasma}}$}$$

Thus the 1 PN approximation (Darwin approximation) corresponding to $k = 2$ is

$$f^{\pm,1\,\mathrm{PN}} = f_0^\pm + c^{-2} f_2^\pm, \quad E^{1\,\mathrm{PN}} = E_0 + c^{-2}E_2, \quad B^{1\,\mathrm{PN}} = c^{-1}B_1. \tag{2.8}$$

This Darwin system is Hamiltonian in the following sense. If the conserved energy

$$\mathcal{E} = \int\int \sqrt{1 + c^{-2}p^2/2}\,(f^+ + f^-)\,dx\,dp + \frac{1}{8\pi}\int (E^2 + B^2)\,dx$$

of (RVM$_c$) is expanded according to (2.4), then the Darwin energy $\mathcal{E}_{\mathrm{D}} = \mathcal{E}_{\mathrm{D,\,kin}} + \mathcal{E}_{\mathrm{D,\,pot}}$ is obtained. Explicitly its kinetic energy and potential energy parts are given by

$$\begin{aligned}
\mathcal{E}_{\mathrm{D,\,kin}} &= \int\int [(p^2/2 - c^{-2}p^4/8)(f_0^+ + f_0^-) + c^{-2}p^2/2(f_2^+ + f_2^-)]\,dx\,dp,\\
\mathcal{E}_{\mathrm{D,\,pot}} &= \frac{1}{8\pi}\int [E_0^2 + 2c^{-2}E_0 \cdot E_2 + c^{-2}B_1^2]\,dx.
\end{aligned}$$

It can be checked that $\mathcal{E}_{\mathrm{D}}$ is conserved along solutions of the 1 PN approximation. The approximation properties of the Darwin system w.r. to solutions of the full Vlasov-Maxwell system are investigated in [BK05]. In this paper it is shown that if we adapt the initial data (2.2) to suit the initial data of the 1 PN approximation, then the solutions are tracked down with an error of order c^{-3}. Hence the naive post-Newtonian expansion is valid up to this order.

For the next level ($k = 3$ or 1.5 PN), $B_3 = \Delta^{-1}(\partial_t^2 B_1) - 4\pi\Delta^{-1}(\nabla \times j_2)$ and also $B_1 = -4\pi\Delta^{-1}(\nabla \times j_0)$ by (2.6). Thus $B_3 = -4\pi\Delta^{-2}(\partial_t^2\nabla \times j_0) - 4\pi\Delta^{-1}(\nabla \times j_2)$ allows for the solution

$$B_3(t,x) = \frac{1}{2}\int |z|\,\partial_t^2\nabla \times j_0(t, x+z)\,dz + \int |z|^{-1}\nabla \times j_2(t, x+z)\,dz, \tag{2.9}$$

where

$$j_2 = \int [p\,(f_2^+ - f_2^-) - (p^2/2)p\,(f_0^+ - f_0^-)]\,dp.$$

It follows that $f^{\pm,1.5\,\mathrm{PN}} = f_0^\pm + c^{-2}f_2^\pm + c^{-3}f_3^\pm = f^{\pm,1\,\mathrm{PN}}$, $E^{1.5\,\mathrm{PN}} = E_0 + c^{-2}E_2 + c^{-3}E_3 = E^{1\,\mathrm{PN}}$, and $B^{1.5\,\mathrm{PN}} = c^{-1}B_1 + c^{-3}B_3 = B^{1\,\mathrm{PN}} + c^{-3}B_3$, due to (2.5) and (2.8). Therefore the energy $\mathcal{E}_{\mathrm{D}}$ from above does not have to be changed in comparison to the 1 PN order. Hence at the 1.5 PN order (corresponding to c^{-3}) we would obtain a Hamiltonian system with no effects due to radiative friction visible. This suggests that the naive post-Newtonian approximation has to be improved in order to resolve such effects which are indeed present in the system. In order to get a clue on how such a refinement should look like it is useful to study the energy which is radiated to future null infinity by the full Vlasov-Maxwell system.

2.2 Dipole Radiation

The starting point for the following calculation is the local energy conservation $\partial_t e + \nabla \cdot \mathcal{P} = 0$ for classical solutions of the Vlasov-Maxwell system. The energy density e and the momentum density $\mathcal{P}$ are given by

$$\begin{aligned}
e(t,x) &= c^2\int \sqrt{1 + c^{-2}p^2}\,(f^+ + f^-)(t,x,p)\,dp + \frac{1}{8\pi}\big(|E(t,x)|^2 + |B(t,x)|^2\big),\\
\mathcal{P}(t,x) &= c^2\int p(f^+ + f^-)(t,x,p)\,dp + \frac{c}{4\pi}\,E(t,x) \times B(t,x).
\end{aligned}$$

Defining the local energy in the ball of radius $r > 0$ as $\mathcal{E}_r(t) = \int_{|x|\le r} e(t,x)\,dx$, this conservation law and the divergence theorem imply that $\frac{d}{dt}\,\mathcal{E}_r(t) = -\int_{|x|=r} \bar{x} \cdot \mathcal{P}(t,x)\,d\sigma(x)$, where $\bar{x} = |x|^{-1}x$

denotes the outer unit normal. Our assumptions on the support of the distribution functions are such that the contribution of $\int p\,(f^+ + f^-)\,dp$ to $\mathcal{P}$ vanishes for $|x| = r$ large. Hence we arrive at

$$\frac{d}{dt}\,\mathcal{E}_r(t) = \frac{c}{4\pi}\int_{|x|=r} \bar{x}\cdot(B\times E)(t,x)\,d\sigma(x).$$

Therefore the energy flux radiated to null infinity at time t is obtained as

$$\lim_{r\to\infty}\frac{c}{4\pi}\int_{|x|=r} \bar{x}\cdot(B\times E)(t+c^{-1}r,x)\,d\sigma(x),$$

where $t+c^{-1}r$ is the advanced time. In [BKReR06, Thm. 1.4] it is shown that for suitable solutions of the Vlasov-Maxwell system which are isolated from incoming radiation in the limit $c\to\infty$ the total amount of radiated energy is given by

$$\frac{2}{3c^3}\,|\ddot{D}(t)|^2, \tag{2.10}$$

where D is the dipole moment of the Newtonian limit ($\mathrm{VP}_{\mathrm{plasma}}$), defined as $D(t) = \int x\rho_0(t,x)\,dx$; see [BKReR06, Thm. 1.4] for the exact statement and Section 2.3.2 below for the retarded system (${}_{\mathrm{ret}}\mathrm{RVM}_c$) which models “no incoming radiation”. This result yields a mathematical formulation and a rigorous proof of the Larmor formula in the case of Vlasov matter. Returning to the approximations, we should therefore introduce into the effective equation a radiation reaction force causing this loss of energy. As already suggested in [KR01a, KR01b] we thus modify the Vlasov equation of the Newtonian distribution by incorporating a small correction into the force term as

$$\partial_t f_0^\pm + p\cdot\nabla_x f_0^\pm \pm \left(E_0 + \frac{2}{3c^3}\dddot{D}\right)\cdot\nabla_p f_0^\pm = 0. \tag{2.11}$$

