Growth rates of orbits in non-periodic twist maps and a theorem by Neishtadt

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Abstract

We consider non-periodic holomorphic twist maps of the form

$$\theta_1 = \theta + \frac{1}{r^\alpha}(\gamma + F_1(\theta, r)), \quad r_1 = r + r^{1-\alpha} F_2(\theta, r),$$

for $\alpha \in]0, 1[$ and $\gamma \in \mathbb{R} \setminus \{0\}$. Under appropriate assumptions on F_1, F_2 and a primitive $\mathfrak{h}$ of $r_1\, d\theta_1 - r\, d\theta$ it is shown that $r_n = \mathcal{O}((\log n)^{1/\alpha})$, if $(\theta_n, r_n)_{n \in \mathbb{N}_0}$ is a forward complete real orbit of the map.

1 Introduction

Over the last years we have examined the dynamics of twist maps with non-periodic angles [2, 3, 4, 5]. Motivated by the Fermi-Ulam ping-pong model, and also by the Littlewood boundedness problem, we have obtained results on the role of the bounded orbits in the general dynamics [6] and also on the improbability of escaping orbits [7, 11, 12]. However, the first result for this class of maps is older and due to Neishtadt [9]. He studied the ping-pong model in the analytic case and proved that for any orbit the velocity v_n after the impact n must satisfy

$$v_n = \mathcal{O}(\log n), \quad n \to \infty. \tag{1.1}$$

In this paper we consider more general holomorphic maps $f : (\theta, r) \mapsto (\theta_1, r_1)$ of the form

$$\theta_1 = \theta + \frac{1}{r^\alpha}(\gamma + F_1(\theta, r)), \quad r_1 = r + r^{1-\alpha} F_2(\theta, r), \tag{1.2}$$

where $\alpha \in]0,1[$ and $\gamma \in \mathbb{R} \setminus \{0\}$. They should be viewed as perturbations of

$$\theta_1 = \theta + \frac{\gamma}{r^\alpha}, \quad r_1 = r.$$

The latter map is well-defined on $\mathbb{R}\times]0,\infty[$ and has a holomorphic extension to the complex domain $\mathbb{C} \times \{r \in \mathbb{C} : \mathrm{Re}\, r > 0\}$. Moreover, it is symplectic, due to $r_1\, d\theta_1 - r\, d\theta = d\mathfrak{h}_0$ for $\mathfrak{h}_0(\theta, r) = -\frac{\alpha\gamma}{1-\alpha}\, r^{1-\alpha}$. We will investigate the dynamics of (1.2), being defined on a set of the type

$$\Omega = \mathbb{R}_\delta \times \{r \in \mathbb{C} : \mathrm{Re}\, r > \underline{r}, |\mathrm{Im}\, r| < \eta|r|\}$$

for some $\delta, \underline{r} > 0$, $\eta \in]0,1[$, with $\mathbb{R}_\delta = \{\theta \in \mathbb{C} : |\mathrm{Im}\, \theta| < \delta\}$ denoting the open strip in the complex plane about $\mathbb{R}$ of width δ. Our main assumptions are:

(i) the smallness of the holomorphic functions F_j on Ω (supposed to map reals into reals), in the sense that $F_j(\theta, r) = \mathcal{O}(r^{-\alpha})$, uniformly in $\theta \in \mathbb{R}_\delta$, for $j = 1, 2$;

(ii) $\mathfrak{h}(\theta, r) = \mathfrak{h}_0(\theta, r) + \mathcal{O}(r^{1-2\alpha})$ uniformly in $\theta \in \mathbb{R}_\delta$, where $r_1\, d\theta_1 - r\, d\theta = d\mathfrak{h}$ holds for (1.2).

Under these hypotheses we are going to show (Theorem 3.1) that there exists a constant $C > 0$ such that if $(\theta_n, r_n)_{n\in\mathbb{N}_0}$ is a forward complete real orbit of (1.2), then there is $n_0 \in \mathbb{N}$ so that

$$r_n \le C(\log n)^{1/\alpha}, \quad n \ge n_0.$$

For the proof, we apply a rescaling $\xi = \varepsilon^{1/\alpha} r$ to put f from (1.2) into the form

$$\psi_\varepsilon : \quad \theta_1 = \theta + \varepsilon R_1(\theta, \xi, \varepsilon), \quad \xi_1 = \xi + \varepsilon R_2(\theta, \xi, \varepsilon), \tag{1.3}$$

where $R_1(\theta, \xi, \varepsilon) = \frac{1}{\xi^\alpha}(\gamma + F_1(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}))$ and $R_2(\theta, \xi, \varepsilon) = \xi^{1-\alpha} F_2(\theta, \frac{\xi}{\varepsilon^{1/\alpha}})$. It turns out that the family of maps $\{\psi_\varepsilon\}$ can be defined on a common domain G_ρ, where $G = \mathbb{R}\times]1,2[$ and

$$G_\rho = \{x = (q,p) \in \mathbb{C}^2 : |\mathrm{Im}\, q| < \rho,\ \mathrm{dist}(p, I) < \rho\}.$$

This leads us to study (see Section 2, also for more discussion of the subtleties) general maps $P_\varepsilon : G_\rho \to \mathbb{C}^2$ given by

$$P_\varepsilon : \quad x_1 = x + \varepsilon l(x, \varepsilon), \quad x_1 = (q_1, p_1), \quad x = (q, p),$$

where l belongs to a certain class of maps $\mathcal{M}_{1,\rho,\sigma}$ that has to be carefully set up in order to account for singularities of l or $\frac{\partial l}{\partial \varepsilon}$ at $\varepsilon = 0$; recall the definition of R_1, R_2 in terms of F_1, F_2 above. Inspired by [5], we call the family of maps $\{P_\varepsilon\}$ E-symplectic, if $p_1\, dq_1 - p\, dq = dh(\cdot, \varepsilon)$ for a function $h \in \mathcal{M}_{1,\rho,\sigma}$ such that, as $\varepsilon \to 0$,

