Rotating, stationary, axially symmetric spacetimes with collisionless matter

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Abstract

The existence of stationary solutions to the Einstein-Vlasov system which are axially symmetric and have non-zero total angular momentum is shown. This provides mathematical models for rotating, general relativistic and asymptotically flat non-vacuum spacetimes. If angular momentum is allowed to be non-zero, the system of equations to solve contains one semilinear elliptic equation which is singular on the axis of rotation. This can be handled very efficiently by recasting the equation as one for an axisymmetric unknown on $\mathbb{R}^5$.

1 Introduction

The geometric features of general relativistic and asymptotically flat spacetimes strongly depend on whether the spacetime has non-trivial total angular momentum or not. A well known point in case is the Kerr family, a two parameter family of stationary vacuum spacetimes which contain a black hole. The two parameters are the ADM mass $\mathcal{M} \geq 0$ and the total angular momentum $\mathcal{L} \geq 0$. If $\mathcal{L} = 0$ one obtains the Schwarzschild spacetime, i.e., a static, spherically symmetric black hole of mass $\mathcal{M}$. Compared to this the case $\mathcal{L} > 0$ exhibits a vastly more complicated geometry, and we refer to [15] for details. However, most astrophysical objects are not exactly spherically symmetric, and many rotate about some axis and have non-trivial total angular momentum. Hence the mathematical difficulties entailed by giving up spherical symmetry and allowing for non-zero angular momentum have to be overcome in order to get closer to physically meaningful models. There are at the moment only two papers where the existence of rotating equilibrium configurations of self-gravitating matter distributions is shown in the framework of General Relativity: these are [10] and [2], where matter is modeled as an ideal fluid and as an elastic body, respectively.

In the present paper we consider matter described as a collisionless gas. In astrophysics, this model is used to analyze galaxies or globular clusters where the stars play the role of the gas particles and collisions among these are sufficiently rare to be neglected. The particles only interact by the gravitational field which the ensemble creates collectively, and the general relativistic description of such an ensemble is given by the Einstein-Vlasov system. The existence of spherically symmetric steady states to this system has for example been shown in [14]. In [4] the present authors proved the existence of static, axially symmetric solutions which are no longer spherically symmetric, but which still have zero angular momentum. In the present paper we also remove the latter restriction, a task which, in view of what was said above, is not trivial.

We shall formulate the Einstein-Vlasov system in standard axial coordinates $t \in \mathbb{R},\ \rho \in [0, \infty[,\ z \in \mathbb{R},\ \varphi \in [0, 2\pi]$. Following [5], we write the metric in the form

$$ds^2 = -c^2 e^{2\nu/c^2} dt^2 + e^{2\mu} d\rho^2 + e^{2\mu} dz^2 + \rho^2 B^2 e^{-2\nu/c^2} (d\varphi - \omega dt)^2 \qquad (1.1)$$

for functions ν, B, μ, ω depending on ρ and z. The reason for keeping the speed of light c as a parameter in the metric will become clear shortly. The metric is to be asymptotically flat in the sense that the boundary values

$$\lim_{|(\rho,z)|\to\infty} (|\nu| + |\mu| + |\omega| + |B-1|)(\rho, z) = 0 \tag{1.2}$$

are attained at spatial infinity with certain rates which are specified later. In addition we require that the metric is locally flat at the axis of symmetry:

$$\nu(0, z)/c^2 + \mu(0, z) = \ln B(0, z), \;\; z \in \mathbb{R}. \tag{1.3}$$

The solutions obtained in [4] are static and have zero total angular momentum. In terms of the metric above this means that $\omega = 0$. The quantity ω is the angular velocity with respect to infinity of the invariantly defined zero angular momentum observers with worldlines perpendicular to the $\{t = \text{const.}\}$ hypersurfaces; see [5]. For a rotating configuration, ω must not vanish identically. As in [4] the solutions to the Einstein-Vlasov system are obtained by perturbing off a spherically symmetric Newtonian steady state using two parameters, $\gamma = 1/c^2$ to turn on general relativity, and λ to turn on angular momentum. In order to apply the implicit function theorem and to make certain solution operators well-defined, it becomes essential to handle the linearized ω-equation

$$\partial_{\rho\rho}\omega + \partial_{zz}\omega + \frac{3}{\rho}\,\partial_\rho\omega = q \tag{1.4}$$

for a suitable class of right-hand sides q. The difficulty with this fairly innocent looking elliptic equation is that the coefficient $3/\rho$ blows up on the axis of symmetry, where the full solution must remain smooth. It is technically *very* demanding to make this equation as it stands fit into the general framework of our approach, cf. the corresponding remark in the appendix. However, this equation is nothing but the Poisson equation on $\mathbb{R}^5$ where both ω and q are axially symmetric, i.e., they depend on $\rho = |(x_1, x_2, x_3, x_4)|$ and $z = x_5$. This observation turns out to make the inclusion of non-trivial total angular momentum quite neat. We are not aware of a physical background for this fact, nor are we aware that this observation has previously been exploited in the area of mathematical relativity. It turns out that an analogous observation applies to the linearized equation for B which can be turned into the Poisson equation on $\mathbb{R}^4$. This simplifies the proof and improves the result also for vanishing angular momentum, when compared with [4]. One should realize that the generalization to non-trivial angular momentum means that one moves to a geometrically more complex spacetime. To appreciate the fact that the resulting complications are of a genuinely relativistic, geometric nature one should notice that in [13], where an analogous strategy was used to obtain axially symmetric steady states in the Newtonian case, i.e., for the Vlasov-Poisson system, one and the same proof gives static solutions with zero total angular momentum and stationary ones which rotate, depending only on which particular ansatz function is chosen.

Let us now give a formulation of the Einstein-Vlasov system. In a kinetic model like the Vlasov equation the particle ensemble is described by its distribution function $f \geq 0$ which is defined on the tangent bundle TM of the spacetime manifold M. Let $g_{\alpha\beta}$ denote the Lorentz metric on the spacetime and let p^α denote the canonical momentum coordinates which correspond to the chosen coordinates x^α on M. The Einstein field equations

$$G_{\alpha\beta} = 8\pi c^{-4} T_{\alpha\beta} \tag{1.5}$$

are then coupled to the Vlasov equation

$$p^\alpha \partial_{x^\alpha} f - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^\alpha} f = 0 \tag{1.6}$$

via the definition of the energy momentum tensor

$$T_{\alpha\beta} = c^{-1} |g|^{1/2} \int p_\alpha p_\beta f \, \frac{dp^0 dp^1 dp^2 dp^3}{m}. \tag{1.7}$$

Here $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols induced by the metric, $|g|$ denotes the modulus of its determinant, and $m > 0$ is the rest mass of the particle with phase space coordinates (x^α, p^β). The characteristic system of the Vlasov equation (1.6) are the geodesic equations written as a first order system on TM. For physical reasons we must require that all particles move forward in time, i.e., p^α is a timelike, future pointing vector on the support of f. Moreover, we make the standard assumption that all particles have the same rest mass which we normalize to unity. The distribution function f is then supported on the mass shell

$$PM = \{g_{\alpha\beta} p^\alpha p^\beta = -c^2 m^2 = -c^2 \text{ and } p^\alpha \text{ is future pointing}\} \subset TM. \tag{1.8}$$

It is now important to realize that due to the presence of ω in the metric, i.e., due to the fact that we want to allow for non-trivial angular momentum of the spacetime, the mass shell condition can in general *not* be used to express p^0 by the remaining variables on TM. It turns out that this can be done if and only if the Killing vector $\partial/\partial t$ which corresponds to the time translation symmetry is timelike everywhere, i.e.,

$$-g(\partial/\partial t, \partial/\partial t) = c^2 e^{2\nu/c^2} - \rho^2 B^2 \omega^2 e^{-2\nu/c^2} > 0. \tag{1.9}$$

For the solutions which we construct we do a priori not know whether this property holds or not. We can if we wish make sure that it does hold so that there is no ergosphere. The question whether among the solutions we construct there are solutions that do have an ergosphere is open. The vector field $\partial/\partial t + \omega\partial/\partial\varphi$ is always timelike and can therefore be used to fix the time orientation of the spacetime. For the solutions we construct,

$$-g(\partial/\partial t + \omega\partial/\partial\varphi, p^\alpha) = c^2 e^{2\nu/c^2} p^0 > 0 \tag{1.10}$$

on the support of f so that all particle trajectories have future pointing tangent vectors as desired. We refer to [3] for more background on the Einstein-Vlasov system and state our main result.

Theorem 1.1 *There exist stationary solutions of the Einstein-Vlasov system (1.5), (1.6), (1.7) with $c = 1$ such that the metric is of the form (1.1) and satisfies the boundary conditions (1.2), (1.3), and where the total angular momentum is non-zero.*

For the proof of this result the following observation is important. The symmetries of the metric imply that the quantities

$$\begin{aligned} E &:= -g(\partial/\partial t, p^\alpha) = c^2 e^{2\nu/c^2} p^0 + \rho^2 B^2 \omega e^{-2\nu/c^2}(p^3 - \omega p^0), \\ L &:= g(\partial/\partial\varphi, p^\alpha) = \rho^2 B^2 e^{-2\nu/c^2}(p^3 - \omega p^0), \end{aligned}$$

are constant along geodesics; notice that

$$E = c^2 e^{2\nu/c^2} p^0 + \omega L.$$

Here E can be thought of as a local or particle energy and L is the angular momentum of a particle with respect to the axis of symmetry. The requirement that p^α be future pointing implies that $E > 0$ on the support of f, provided (1.9) holds, i.e., provided there is no ergosphere. Up to regularity issues a distribution function f satisfies the Vlasov equation if and only if it is constant along geodesics. Hence we make the ansatz

$$f = \Phi(E, L)\delta(m - 1),$$

and the Vlasov equation (1.6) holds. The δ distribution on the right hand side is to restrict f to particles with rest mass equal to unity; notice that the rest mass m is conserved along geodesics as well. If we insert this ansatz into the definition (1.7) of the energy momentum tensor the latter becomes a functional $T_{\alpha\beta} = T_{\alpha\beta}(\nu, B, \mu, \omega)$ of the yet unknown metric functions ν, B, μ, ω, and we are left with the problem of solving the field Einstein equations (1.5) with this right hand side.