The additional term is the generalization of the radiation reaction force used in particle models; see [J99, (16.8)]. We also note that for this system the quantity

$$\mathcal{E}_{\mathrm{S}}(t) = \frac{1}{2}\int\!\!\int p^2(f_0^+ + f_0^-)(t,x,p)\,dx\,dp + \frac{1}{8\pi}\int |E_0(t,x)|^2\,dx - \frac{2}{3c^3}\,\dot{D}(t)\cdot\ddot{D}(t) \tag{2.12}$$

is decreasing. More precisely one obtains $\frac{d}{dt}\,\mathcal{E}_{\mathrm{S}}(t) = -\frac{2}{3c^3}|\ddot{D}(t)|^2$, cf. (2.10). The subscript S refers to Schott who considered similar quantities for particle models; see [Sp04]. Although $\mathcal{E}_{\mathrm{S}}$ has no definite sign, its decrease can be attributed to the effect of radiation damping. If instead of $\mathcal{E}_{\mathrm{S}}$ the usual positive energy $\mathcal{E}_{\mathrm{VP}} = \frac{1}{2}\int\int p^2(f_0^+ + f_0^-)\,dx\,dp + \frac{1}{8\pi}\int |E_0|^2\,dx$ of the Vlasov-Poisson system is considered, then along solutions of (2.11) the relation $\frac{d}{dt}\,\mathcal{E}_{\mathrm{VP}} = \frac{2}{3c^3}(\dot{D}\cdot\dddot{D} - |\ddot{D}|^2)$ is obtained which has no straightforward interpretation.

Thus we consider (2.11) to be a promising candidate of an effective equation for the relativistic Vlasov-Maxwell system. However, one immediately runs into the problem that initial data have to be supplied for $D(0)$, $\dot{D}(0)$, and $\ddot{D}(0)$, as third derivatives of D occur in the equations. Since only $D(0)$ and $\dot{D}(0)$ are determined by $f_0^{\circ,\pm}$ and since there is no obvious way to extract the missing information from the approximation scheme, additional degrees of freedom seem to be generated. This phenomenon is also known from the theory of accelerated single charges and leads to a multitude of unphysical (so-called run-away) solutions. In [KSp01] it has been observed that in the particle model this problem has the structure of a geometric singular perturbation problem, and the “physical” dynamics are obtained on a center-like manifold of the full dynamics.

In order to adopt this language to the model under consideration, we assume that we are supplied with a (local in time) classical solution $(f_0^\pm, E_0)$ of (2.11) and assume that the support of $f_0^\pm(t,\cdot,\cdot)$ remains compact for all t in the interval of existence of the solution. We define the bare mass by $M(t) = \int\int (f_0^+ + f_0^-)(t,x,p)\,dx\,dp$. Then (2.11) yields mass conservation $\partial_t M = 0$ as well as charge conservation for both species $\partial_t \rho_0^\pm + \nabla \cdot j_0^\pm = 0$, where $j_0^\pm = \int p f_0^\pm \,dp$ and $\rho_0^\pm = \int f_0^\pm\,dp$. From $D = \int x\rho_0\,dx$ and (2.11) we then find that $\dot{D} = \int\int p\,(f_0^+ - f_0^-)\,dx\,dp$ and $\ddot{D} = D^{[2]} + \frac{2}{3c^3} M \dddot{D}$, where

$$D^{[2]}(t) = \int\int E_0(t,x)(f_0^+ + f_0^-)(t,x,p)\,dx\,dp. \tag{2.13}$$

Defining $y = \ddot{D}$ and $\eta = \frac{2}{3c^3}M$, this can be rewritten as $y = D^{[2]} + \eta \dot{y}$. Putting

$$\begin{aligned} F^\pm(f_0^\pm, y) &= -p\cdot\nabla_x f_0^\pm \mp (E_0 + M^{-1}(y - D^{[2]}))\cdot\nabla_p f_0^\pm, \\ G(f_0^\pm, y) &= y - D^{[2]}, \end{aligned}$$

it follows that (2.11) can be recast as the singular perturbation problem

$$\left.\begin{aligned} \dot{f}_0^\pm &= F^\pm(f_0^\pm, y) \\ \eta\dot{y} &= G(f_0^\pm, y) \end{aligned}\right\} \tag{SGPP$_\eta$}$$

In contrast to [KSp01] we are dealing with a phase space of infinite dimension. Thus the proof of the existence of an invariant manifold is hard. We shall return to that question in a forthcoming paper. For the moment we shall take the existence of a smooth invariant manifold for granted and assume that it is given as a smooth graph $h_\eta = h_\eta(f_0^\circ)$, where h_η is defined on (a subset of) $C_0^\infty(\mathbb{R}^3\times\mathbb{R}^3)\times C_0^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ and takes values in $\mathbb{R}^3$. For the moment f_0° denotes $(f_0^{\circ,+}, f_0^{\circ,-})$, and similarly $f_0 = (f_0^+, f_0^-)$ and $F = (F^+, F^-)$. The manifold $\mathcal{M}_\eta = \{(f_0^\circ, h_\eta(f_0^\circ))\}$ is invariant under the flow of (SGPP$_\eta$) if the solution of (SGPP$_\eta$) subject to the initial conditions $(f_0(0), y(0)) = (f_0^\circ, h_\eta(f_0^\circ))$ satisfies

$$y(t) = h_\eta(f_0(t,\cdot,\cdot)). \tag{2.14}$$

We want to determine a system of Vlasov-Poisson type which is a good approximation of the dynamics on the manifold. For this reason we assume that we can expand h_η in η about 0 as $h_\eta = h_0 + \eta h_1 + \mathcal{O}(\eta^2)$. Setting $\eta = 0$ in the second relation of (SGPP$_\eta$), $0 = G(f_0, h_0(f_0)) = h_0(f_0) - D^{[2]}$ is obtained, so that $h_0 = D^{[2]}$; note that $D^{[2]}$ depends on f_0. Thus

$$\eta\dot{y} = G(f_0, h_0(f_0) + \eta h_1(f_0) + \mathcal{O}(\eta^2)) = h_0(f_0) + \eta h_1(f_0) + \mathcal{O}(\eta^2) - D^{[2]} = \eta h_1(f_0) + \mathcal{O}(\eta^2).$$

On the other hand, differentiating (2.14) yields

$$\eta\dot{y} = \eta\,\langle h_\eta'(f_0), \dot{f}_0\rangle = \eta\,\langle h_0'(f_0), \dot{f}_0\rangle + \mathcal{O}(\eta^2),$$

and consequently $h_1(f_0) = \langle h_0'(f_0), F(f_0, h_0(f_0))\rangle$ by (SGPP$_\eta$). Explicitly,

$$\langle h_0'(f_0), F(f_0, h_0(f_0))\rangle = h_0'(f_0)\cdot\int\int\Big(-p\cdot\nabla_x(f_0^+ - f_0^-) - E_0\cdot\nabla_p(f_0^+ + f_0^-)\Big)(\cdot,x,p)\,dx\,dp.$$