$$h(q, p, \varepsilon) = \varepsilon\, \mathfrak{m}(q, p) + \mathcal{O}(\varepsilon^2), \quad \frac{\partial h}{\partial \varepsilon}(q, p, \varepsilon) = \mathfrak{m}(q, p) + \mathcal{O}(\varepsilon),$$

uniformly in $(q, p) \in G_\rho$ for a bounded function $\mathfrak{m} : G_\rho \to \mathbb{C}$. It turns out that all these conditions can be verified for (1.3) after rescaling $\mathfrak{h}$ from (ii) to h. Furthermore, it is possible to construct a function $E = E(x)$ satisfying $J\nabla E(x) = l(x, 0)$, where J denotes the standard symplectic matrix; in fact $E(\theta, \xi) = E(\xi) = \frac{\gamma}{1-\alpha}\, \xi^{1-\alpha}$ for the maps from (1.3). The function E

should be thought of as an approximate first integral (adiabatic invariant) for the family $\{P_\varepsilon\}$. This means that the variation of E along the orbit remains small for an exponentially large time. More precisely, Theorem 2.5 ensures that if

$$(x_n)_{0\le n\le N} = (P_\varepsilon^n(x_0))_{0\le n\le N}$$

is a real forward orbit piece of P_ε so that $x_n \in G$ for all $0 \le n \le N$, then

$$|E(x_n) - E(x_0)| \le \hat{C}\varepsilon, \quad 0 \le n \le \min\{N, N_\varepsilon\}, \quad N_\varepsilon = [e^{\hat{D}/\varepsilon}], \tag{1.4}$$

for constants $\hat{C}, \hat{D} > 0$ and if $\varepsilon > 0$ is small enough (all independent of the orbit). Going back to the original variables (θ, r), it follows that

$$|s_n - s_m| \le C s_m^\beta, \quad m \le n \le m + [e^{s_m^\delta}],$$

where $s_n \sim r_n^{1-\alpha}$ (up to a multiplicative constant), $\beta = \frac{1-2\alpha}{1-\alpha} < 1$ and $\delta = \frac{\alpha}{1-\alpha} > 0$. Then to complete the proof of Theorem 3.1 we need to show that $\limsup_{m\to\infty} \frac{s_m}{(\log m)^{1/\delta}} \le C_1$. This is accomplished in a clean way by using Lemma 3.4, which is related to upper and lower solutions to the difference equation $x_{n+1} = x_n + C x_n^\beta$.

Section 4 concerns the ping-pong map. This important example was analyzed in [9] and we revisit it to illustrate the applicability of our results. Our proof is substantially different from the proof in [9], since the change of variables and the adiabatic invariant we are going to use seem to be new. Note that $\alpha = 1/2$ for the ping-pong map, but in the notation of the main theorem (Theorem 3.1) as mentioned above $r_n = E_n = v_n^2/2$ corresponds to energy, not velocity, and hence we recover (1.1). An important issue here is how to extend the map to the analytic setup. We also remark that the result comes with some uniformity, in the sense that it leads to the estimate

$$\limsup_{n\to\infty} \frac{v_n}{\log n} \le C_0$$

for a constant $C_0 > 0$ that is independent of the chosen orbit.

It remains an open question, if the logarithmic bound, as provided by Theorem 3.1, is optimal. In Section 5 we will give an example for $\alpha = 1/2$ such that for every $(\theta_0, r_0) \in \mathbb{R}^2$ so that $\theta_0 > 0$ and $r_0 > 0$ the forward complete orbit $(\theta_n, r_n)_{n\in\mathbb{N}_0}$ does exist and satisfies

$$0 < \liminf_{n\to\infty} \frac{r_n}{(\log n)^2} \le \limsup_{n\to\infty} \frac{r_n}{(\log n)^2} < \infty.$$

However, this example will barely fail one of the assumptions of Theorem 3.1 (the bound (3.4) does only hold uniformly for θ in bounded sets, but not for $\theta \in \mathbb{R}_\delta$), and hence it does not yield optimality. This indicates that maybe some assumption of Theorem 3.1 could be relaxed, or in some examples unbounded orbits could exist. However, given the advanced technical machinery that is used to establish Theorem 3.1, both points seem to be difficult to address.

2 E-symplectic families of maps

An important observation in [9] is the existence of adiabatic invariants for families of analytic canonical maps close to the identity. Given a convex domain $G \subset \mathbb{R}^N \times \mathbb{R}^N$ and a family of symplectic maps

$$P_\varepsilon : G \to \mathbb{R}^N \times \mathbb{R}^N, \quad x_1 = x + \varepsilon l(x, \varepsilon),$$

it is possible to construct a function $E = E(x)$ satisfying

$$J\nabla E(x) = l(x, 0), \tag{2.1}$$

where $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$. For small ε the iteration $x_{n+1} = P_\varepsilon(x_n)$ can be interpreted as a numerical integration method for the Hamiltonian system $\dot{x} = J\nabla E(x)$. This fact suggests that $E(x)$ should be an adiabatic invariant for P_ε, meaning that

$$|E(P_\varepsilon^n(x)) - E(x)| \leq C\varepsilon, \quad 0 \leq n \leq N_\varepsilon, \tag{2.2}$$

where N_ε is of the order $e^{D/\varepsilon}$; the constants $C, D > 0$ should only depend upon an appropriate norm of l. In essence this is discussed in Remark 5 and Proposition 3 of [9]. Additional details can be found in [1], in particular in the case of bounded domains.

However, the previous statements must be taken with some caution in the case where the underlying domain is unbounded. As a counter-example we consider the family of translations

$$x_1 = x + \varepsilon Jv + \varepsilon^2 v,$$

defined on the whole space $G = \mathbb{R}^N \times \mathbb{R}^N$. Here $v \neq 0$ is a fixed vector and $E(x) = \langle x, v\rangle$ satisfies (2.1), since $l(x, \varepsilon) = Jv + \varepsilon v$. Due to $P_\varepsilon^n(x) = x + n\varepsilon l(x, \varepsilon)$ we obtain

$$|E(P_\varepsilon^n(x)) - E(x)| = \varepsilon^2 n|v|^2.$$

Therefore (2.2) does hold only for $n \leq N_\varepsilon = \mathcal{O}(1/\varepsilon)$ many steps.