We will obtain the solutions by perturbing off spherically symmetric steady states of the Vlasov-Poisson system via the implicit function theorem; the latter system arises as the Newtonian limit of the Einstein-Vlasov system. We will specify conditions on the ansatz function Φ above such that a two parameter family of axially symmetric solutions of the Einstein-Vlasov system passes through the corresponding spherically symmetric, Newtonian steady state. The parameter $\gamma = 1/c^2$ turns on general relativity and a second parameter λ turns on the dependence on L and hence axial symmetry; L is not invariant under arbitrary rotations about the origin, so if f depends on L the solution is not spherically symmetric. Moreover, suitable assumptions on the ansatz function Φ will force ω to be non-trivial so that the solution rotates about the axis $\rho = 0$ and has non-zero total angular momentum. One should be careful to notice here that there are always particles with non-zero angular momentum L provided f is non-trivial and smooth on the mass shell, but in general this does not imply that the total angular momentum of the whole spacetime is non-trivial. The scaling symmetry of the Einstein-Vlasov system can then be used to obtain the desired solutions for the physically correct value of c, and not only for large c.

The idea of employing the implicit function theorem to obtain new equilibrium configurations of self-gravitating matter distributions from known ones can be traced back to L. Lichtenstein, who investigated the existence of non-relativistic, axially symmetric, stationary, self-gravitating fluid balls [11, 12]. His arguments were put into a rigorous and modern framework in [9] and extended to the general relativistic set-up in [10]. As mentioned above, rotating elastic bodies were considered in [2]. Our approach significantly differs from [2, 10] not only in the matter model, but also in that we use the explicit form of the metric stated in (1.1), together with a reduced version of the Einstein field equations.

The outline of the paper is as follows. The detailed formulation of our main result and the set-up for the application of the implicit function theorem are stated in the next section. In Section 3 we then give a detailed outline of its proof. The proof consists of several steps and some of them are more or less identical to the corresponding steps in [4] and need not be repeated. However, the logical structure of the present proof will be given in full detail. In Section 4 we collect some properties of the matter terms which will be needed throughout. Section 5 contains information on certain Newton potentials which is then used to show that the operator to which we apply the implicit function theorem is well defined. Section 6 explains how a solution of the reduced field equations leads to a solution of all the field equations. In an appendix we collect a few general results on the regularity of axially symmetric functions, and we comment on solving the equation (1.4) without resorting to the device of moving it into a higher dimension.

2 Basic set-up and the precise result

We introduce the parameter $\gamma = 1/c^2 \in [0,\infty[$. In order to handle the mass shell condition effectively it is useful to introduce new momentum variables

$$v^0 = e^{\gamma\nu}p^0,\ v^1 = e^{\mu}p^1,\ v^2 = e^{\mu}p^2,\ v^3 = \rho\, Be^{-\gamma\nu}(p^3 - \omega p^0). \tag{2.1}$$

This turns the mass shell condition for general m into

$$-c^2m^2 = -c^2(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2 \ \text{ or } \ (v^0)^2 = m^2 + \gamma|v|^2$$

where $v = (v^1, v^2, v^3) \in \mathbb{R}^3$ and $|v|$ is the Euclidean norm on $\mathbb{R}^3$. We eliminate v^0 by choosing the positive root which makes sure that (1.10) holds, i.e., all particles move forward in time. With $m = 1$ we find that

$$E = c^2e^{\gamma\nu}\sqrt{1+\gamma|v|^2} + \omega L,\ L = \rho Be^{-\gamma\nu}v^3.$$

In particular

$$E > c^2e^{\gamma\nu}|v^3|/c - |\rho\omega Be^{-\gamma\nu}||v^3| \geq 0$$

provided the no-ergosphere condition (1.9) holds. The formula (1.7) for the energy-momentum tensor turns into

$$T_{\alpha\beta} = \int_{\mathbb{R}^3} p_\alpha p_\beta \Phi(E,L)\frac{d^3v}{\sqrt{1+\gamma|v|^2}}, \tag{2.2}$$

where p_α has to be expressed via (2.1). Here we first express the four dimensional integral in (1.7) in terms of $(v^0,\dots,v^3)$, replace the integration variable v^0 by m, and then use the fact that f is δ distributed with respect to m. In what follows we view f as a function on the mass shell.

In order to turn on or off angular momentum we introduce a second parameter $\lambda \in \mathbb{R}$, and in order to obtain the correct Newtonian limit for $\gamma = 0$ we adjust the ansatz for f as follows:

$$f = \phi\,(E - 1/\gamma)\,\psi(\lambda, L). \tag{2.3}$$

The important point here is that

$$E - 1/\gamma = \frac{e^{\gamma\nu}\sqrt{1+\gamma|v|^2} - 1}{\gamma} + \omega L \to \frac{1}{2}|v|^2 + \nu + \omega L \text{ as } \gamma \to 0; \tag{2.4}$$

see (ϕ2) below. For $\gamma = 0$ this limit is to replace the argument of ϕ in (2.3). We specify the conditions on the functions ϕ and ψ.

Conditions on ϕ and ψ.

(ϕ1) $\phi \in C^1(\mathbb{R})$ and there exists $E_0 > 0$ such that $\phi(\eta) = 0$ for $\eta \geq E_0$ and $\phi(\eta) > 0$ for $\eta < E_0$.

(ϕ2) The ansatz $f(x,v) = \phi\left(\frac{1}{2}|v|^2 + U(x)\right)$, $x, v \in \mathbb{R}^3$, leads to a compactly supported steady state f_N of the Vlasov-Poisson system, i.e., there exists a solution $U = U_N \in C^2(\mathbb{R}^3)$ of the semilinear Poisson equation

$$\Delta U = 4\pi\rho_N = 4\pi \int \phi\left(\frac{1}{2}|v|^2 + U\right) dv,\ U(0) = 0,$$

$U_N(x) = U_N(|x|)$ is spherically symmetric, and the support of $\rho_N \in C_c^2(\mathbb{R}^3)$ is the closed ball $\overline{B}_{R_N}(0)$ where $U_N(R_N) = E_0$ and $U_N(r) < E_0$ for $0 \leq r < R_N$, $U_N(r) > E_0$ for $r > R_N$.

(ϕ3) We have

$$6 + 4\pi r^2 a_N(r) > 0, \quad r \in [0, \infty[,$$

where

$$a_N(r) := \int_{\mathbb{R}^3} \phi'\left(\frac{1}{2}\,|v|^2 + U_N(r)\right) dv.$$

(ψ) $\psi \in C_c^\infty(\mathbb{R}^2)$ is compactly supported, $\psi \geq 0$, $\partial_L \psi(\lambda, 0) = 0$ for $\lambda \in \mathbb{R}$, and $\psi(0, L) = 1$ on an open neighborhood of the set

$$\{L = L(x,v) \mid (x,v) \in \operatorname{supp} f_N\}.$$

For the Newtonian steady state

$$\lim_{|x|\to\infty} U_N(x) = U_N(\infty) > E_0.$$

The normalization condition $U_N(0) = 0$ instead of $U_N(\infty) = 0$ is unconventional from the physics point of view, but it has technical advantages. Examples for ansatz functions ϕ which satisfy (ϕ1) and (ϕ2) are found in [6, 14], the most well-known ones being the polytropes

$$\phi(E) := (E_0 - E)_+^k$$

for $1 < k < 7/2$; here $E_0 > 0$ and $(\cdot)_+$ denotes the positive part. In [4, Sect. 7] it is shown that (ϕ3) holds for a subclass of the polytropes.

We recall that the metric (1.1) was written in terms of the axial coordinates $\rho \in [0, \infty[$, $z \in \mathbb{R}$, $\varphi \in [0, 2\pi]$. In what follows we shall also use the corresponding Cartesian coordinates

$$x = (\rho \cos \varphi, \rho \sin \varphi, z) \in \mathbb{R}^3,$$

and by abuse of notation we write $\nu(\rho, z) = \nu(x)$ etc. It should be noted that tensor indices always refer to the spacetime coordinates t, ρ, z, φ. In Section 7 we collect the relevant information on the relation between regularity properties of axially symmetric functions expressed in these different variables. We can now give a detailed formulation of our main result.

Theorem 2.1 *There exist $\delta > 0$ and a two parameter family*

$$(\nu_{\gamma,\lambda}, B_{\gamma,\lambda}, \mu_{\gamma,\lambda}, \omega_{\gamma,\lambda})_{(\gamma,\lambda)\in[0,\delta[\times]-\delta,\delta[} \subset C^2(\mathbb{R}^3)^4$$

with the following properties:

(i) $(\nu_{0,0}, B_{0,0}, \mu_{0,0}, \omega_{0,0}) = (U_N, 1, 0, 0)$ *where U_N is the potential of the Newtonian steady state specified in (ϕ2).*

(ii) If for $\gamma > 0$ a distribution function is defined by Eqn. (2.3) and a Lorentz metric by (1.1) with $c = 1/\sqrt{\gamma}$ then we obtain a solution of the Einstein-Vlasov system (1.5), (1.6), (1.7) which satisfies the boundary condition (1.3) and is asymptotically flat. For $\lambda \neq 0$ this solution is not spherically symmetric, and for appropriate choices of ψ its total angular momentum is non-zero.