From (2.14) and the above it follows that $y = h_\eta(f_0) = h_0(f_0) + \eta h_1(f_0) + \mathcal{O}(\eta^2) = D^{[2]} + \eta\,\langle D^{[2]\prime}(f_0), F(f_0, D^{[2]}(f_0))\rangle + \mathcal{O}(\eta^2)$. After a straightforward computation we obtain

$$\langle D^{[2]\prime}(f_0), F(f_0, D^{[2]}(f_0))\rangle = D^{[3]}, \tag{2.15}$$

where

$$D^{[3]}(t) = 2\int \Big(H^+(t,x)j_0^-(t,x) - H^-(t,x)j_0^+(t,x)\Big)\,dx, \tag{2.16}$$

defining

$$H^\pm(t,x) := \oint |z|^{-3}(-3\bar{z}\otimes\bar{z} + \mathrm{id})\rho_0^\pm(t,x+z)\,dz \in \mathbb{R}^{3\times 3}. \tag{2.17}$$

Note that $H(z) = -3(\bar{z}\otimes\bar{z}) + \mathrm{id}$ is bounded on $\mathbb{R}^3\setminus\{0\}$, homogeneous of degree zero, and satisfies $\int_{|z|=1} H(z)\,d\sigma(z) = 0$. Formally, $D^{[3]}$ is close to $\dddot{D}$, since $\dddot{D} = \frac{d}{dt}(D^{[2]} + \frac{2}{3c^3}M\dddot{D}) = \frac{d}{dt}D^{[2]} + \mathcal{O}(c^{-3}) = D^{[3]} + \mathcal{O}(c^{-3})$ by (2.15).

We introduce the "reduced radiating Vlasov-Poisson system" as

$$\left.\begin{aligned}
\partial_t f_0^\pm + p\cdot\nabla_x f_0^\pm \pm \Big(E_0 + \frac{2}{3c^3}\,D^{[3]}\Big)\cdot\nabla_p f_0^\pm &= 0,\\
E_0(t,x) &= -\int |z|^{-2}\bar{z}\,\rho_0(t,x+z)\,dz,\\
\rho_0 &= \int (f_0^+ - f_0^-)\,dp,\\
f_0^\pm(0,x,p) &= f_0^{\circ,\pm}(x,p),
\end{aligned}\right\} \tag{rrVP$_c$}$$

where $D^{[3]}$ is defined by (2.16) and (2.17). The next proposition addresses the existence and uniqueness of local classical solutions of (rrVP$_c$). Furthermore, it provides some useful estimates. For the initial data we assume

$$\begin{aligned}
&f_0^{\circ,\pm} \in C_0^\infty(\mathbb{R}^3\times\mathbb{R}^3), \quad f_0^{\circ,\pm} \geq 0,\\
&f_0^{\circ,\pm}(x,p) = 0 \quad \text{for} \quad |x| \geq r_0 \quad \text{or} \quad |p| \geq r_0, \quad \|f_0^{\circ,\pm}\|_{W^{4,\infty}} \leq S_0,
\end{aligned} \tag{2.18}$$

with some $r_0, S_0 > 0$ fixed.

Proposition 2.1 *If $f_0^{\circ,\pm}$ satisfies the above hypotheses, then there exists $0 < \tilde{T} \leq \infty$ such that the following holds for $c \geq 1$.*

(a) There is a unique classical solution $(f_0^\pm, E_0)$ of (rrVP$_c$) existing on a time interval $[0, T_c[$ with $\tilde{T} \leq T_c \leq \infty$. In addition, $\frac{d}{dt}D^{[2]} = D^{[3]}$, where $D^{[2]}$ is defined by (2.13).

(b) For every $T < \tilde{T}$ there is a constant $M_1(T) > 0$ such that for all $0 \leq t \leq T$,

$$f_0^\pm(t,x,p) = 0 \quad \textit{if} \quad |x| \geq M_1(T) \quad \textit{or} \quad |p| \geq M_1(T).$$

(c) Even $f_0^\pm \in C^\infty$ holds, and for every $T < \tilde{T}$ there is constant $M_2(T) > 0$ such that for all $0 \leq t \leq T$,

$$|\partial^\alpha f_0^\pm(t,x,p)| + |\partial_t^\beta E_0(t,x)| + |\partial_t^\gamma D^{[3]}(t)| \leq M_2(T)$$

for every $x \in \mathbb{R}^3$, $p \in \mathbb{R}^3$, $|\alpha| \leq 4$, $\beta \leq 2$, and $\gamma \leq 1$.

See [B06a] for the proof. Note that the constants $\tilde{T}$, $M_1(T)$, and $M_2(T)$ do only depend on the "basic" constants r_0 and S_0. In particular $\tilde{T}$, $M_1(T)$, and $M_2(T)$ are independent of c. Since the second moment $\int\int p^2(f_0^+ + f_0^-)\,dx\,dp$ cannot be bounded a priori by using energy conservation, it seems difficult to prove global existence of classical solutions to (rrVP$_c$). Recall that both methods yielding global existence of Vlasov-Poisson type systems essentially relied on such an a bound; see [P92, S91, LPe91].

By means of (rrVP$_c$) the approximations are improved by replacing at order zero the solution to (VP$_{\mathrm{plasma}}$) by the solution to (rrVP$_c$). Thus let (f_0, E_0) be the solution of (rrVP$_c$); note that this solutions depends on c, as opposed to the solution of (VP$_{\mathrm{plasma}}$). Next define B_1, $(f_2^{\pm}, E_2)$, and B_3 according to (2.7), (LVP$_{\mathrm{plasma}}$), and (2.9), respectively. We remark that solutions to (LVP$_{\mathrm{plasma}}$) do exist on $[0, T_c[$ (where T_c is from Proposition 2.1) and enjoy the usual properties, provided that both $f_0^{\circ,\pm}$ and $f_2^{\circ,\pm}$ satisfy the assumptions (2.18); see [B06a].

In the following section we are going to explain that

$$\begin{aligned} f^{\pm,\mathrm{R}} &= f_0^{\pm} + c^{-2} f_2^{\pm}, \\ E^{\mathrm{R}} &= E_0 + c^{-2} E_2 + (2/3) c^{-3} D^{[3]}, \\ B^{\mathrm{R}} &= c^{-1} B_1 + c^{-3} B_3, \end{aligned} \tag{2.19}$$

yields a higher order pointwise approximation of (RVM$_c$) than the Vlasov-Poisson or the Darwin system considered in [BK05]. We call (2.19) the radiation approximation. In the terminology of post-Newtonian approximations it is the 1.5 PN approximation.

Using the Vlasov equation and integration by parts the following formulas are found.