To overcome this inherent difficulty, Benettin and Giorgilli in [1] considered an unbounded domain G and a family of maps derived from a symplectic integration algorithm for a Newtonian system of the type $\ddot{q} = -\nabla V(q)$. Then they impose some growth conditions on $V(q)$ as $|q| \to \infty$. We will follow a different approach and assume that our family $\{P_\varepsilon\}$ satisfies a condition inspired by the notion of an exact symplectic map (called E-symplectic), as it was understood in our previous work [5]. Furthermore, to simplify matters, we will restrict ourselves to the case of direct interest to us for applications. Throughout we will take

$$N = 1 \quad \text{and} \quad G = \mathbb{R} \times I,$$

where $I \subset \mathbb{R}$ is an open and bounded interval. Our goal will be to understand the dynamics of a map on the plane $(\theta, r) \mapsto (\theta_1, r_1)$ when $r \to \infty$. For this reason our family of maps $\{P_\varepsilon\}$, $P_\varepsilon : (q, p) \mapsto (q_1, p_1)$, will be obtained after a rescaling $q = \theta$, $p = \varepsilon r$ with $q \in \mathbb{R}$ and $p \in]1, 2[$. This procedure will lead to functions $l(x, \varepsilon)$ that are analytic in x, but not necessarily smooth in ε; a prototype can be the function $l(x, \varepsilon) = h(x/\varepsilon^2)$, where h is real analytic in $[1, \infty[$ and $h(\zeta) \to 0$ as $\zeta \to \infty$. Then l is continuous as a function of the two variables (x, ε), but the partial derivatives $\partial_\varepsilon^k l$ do not always exist at $\varepsilon = 0$.

The following definitions are motivated by the previous discussions. In general, for the norms on $\mathbb{C}^d$ and $\mathbb{C}^{d_1 \times d_2}$ we will take $|x| = \max_{1\leq i\leq d} |x_i|$ and $|A| = \max_{1\leq i\leq d_1, 1\leq j\leq d_2} |a_{ij}|$, respectively. Note that for $A \in \mathbb{C}^{d\times d}$, $x \in \mathbb{C}^d$, $A_1 \in \mathbb{C}^{d_1\times d}$ and $A_2 \in \mathbb{C}^{d\times d_2}$ this implies

$$|Ax| \leq d|A||x|, \quad |A_1 A_2| \leq d|A_1||A_2|.$$

The points in $G = \mathbb{R} \times I$ will be denoted by $x = (q, p)$. For $\rho > 0$ we will write

$$G_\rho = \{x = (q, p) \in \mathbb{C}^2 : |\operatorname{Im} q| < \rho,\ \operatorname{dist}(p, I) < \rho\}.$$

Given $\varphi : G_\rho \to \mathbb{C}$ holomorphic, let

$$\|\varphi\|_\rho = \sup \{|\varphi(x)| : x \in G_\rho\}.$$

If $0 < r < \rho$, then by the Cauchy integral formula one has

$$\|D\varphi\|_r \leq \frac{1}{\rho - r} \|\varphi\|_\rho,$$

where $D\varphi$ is the Jacobian.

Definition 2.1 (The classes $\mathcal{M}_{\rho,\sigma}$ and $\mathcal{M}_{1,\rho,\sigma}$) *Let $\rho > 0$ and $\sigma \in]0, 1[$.*

(i) The class $\mathcal{M}_{\rho,\sigma}$ consists of those continuous maps $l : G_\rho \times [0, \sigma] \to \mathbb{C}^2$, $l = l(x, \varepsilon)$, which satisfy:

(a) l maps real into reals; and

(b) for every $\varepsilon \in [0, \sigma]$ the map $l(\cdot, \varepsilon)$ is holomorphic on G_ρ and

$$\|l\|_{\rho,\sigma} = \sup \{\|l(\cdot, \varepsilon)\|_\rho : \varepsilon \in [0, \sigma]\} < \infty.$$

(ii) The class $\mathcal{M}_{1,\rho,\sigma}$ consists of those continuous maps $l : G_\rho \times [0, \sigma] \to \mathbb{C}^2$, $l = l(x, \varepsilon)$, satisfying

(a) l maps real into reals;

(b) l is C^∞ in $G_\rho \times]0, \sigma]$;

(c) for every $\varepsilon \in [0, \sigma]$ the map $l(\cdot, \varepsilon)$ is holomorphic on G_ρ;

(d) one has

$$\|l\|_{1,\rho,\sigma} = \|l\|_{\rho,\sigma} + \sup \left\{ \left\| \frac{\partial l}{\partial \varepsilon}(\cdot, \varepsilon) \right\|_\rho : \varepsilon \in]0, \sigma] \right\} < \infty.$$

Remark 2.2 Note that, for a map $l \in \mathcal{M}_{\rho,\sigma}$ or $l \in \mathcal{M}_{1,\rho,\sigma}$, all the derivatives $\partial_x^\alpha \partial_\varepsilon^k l(\cdot, \varepsilon) : G_\rho \to \mathbb{C}^2$ for $\varepsilon \in]0, \sigma]$ are holomorphic, where $\alpha \in \mathbb{N}_0^2$ and $k \in \mathbb{N}_0$. Similarly, all the $\partial_x^\alpha l : G_\rho \times [0, \sigma] \to \mathbb{C}^2$ are continuous functions of both variables. This follows from the Cauchy integral formula and the continuity of l. Furthermore, the derivatives can be interchanged: $\partial_x^\alpha \partial_\varepsilon^k l(\cdot, \varepsilon) = \partial_\varepsilon^k \partial_x^\alpha l(\cdot, \varepsilon)$.