(iii) If for $\gamma = 0$ a distribution function is defined by Eqn. (2.3), observing (2.4), this yields a steady state of the Vlasov-Poisson system with gravitational potential $\nu_{0,\lambda}$ which is not spherically symmetric for $\lambda \neq 0$.

(iv) In all cases the matter distribution is compactly supported both in phase space and in space.

Remarks.

(a) The smallness restriction to λ implies that the solutions obtained are close to being spherically symmetric, and that their total angular momentum is small.

(b) The smallness restriction to $\gamma = 1/c^2$ is undesired because c is, in a given set of units, a definite number. However, if (f, ν, B, μ, ω) is a solution for some choice of $c \in]0, \infty[$ then the rescaling

$$\begin{aligned}
\tilde{f}(\rho, z, p^1, p^2, p^3) &= c^{-3} f(c\rho, cz, cp^1, cp^2, p^3), \\
\tilde{\nu}(\rho, z) &= c^{-2}\nu(c\rho, cz), \\
\tilde{B}(\rho, z) &= B(c\rho, cz), \\
\tilde{\mu}(\rho, z) &= \mu(c\rho, cz), \\
\tilde{\omega}(\rho, z) &= \omega(c\rho, cz),
\end{aligned}$$

yields a solution of the Einstein-Vlasov system with $c = 1$; notice that this rescaling turns (1.9) into the corresponding condition with $c = 1$.

(c) The metric functions ν and μ do not satisfy the boundary conditions (1.2), but

$$\lim_{|(\rho,z)|\to\infty} \nu(\rho, z) = \nu_\infty, \quad \lim_{|(\rho,z)|\to\infty} \mu(\rho, z) = -\nu_\infty/c^2, \tag{2.5}$$

see Proposition 3.2. If we now abuse notation and redefine $\nu = \nu - \nu_\infty$, $\mu = \mu + \nu_\infty/c^2$ and $\omega = \omega e^{-\nu_\infty/c^2}$ then the original condition (1.2) is restored and the metric (1.1) takes the form

$$\begin{aligned}
ds^2 = &-c^2 e^{2\nu/c^2} c_1^2 dt^2 \\
&+ c_2^2 \left(e^{2\mu} d\rho^2 + e^{2\mu} dz^2 + \rho^2 B^2 e^{-2\nu/c^2} (d\varphi - \omega c_1 dt)^2 \right)
\end{aligned}$$

with constants $c_1, c_2 > 0$ which simply amounts to a choice of different units of time and space. By general covariance of the Einstein-Vlasov system (1.5), (1.6), (1.7) the equations still hold.

(d) In the course of the proof of the theorem additional regularity properties and specific rates at which the boundary values at infinity are approached will emerge.

We will transform the problem of finding the desired solutions into the problem of finding zeros of a suitably defined operator. The Newtonian steady state specified in (ϕ2) will be a zero of this operator for $\gamma = \lambda = 0$, and the implicit function theorem will yield our result.

The Einstein field equations are overdetermined, and we need to identify a suitable subset of (a combination of) these equations which suffice to determine ν, B, μ, ω, and which are such that at the end of the day all the field equations hold once the reduced system is solved. To do so we introduce the auxiliary metric function

$$\xi = \gamma\nu + \mu.$$

Let Δ and ∇ denote the Cartesian Laplace and gradient operator respectively. Taking suitable combinations of the field equations one finds that

$$\Delta\nu + \frac{\nabla B}{B} \cdot \nabla\nu - \frac{1}{2}\rho^2 B^2 e^{-4\gamma\nu} |\nabla\omega|^2 = 4\pi \Bigg[\gamma^2 e^{(2\xi - 4\gamma\nu)} \left(T_{00} + 2\omega T_{03}\right) + \gamma(T_{11} + T_{22}) + e^{2\xi} \left(\frac{\gamma}{\rho^2 B^2} + \gamma^2 \omega^2 e^{-4\gamma\nu} \right) T_{33} \Bigg], \tag{2.6}$$

$$\Delta B + \frac{\nabla\rho}{\rho} \cdot \nabla B = 8\pi\gamma^2 B \left(T_{11} + T_{22}\right), \tag{2.7}$$

$$\Delta\omega + \left(2\frac{\nabla\rho}{\rho} + 3\frac{\nabla B}{B} - 4\gamma\nabla\nu \right) \cdot \nabla\omega = \frac{16\pi\gamma^2 e^{2\xi}}{\rho^2 B^2} \left(T_{03} + \omega T_{33}\right), \tag{2.8}$$

$$\begin{aligned} &\left(1 + \rho\frac{\partial_\rho B}{B}\right) \partial_\rho\xi - \rho\frac{\partial_z B}{B}\partial_z\xi \\ &\qquad = \frac{1}{2\rho B}\partial_\rho(\rho^2 \partial_\rho B) - \frac{\rho}{2B}\partial_{zz}B + \gamma^2 \rho \left((\partial_\rho\nu)^2 - (\partial_z\nu)^2\right) \\ &\qquad\quad - \gamma\rho^3 B^2 e^{-4\gamma\nu} \left((\partial_\rho\omega)^2 - (\partial_z\omega)^2\right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\left(1 + \rho\frac{\partial_\rho B}{B}\right) \partial_z\xi + \rho\frac{\partial_z B}{B}\partial_\rho\xi \\ &\qquad = \frac{\partial_\rho(\rho\partial_z B)}{B} + 2\gamma^2 \rho\, \partial_\rho\nu\partial_z\nu + \frac{1}{2}\gamma\rho^3 B^2 e^{-4\gamma\nu} \partial_\rho\omega\partial_z\omega. \end{aligned} \tag{2.10}$$

We write

$$B = 1 + b.$$

By taking a suitable combination of (2.9) and (2.10) we obtain equations which contain only $\partial_\rho\xi$ or $\partial_z\xi$ respectively, and we chose the former. In the above equations the terms $T_{\alpha\beta}$ are functions of the unknown quantities $\nu, b, \omega, \xi = \gamma\nu + \mu$ for which we therefore have obtained the following reduced system of equations; throughout, $\Phi_{ij} = \Phi_{ij}(\nu, B, \xi, \omega, \rho; \gamma, \lambda)$:

$$\Delta\nu = 4\pi \left[\Phi_{00} + \gamma\Phi_{11} + 2\gamma\omega\Phi_{03} + e^{2\xi} \left(\gamma \frac{1}{\rho^2 B^2} + \gamma^2 \omega^2 e^{-4\gamma\nu} \right) \Phi_{33} \right] \\ - \frac{1}{B} \nabla b \cdot \nabla \nu + \frac{1}{2} \rho^2 B^2 e^{-4\gamma\nu} |\nabla\omega|^2, \tag{2.11}$$

$$\Delta b + \frac{\nabla\rho}{\rho} \cdot \nabla b = 8\pi\gamma^2 B \Phi_{11}, \tag{2.12}$$

$$\begin{aligned} &\left((1 + b + \rho\partial_\rho b)^2 + (\rho\partial_z b)^2 \right) \partial_\rho \xi \\ &\quad = \rho\partial_z b \left(\partial_z b + \rho\partial_{z\rho} b + 2\gamma^2 \rho B \partial_\rho \nu \partial_z \nu + \frac{1}{2} \gamma \rho^3 B^3 e^{-4\gamma\nu} \partial_\rho \omega \partial_z \omega \right) \\ &\qquad + (1 + b + \rho\partial_\rho b) \left(\frac{\rho}{2} (\partial_{\rho\rho} b + \frac{2}{\rho} b - \partial_{zz} b) + \gamma^2 \rho B \left((\partial_\rho \nu)^2 - (\partial_z \nu)^2 \right) \right. \\ &\qquad\qquad \left. - \gamma \rho^3 B^3 e^{-4\gamma\nu} \left((\partial_\rho \omega)^2 - (\partial_z \omega)^2 \right) \right), \end{aligned} \tag{2.13}$$

$$\Delta\omega + 2\frac{\nabla\rho}{\rho} \cdot \nabla\omega = \frac{16\pi}{\rho^2 B^2} e^{2\xi} \left(\gamma\Phi_{03} + \gamma^2 \omega \Phi_{33} \right) - \left(3\frac{\nabla b}{B} - 4\gamma\nabla\nu \right) \cdot \nabla\omega. \tag{2.14}$$

We supplement this system with the boundary condition (1.3), which in terms of the new unknowns reads

$$\xi(0, z) = \ln\left(1 + b(0, z)\right). \tag{2.15}$$

It remains to determine precisely the dependence of the functions $\Phi_{\alpha\beta}$ on the unknown quantities ν, b, ξ, ω. The corresponding computation uses the new integration variables

$$\eta = \frac{e^{\gamma\nu}\sqrt{1 + \gamma|v|^2} - 1}{\gamma},\ s = Be^{-\gamma\nu} v^3.$$