Proposition 2.2 *The fields E^{R} and B^{R} can be written as*

$$\begin{aligned} E^{\mathrm{R}}(t,x) &= -\int |z|^{-2} \bar z (\rho_0 + c^{-2}\rho_2)(t, x+z)\, dz \\ &\quad + \frac{1}{2} c^{-2} \int\int |z|^{-2} \Big\{ 3(\bar z \cdot p)^2 \bar z - p^2 \bar z \Big\} (f_0^+ - f_0^-)(t, x+z, p)\, dz\, dp \\ &\quad - c^{-2} \int\int |z|^{-1} \Big\{ \bar z \otimes \bar z + 1 \Big\} \\ &\qquad\qquad \Big(E_0(t, x+z) + (2/3) c^{-3} D^{[3]}(t) \Big) (f_0^+ + f_0^-)(t, x+z, p)\, dz\, dp \\ &\quad + \frac{2}{3} c^{-3} D^{[3]}(t), \end{aligned} \tag{2.20a}$$

and

$$\begin{aligned} B^{\mathrm{R}}(t,x) &= c^{-1} \int\int |z|^{-2} (\bar z \wedge p)(f^{+,\mathrm{R}} - f^{-,\mathrm{R}})(t, x+z, p)\, dz\, dp \\ &\quad - \frac{3}{2} c^{-3} \int\int |z|^{-2} (\bar z \cdot p)^2 (\bar z \wedge p)(f_0^+ - f_0^-)(t, x+z, p)\, dz\, dp \\ &\quad + \frac{1}{2} c^{-3} \int\int |z|^{-1} \Big\{ (\bar z \wedge p) \otimes \bar z + (\bar z \cdot p) \bar z \wedge (\cdots) \Big\} \\ &\qquad\qquad \Big(E_0(t, x+z) + (2/3) c^{-3} D^{[3]}(t) \Big) (f_0^+ + f_0^-)(t, x+z, p)\, dz\, dp \\ &\quad - c^{-3} \int \bar z \wedge \Big(H^+(t, x+z) j_0^-(t, x+z) - H^-(t, x+z) j_0^+(t, x+z) \Big)\, dz. \end{aligned} \tag{2.20b}$$

At the end of this section we want to mention that there is another variant of a damped Vlasov Poisson type system, considered in [KR01a, KR01b]. While for that system a global solution theory is at hand, the authors did not compare approximations based on their solutions to solutions of the full system.

2.3 1.5 PN comparison dynamics

For the 1.5 PN comparison dynamics, as outlined in Section 2.2, the initial data $(f_0^{\circ,\pm}, f_2^{\circ,\pm})$ are given. The field quantities are to be computed from the resulting densities $(f_0^{\pm}, f_2^{\pm})$ by means of (2.20a), (2.20b), $f^{\pm,\mathrm{R}} = f_0^{\pm} + c^{-2} f_2^{\pm}$, and (2.17). In comparison to that, for the Cauchy problem of (RVM$_c$) the initial fields E° and B° also have to be specified; see (2.2). Thus it is the question for which choice of initial data (2.19) yields a good comparison dynamics and for which not. Certainly it is possible to choose initial data for the fields such that the densities of the two dynamics evolve in a completely different way.

In Section 2.3.1 we therefore fix the initial fields for the Vlasov-Maxwell dynamics from the comparison dynamics; see formula (IC) below. From a mathematical viewpoint this procedure comes with the additional advantage that results on the existence and uniqueness of local-in-time solutions for both the full dynamics and the comparison dynamics have been established; see [GSt86, S86, B06a]. This way it is possible in Theorem 2.4 to obtain a pointwise approximation up to the order $\mathcal{O}(c^{-4})$. It should be mentioned that in [S86, D86, BK05] the fields are adapted in the same way up to the relevant orders.

There are two drawbacks of this method. In essence post-Newtonian expansion is an expansion of the relativistic velocity $\hat{p}$ and the retarded time $t - c^{-1}|x-y|$. It is clear that assuming localized sources the expansion of the retarded time is only a good approximation in the near zone of the source where $|x-y| \ll c$. This is reflected in the fact that the estimates for the fields in Theorem 2.4 and Theorem 2.6 are only local in the space variable x. Thus also the adapted initial fields are only reliable in the near zone, as is underlined by the fact that they have infinite energy. From a more physical point of view it is moreover questionable to use the Cauchy problem at all. For post-Newtonian expansions the main interest lies in localized systems which are isolated from the rest of the world and which have already evolved for a long time with small velocities. Therefore the Cauchy problem might not be the right formulation since it is not clear how to incorporate these properties into the initial fields.

In physics textbooks isolated systems are characterized by the absence of incoming radiation, i.e., there is no energy coming into the system from past null infinity; see [C04]. Here past null infinity is that region of spacetime which is reached along backward light cones. In case that the sources are given, fields which are free of incoming radiation are usually calculated by means of retarded potentials. In Section 2.3.2 we consider a family of solutions of (RVM$_c$), parametrized by c, passing through $f^{\circ,\pm}$ at time $t = 0$; in fact the initial data may also depend on c according to (IC), but this dependence is suppressed in our notation. Then in contrast to the Cauchy problem for (RVM$_c$) the electromagnetic fields are just computed by means of the retarded potentials; see ($_{\mathrm{ret}}$RVM$_c$). The underlying physical picture is that in the absence of incoming radiation every solution of (RVM$_c$) will approach a solution of ($_{\mathrm{ret}}$RVM$_c$), i.e., solutions of ($_{\mathrm{ret}}$RVM$_c$) form a kind of initial layer. Since it is our goal to model slow systems we assume that the momenta are bounded uniformly in $c \geq 1$ and time $t \in \mathbb{R}$. It is beyond the scope of the present survey paper to investigate the existence of solutions with mathematical rigor. Instead we simply introduce Assumption 2.5 below which summarizes all the properties needed. Note however that in [C04] the existence of such global solutions is proved for small $f^{\circ,\pm}$ and also uniqueness is discussed.

2.3.1 Vlasov-Maxwell dynamics with adapted initial data

To achieve the improved approximation accuracy we match the initial data of (RVM$_c$) by the data for the radiation system. For prescribed initial densities $f_0^{\circ,\pm}$ and $f_2^{\circ,\pm}$ we determine $(f_0^{\pm}, E_0)$,

B_1, $(f_2^\pm, E_2)$, and B_3 according to what has been explained above. Then we consider the Cauchy problem for (RVM$_c$), where the initial data are taken as

$$\left.\begin{array}{rcl} f^\pm(0,x,p) &=& f^{\circ,\pm}(x,p) = f_0^{\circ,\pm}(x,p) + c^{-2} f_2^{\circ,\pm}(x,p) + c^{-4} f_{c,\,\mathrm{free}}^{\circ,\pm}(x,p), \\ E(0,x) &=& E^\circ(x) = E_0(0,x) + c^{-2}E_2(0,x) + (2/3)c^{-3}D^{[3]}(0) + c^{-4}E^\circ_{c,\,\mathrm{free}}(x), \\ B(0,x) &=& B^\circ(x) = c^{-1}B_1(0,x) + c^{-3}B_3(0,x) + c^{-4}B^\circ_{c,\,\mathrm{free}}(x). \end{array}\right\} \qquad \text{(IC)}$$