Definition 2.3 *Suppose that $l \in \mathcal{M}_{1,\rho,\sigma}$, and for $\varepsilon \in [0, \sigma]$ consider the family of maps $P_\varepsilon : G_\rho \to \mathbb{C}^2$ given by*

$$P_\varepsilon : \quad x_1 = x + \varepsilon l(x, \varepsilon), \quad x_1 = (q_1, p_1), \quad x = (q, p). \tag{2.3}$$

We say that the family $\{P_\varepsilon\}$ is E-symplectic, if there is a function $h \in \mathcal{M}_{1,\rho,\sigma}$ such that

$$p_1\,dq_1 - p\,dq = dh(\cdot,\varepsilon) \tag{2.4}$$

and there exists a bounded function $\mathfrak{m} : G_\rho \to \mathbb{C}$ satisfying

$$h(q,p,\varepsilon) = \varepsilon\,\mathfrak{m}(q,p) + \mathcal{O}(\varepsilon^2) \quad as \quad \varepsilon \to 0 \tag{2.5}$$

and

$$\frac{\partial h}{\partial \varepsilon}(q,p,\varepsilon) = \mathfrak{m}(q,p) + \mathcal{O}(\varepsilon) \quad as \quad \varepsilon \to 0 \tag{2.6}$$

uniformly in $(q,p) \in G_\rho$.

Remark 2.4 (a) $\mathfrak{m}$ is holomorphic in G_ρ. To see this, note that $\frac{\partial h}{\partial \varepsilon}(\cdot,\varepsilon)$ is holomorphic for $\varepsilon > 0$ by Remark 2.2. Since $\mathfrak{m}$ is the uniform limit of $\int_0^1 \frac{\partial h}{\partial \varepsilon}(q,p,t\varepsilon)\,dt$ as $\varepsilon \to 0$, it is holomorphic itself.
(b) $\mathfrak{m}$ satisfies

$$\frac{\partial \mathfrak{m}}{\partial q}(q,p) = p\,\frac{\partial l_1}{\partial q}(q,p,0) + l_2(q,p,0), \quad \frac{\partial \mathfrak{m}}{\partial p}(q,p) = p\,\frac{\partial l_1}{\partial p}(q,p,0), \tag{2.7}$$

where $l = (l_1, l_2)$. For, we observe from (2.5) that $\varepsilon^{-1} h \to \mathfrak{m}$ uniformly on G_ρ. Therefore also the derivatives converge, uniformly on compact subsets of G_ρ. From (2.4),

$$\varepsilon^{-1}\frac{\partial h}{\partial q} = l_2 + p\,\frac{\partial l_1}{\partial q} + \varepsilon\, l_2\,\frac{\partial l_1}{\partial q}, \quad \varepsilon^{-1}\frac{\partial h}{\partial p} = p\,\frac{\partial l_1}{\partial p} + \varepsilon\, l_2\,\frac{\partial l_1}{\partial p}.$$

Thus it remains to pass to the limit $\varepsilon \to 0$ and use Remark 2.2. Relation (2.7) can also be stated as

$$\nabla \mathfrak{m}(x) = p\,\nabla l_1(x,0) + \begin{pmatrix} l_2(x,0) \\ 0 \end{pmatrix}, \quad x = (q,p). \tag{2.8}$$

(c) One has

$$\frac{\partial l_1}{\partial q}(q,p,0) + \frac{\partial l_2}{\partial p}(q,p,0) = 0, \tag{2.9}$$

as follows from $\frac{\partial^2 \mathfrak{m}}{\partial q \partial p} = \frac{\partial^2 \mathfrak{m}}{\partial p \partial q}$. Relation (2.9) implies that the Jacobian matrix $Dl(x,0)$ is Hamiltonian, i.e., it satisfies $Dl(x,0)^* J + J Dl(x,0) = 0$, or equivalently, $J Dl(x,0)$ is symmetric. Since G_ρ is simply connected, we conclude that there is a holomorphic function $E : G_\rho \to \mathbb{C}$ such that $J\nabla E = l(\cdot,0)$, i.e., (2.1) holds. Actually, (2.8) shows that we can take

$$E(x) = l_1(x,0)p - \mathfrak{m}(x), \quad x = (q,p). \tag{2.10}$$

(d) The relation $J\nabla E = l(\cdot,0)$ yields

$$dE = \frac{\partial E}{\partial q}\,dq + \frac{\partial E}{\partial p}\,dp = -l_2\,dq + l_1\,dp.$$

Hence $E(x) = E(x_0) + \int_\gamma (-l_2\,dq + l_1\,dp)$ for every path γ that connects a fixed $x_0 \in G$ to x. This observation makes the connection to the formula for E given in [9] below (2.7).

(e) Condition (2.6) does not follow from (2.5), as the example

$$h(q, p, \varepsilon) = \varepsilon\,\mathfrak{m}(q, p) + \varepsilon^2 \sin\left(\frac{1}{\varepsilon}\right)$$

shows.

The proof of our main result (see Theorem 3.1 below) relies on the following theorem, which should be compared to [9, (2.7), p. 135 and Prop. 3, p. 136]. It can be established along the lines as indicated in [9], cf. [8] for more discussion and full details.

Theorem 2.5 *Suppose that $l \in \mathcal{M}_{1,\rho,\sigma}$, and for $\varepsilon \in [0, \sigma]$ consider the family of maps $P_\varepsilon : G_\rho \to \mathbb{C}^2$ given by*

$$P_\varepsilon : \quad x_1 = x + \varepsilon l(x, \varepsilon). \tag{2.11}$$

Let the family $\{P_\varepsilon\}$ be E-symplectic. Then there exist $\hat{\sigma} \in]0, \sigma]$ and constants $\hat{C}, \hat{D} > 0$ (depending upon ρ, σ, $\|l\|_{1,\rho,\sigma}$, the interval I, $\|h\|_{1,\rho,\sigma}$ and $\sup_{\varepsilon\in]0,\sigma]} \|\varepsilon^{-1}(\frac{\partial h}{\partial\varepsilon}(\cdot, \varepsilon) - \mathfrak{m})\|_\rho$) such that if

$$(x_n)_{0\le n\le N} = (P_\varepsilon^n(x_0))_{0\le n\le N}$$

is a real forward orbit piece of P_ε so that $x_n \in G$ for all $0 \le n \le N$, then

$$|E(x_n) - E(x_0)| \le \hat{C}\varepsilon, \quad 0 \le n \le \min\{N, N_\varepsilon\}, \quad N_\varepsilon = [e^{\hat{D}/\varepsilon}]. \tag{2.12}$$