We also introduce the notation

$$m = m(\eta, B, \nu, \gamma) = Be^{-\gamma\nu} \sqrt{\frac{e^{-2\gamma\nu}(1 + \gamma\eta)^2 - 1}{\gamma}},$$
$$l = l(s, B, \nu, \gamma) = \frac{1}{\gamma} \left(e^{\gamma\nu} \sqrt{1 + \gamma \frac{e^{2\gamma\nu} s^2}{B^2}} - 1 \right),$$

and we obtain

$$\Phi_{00}(\nu, B, \xi, \omega, \rho; \gamma, \lambda) = \gamma^2 e^{2\xi - 4\gamma\nu} T_{00} \tag{2.16}$$
$$= \frac{2\pi}{B} e^{2\xi - 4\gamma\nu} \int_{(e^{\gamma\nu}-1)/\gamma}^{\infty} \int_{-m}^{m} (1 + \gamma\eta + \gamma\rho s\omega)^2 \phi(\eta + \rho s\omega)\psi(\lambda, \rho s)\, ds\, d\eta$$
$$= \frac{2\pi}{B} e^{2\xi - 4\gamma\nu} \int_{-\infty}^{\infty} \int_{l+\rho\omega s}^{\infty} (1 + \gamma\eta)^2 \phi(\eta)\psi(\lambda, \rho s)\, d\eta\, ds,$$

$$\Phi_{11}(\nu, B, \xi, \omega, \rho; \gamma, \lambda) = T_{11} + T_{22} \tag{2.17}$$
$$= \frac{2\pi}{B^3} e^{2\xi} \int_{(e^{\gamma\nu}-1)/\gamma}^{\infty} \int_{-m}^{m} (m^2 - s^2)\, \phi(\eta + \rho s\omega)\psi(\lambda, \rho s) ds\, d\eta$$
$$= \frac{2\pi}{B^3} e^{2\xi} \int_{-\infty}^{\infty} \int_{l+\rho\omega s}^{\infty} (m^2(\eta - \rho\omega s, B, \nu, \gamma) - s^2)\, \phi(\eta)\psi(\lambda, \rho s) d\eta\, ds,$$

$$\Phi_{33}(\nu, B, \xi, \omega, \rho; \gamma, \lambda) = e^{2\xi} T_{33} \tag{2.18}$$
$$= \frac{2\pi\rho^2}{B}\, e^{2\xi} \int_{(e^{\gamma\nu}-1)/\gamma}^{\infty} \int_{-m}^{m} s^2 \phi(\eta + \rho s\omega)\psi(\lambda, \rho s)\, ds\, d\eta$$
$$= \frac{2\pi\rho^2}{B}\, e^{2\xi} \int_{-\infty}^{\infty} \int_{l+\rho\omega s}^{\infty} s^2 \phi(\eta)\psi(\lambda, \rho s)\, d\eta\, ds, \tag{2.19}$$

$$\Phi_{03}(\nu, B, \xi, \omega, \rho; \gamma, \lambda) = \gamma e^{2\xi} T_{03} \tag{2.20}$$
$$= -\frac{2\pi\rho}{B}\, e^{2\xi} \int_{(e^{\gamma\nu}-1)/\gamma}^{\infty} \int_{-m}^{m} s\, (1 + \gamma\eta + \gamma\rho s\omega)\, \phi(\eta + \rho s\omega)\psi(\lambda, \rho s)\, ds\, d\eta$$
$$= -\frac{2\pi\rho}{B}\, e^{2\xi} \int_{-\infty}^{\infty} \int_{l+\rho\omega s}^{\infty} s\, (1 + \gamma\eta)\, \phi(\eta)\psi(\lambda, \rho s)\, d\eta\, ds;$$

we recall that $T_{11} = T_{22}$. In the course of the proof we will benefit from both of these two different representations of the matter terms.

We now define the function spaces in which we will obtain the solutions of the system (2.11)–(2.15). By abuse of notation we write axially symmetric functions as functions of $x \in \mathbb{R}^3$ or alternatively of $\rho \geq 0$, $z \in \mathbb{R}$. We fix $0 < \alpha < 1/2$, $0 < \beta < 1$ and consider the Banach spaces

$$\mathcal{X}_1 := \Big\{\nu \in C^{2,\alpha}(\mathbb{R}^3) \mid \nu(x) = \nu(\rho, z) \text{ and } \|\nu\|_{\mathcal{X}_1} < \infty\Big\},$$
$$\mathcal{X}_2 := \Big\{b \in C^{3,\alpha}(\mathbb{R}^3) \mid b(x) = b(\rho, z) \text{ and } \|b\|_{\mathcal{X}_2} < \infty\Big\},$$
$$\mathcal{X}_3 := \Big\{\xi \in C^{1,\alpha}(Z_R) \mid \xi(x) = \xi(\rho, z) \text{ and } \|\xi\|_{\mathcal{X}_3} < \infty\Big\},$$
$$\mathcal{X}_4 := \Big\{\omega \in C^{2,\alpha}(\mathbb{R}^3) \mid \omega(x) = \omega(\rho, z) \text{ and } \|\omega\|_{\mathcal{X}_4} < \infty\Big\},$$

where

$$Z_R := \{x \in \mathbb{R}^3 \mid \rho < R\}$$

is the cylinder of radius $R > 0$, the quantity R being defined in (2.21) below, and the norms are defined by

$$\begin{aligned}
\|\nu\|_{\mathcal{X}_1} &:= \|\nu\|_{C^{2,\alpha}(\mathbb{R}^3)} + \|(1+|x|)^{1+\beta}\nabla\nu\|_\infty, \\
\|b\|_{\mathcal{X}_2} &:= \|b\|_{C^{3,\alpha}(\mathbb{R}^3)} + \|(1+|x|)^{3}\nabla b\|_\infty, \\
\|\xi\|_{\mathcal{X}_3} &:= \|\xi\|_{C^{1,\alpha}(Z_R)}, \\
\|\omega\|_{\mathcal{X}_4} &:= \|\omega\|_{C^{2,\alpha}(\mathbb{R}^3)} + \|(1+|x|)^{4}\nabla\omega\|_\infty,
\end{aligned}$$

and

$$\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3 \times \mathcal{X}_4, \quad \|(\nu, b, \xi, \omega)\|_{\mathcal{X}} := \|\nu\|_{\mathcal{X}_1} + \|b\|_{\mathcal{X}_2} + \|\xi\|_{\mathcal{X}_3} + \|\omega\|_{\mathcal{X}_4}.$$

Here $\|\cdot\|_\infty$ denotes the L^∞-norm, functions in $C^{k,\alpha}(\Omega)$ have by definition continuous derivatives up to order k and all their highest order derivatives are Hölder continuous with exponent α, and

$$\|g\|_{C^{k,\alpha}(\Omega)} := \sum_{|\sigma|\le k} \|D^\sigma g\|_\infty + \sum_{|\sigma|=k} \sup_{x,y\in\Omega,\, x\neq y} \frac{|D^\sigma g(x) - D^\sigma g(y)|}{|x-y|^\alpha},$$

where D^σ denotes the derivative corresponding to a multi-index $\sigma \in \mathbb{N}_0^3$. It will be straightforward to extend ξ to $\mathbb{R}^3$ once a solution is obtained in the above space.

The condition ($\phi 2$) on the Newtonian steady state implies that there exists $R > R_N > 0$ such that

$$U_N(r) > (E_0 + U_N(\infty))/2 \text{ for all } r > R. \tag{2.21}$$

If

$$\|\nu - U_N\|_\infty < |E_0 - U_N(\infty)|/4 \text{ and } 0 \le \gamma < \gamma_0,$$

with $\gamma_0 > 0$ sufficiently small, depending on E_0 and U_N, then

$$\frac{e^{\gamma\nu(x)} - 1}{\gamma} > E_0 \text{ for all } |x| > R.$$

Since L is bounded on the support of ψ this implies that there exists some $\delta_0 > 0$ such that for all $(\nu, b, \xi, \omega; \gamma, \lambda) \in \mathcal{U}$ the matter terms resulting from (2.16)–(2.18) are compactly supported in $B_R(0)$, where

$$\begin{aligned}
\mathcal{U} := \{(\nu, b, \xi, \omega; \gamma, \lambda) \in \mathcal{X} \times [0, \delta_0[\times] - \delta_0, \delta_0[\,| \\
\|(\nu, b, \xi, \omega) - (U_N, 0, 0, 0)\|_{\mathcal{X}} < \delta_0\}.
\end{aligned} \tag{2.22}$$

In addition we require that $\delta_0 > 0$ is sufficiently small so that $B = 1+b > 1/2$ for all elements in $\mathcal{U}$ and the factor in front of $\partial_\rho \xi$ in (2.13) is larger than 1/2.

Remark. If we want to make sure that the no-ergosphere condition (1.9) holds for the solutions we construct we redefine

$$\|\omega\|_{\mathcal{X}_4} := \|\omega\|_{C^{2,\alpha}(\mathbb{R}^3)} + \|(1+|x|)^3\omega\|_\infty + \|(1+|x|)^4\nabla\omega\|_\infty. \tag{2.23}$$

If $\delta_0 > 0$ is sufficiently small then ν is close to the given Newtonian potential U_N, B is close to 1, and due to the redefined norm $\rho\omega$ is bounded so that (1.9) holds if γ is sufficiently small.

Now we substitute an element $(\nu, b, \xi, \omega; \gamma, \lambda) \in \mathcal{U}$ into the matter terms defined in (2.16)–(2.20). With the right hand sides obtained in this way the equations (2.11)–(2.14) can then be solved, observing the boundary condition (2.15). In order to do so we need to be a little careful with the definition of axial symmetry, because we will rewrite different equations as equations on $\mathbb{R}^n$ with suitable, different dimensions $n \geq 3$.