In contrast to the contributions at orders 0 to 3, which are fixed by the values of the approximations, $(f_{c,\,\mathrm{free}}^{\circ,\pm}, E^\circ_{c,\,\mathrm{free}}, B^\circ_{c,\,\mathrm{free}})$ can be chosen freely. They are only subjected to the constraints $\nabla\cdot E^\circ_{c,\,\mathrm{free}} = 4\pi\int(f_{c,\,\mathrm{free}}^{\circ,+} - f_{c,\,\mathrm{free}}^{\circ,-})\,dp$ and $\nabla\cdot B^\circ_{c,\,\mathrm{free}} = 0$. Note that the constraint equations at the lower orders are satisfied by fiat. Furthermore, we shall assume that

$$\begin{gathered} f_{c,\,\mathrm{free}}^{\circ,\pm} \in C^\infty(\mathbb{R}^3\times\mathbb{R}^3), \quad E^\circ_{c,\,\mathrm{free}}, B^\circ_{c,\,\mathrm{free}} \in C_0^\infty(\mathbb{R}^3), \\ f_{c,\,\mathrm{free}}^{\circ,\pm} = 0 \quad \text{if} \quad |x| \ge r_0 \quad \text{or} \quad |p| \ge r_0, \\ \|f_{c,\,\mathrm{free}}^{\circ,\pm}\|_{L^\infty} \le S_0, \quad \|E^\circ_{c,\,\mathrm{free}}\|_{W^{1,\infty}} + \|B^\circ_{c,\,\mathrm{free}}\|_{W^{1,\infty}} \le S_0, \end{gathered} \qquad (2.21)$$

holds uniformly in c, with some $r_0, S_0 > 0$ fixed. Before we formulate the approximation result, let us recall that solutions of (RVM$_c$) with initial data (IC) exist at least on some time interval $[0,\hat{T}[$ which is independent of $c \ge 1$; see [S86, Thm. 1].

Proposition 2.3 *Assume that $f_0^{\circ,\pm}$ and $f_2^{\circ,\pm}$ satisfy (2.18). If $f^{\circ,\pm}$, E°, and B° are defined according to (IC), then there exists $0 < \hat{T} \le \infty$ (independent of c) such that for all $c \ge 1$ there is a unique smooth solution $(f^\pm, E, B)$ of (RVM$_c$) with initial data (IC) on the time interval $[0,\hat{T}[$. In addition, for every $T < \hat{T}$ there are constants $M_3(T), M_4(T) > 0$ such that for all $0 \le t \le T$,*

$$\begin{gathered} f^\pm(t,x,p) = 0 \quad \textit{if} \quad |x| \ge M_3(T) \quad \textit{or} \quad |p| \ge M_3(T), \\ |f^\pm(t,x,p)| + |E(t,x)| + |B(t,x)| \le M_4(T), \end{gathered}$$

for every $x, p \in \mathbb{R}^3$ and $c \ge 1$.

Actually in [S86, Thm. 1] E° and B° do not depend on c, but an inspection of the proof shows that the assertions remain valid for initial fields as defined by (IC).

The first main approximation result at 1.5 PN is as follows; see [B06b].

Theorem 2.4 *Assume that $f_0^{\circ,\pm}$ and $f_2^{\circ,\pm}$ satisfy (2.18). Then calculate $(f_0^\pm, E_0)$, B_1, $(f_2^\pm, E_2)$, and B_3 by means of (rrVP$_c$), (2.7), (LVP$_{plasma}$), and (2.9), respectively. Thereafter choose the initial data $(f^{\circ,\pm}, E^\circ, B^\circ)$ for (RVM$_c$) according to (IC) and (2.21). Let (f, E, B) denote the solution of (RVM$_c$) with initial data (IC) and let $(f^{\pm,\mathrm{R}}, E^{\mathrm{R}}, B^{\mathrm{R}})$ be defined by (2.19). Then for every $T < \min\{\tilde{T}, \hat{T}\}$ and $r > 0$ there are constants $M(T) > 0$ and $M(T,r) > 0$ such that for all $0 \le t \le T$,*

$$\begin{array}{rcll} |f^\pm(t,x,p) - f^{\pm,\mathrm{R}}(t,x,p)| &\le& M(T)c^{-4} & (x \in \mathbb{R}^3), \\ |E(t,x) - E^{\mathrm{R}}(t,x)| &\le& M(T,r)c^{-4} & (|x| \le r), \\ |B(t,x) - B^{\mathrm{R}}(t,x)| &\le& M(T,r)c^{-4} & (|x| \le r), \end{array}$$

for every $p \in \mathbb{R}^3$ and $c \ge 1$.

The constants $M(T)$ and $M(T,r)$ are independent of $c \geq 1$, but they do depend on the basic constants r_0 and S_0. Note that if (RVM$_c$) is compared to the Vlasov-Poisson system (VP$_{\text{plasma}}$) only, one obtains the Newtonian approximation

$$|f^{\pm}(t,x,p) - f_0^{\pm}(t,x,p)| + |E(t,x) - E_0(t,x)| + |B(t,x)| \leq M(T)c^{-1},$$

see [S86, Thm. 2B]. If it is compared to the Darwin system the estimates

$$\begin{aligned}
&|f^{\pm}(t,x,p) - f^{\pm,1\,\mathrm{PN}}(t,x,p)| + |B(t,x) - B^{1\,\mathrm{PN}}(t,x)| \leq M(T)c^{-3},\\
&|E(t,x) - E^{1\,\mathrm{PN}}(t,x)| \leq M(T,r)c^{-3},
\end{aligned}$$

are found; see [BK05, Thm. 1.1] and recall (2.8). At first glance it could seem it is a strong limitation to Theorem 2.4 that the time interval $[0,T] \subset [0, \min\{\tilde{T}, \hat{T}\}[$ might be very small, as $\tilde{T}$ and $\hat{T}$ might be very small. Regarding this point we remind the rescaling from Section 2.1 which allows to reformulate the result in an ε-dependent fashion on the time interval $[0, \varepsilon^{-3/2}T]$ as $\varepsilon \to 0$, i.e., for long times.