3 Main result

To motivate our main result let $\alpha \in]0, 1[$ and $\gamma \in \mathbb{R} \setminus \{0\}$. Consider the map $(\theta, r) \mapsto (\theta_1, r_1)$ given by

$$\theta_1 = \theta + \frac{\gamma}{r^\alpha}, \quad r_1 = r.$$

It is well-defined on $\mathbb{R}\times]0, \infty[$ and has a holomorphic extension to the complex domain $\mathbb{C}\times\{r \in \mathbb{C} : \mathrm{Re}\, r > 0\}$. Moreover, the map is symplectic, since it satisfies

$$r_1\,d\theta_1 - r\,d\theta = d\mathfrak{h}_0$$

for

$$\mathfrak{h}_0(\theta, r) = -\frac{\alpha\gamma}{1-\alpha}\, r^{1-\alpha}. \tag{3.1}$$

We will consider perturbations of this map on a sub-domain of $\mathbb{C}^2$ of the type

$$\Omega = \mathbb{R}_\delta \times \{r \in \mathbb{C} : \mathrm{Re}\, r > \underline{r}, |\mathrm{Im}\, r| < \eta|r|\}, \tag{3.2}$$

where $\delta, \underline{r} > 0$, $\eta \in]0, 1[$, and $\mathbb{R}_\delta = \{\theta \in \mathbb{C} : |\mathrm{Im}\, \theta| < \delta\}$ denotes the open strip in the complex plane about $\mathbb{R}$ of width δ.

Theorem 3.1 *Consider the map $f : (\theta, r) \mapsto (\theta_1, r_1)$ given by*

$$\theta_1 = \theta + \frac{1}{r^\alpha}(\gamma + F_1(\theta, r)), \quad r_1 = r + r^{1-\alpha} F_2(\theta, r), \tag{3.3}$$

under the following hypotheses:

(a) F_1 and F_2 are holomorphic in Ω from (3.2).

(b) If $(\theta, r) \in \Omega \cap \mathbb{R}^2$, then $F_1(\theta, r), F_2(\theta, r) \in \mathbb{R}$.

(c) $F_j(\theta, r) = \mathcal{O}(r^{-\alpha})$, uniformly in $\theta \in \mathbb{R}_\delta$ and for $j = 1, 2$.

(d) There is a holomorphic function $\mathfrak{h} : \Omega \to \mathbb{C}$ that maps reals into reals and such that $r_1\, d\theta_1 - r\, d\theta = d\mathfrak{h}$ as well as

$$\mathfrak{h}(\theta, r) = \mathfrak{h}_0(\theta, r) + \mathcal{O}(r^{1-2\alpha}), \tag{3.4}$$

uniformly in $\theta \in \mathbb{R}_\delta$, where $\mathfrak{h}_0$ is defined in (3.1).

Then there exists a constant $C > 0$ such that if $(\theta_n, r_n)_{n \in \mathbb{N}_0}$ is a forward complete real orbit of f, then there is $n_0 \in \mathbb{N}$ so that

$$r_n \leq C (\log n)^{1/\alpha}, \quad n \geq n_0.$$

Remark 3.2 (a) The dependence of C with respect to the parameters will be discussed along the proof; C will be obtained from a sequence of constants $C_1, \dots C_{13}$.
(b) If the functions F_1 and F_2 are 2π-periodic in θ, then f can be defined on a cylinder and the conclusion can be improved to $r_n = \mathcal{O}(1)$ as $n \to \infty$ for each complete real orbit. This is a consequence of the Small Twist Theorem, see [13, Chapter III]. In fact, after the rescaling $\rho = \varepsilon r$ with $\rho \in [1, 2]$, the map f has an expansion of the form

$$\theta_1 = \theta + \varepsilon^\alpha \frac{\gamma}{\rho^\alpha} + \mathcal{O}(\varepsilon^{2\alpha}), \quad \rho_1 = \rho + \mathcal{O}(\varepsilon^{2\alpha}),$$

as $\varepsilon \to 0$, uniformly in $\theta \in \mathbb{R}_\delta$. Taking a sequence $\varepsilon_n \to 0$, we find corresponding invariant curves $r = \psi_n(\theta)$ such that $1/\varepsilon_n \leq \psi_n(\theta) \leq 2/\varepsilon_n$ for $\theta \in \mathbb{R}$. These curves are closed in the cylinder and act as barriers for all real orbits, preventing them to escape. The same conclusion is valid if then dependence on θ is quasiperiodic and the frequencies satisfy a Diophantine condition, cf. [14].
(c) Without any further assumptions, for a map f which satisfies (a)-(d), there are infinitely many forward complete real orbits such that $r_n = \mathcal{O}(1)$ along the orbit. This is a consequence of the results in [3]. To establish the claim, we first observe that (c) yields for $r \in \mathbb{R}$ the bound

$$\frac{\partial F_1}{\partial r}(\theta, r) = \mathcal{O}(r^{-(1+\alpha)}), \quad r \to \infty, \tag{3.5}$$

uniformly in $\theta \in \mathbb{R}$; (3.5) follows from the Cauchy formula, see the proof of Theorem 3.1 below. According to [5], the latter estimate is sufficient to guarantee the existence of a generating function $h = h(\theta, \theta_1)$ associated to f, i.e., $r = \frac{\partial h}{\partial \theta}$ and $r_1 = -\frac{\partial h}{\partial \theta_1}$ are verified. Actually one can take $h(\theta, \theta_1) = -\mathfrak{h}(\theta, R(\theta, \theta_1))$, where $r = R(\theta, \theta_1)$ is implicitly defined by the first equation in (3.3). Some computations then show that

$$R(\theta, \theta_1) \sim \gamma^{1/\alpha}(\theta_1 - \theta)^{-1/\alpha}, \quad h(\theta, \theta_1) \sim \frac{\alpha\gamma^{1/\alpha}}{1-\alpha}\,(\theta_1 - \theta)^{-\frac{1-\alpha}{\alpha}},$$

as $\theta_1 - \theta \to 0^+$, where as usual $F(x) \sim G(x)$ as $x \to x_0$ means that $\lim_{x\to x_0} F(x)/G(x) = 1$. Hence we can invoke [3, Thm. 2.5] or [5, Exercise 5.6] to deduce that for each $\hat{r} > 0$ the map f has an orbit $(\theta_n, r_n)_{n\in\mathbb{N}_0}$ such that $r_n \geq \hat{r}$ for all $n \in \mathbb{N}_0$ and furthermore $\sup_n r_n < \infty$.