We call a function $u : \mathbb{R}^n \to \mathbb{R}$ *axially symmetric* if it is invariant under all rotations about the x_n-axis, or equivalently, if there exists a function $\tilde{u} : [0,\infty[\times\mathbb{R} \to \mathbb{R}$ such that

$$u(x) = \tilde{u}(\rho, z), \text{ where } \rho = \sqrt{x_1^1 + \ldots + x_{n-1}^2} \text{ and } z = x_n \text{ for } x \in \mathbb{R}^n.$$

Of course we will identify u and $\tilde{u}$. For what follows it is important that we can view an axially symmetric function u as a function defined on *any* $\mathbb{R}^n$ with $n \geq 3$. In particular, we remark that

$$\Delta_n u = \partial_{\rho\rho} u + \frac{n-2}{\rho}\partial_\rho u + \partial_{zz} u = \Delta u + (n-3)\frac{\nabla\rho}{\rho}\cdot\nabla u,$$

where the left hand side refers to Cartesian coordinates on $\mathbb{R}^n$ and the right hand side to $\mathbb{R}^3$. In view of this relation, equation (2.14), when rewritten in terms of ρ and z, takes the form (1.4). The latter equation can be handled directly in these variables, cf. Lemma 7.2, but it is much more efficient to observe that this equation is nothing but a semilinear Poisson equation on $\mathbb{R}^5$ for an axially symmetric function. Similarly, (2.12) is nothing but a semilinear Poisson equation on $\mathbb{R}^4$.

Equation (2.11) and the properly rewritten equations (2.12) and (2.14) are solved (in terms of the right hand sides which of course contain the unknowns) by the corresponding Newton potentials, and (2.13) can simply be integrated in ρ. We define the corresponding solution operators by

$$\begin{aligned}
G_1(\nu, b, \xi, \omega; \gamma, \lambda)(x) &:= -\int_{\mathbb{R}^3} \left(\frac{1}{|x-y|} - \frac{1}{|y|}\right) M_1(y)\, dy \\
&\quad + \frac{1}{4\pi}\int_{\mathbb{R}^3} \left[\frac{\nabla b \cdot \nabla \nu}{B} - \frac{1}{2}\rho^2 B^2 e^{-4\gamma\nu}|\nabla\omega|^2\right](y)\frac{dy}{|x-y|},\ x \in \mathbb{R}^3, \quad (2.24)\\
G_2(\nu, b, \xi, \omega; \gamma, \lambda)(x) &:= -\frac{1}{\pi}\int_{\mathbb{R}^4} M_2(y)\frac{dy}{|x-y|^2},\ x \in \mathbb{R}^4, \quad (2.25)\\
G_3(\nu, b, \xi, \omega; \gamma, \lambda)(\rho, z) &:= \ln\left(1 + b(0,z)\right) + \int_0^\rho g(s,z)\, ds,\ 0 \le \rho < R, \quad (2.26)\\
G_4(\nu, b, \xi, \omega; \gamma, \lambda)(x) &:= \frac{1}{8\pi^2}\int_{\mathbb{R}^5}\left[\frac{\nabla b}{B}\cdot\nabla\omega + 4\gamma\nabla\nu\cdot\nabla\omega - M_3\right](y)\frac{dy}{|x-y|^3},\\
&\qquad x \in \mathbb{R}^5. \quad (2.27)
\end{aligned}$$

Here we put

$$\begin{aligned}
M_1 &:= \Big(\Phi_{00} + \gamma\Phi_{11} + 2\gamma\omega\Phi_{03} \\
&\qquad + \Big(\gamma\,\frac{1}{\rho^2 B^2} + \gamma^2\omega^2 e^{-4\gamma\nu}\Big)\Phi_{33}\Big)(\nu, B, \xi, \omega, \rho; \gamma, \lambda), \quad (2.28)\\
M_2 &:= \gamma^2 B\, \Phi_{11}(\nu, B, \xi, \omega, \rho; \gamma, \lambda), \quad (2.29)\\
M_3 &:= \frac{16\pi}{\rho^2 B^2}(\gamma\Phi_{03} + \gamma^2\omega\Phi_{33})(\nu, B, \xi, \omega, \rho; \gamma, \lambda), \quad (2.30)
\end{aligned}$$

and

$$\begin{aligned}
g := &\left((1 + b + \rho\partial_\rho b)^2 + (\rho\partial_z b)^2\right)^{-1}\\
&\times\Big[\rho\partial_z b\Big(\partial_z b + \rho\partial_{z\rho} b + 2\gamma^2\rho B\partial_\rho\nu\partial_z\nu + \frac{1}{2}\,\gamma\rho^3 B^3 e^{-4\gamma\nu}\partial_\rho\omega\partial_z\omega\Big)\\
&\quad + (1 + b + \rho\partial_\rho b)\Big(\frac{\rho}{2}(\partial_{\rho\rho} b + \frac{2}{\rho}\partial_\rho b - \partial_{zz} b) + \gamma^2\rho B\left((\partial_\rho\nu)^2 - (\partial_z\nu)^2\right)\\
&\qquad - \gamma\rho^3 B^3 e^{-4\gamma\nu}((\partial_\rho\omega)^2 - (\partial_z\omega)^2)\Big)\Big].
\end{aligned}$$

Finally we define the mapping to which we are going to apply the implicit function theorem as

$$\mathcal{F} : \mathcal{U} \to \mathcal{X},\ (\nu, b, \xi, \omega; \gamma, \lambda) \mapsto (\nu, b, \xi, \omega) - (G_1, G_2, G_3, G_4)(\nu, b, \xi, \omega; \gamma, \lambda).$$

In the next section we obtain the solutions in Theorem 2.1 as a two parameter family of zeros of this mapping. It should be noted that functions resulting from the operators G_i will all be axially symmetric so that they can all be viewed as functions on $\mathbb{R}^3$ even if they are at first defined on different $\mathbb{R}^n$'s.

3 Outline of the proof

We check in a number of steps that the mapping $\mathcal{F}$ satisfies the conditions for applying the implicit function theorem. Some of these steps turn out to be identical, or almost identical, to the corresponding steps in [4] in which cases the details will be left out.

Step 1.
We need to check that the mapping $\mathcal{F}$ is well defined. Since in this step the presence of ω and also the somewhat different set-up for the space $\mathcal{X}$ causes some differences compared with the analysis in [4], we deal with this issue in some detail in Section 5.

Step 2.
The next step is to see that

$$\mathcal{F}(U_N, 0, 0, 0; 0, 0) = 0.$$

This is due to the fact that for $\gamma = \lambda = 0$ the choice $b = \xi = \omega = 0$ trivially satisfies (2.12), (2.13), (2.14), while (2.11) reduces to

$$\Delta\nu = 4\pi\Phi_{00}(\nu, 1, 0, 0; 0, 0)$$

with

$$\Phi_{00}(\nu, 1, 0, 0; 0, 0) = 4\pi \int_{\nu}^{\infty} \phi(\eta)\sqrt{2(\eta-\nu)}\, d\eta = \int_{\mathbb{R}^3} \phi\left(\frac{1}{2}|v|^2 + \nu\right) dv;$$

notice that $b = 0$ implies that $B = 1$. By (ϕ2), $\nu = U_N$ is a solution of this equation, and the fact that $U_N \in \mathcal{X}_1$ is part of what was shown in the previous step. Notice further that $\psi(0, L) = 1$ on the support of f_N so that this factor, which vanishes for large L and formally makes the ansatz depend on L also in the Newtonian case, does not affect the Newtonian steady state at all.

Step 3.
The mapping $\mathcal{F}$ is continuous, and continuously Fréchet differentiable with respect to (ν, b, ξ, ω). Since the new element ω does not affect the proof for

the static case in any essential way we refer to [4, Sect. 5] for the details of this step.

Step 4.
We have to show that the Fréchet derivative

$$\mathcal{L} := D\mathcal{F}(U_N, 0, 0, 0; 0, 0) : \mathcal{X} \to \mathcal{X}$$

is one-to-one and onto. Indeed,

$$\mathcal{L}(\delta\nu, \delta b, \delta\xi, \delta\omega) = \Big(\delta\nu - L_1(\delta\nu) - L_2(\delta b, \delta\xi), \delta b, \delta\xi - L_3(\delta b), \delta\omega\Big)$$

where

$$\begin{aligned} L_1(\delta\nu)(x) &:= -\int_{\mathbb{R}^3}\left(\frac{1}{|x-y|} - \frac{1}{|y|}\right) a_N(y)\delta\nu(y)\,dy, \\ L_2(\delta b, \delta\xi)(x) &:= \frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla\delta b(y)\cdot\nabla U_N(y)\,\frac{dy}{|x-y|}, \\ &\quad - 2\int_{\mathbb{R}^3}\left(\frac{1}{|x-y|} - \frac{1}{|y|}\right)\rho_N(y)\delta\xi(y)\,dy, \\ L_3(\delta b)(x) &:= \delta b(0,z) + \frac{1}{2}\int_0^\rho s\,\left(\partial_{\rho\rho}\delta b + \frac{2}{\rho}\partial_\rho\delta b - \partial_{zz}\delta b\right)(s,z)\,ds, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le \rho < R, \end{aligned}$$

with a_N as defined in (ϕ3). To see that $\mathcal{L}$ is one-to-one let $\mathcal{L}(\delta\nu, \delta b, \delta\xi, \delta\omega) = 0$. The second and the last component of this identity imply that $\delta b = 0 = \delta\omega$, and hence also $\delta\xi = 0$ by the third component. It therefore remains to show that $\delta\nu = 0$ is the only solution of the equation $\delta\nu = L_1(\delta\nu)$, i.e., of the equation

$$\Delta\delta\nu = 4\pi a_N\delta\nu,\ \ \delta\nu(0) = 0,$$

in the space $\mathcal{X}_1$. Under the assumption on a_N stated in (ϕ3) this was established in [4, Sect. 7].