2.3.2 The retarded Vlasov-Maxwell dynamics

Following [C04] we introduce the retarded relativistic Vlasov-Maxwell system as

$$\left.\begin{aligned}
\partial_t f^{\pm} + \hat{p}\cdot\nabla_x f^{\pm} \pm (E + c^{-1}\hat{p}\times B)\cdot\nabla_p f^{\pm} = 0,&\\
E(t,x) = -\int \frac{dy}{|x-y|}\big(\nabla\rho + c^{-2}\partial_t j\big)(t - c^{-1}|x-y|, y),&\\
B(t,x) = c^{-1}\int \frac{dy}{|x-y|}\nabla\times j(t - c^{-1}|x-y|, y),&\\
\rho = \int (f^+ - f^-)\,dp, \qquad j = \int \hat{p}\,(f^+ - f^-)\,dp.&
\end{aligned}\right\} \qquad ({}_{\text{ret}}\mathrm{RVM}_c)$$

If we assume that $(f^{\pm}, E, B)$ is a smooth solution of (${}_{\text{ret}}$RVM$_c$), then ρ and j satisfy the continuity equation $\partial_t\rho + \nabla\cdot j = 0$. Therefore the retarded fields are a solution of the Maxwell equations, i.e., (f, E, B) also solves (RVM$_c$). Note that it is necessary to know the densities for all times $]-\infty, t]$ in order to compute the fields at time t. Hence there is no sense to the notation of a local solution of this system. As in the case of the Cauchy problem every solution of (${}_{\text{ret}}$RVM$_c$) satisfies $f^{\pm}(t,x,p) = f^{\pm}(0, X^{\pm}(0;t,x,p), P^{\pm}(0;t,x,p))$, where $s \mapsto (X^{\pm}(s;t,x,p), P^{\pm}(s;t,x,p))$ solves the characteristic system

$$\dot{X} = \hat{P}, \quad \dot{P} = \pm(E + c^{-1}\hat{P}\times B), \tag{2.23}$$

with initial data $X^{\pm}(t;t,x,p) = x$ and $P^{\pm}(t;t,x,p) = p$. Thus $0 \leq f^{\pm}(t,x,p) \leq \|f^{\pm}(0,\cdot,\cdot)\|_\infty$ holds.

As before let $f_0^{\circ,\pm}$, $f_2^{\circ,\pm}$, and $f_{c,\,\text{free}}^{\circ,\pm}$ satisfy (2.18) with some $r_0, S_0 > 0$. Put

$$f^{\circ,\pm} = f_0^{\circ,\pm} + c^{-2}f_2^{\circ,\pm} + c^{-4}f_{c,\,\text{free}}^{\circ,\pm}.$$

We make the following

Assumption 2.5 *(a) For every $c \geq 1$ there is a global solution $f^{\pm} \in C^4(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$ of (${}_{\text{ret}}$RVM$_c$) passing through $f^{\circ,\pm}$ at time $t = 0$, i.e., $f^{\pm}(0,x,p) = f^{\circ,\pm}(x,p)$ for $x, p \in \mathbb{R}^3$.*

(b) There is a constant $P_1 > 0$ such that $f^{\pm}(t,x,p) = 0$ for $|p| \geq P_1$ and $c \geq 1$. In particular, $f^{\pm}(t,x,p) = 0$ for $|x| \geq r_0 + P_1|t|$ by (2.23).

(c) For every $T > 0$, $R > 0$, and $P > 0$ there is a constant $M_5(T,R,P) > 0$ such that

$$|\partial_t^{\alpha+1} f^{\pm}(t,x,p)| + |\partial_t^{\alpha} \nabla_x f^{\pm}(t,x,p)| \leq M_5(T,R,P)$$

for $|t| \leq T$, $|x| \leq R$, $|p| \leq P$, and $|\alpha| \leq 3$, uniformly in $c \geq 1$.

Our second main approximation result at 1.5 PN is taken from [B06b].

Theorem 2.6 *Assume that $(f^{\pm}, E, B)$ is a family of solutions of $({}_{\mathrm{ret}}RVM_c)$ satisfying Assumption 2.5 with constants P_1 and $M_5(T,R,P)$. Take $\tilde{T} > 0$ from Proposition 2.1. Then for every $T < \tilde{T}$ and $r > 0$ there are constants $M(T) > 0$ and $M(T,r) > 0$ such that for all $0 \leq t \leq T$,*

$$\begin{aligned}
|f^{\pm}(t,x,p) - f^{\pm,\mathrm{R}}(t,x,p)| &\leq M(T)c^{-4} \quad (x \in \mathbb{R}^3), \\
|E(t,x) - E^{\mathrm{R}}(t,x)| &\leq M(T,r)c^{-4} \quad (|x| \leq r), \\
|B(t,x) - B^{\mathrm{R}}(t,x)| &\leq M(T,r)c^{-4} \quad (|x| \leq r),
\end{aligned}$$

for every $p \in \mathbb{R}^3$ and $c \geq 2P_1$. The constants $M(T)$ and $M(T,r)$ do only depend on r_0, S_0, P_1, and $M_5(\cdot,\cdot,\cdot)$. In particular they are independent of $c \geq 2P_1$.

2.4 Comparison to the particle model

We shall compare our results for the continuous density Vlasov models to the corresponding results for individual particle models. Usually the latter are denoted the Abraham-Lorentz system; see [KSp00, KSp01, Sp04]. Both systems are quite similar for the Hamiltonian approximations up to 1 PN. In addition, dissipative corrections at 1.5 PN make it necessary to have a closer look at the underlying phase space. The true comparison dynamics lives on a center-like manifold in an extended phase space. In addition, the dynamics on this manifold can be approximated by a modified Vlasov-Poisson system coupled to a second order equation, as in (SGPP$_\eta$). In [KR01b, Sect. 3] it is argued that the force term of the 1.5 PN approximation used in that paper can be obtained formally in the limit "number of particles $\to \infty$" from the individual particle models. In comparison to [KSp00, KSp01] the main difference to the present paper lies in the treatment of the initial data. For the individual particle model the initial data for the fields are supposed to be of "charged soliton" type. One can think of these fields as generated by charges forced to move freely for $-\infty < t \leq 0$ with their initial velocities. For the approximation this leads to an initial time slip t_0 which the charges need to "forget" their initial data. The initial data for the approximation are then fixed by matching the data of the full system at time t_0. Therefore the initial data for the approximation are given only implicitly, since first one has to compute a solution of the full system over $[0, t_0]$. Regarding the Cauchy problem for the Vlasov-Maxwell system, the matching is done the other way round. For a given initial density one computes the fields of the approximations and imposes their values at $t = 0$ as initial data for the fields of the full system. Hence these initial data are given more explicitly. In fact it is possible to calculate them by fixing only $f_0^{\circ,\pm}$ and $f_2^{\circ,\pm}$; see (2.20a) and (2.20b). Also Theorem 2.4 and Theorem 2.6 seem to be stronger than the results obtained for the particle model, as the passage from 1 PN to 1.5 PN did improve the approximation only in certain directions [KSp01, (3.21), (3.32)]. It seems reasonable to expect that a matching of the initial data at $t = 0$ as described above could also improve the earlier results on the Abraham-Lorentz system.