To prepare for the proof of the theorem, we are going to discuss some aspects of the method of upper and lower solutions for the difference equation

$$x_{n+1} = g(x_n), \tag{3.6}$$

where $g : I \to \mathbb{R}$ is an increasing function that is defined on an interval $I \subset \mathbb{R}$.

A sequence $(\gamma_n)_{0\leq n\leq N} \subset I$ is called a lower solution of (3.6), if $\gamma_{n+1} \leq g(\gamma_n)$ for $n = 0, \ldots, N-1$. An upper solution is defined by reversing the previous inequality.

Lemma 3.3 *Let (γ_n) and (Γ_n) be a lower solution and an upper solution of (3.6). If $\gamma_0 \leq \Gamma_0$, then $\gamma_n \leq \Gamma_n$ for all n.*

Proof: This follows by induction from the monotonicity of g. □

Next we will show how to construct lower and upper solutions for an equation that will be important for the proof of Theorem 3.1. Consider

$$x_{n+1} = x_n + Cx_n^\beta,$$

where $C > 0$ and $\beta < 1$. The function $g(x) = x + Cx^\beta$ is increasing in $I = [0, \infty[$, if $\beta \geq 0$, and it is increasing in $I = [(C|\beta|)^{\frac{1}{1-\beta}}, \infty[$, if $\beta < 0$. Inspired by the general solution of the differential equation $\dot{x} = Cx^\beta$, we test sequences of the type

$$\gamma_n = (A + Bn)^{\frac{1}{1-\beta}}, \quad n \geq 0,$$

for some $A, B > 0$; the condition $A \geq (C|\beta|)^{\frac{1}{1-\beta}}$ is also assumed, if $\beta < 0$, to make sure that $\gamma_n \in I$. From the mean value theorem we obtain

$$\gamma_{n+1} - \gamma_n = \frac{B}{1-\beta}\,(A + B(n + \zeta_n))^{\frac{\beta}{1-\beta}}$$

for some $\zeta_n \in]0, 1[$.

Let us first look at the case where $\beta \in [0, 1[$. Here we deduce that

$$\frac{B}{1-\beta}\,\gamma_n^\beta \leq \gamma_{n+1} - \gamma_n \leq \frac{B}{1-\beta}\,\gamma_{n+1}^\beta. \tag{3.7}$$

Hence γ_n will be an upper solution, as soon as $B \geq C(1-\beta)$. To get a lower solution, we observe that

$$\frac{\gamma_{n+1}^{1-\beta}}{\gamma_n^{1-\beta}} = 1 + \frac{B}{A+Bn} \leq 1 + \frac{B}{A}.$$

Therefore we get

$$\gamma_{n+1}^{\beta} \leq \Big(1 + \frac{B}{A}\Big)^{\frac{\beta}{1-\beta}} \gamma_n^{\beta},$$

and thus, due to (3.7), γ_n will be a lower solution, if $B(1+\frac{B}{A})^{\frac{\beta}{1-\beta}} \leq C(1-\beta)$.

For $\beta < 0$ the inequality (3.7) is reversed. As a consequence, γ_n will be a lower solution, if $A \geq (C|\beta|)^{\frac{1}{1-\beta}}$ and $B \leq C(1-\beta)$, and it is an upper solution for $A \geq (C|\beta|)^{\frac{1}{1-\beta}}$ and $B(1+\frac{B}{A})^{\frac{\beta}{1-\beta}} \geq C(1-\beta)$. Thus to summarize:

(a) If $\beta < 1$ is fixed and $B = \frac{1}{2}C(1-\beta)$, then $\Gamma_n = (A+Bn)^{\frac{1}{1-\beta}}$ will be a lower solution, if $A > 0$ is taken sufficiently large (depending on C and β); this fact won't be needed in what follows.

(b) If $\beta < 1$ is fixed and $B = 2C(1-\beta)$, then $\Gamma_n = (A+Bn)^{\frac{1}{1-\beta}}$ will be an upper solution, if $A > 0$ is taken sufficiently large (depending on C and β).

Returning to the general setup, let us now assume that the interval I is of the type $I =]b, \infty[$ and let $h : \mathbb{N}_0 \to \mathbb{N}_0$ be a given function with the property that

$$h(n) \geq n+1, \quad n \in \mathbb{N}_0. \tag{3.8}$$

Let $(\gamma_n)_{n\in\mathbb{N}_0} \subset I$ be a sequence such that

$$\gamma_m \leq g(\gamma_n), \quad 0 \leq n \leq m \leq h(n). \tag{3.9}$$

This sequence is a lower solution of (3.6), but it has additional favorable properties; in this case it will be possible to sharpen the conclusion of Lemma 3.3 as follows.

Lemma 3.4 *Let $(\gamma_n)_{n\in\mathbb{N}_0} \subset I$ and $(\Gamma_n)_{n\in\mathbb{N}_0} \subset I$ be such that:*

(a) (γ_n) satisfies (3.9),

(b) (Γ_n) is an upper solution to (3.6),

(c) $\gamma_0 \leq \Gamma_0$,

(d) (Γ_n) is increasing and $\limsup_{n\to\infty} \gamma_n = \infty$.