To see that $\mathcal{L}$ is onto let $(g_1, g_2, g_3, g_4) \in \mathcal{X}$ be given. We need to verify that there exists $(\delta\nu, \delta b, \delta\xi, \delta\omega) \in \mathcal{X}$ such that $\mathcal{L}(\delta\nu, \delta b, \delta\xi, \delta\omega) = (g_1, g_2, g_3, g_4)$. The second and fourth components of this equation simply say that $\delta b = g_2$ and that $\delta\omega = g_4$. Now $\delta b \in \mathcal{X}_2$ implies that $L_3(\delta b) \in \mathcal{X}_3$ by Lemma 7.1(b). Hence we can set $\delta\xi = g_3 + L_3(\delta b)$ to satisfy the third component of the 'onto' equation, and it remains to show that

$$\delta\nu - L_1(\delta\nu) = g_1 + L_2(\delta b, \delta\xi)$$

has a solution $\delta\nu \in \mathcal{X}_1$. Firstly, $L_2(\delta b, \delta\xi) \in \mathcal{X}_1$. The assertion therefore follows from the fact that $L_1 : \mathcal{X}_1 \to \mathcal{X}_1$ is compact. We refer to [4, Lemma 6.2] for the proof of this property. It is at this point that we use the fact that in $\mathcal{X}_1$ the decay assumption is weaker than what we actually get for G_1 and that $0 < \alpha < 1/2$: L_1 gains some Hölder regularity which is the source for the compactness.

In view of the steps above we can now apply the implicit function theorem, cf. [7, Thm. 15.1], to the mapping $\mathcal{F} : \mathcal{U} \to \mathcal{X}$.

Theorem 3.1 *There exists $\delta_1, \delta_2 \in]0, \delta[$ and a unique, continuous solution map*

$$S : [0, \delta_1[\times] - \delta_1, \delta_1[\to B_{\delta_2}(U_N, 0, 0, 0) \subset \mathcal{X}$$

such that $S(0,0) = (U_N, 0, 0, 0)$ and

$$\mathcal{F}(S(\gamma,\lambda);\gamma,\lambda) = 0 \textit{ for all } (\gamma,\lambda) \in [0,\delta_1[\times] - \delta_1, \delta_1[.$$

The definition of $\mathcal{F}$ implies that for any (γ, λ) the functions $(\nu, b, \xi, \omega) = S(\gamma, \lambda)$ solve the equations (2.11)–(2.14) where the last equation deserves some explanation. By construction, ω satisfies the equation $\Delta\omega = q$ on $\mathbb{R}^5$ where q is an abbreviation for the right hand side of (2.14). Both ω and q are axially symmetric, cf. Lemma 5.1. Hence ω and q can be viewed as functions of ρ and z, and as such they satisfy (1.4) which in turn implies that as functions on $\mathbb{R}^3$ they satisfy (2.14). An analogous argument applies to (2.12).

If f is defined by (2.3) then the equations (2.6)–(2.8), (2.13) hold with the induced energy momentum tensor. We can extend ξ to the whole space using the solution operator G_3 for all $x \in \mathbb{R}^3$. Also, the boundary condition (1.3) on the axis of symmetry is satisfied:

$$\xi(0, z) = G_3(\nu, b, \xi)(0, z) = \ln(1 + b(0, z)) = \ln B(0, z);$$

recall that $\xi = \gamma\nu + \mu$. The solutions are asymptotically flat in view of Remark (c) given after the formulation of Theorem 2.1; also see Proposition 3.2.

The solutions will in general have non-zero total angular momentum for $\gamma \neq 0$. To prove this assume that $\omega = 0$. Then

$$T_{03} = -\frac{2\pi\rho}{\gamma B} \int_{(e^{\gamma\nu}-1)/\gamma}^{\infty} (1 + \gamma\eta)\, \phi(\eta) \int_{-m}^{m} s\, \psi(\lambda, \rho s)\, ds\, d\eta.$$

It is easy to see that there are functions ψ satisfying the condition (ψ) such that this integral is non-zero in contradiction to (2.14) and $\omega = 0$, e.g., choose ψ such that on the support of ψ, $\psi(\lambda, L) > \psi(\lambda, -L)$ for $L > 0$ and $\lambda \neq 0$.

For $\gamma = 0$ we conclude first that $b = 0$, cf. (2.12) and the G_2-part of the solution operator, respectively. Then the G_3-part implies that $\xi = 0$ and the G_4-part yields $\omega = 0$, so that the solution reduces to $(\nu, 0, 0, 0)$, where ν solves

$$\Delta\nu = 4\pi\Phi_{00}(\nu, 1, 0, 0, \rho; 0, \lambda).$$

Since

$$\Phi_{00}(\nu, 1, 0, 0, \rho; 0, \lambda) = 4\pi \int_\nu^\infty \phi(\eta + \rho s\omega) \int_0^{\sqrt{2(\eta-\nu)}} \psi(\lambda, \rho s)\, ds\, d\eta$$

coincides with the spatial density induced by the ansatz (2.3) for the Newtonian case, cf. [13, Lemma 2.1], part (iii) of Theorem 2.1 is established. The resulting Newtonian steady state may or may not rotate, depending on the properties of ψ; see [13, Remark (b), p. 324].

The matter terms are compactly supported in view of of the discussion following (2.21).

To complete the proof of Theorem 2.1 we must show that all the field equations are satisfied by the obtained metric (1.1). The argument relies on the Bianchi identity $\nabla_\alpha G^{\alpha\beta} = 0$ which holds for the Einstein tensor induced by any (sufficiently regular) metric, and on the identity $\nabla_\alpha T^{\alpha\beta} = 0$ which is a direct consequence of the Vlasov equation (1.6); ∇_α denotes the covariant derivative corresponding to the metric (1.1). Due to the inclusion of the ω-equation we cannot refer to the corresponding argument in [4], and we provide the details of the argument in Section 6.

We conclude this outline of the proof of our main result by collecting some additional information on the solution which we obtain in the course of the proof and which shows that the solutions are asymptotically flat.

Proposition 3.2 *Let $(\nu, b, \xi, \omega) = S(\gamma, \lambda)$ be any of the solutions obtained in Theorem 3.1 and define $\mu := \xi - \nu/c^2$ and $B := 1 + b$. Then $\xi \in C^{2,\alpha}(\mathbb{R}^3)$, the limit $\nu_\infty := \lim_{|x|\to\infty} \nu(x)$ exists, and for all $\sigma \in \mathbb{N}_0^3$ with $|\sigma| \leq 2$ and $x \in \mathbb{R}^3$ the following estimates hold:*

$$\begin{aligned}
|D^\sigma(\nu(x) - \nu_\infty)| &\leq C(1 + |x|)^{-(1+|\sigma|)},\\
|D^\sigma(B - 1)(x)| &\leq C(1 + |x|)^{-(2+|\sigma|)},\\
|D^\sigma\xi(x)| &\leq C(1 + |x|)^{-(2+|\sigma|)},\\
|D^\sigma\omega(x)| &\leq C(1 + |x|)^{-(3+|\sigma|)}.
\end{aligned}$$

In particular, the spacetime equipped with the metric (1.1) is asymptotically flat in the sense that (2.5) and, after a trivial change of coordinates, also (1.2) holds; see Remark (c) after Theorem 2.1.

The proof follows easily from the decay estimates for various Newton potentials which we establish in Section 5, and from the fact that ξ now satisfies the equations (2.9) and (2.10); see also [4, Prop. 2.3]. In passing we remark that the asymptotic behavior stated in Proposition 3.2 agrees with what is given in [5]. The outline of the proof of our main result, Theorem 2.1, is now complete.

4 Regularity of the matter terms

In this section we investigate the regularity properties of the functions Φ_{00}, Φ_{11}, Φ_{33}, and Φ_{03}, and of the induced matter terms M_1, M_2, and M_3 from (2.28), (2.29), and (2.30).

Lemma 4.1 *Let ϕ and ψ satisfy the conditions (ϕ1), (ϕ2), and (ψ).*

(a) The functions Φ_{00}, Φ_{33}, and Φ_{03} have derivatives with respect to ξ, ν, ω, ρ, and $B \in]1/2, 3/2[$ up to order two, and these are continuous in $\nu, \xi, B, \omega, \rho, \gamma, \lambda$. For Φ_{11} the same is true with derivatives up to order three.

(b) For $(\nu, \xi, b, \omega, \gamma, \lambda) \in \mathcal{U}$ we have M_1, $M_2, \in C_c^{1,\alpha}(\mathbb{R}^3)$, $M_3 \in C_c^{0,\alpha}(\mathbb{R}^3)$, and M_1, M_2, M_3 are axially symmetric.

Proof. (a) Differentiability with respect to ξ is obvious to any order. Concerning differentiability with respect to ν, ω, B, and ρ, the integrals in the formulae for Φ_{ij} expressed by means of l gain one derivative. Since $\phi \in C^1(\mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R}^2)$, the Φ_{ij} have the desired regularity. For Φ_{11} we have to observe the following fact. If we differentiate the integrand in the second form of (2.17) with respect to one of the relevant parameters we obtain an expression which has the same structure as the Φ_{ij} in general. If we differentiate the integration boundary $l + \rho\omega s$ this gets substituted for η in the integrand and the term $m^2 - s^2$ vanishes.