3 The Vlasov-Nordström system

Recently there has been some interest in a simplified but still relativistic model of gravitation in which Vlasov matter is coupled to a scalar theory of gravitation; the latter essentially goes back to Nordström [N13]. The metric tensor used in General Relativity is replaced by a scalar function and the Einstein equations are replaced by a wave equation. In [CRe03] the following system has been considered.

$$\left.\begin{aligned} S(f) - \Big(S(\varphi)p + \gamma c^2 \nabla_x \varphi\Big) \cdot \nabla_p f &= 4S(\varphi) f, \\ \mu &= \int \gamma f \, dp, \\ -\partial_t^2 \varphi + c^2 \Delta_x \varphi &= 4\pi\mu. \end{aligned}\right\} \qquad (\mathrm{VN}_c)$$

Once again $f = f(t,x,p)$ denotes the density to find a particle at time t at position x with momentum p, where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, and $p \in \mathbb{R}^3$. The scalar gravitational potential $\varphi = \varphi(t,x)$ is generated by the particles via the source μ, and c denotes the speed of light. In addition,

$$p^2 = |p|^2, \quad \gamma = (1 + c^{-2}p^2)^{-1/2}, \quad \hat{p} = \gamma p, \quad \text{and} \quad S = \partial_t + \hat{p} \cdot \nabla_x.$$

The initial data are

$$f(0,x,p) = f^\circ(x,p), \quad \varphi(0,x) = \varphi^0(x), \quad \partial_t \varphi(0,x) = \varphi^1(x).$$

For a physical interpretation and a derivation of this system see [CLe04]. In this formulation (VN_c) exhibits many similarities to the relativistic Vlasov-Maxwell system. Thus it is not surprising that many techniques developed for the Vlasov-Maxwell system also apply to the Vlasov-Nordström system. However, regarding basic questions like existence and uniqueness of solutions (VN_c) is by far better understood than (RVM_c). Global existence of classical solutions for unrestricted data is proved in [C06], also see [Le05] for the 2D case.

Again we are concerned with the non-relativistic limit $c \to \infty$ of (VN_c). Under certain circumstances it has been made rigorous in [R94] that the (gravitational) Vlasov-Poisson system is the non-relativistic limit of the full Einstein-Vlasov system. In [CLe04] it has been shown that as $c \to \infty$ also solutions of (VN_c) converge to a solution of a Vlasov-Poisson system with an error of the order $\mathcal{O}(c^{-1})$. These facts support the belief that (VN_c) may serve well as a model problem for Einstein-Vlasov.

To derive the higher order (post-Newtonian) approximations we follow the naive approach from Section 2.1 and first expand all relevant quantities in powers of c^{-1},

$$\begin{aligned} f &= f_0 + c^{-1} f_1 + c^{-2} f_2 + \dots, \\ \mu &= \mu_0 + c^{-1}\mu_1 + c^{-2}\mu_2 + \dots, \\ \varphi &= \varphi_0 + c^{-1}\varphi_1 + c^{-2}\varphi_2 + \dots. \end{aligned}$$

Comparing coefficients yields the equations $-\Delta_x \varphi_0 = 0$ and $-\Delta_x \varphi_1 = 0$. Thus we set $\varphi_0 = \varphi_1 = 0$. As mentioned above, at order zero the gravitational Vlasov-Poisson system

$$\left.\begin{aligned} \partial_t f_0 + p \cdot \nabla_x f_0 - \nabla_x \varphi_2 \cdot \nabla_p f_0 = 0, \quad \mu_0 = \int f_0 \, dp, \\ \varphi_2(t,x) = -\int |z|^{-1} \mu_0(t, x+z) \, dz, \quad f_0(0,x,p) = f^\circ(x,p), \end{aligned}\right\} \qquad (\mathrm{VP}_{\mathrm{grav}})$$

is obtained; see [CLe04] for a proof including the necessary error estimates. At the first order the linearized Vlasov-Poisson system

$$\left.\begin{aligned} \partial_t f_1 + p\cdot\nabla_x f_1 - \nabla_x\varphi_3\cdot\nabla_p f_0 - \nabla_x\varphi_2\cdot\nabla_p f_1 = 0, \\ \mu_1 = \int f_1\,dp, \quad \Delta_x\varphi_3 = 4\pi\mu_1 + \partial_t^2\varphi_1, \end{aligned}\right\}$$

appears. Hence if we suppose that $f_1(0,x,p)=0$, then we can set $f_1=0$ and $\varphi_3=0$, which also yields $\mu_1=0$. For the second order one derives an inhomogeneous Vlasov equation coupled to a Poisson equation,

$$\left.\begin{aligned} &\partial_t f_2 + p\cdot\nabla_x f_2 - \nabla_x\varphi_2\cdot\nabla_p f_2 - \nabla_x\varphi_4\cdot\nabla_p f_0 \\ &\quad = 4f_0\tilde S(\varphi_2) + (p^2/2)\,p\cdot\nabla_x f_0 + \Big(\tilde S(\varphi_2)p - (p^2/2)\nabla_x\varphi_2\Big)\cdot\nabla_p f_0, \\ &\mu_2 = \int (f_2 - (p^2/2)f_0)\,dp, \quad \Delta_x\varphi_4 = 4\pi\mu_2 + \partial_t^2\varphi_2, \end{aligned}\right\} \qquad (\mathrm{LVP}_{\mathrm{grav}})$$

where $\tilde S = \partial_t + p\cdot\nabla_x$. For ($\mathrm{LVP}_{\mathrm{grav}}$) we choose homogeneous initial data $f_2(0,x,p)=0$.

We now follows the route of adapted initial data as in Section 2.3.1. Similarly to Section 2.3.2 it would also be possible to approximate solutions to the retarded Vlasov-Nordström system, the latter being defined analogously to (${}_{\mathrm{ret}}\mathrm{RVM}_c$). For the retarded Vlasov-Nordström system the leading order radiation contribution can be determined explicitly. It is due to monopole radiation and does not vanish for spherically symmetric solutions [BKReR06, ShT93].

For the adapted initial data let f° be given. Then we calculate (f_0,φ_2) and (f_2,φ_4) according to ($\mathrm{VP}_{\mathrm{grav}}$) and ($\mathrm{LVP}_{\mathrm{grav}}$). Now we consider ($\mathrm{VN}_c$) with initial data

$$\begin{aligned} f(0,x,p) &= f^\circ(x,p), \\ \varphi^0(x) = \varphi(0,x) &= c^{-2}\varphi_2(0,x) + c^{-4}\varphi_4(0,x) + c^{-6}\varphi^0_{\mathrm{free}}(x), \\ \varphi^1(x) = \partial_t\varphi(0,x) &= c^{-2}\partial_t\varphi_2(0,x) + c^{-4}\partial_t\varphi_4(0,x) + c^{-6}\varphi^1_{\mathrm{free}}(x), \end{aligned} \qquad (3.24)$$

where $\varphi^0_{\mathrm{free}}$, $\varphi^1_{\mathrm{free}} \in C_0^\infty(\mathbb{R}^3)$. The following approximation theorem from [B05] shows that the Darwin approximation

$$f^{\mathrm{D}} = f_0 + c^{-2}f_2, \quad \varphi^{\mathrm{D}} = c^{-2}\varphi_2 + c^{-4}\varphi_4, \qquad (3.25)$$

yields a higher order pointwise approximation of the Vlasov-Nordström (VN_c). Before we formulate this result let us recall that solutions of (VN_c) with initial data (3.24) do exist at least on some time interval $[0,T]$ which is independent of $c\ge 1$; see [CLe04, Thm. 3].