Then there is a non-decreasing function $\sigma : \mathbb{N}_0 \to \mathbb{N}$ such that

$$\gamma_{\sigma(n)} > \Gamma_n \quad \textit{and} \quad \gamma_m \leq \Gamma_n, \quad m \in \{0, \ldots, \sigma(n)-1\}. \tag{3.10}$$

In addition,

$$\sigma(n+1) > h(\sigma(n)-1), \quad n \in \mathbb{N}_0. \tag{3.11}$$

Proof: Define
$$\sigma(n) = \min\{k \in \mathbb{N}_0 : \gamma_k > \Gamma_n\}.$$
It follows from (d) that σ is well-defined and monotone increasing. Also, thanks to (c), we have $\sigma(0) \geq 1$ and accordingly $\sigma(n) \geq 1$ for all $n \in \mathbb{N}_0$; in particular, the statement (3.11) makes sense. The relations in (3.10) are obtained directly from the definition of σ, and we are going to prove (3.11) by contradiction. So assume that for some $n \in \mathbb{N}_0$ we have $\sigma(n+1) \leq h(\sigma(n)-1)$. Then
$$h(\sigma(n)-1) \geq \sigma(n+1) \geq \sigma(n) > \sigma(n) - 1$$
shows that we can use (3.9), with n replaced by $\sigma(n)-1$ and m replaced by $\sigma(n+1)$, to deduce that $\gamma_{\sigma(n+1)} \leq g(\gamma_{\sigma(n)-1})$. Since g is increasing and due to (b), this would yield
$$\gamma_{\sigma(n+1)} \leq g(\gamma_{\sigma(n)-1}) \leq g(\Gamma_n) \leq \Gamma_{n+1},$$
which is impossible by (3.10). □

Remark 3.5 The previous proof is still valid, if the sequence (γ_n) does not lie in I, but satisfies a modified version of (3.9). Assume that there a numbers $b^* > b_* > b$ such that $\Gamma_0 \geq b^*$ and
$$\gamma_{n+1} \geq b^* \quad \Longrightarrow \quad \gamma_n \geq b_*. \tag{3.12}$$
Then γ_n is required to have the property that
$$\gamma_n \geq b_* \quad \Longrightarrow \quad b \leq \gamma_m \leq g(\gamma_n), \quad 0 \leq n \leq m \leq h(n). \tag{3.13}$$

Proof of Theorem 3.1: Step 1: Some estimates. We will show that, after restricting the size of Ω, the functions F_1 and F_2 will satisfy some additional estimates. From (c) we know that there are numbers $C_j > 0$ such that
$$|F_j(\theta, r)| \leq C_j r^{-\alpha}, \quad (\theta, r) \in \Omega, \quad j = 1, 2. \tag{3.14}$$
We consider the smaller region
$$\Omega_* = \mathbb{R}_\delta \times \left\{ r \in \mathbb{C} : \operatorname{Re} r > 2\underline{\mathrm{r}}, |\operatorname{Im} r| < \frac{\eta}{2}|r| \right\}$$
and claim that there are constants $C_j^{(1)} > 0$ for $j = 1, 2$ such that
$$\left| \frac{\partial F_j}{\partial r}(\theta, r) \right| \leq C_j^{(1)} r^{-(\alpha+1)}, \quad (\theta, r) \in \Omega_*, \quad j = 1, 2, \tag{3.15}$$
where $C_j^{(1)}$ only depends upon $\underline{\mathrm{r}}$, η and C_j. To prove this we first use an elementary geometric argument to find a constant $\kappa \in]0,1[$, depending upon $\underline{\mathrm{r}}$ and η, such that if $(\theta, r) \in \Omega_*$, then all points (θ, ρ) with $\theta \in \mathbb{R}_\delta$ and $|\rho - r| \leq \kappa |r|$ will belong to Ω. Now it is possible to use the Cauchy formula
$$\frac{\partial F_j}{\partial r}(\theta, r) = \frac{1}{2\pi i} \int_\gamma \frac{F_j(\theta, \rho)}{(\rho - r)^2}\, d\rho,$$

where γ is a circle with center r and radius $\kappa|r|$. Then (3.14) leads, after a short computation, to (3.15). The same kind of arguments in conjunction with (d) yields the following bounds for $\mathfrak{h}$:

$$|\mathfrak{h}(\theta,r)-\mathfrak{h}_0(\theta,r)|\le C_3\, r^{1-2\alpha},\quad (\theta,r)\in\Omega, \tag{3.16}$$

$$\Big|\frac{\partial\mathfrak{h}}{\partial r}(\theta,r)-\frac{\partial\mathfrak{h}_0}{\partial r}(\theta,r)\Big|\le C_3^{(1)}r^{-2\alpha},\quad (\theta,r)\in\Omega_*, \tag{3.17}$$

where $C_3>0$ and $C_3^{(1)}>0$ are suitable constants. From now on the domain Ω will be replaced by Ω_*. To simplify notation, we will assume that already Ω is a domain on which the estimates (3.14), (3.15), (3.16) and (3.17) are verified.

Step 2: Rescaling. Under the transformation $\xi=\varepsilon^{1/\alpha}r$ the map f becomes

$$\psi_\varepsilon:\quad \theta_1=\theta+\varepsilon R_1(\theta,\xi,\varepsilon),\quad \xi_1=\xi+\varepsilon R_2(\theta,\xi,\varepsilon),$$

where

$$R_1(\theta,\xi,\varepsilon)=\frac{1}{\xi^\alpha}\Big(\gamma+F_1\Big(\theta,\frac{\xi}{\varepsilon^{1/\alpha}}\Big)\Big),\quad R_2(\theta,\xi,\varepsilon)=\xi^{1-\alpha}F_2\Big(\theta,\frac{\xi}{\varepsilon^{1/\alpha}}\Big).$$

According to (a), ψ_ε is defined on

$$\Sigma_\varepsilon=\mathbb{R}_\delta\times\{\xi\in\mathbb{C}:|\mathrm{Im}\,\xi|<\eta|\xi|,\mathrm{Re}\,\xi>\varepsilon^{1/\alpha}\underline{\mathrm{r}}\}.$$