(b) By the choice of R in (2.21) and δ_0 in (2.22) the matter terms which result by substituting an element from $\mathcal{U}$ into the Φ_{ij} are compactly supported. By the definitions of the spaces $\mathcal{X}_j$ the functions which are substituted into Φ_{ij} are axially symmetric and at least in $C^{1,\alpha}(\mathbb{R}^3)$. The expression Φ_{33} contains the factor ρ^2 so that the term Φ_{33}/ρ^2, which is present in both

M_1 and M_3, lies in $C^{1,\alpha}(\mathbb{R}^3)$. Thus M_1 and M_2 belong to $C^{1,\alpha}_c(\mathbb{R}^3)$. In order to establish the assertion for M_3 we only need to consider the expression $N(\rho,z)/\rho$ where

$$N(\rho,z)=\int_{-\infty}^{\infty}\int_{l+\rho\omega s}^{\infty} s\,(1+\gamma\eta)\,\phi(\eta)\psi(\lambda,\rho s)\,d\eta\,ds.$$

We now think of ν, B, ω as functions of $\rho\in\mathbb{R}$ and $z\in\mathbb{R}$ which are even in ρ and lie in $C^2(\mathbb{R}^2)$. Hence $N\in C^2_c(\mathbb{R}^2)$, and N is odd with respect to $\rho\in\mathbb{R}$ as can be seen by the change of variables $s\mapsto -s$. This easily implies that N/ρ is in $C^1_c(\mathbb{R}^3)$, cf. [4, Lemma 3.2], and hence $M_3\in C^1_c(\mathbb{R}^3)\subset C^{0,\alpha}_c(\mathbb{R}^3)$ as claimed; notice also Lemma 7.1. □

Remark. The additional regularity of Φ_{11} is needed for the Fréchet differentiability of G_2; notice that G_2 maps into $C^{3,\alpha}$.

5 $\mathcal{F}$ is well defined

In this section we show that $\mathcal{F}(\nu,b,\xi,\omega;\gamma,\lambda)\in\mathcal{X}$ for $(\nu,b,\xi,\omega;\gamma,\lambda)\in\mathcal{U}$. For the most part this is an assertion on certain Newton potentials. The following lemma collects the necessary information.

Lemma 5.1 *Let $0<\alpha,\delta<1$, $n\in\mathbb{N}$ with $n\geq 3$, and $g\in C^{0,\alpha}(\mathbb{R}^n)$ with*

$$|g(x)|\leq C(1+|x|)^{-n-\delta},\ x\in\mathbb{R}^n.$$

Define

$$U(x):=-\int_{\mathbb{R}^n}\frac{g(y)}{|x-y|^{n-2}}\,dy,\ x\in\mathbb{R}^n.$$

Then $U\in C^{2,\alpha}(\mathbb{R}^n)$, and for any $\sigma\in\mathbb{N}_0^n$ with $|\sigma|\leq 2$,

$$|D^\sigma U(x)|\leq C(1+|x|)^{2-n-|\sigma|},\ x\in\mathbb{R}^n.$$

If g is axially symmetric, then so is U.

Proof. Since $g\in L^1\cap L^\infty(\mathbb{R}^n)$ and Hölder continuous, $U\in C^{2,\alpha}(\mathbb{R}^n)$ with

$$\nabla U(x)=(n-2)\int\frac{x-y}{|x-y|^n}\,g(y)\,dy$$

and

$$\begin{aligned}\partial_{x_i}\partial_{x_j}U(x)&=(n-2)\alpha_n\delta_{ij}g(x)\\&\quad+(n-2)\int_{|x-y|\leq d}\partial_{x_j}\left(\frac{x_i-y_i}{|x-y|^n}\right)(g(y)-g(x))\,dy\\&\quad+(n-2)\int_{|x-y|>d}\partial_{x_j}\left(\frac{x_i-y_i}{|x-y|^n}\right)g(y)\,dy\end{aligned}$$

for $i, j \in \{1, \ldots, n\}$, $x \in \mathbb{R}^n$, and $d > 0$, cf. [8]; α_n denotes the volume of the unit ball in $\mathbb{R}^n$. We consider the decay of the gradient first:

$$
\begin{aligned}
|\nabla U(x)| &\le (n-2) \int_{|x-y|<|x|/2} \frac{|g(y)|}{|x-y|^{n-1}} dy + (n-2) \int_{|x-y|\ge|x|/2} \cdots \\
&\le C \int_{|x-y|<|x|/2} (1+|y|)^{-n-\delta} \frac{dy}{|x-y|^{n-1}} + C\|g\|_1 |x|^{1-n} \\
&\le C(1+|x|/2)^{-n-\delta} \int_{|x-y|<|x|/2} \frac{dy}{|x-y|^{n-1}} + C|x|^{1-n} \\
&\le C(1+|x|)^{-n-\delta}|x|/2 + C|x|^{1-n} \le C|x|^{1-n}.
\end{aligned}
$$

The decay for U follows in the same way. For the second order derivatives we observe that the first term in the formula above decays as required by assumption on g. We choose $d = (1+|x|)^{-n/\alpha}$. Then

$$
\begin{aligned}
&\left| \int_{|x-y|\le d} \partial_{x_j} \left(\frac{x_i - y_i}{|x-y|^n} \right) (g(y) - g(x))\, dy \right| \\
&\qquad \le C\|g\|_{C^{0,\alpha}(\mathbb{R}^3)} \int_{|x-y|\le d} \frac{dy}{|x-y|^{n-\alpha}} = C \int_0^d r^{\alpha-1} dr = C d^\alpha \\
&\qquad = C(1+|x|)^{-n}.
\end{aligned}
$$

In order to estimate the remaining term we consider $x \in \mathbb{R}^n$ with $|x|$ sufficiently large: $|x|/2 \ge (1+|x|)^{-n/\alpha}$. Then

$$
\begin{aligned}
&\left| \int_{|x-y|>d} \partial_{x_j} \left(\frac{x_i - y_i}{|x-y|^n} \right) g(y)\, dy \right| \\
&\qquad \le C \int_{d<|x-y|<|x|/2} \frac{|g(y)|\, dy}{|x-y|^n} + C \int_{|x-y|\ge|x|/2} \cdots \\
&\qquad \le C \int_{d<|x-y|<|x|/2} (1+|y|)^{-n-\delta} \frac{dy}{|x-y|^n} + C\|g\|_1 |x|^{-n} \\
&\qquad \le C(1+|x|/2)^{-n-\delta} \int_{d<|x-y|<|x|/2} \frac{dy}{|x-y|^n} + C|x|^{-n} \\
&\qquad \le C(1+|x|/2)^{-n-\delta} \ln\left(\frac{|x|}{2} (1+|x|)^{n/\alpha} \right) + C|x|^{-n} \le C|x|^{-n}.
\end{aligned}
$$

Assume now that g is axially symmetric, i.e., the function is invariant under rotations about the x_n-axis. Then the equation $\Delta U = n(n-2)\alpha_n g$, which is satisfied by U on $\mathbb{R}^n$, is invariant under these rotations. Since U vanishes at infinity, it is the unique solution of this equation and hence axially symmetric as well. $\square$

Lemma 5.2 *Let $(\nu, b, \xi, \omega; \gamma, \lambda) \in \mathcal{U}$ and let $G_1 = G_1(\nu, b, \xi, \omega; \gamma, \lambda)$ be as defined in (2.24). Then $G_1 \in \mathcal{X}_1$*

Proof. The source term M_1 of the first part of G_1 is in $C_c^{0,\alpha}(\mathbb{R}^3)$ by Lemma 4.1 (b). The definitions of $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_4$ imply that the source term

$$\frac{1}{B}\nabla b \cdot \nabla \nu - \frac{1}{2}\rho^2 B^2 e^{-4\gamma\nu}|\nabla\omega|^2$$

of the second part of G_1 lies in $C^{0,\alpha}(\mathbb{R}^3)$ and decays like $(1 + |x|)^{-3-\beta}$. Moreover, both source terms are axially symmetric. Hence Lemma 5.1 with $n = 3$ implies the assertion. $\square$

We notice that in the proof above we did not need to use the full available regularity.

Lemma 5.3 *Let $(\nu, b, \xi, \omega; \gamma, \lambda) \in \mathcal{U}$ and let $G_4 = G_4(\nu, b, \xi, \omega; \gamma, \lambda)$ be as defined in (2.27). Then $G_4 \in \mathcal{X}_4$.*

Proof. We can view the source term

$$q := M_3 - 3\frac{\nabla b}{B} \cdot \nabla\omega + 4\gamma\nabla\nu \cdot \nabla\omega$$

as an axially symmetric function both on $\mathbb{R}^3$ and on $\mathbb{R}^5$. By Lemma 4.1 (b) and the definitions of $\mathcal{X}_1$, $\mathcal{X}_2$, and $\mathcal{X}_4$, $q \in C^{0,\alpha}(\mathbb{R}^3)$ and hence also $q \in C^{0,\alpha}(\mathbb{R}^5)$. Moreover, the compact support of M_3 and the decay estimates for $\nabla\nu$, ∇b, and $\nabla\omega$ in the corresponding spaces imply that q decays like

$$(1 + |x|)^{-5-\beta} \sim (1 + \rho + |z|)^{-5-\beta}$$

where $x \in \mathbb{R}^3$ or in $\mathbb{R}^5$ has axial coordinates ρ and z. Lemma 5.1 with $n = 5$ implies the assertion. The latter is true with the original definition of the norm in $\mathcal{X}_4$ as well as with (2.23). $\square$

Lemma 5.4 *Let $(\nu, b, \xi, \omega; \gamma, \lambda) \in \mathcal{U}$ and let $G_2 = G_2(\nu, b, \xi, \omega; \gamma, \lambda)$ be as defined in (2.25). Then $G_2 \in \mathcal{X}_2$.*

Proof. The source term M_2 lies in $C_c^{1,\alpha}(\mathbb{R}^3)$, and since it is axially symmetric we can equally well view it as a function in $C_c^{1,\alpha}(\mathbb{R}^4)$. Thus Lemma 5.1 with $n = 4$ implies the assertion; notice that here we can throw one derivative onto the source term so that G_2 ends up in $C^{3,\alpha}$. $\square$

It remains to see that also $G_3 \in \mathcal{X}_3$, but the corresponding proof is identical to the one in [4, Lemma 4.2 (c)].