Theorem 3.1 *Assume that $f^\circ \in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ is nonnegative and compactly supported. From f° calculate (f_0,φ_2) and (f_2,φ_4). Thereafter introduce the initial data for (VN_c) by (3.24). Let (f,φ) denote the solution of (VN_c) with initial data (3.24) and let $(f^{\mathrm{D}},\varphi^{\mathrm{D}})$ be defined by (3.25). Then there is a constant $M(T)>0$, and for every $r>0$ one can select $M(T,r)>0$, such that for all $0\le t\le T$,*

$$\begin{aligned} |f(t,x,p) - f^{\mathrm{D}}(t,x,p)| &\le M(T)c^{-4} \quad (x\in\mathbb{R}^3), \\ |\varphi(t,x) - \varphi^{\mathrm{D}}(t,x)| &\le M(T,r)c^{-4} \quad (|x|\le r), \\ |\partial_t\varphi(t,x) - \partial_t\varphi^{\mathrm{D}}(t,x)| &\le M(T)c^{-4} \quad (x\in\mathbb{R}^3), \\ |\nabla_x\varphi(t,x) - \nabla_x\varphi^{\mathrm{D}}(t,x)| &\le M(T,r)c^{-6} \quad (|x|\le r), \end{aligned}$$

for every $p\in\mathbb{R}^3$ and $c\ge 1$. The constants $M(T)$ and $M(T,r)$ are independent of $c\ge 1$.

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References

[AU86] K. Asano and S. Ukai. On the Vlasov-Poisson limit of the Vlasov-Maxwell equation. In T. Nishida, M. Mimura M., and H. Fujii, editors, *Patterns and Waves*, Stud. Math. Appl. Volume 18, pages 369-383, Amsterdam, 1986, North-Holland.

[B05] S. Bauer. Post-Newtonian approximation of the Vlasov-Nordström system. *Comm. Partial Differential Equations*, 30:957-985, 2005.

[B06a] S. Bauer. The reduced Vlasov-Poisson system with radiation damping. In preparation, 2006.

[B06b] S. Bauer. Post-Newtonian dynamics at order 1.5 in the Vlasov-Maxwell system. Preprint `arXiv math-ph/0602031`, 2006.

[BK05] S. Bauer and M. Kunze. The Darwin approximation of the relativistic Vlasov-Maxwell system. *Ann. Henri Poincaré*, 6:283-308, 2005.

[BKReR06] S. Bauer, M. Kunze, G. Rein, and A.D. Rendall. Multipole radiation in a collisionless gas coupled to electromagnetism or scalar gravitation. Preprint `arXiv math-ph/0508057`. To appear in *Comm. Math. Phys.*, 2006.

[BeFLaSo03] S. Benachour, F. Filbet, Ph. Laurençot, and E. Sonnendrücker. Global existence for the Vlasov-Darwin system in $\mathbb{R}^3$ for small initial data. *Math. Methods Appl. Sci.*, 26:297-319, 2003.

[Br04] C. Bradaschia, editor, Proceedings of the 5th Edoardo Amaldi Conference on Gravitational Waves. In *Classical Quantum Gravity*, 21:S377-S1263, 2004.

[C03] S. Calogero. Spherically symmetric steady states of galactic dynamics in scalar gravity. *Classical Quantum Gravity*, 20:1729-1742, 2003.

[C04] S. Calogero. Global small solutions of the Vlasov-Maxwell system in the absence of incoming radiation. *Indiana Univ. Math. J.*, 53:1331-1364, 2004.

[C06] S. Calogero. Global classical solutions to the 3D Nordström-Vlasov system. Preprint `arXiv math-ph/0507030`, 2006.

[CLe04] S. Calogero and H. Lee. The non-relativistic limit of the Nordström-Vlasov system. *Comm. Math. Sci.*, 2:19-34, 2004.

[CRe03] S. Calogero and G. Rein. On classical solutions of the Nordström-Vlasov system. *Comm. Partial Differential Equations*, 28:1-29, 2003.

[D86] P. Degond. Local existence of solutions of the Vlasov-Maxwell equations and convergence to the Vlasov-Poisson equation for infinite light velocity. *Math. Methods Appl. Sci.*, 8:533-558, 1986.

[DRa92] P. Degond and P. Raviart. An analysis of the Darwin model of approximation to Maxwell's equations. *Forum Math.*, 4:13-44, 1992.

[GSt86] R.T. Glassey and W. Strauss. Singularity formation in a collisionless plasma could occur only at high velocities. *Arch. Rat. Mech. Anal.*, 92:59-90, 1986.

[J99] J.D. Jackson. *Classical Electrodynamics.* 3rd edition, Wiley, New York, 1999.

[KR01a] M. Kunze and A.D. Rendall. The Vlasov-Poisson system with radiation damping. *Ann. Henri Poincaré*, 2:857-886, 2001.

[KR01b] M. Kunze and A.D. Rendall. Simplified models of electromagnetic and gravitational radiation damping *Classical Quantum Gravity*, 18:3573-3587, 2001.

[KSp00] M. Kunze and H. Spohn. Slow motion of charges interacting through the Maxwell field. *Comm. Math. Phys.*, 212:437-467, 2000.

[KSp01] M. Kunze and H. Spohn. Post-Coulombian dynamics at order c^{-3}. *J. Nonlinear Science*, 11:321-396, 2001.

[Le04] H. Lee. The classical limit of the relativistic Vlasov-Maxwell system in two space dimensions. *Math. Methods Appl. Sci.*, 27:249-287, 2004.

[Le05] H. Lee. Global existence of solutions of the Nordström-Vlasov system in two space dimensions. *Comm. Partial Differential Equations*, 30:663-687, 2005. Erratum: *Comm. Partial Differential Equations*, 30:1261-1262, 2005.

[LPe91] P.-L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.*, 105:415-430, 1991.

[N13] G. Nordström. Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips. *Ann. Phys. Lpz.*, 42:533, 1913.

[PaSpT06] G. Panati, H. Spohn, and S. Teufel. Motion of electrons in adiabatically perturbed periodic structures, in this volume.

[P92] K. Pfaffelmoser. Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Differential Equations*, 95:281-303, 1992.

[R92] A.D. Rendall. On the definition of post-Newtonian approximation. *Proc. R. Soc. Lond. A*, 438:341-360, 1992.

[R94] A.D. Rendall. The Newtonian limit for asymptotically flat solutions of the Vlasov-Einstein system. *Comm. Math. Phys.*, 163:89-112, 1994.

[S86] J. Schaeffer. The classical limit of the relativistic Vlasov-Maxwell system. *Comm. Math. Phys.*, 104:403-421, 1986.

[S91] J. Schaeffer. Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Commun. Partial Differ. Equations*, 16:1313-1335, 1991.

[ShT93] S.L. Shapiro and S.A. Teukolsky. Scalar gravitation: A laboratory for numerical relativity. *Phys. Rev. D*, 47:1529-1540, 1993.

[Sp04] H. Spohn. *Dynamics of Charged Particles and Their Radiation Field.* Cambridge University Press, Cambridge, 2004.