6 All field equations hold

Lemma 6.1 *Let $(\nu, b, \mu = \xi - \gamma\nu, \omega)$ be one of the solutions obtained in Theorem 3.1. Then the metric (1.1) together with f defined by (2.3) solve the full Einstein-Vlasov system (1.5), (1.6), and (1.7).*

Proof. For a metric of the form (1.1) the components 00, 11, 22, 33, 03, and 12 of the field equations are nontrivial. We have so far obtained a solution ν, B, ξ, ω of the reduced system (2.11), (2.14), (2.12), (2.13). We define $E_{\alpha\beta} := G_{\alpha\beta} - 8\pi c^{-4} T_{\alpha\beta}$ so that the Einstein field equations become $E_{\alpha\beta} = 0$. By the reduced system,

$$E_{00} + c^2 e^{-2(\mu-\nu/c^2)}(E_{11}+E_{22}) + 2\omega E_{03} + \left(\frac{c^2}{\rho^2 B^2} e^{4\nu/c^2} + \omega^2\right) E_{33} = 0, \tag{6.1}$$

$$E_{03} + \omega E_{33} = 0, \tag{6.2}$$

$$E_{11} + E_{22} = 0, \tag{6.3}$$

$$\left(1 + \rho\frac{\partial_\rho B}{B}\right)(E_{11} - E_{22}) + \rho\frac{\partial_z B}{B} E_{12} = 0. \tag{6.4}$$

The Vlasov equation implies that $\nabla_\alpha T^{\alpha\beta} = 0$, and $\nabla_\alpha G^{\alpha\beta} = 0$ due to the contracted Bianchi identity where ∇_α denotes the covariant derivative corresponding to the metric (1.1). Hence $\nabla_\alpha E^{\alpha\beta} = 0$. We want to use these relations for $\beta = 1$ and $\beta = 2$ together with (6.1)–(6.4) to show that $E_{\alpha\beta} = 0$. To do so we first rewrite the equations (6.1)–(6.4) in terms of $E^{\alpha\beta}$. Then (6.2) and (6.3) turn into

$$\omega E^{00} - E^{03} = 0, \tag{6.5}$$

$$E^{11} + E^{22} = 0. \tag{6.6}$$

Using these to eliminate E^{22} and E^{03} the equations (6.1) and (6.4) become

$$\left(c^2 e^{4\nu/c^2} - \rho^2 B^2 \omega^2\right) E^{00} + \rho^2 B^2 E^{33} = 0, \tag{6.7}$$

$$2\left(1 + \rho\frac{\partial_\rho B}{B}\right) E^{11} + \rho\frac{\partial_z B}{B} E^{12} = 0. \tag{6.8}$$

The two Bianchi equations mentioned above can be written in the form

$$\partial_\rho(\rho E^{11}) + \partial_z(\rho E^{12}) + \left(4\partial_\rho\mu + \frac{\partial_\rho B}{B}\right)(\rho E^{11}) + \left(4\partial_z\mu + \frac{\partial_z B}{B}\right)(\rho E^{12}) + \rho\left(\Gamma^1_{00} + 2\omega\Gamma^1_{03}\right)E^{00} + \rho\Gamma^1_{33}E^{33} = 0, \tag{6.9}$$

$$\partial_\rho(\rho E^{12}) - \partial_z(\rho E^{11}) + \left(4\partial_\rho\mu + \frac{\partial_\rho B}{B}\right)(\rho E^{12}) - \left(4\partial_z\mu + \frac{\partial_z B}{B}\right)(\rho E^{11}) + \rho\left(\Gamma^2_{00} + 2\omega\Gamma^2_{03}\right)E^{00} + \rho\Gamma^2_{33}E^{33} = 0, \tag{6.10}$$

where (6.5) and (6.6) were used to eliminate E^{03} and E^{22}.

At this point there is a small subtlety concerning regularity. By definition of $\mathcal{X}_3$ we have $\xi \in C^{1,\alpha}$, and since this function satisfies (2.13) also $\partial_\rho\xi \in C^{1,\alpha}$. Hence $\partial_\rho(\rho E^{11})$ and $\partial_\rho(\rho E^{12})$ exist classically, but a priori this need not be true for $\partial_z(\rho E^{11})$ and $\partial_z(\rho E^{12})$. However, approximating ξ by smooth functions the corresponding Bianchi identities again hold, and passing to the limit, (6.9) and (6.10) hold in the sense of distributions. With the possible exception of $\partial_z(\rho E^{11})$ and $\partial_z(\rho E^{12})$ all the terms in (6.9) and (6.10) are continuous so that these identities show that the latter terms are classical, continuous derivatives as well.

We use (6.7) and (6.8) to eliminate E^{00} and E^{11} from the two Bianchi identities (6.9) and (6.10). The resulting two equations contain only E^{12} and its first order derivatives and E^{33}. Eliminating the latter finally yields the following first order partial differential equation for ρE^{12}:

$$\left(\left(1 + \rho\frac{\partial_\rho B}{B}\right)^2 + \frac{1}{2}\left(\rho\frac{\partial_z B}{B}\right)^2\right)\partial_\rho(\rho E^{12}) - \frac{1}{2}\left(1 + \rho\frac{\partial_\rho B}{B}\right)\left(\rho\frac{\partial_z B}{B}\right)\partial_z(\rho E^{12}) + c(\rho, z)\ (\rho E^{12}) = 0.$$

Here $c = c(\rho, z)$ is a continuous function on $[0, \infty[\times\mathbb{R}$ the form of which is of no further interest, and the equation holds for $\rho \geq 0$.

We recall that by definition (2.22) of the set $\mathcal{U}$ the quantity $1 + \rho\partial_\rho B/B$ is bounded away from zero so that any characteristic curve of the above equation must intersect the axis $\rho = 0$ where $\rho E^{12} = e^{-2\mu}(\partial_z B/B - \gamma\partial_z\nu - \partial_z\mu)$ vanishes due to (1.3). Hence $E^{12} = 0$ on $[0, \infty[\times\mathbb{R}$. By (6.8) also $E^{11} = 0$, and by (6.6) the same is true for E^{11}. If we eliminate E^{00} from (6.9) using (6.7) we find an equation of the form $(1+\rho\partial_\rho B/B)E^{33} = 0$ so that $E^{33} = 0$. By (6.7) this implies that $E^{00} = 0$ provided we choose the smallness parameter in the definition (2.22) of the set $\mathcal{U}$ such that the coefficient of E^{00} in (6.7) does not vanish which is possible due to the estimates in

Proposition 3.2. By (6.5) we finally find that $E^{03} = 0$, and hence all the Einstein equations $E_{\alpha\beta} = 0$ hold. $\square$

7 Appendix: On the regularity of axially symmetric functions, and a remark on the ω equation

We first collect a some remarks concerning the regularity of axially symmetric functions on $\mathbb{R}^n$, $n \geq 3$.

Lemma 7.1 *Let $u : \mathbb{R}^n \to \mathbb{R}$ be axially symmetric and $u(x) = \tilde{u}(\rho, z)$ where $\tilde{u} : [0, \infty[\times\mathbb{R} \to \mathbb{R}$. Let $k \in \{1, 2, 3\}$ and $\alpha \in]0, 1[$.*

(a) $u \in C^k(\mathbb{R}^n) \iff \tilde{u} \in C^k([0, \infty[\times\mathbb{R})$ and all derivatives of $\tilde{u}$ of order up to k which are of odd order in ρ vanish for $\rho = 0$.

(b) $u \in C^{0,\alpha}(\mathbb{R}^n) \iff \tilde{u} \in C^{0,\alpha}([0, \infty[\times\mathbb{R})$.

The case $n = 3$ is already stated in [4, Lemma 3.1], and the proof does not depend on the space dimension.

We conclude this paper by pointing out that the ω equation (1.4) can also be solved directly in the original variables ρ and z. However, as the authors had to learn, basing the analysis on the following result makes it *much* harder.

Lemma 7.2 *A solution to the equation (1.4) is given by*

$$\omega(\rho, z) = \int_0^\infty \int_{\mathbb{R}} \mathcal{K}(\rho, z, \tilde{\rho}, \tilde{z})\, q(\tilde{\rho}, \tilde{z})\, d\tilde{z}\, d\tilde{\rho},$$

where

$$\mathcal{K}(\rho, z, \tilde{\rho}, \tilde{z}) = -\frac{1}{2\pi} \left(\frac{\tilde{\rho}}{\rho}\right)^{3/2} Q_{1/2}\left(\frac{\rho^2 + \tilde{\rho}^2 + (z - \tilde{z})^2}{2\rho\tilde{\rho}}\right),$$

and $Q_{1/2}$ is a half-integer Legendre function of the second kind.

Sketch of proof. As explained above, we can interpret q as an axisymmetric function on $\mathbb{R}^5$ and rewrite (1.4) as the Poisson equation on $\mathbb{R}^5$. Hence the solution can be represented as

$$\omega(x) = -\frac{1}{8\pi^2} \int_{\mathbb{R}^5} \frac{q(y)}{|x - y|^3} dy, \;\; x \in \mathbb{R}^5.$$

If we let $x = (0, 0, 0, \rho, z)$ and introduce polar coordinates for the integration, two integrals can be carried out explicitly, and the fact that

$$Q_{1/2}(\chi) = \int_0^\pi \frac{\sin^2 \eta}{(\chi - \cos \eta)^{3/2}} d\eta$$

gives the formula in terms of $\mathcal{K}$. $\square$

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