We intend to apply Theorem 2.5 to the family of maps $\{\psi_\varepsilon\}$, and the first task will be to determine a common domain. Let us fix $I=]1,2[$ and define $G=\mathbb{R}\times I$. A generic point in G will be denoted by $x=(\theta,\xi)$ and we also recall that $|x|=\max\{|\theta|,|\xi|\}$ will be taken as the norm on $\mathbb{C}^2$. Elementary geometric considerations show that it is possible to select $\rho\in]0,\min\{1/2,\delta\}[$ and $\sigma>0$ such that $G_\rho\subset\Sigma_\varepsilon$ for $\varepsilon\in[0,\sigma]$. The next step is to show that $l=(R_1,R_2)$ belongs to $\mathcal{M}_{1,\rho,\sigma}$. Note that we are extending this map to $\varepsilon=0$ by letting

$$R_1(\theta,\xi,0)=\frac{\gamma}{\xi^\alpha},\quad R_2(\theta,\xi,0)=0.$$

The functions $R_i(\cdot,\cdot,0)$ are obviously continuous on G_ρ. We are going to show that

$$R_1(\theta,\xi,\varepsilon)\to\frac{\gamma}{\xi^\alpha},\quad R_2(\theta,\xi,\varepsilon)\to 0, \tag{3.18}$$

as $\varepsilon\to0$, uniformly in G_ρ. This implies that the extension of R_i to $G_\rho\times[0,\sigma]$ is continuous, and hence the same holds for l. The limits in (3.18) are a consequence of (3.14) and the bounds $1/2\le 1-\rho\le|\xi|\le 2+\rho\le 5/2$ for $(\theta,\xi)\in G_\rho$.

Now that we know that l is continuous in $G_\rho\times[0,\sigma]$, the conditions (a), (b) and (c) from the definition of the class $\mathcal{M}_{1,\rho,\sigma}$ follow directly from the assumptions on F_1 and F_2. To establish (d), we first consider $\|l\|_{\rho,\sigma}$. Here

$$\|R_1(\cdot,\cdot,\varepsilon)\|_\rho\le 2^\alpha(|\gamma|+2^\alpha C_1\,\varepsilon),\quad \|R_2(\cdot,\cdot,\varepsilon)\|_\rho\le m_\alpha\, C_2\,\varepsilon, \tag{3.19}$$

for $m_\alpha = \max\{(\frac{5}{2})^{1-2\alpha}, 2^{2\alpha-1}\}$, is derived from $1/2 \le |\xi| \le 5/2$ and (3.14), so that $\|l\|_{\rho,\sigma} < \infty$. For the derivatives w.r. to ε, we have

$$\frac{\partial R_1}{\partial \varepsilon}(\theta,\xi,\varepsilon) = -\frac{1}{\alpha}\,\frac{\xi^{1-\alpha}}{\varepsilon^{1+1/\alpha}}\,\frac{\partial F_1}{\partial r}\Big(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\Big), \quad \frac{\partial R_2}{\partial \varepsilon}(\theta,\xi,\varepsilon) = -\frac{1}{\alpha}\,\frac{\xi^{2-\alpha}}{\varepsilon^{1+1/\alpha}}\,\frac{\partial F_2}{\partial r}\Big(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\Big),$$

for $\varepsilon \in]0,\sigma]$. Using (3.15), we deduce that

$$\Big\|\frac{\partial R_1}{\partial \varepsilon}(\cdot,\cdot,\varepsilon)\Big\|_\rho \le \frac{2^{2\alpha}}{\alpha}\,C_1^{(1)}, \quad \Big\|\frac{\partial R_2}{\partial \varepsilon}(\cdot,\cdot,\varepsilon)\Big\|_\rho \le \frac{m_\alpha}{\alpha}\,C_2^{(1)},$$

so that $\|l\|_{1,\rho,\sigma} < \infty$ and therefore $l \in \mathcal{M}_{1,\rho,\sigma}$.

Step 3: The symplectic condition. We apply assumption (d) to observe that

$$h(\theta,\xi,\varepsilon) = \varepsilon^{1/\alpha}\,\mathfrak{h}\Big(\theta, \frac{\xi}{\varepsilon^{1/\alpha}}\Big) \tag{3.20}$$

is a potential for ψ_ε on G_ρ, i.e., $\xi_1\,d\theta_1 - \xi\,d\theta = dh(\cdot,\varepsilon)$ is satisfied. Moreover, from (3.16), we obtain that

$$h(\theta,\xi,\varepsilon) = \varepsilon\,\mathfrak{m}(\theta,\xi) + \mathcal{O}(\varepsilon^2) \quad \text{as} \quad \varepsilon \to 0, \tag{3.21}$$

uniformly in $(\theta,\xi) \in G_\rho$, where $\mathfrak{m}(\theta,\xi) = \mathfrak{h}_0(\theta,\xi)$; for this note that $\mathfrak{m}$ is homogeneous in ξ of degree $1-\alpha$. The function $\mathfrak{m}$ is bounded on G_ρ. Next, condition (2.6) follows from (3.17), and it should be observed that the bound on $\|\varepsilon^{-1}(\frac{\partial h}{\partial \varepsilon} - \mathfrak{m})\|_\rho$ then only depends upon C_3, $C_3^{(1)}$ and α. It remains to prove that $h \in \mathcal{M}_{1,\rho,\sigma}$ in order to conclude that the family $\{\psi_\varepsilon\}$ is E-symplectic; note that h is extended to $\varepsilon = 0$ by $h(\theta,\xi,0) = 0$. From (3.21) we get the continuity of h on $G_\rho \times [0,\sigma]$. Next we are going to show that condition (d) in the definition of $\mathcal{M}_{1,\rho,\sigma}$ (see Definition 2.1(ii)) also holds. The definition of $\mathfrak{h}_0$ and (3.16), (3.17) imply that

$$\mathfrak{h}(\theta,r) = \mathcal{O}(r^{1-\alpha}) \quad \text{and} \quad \frac{\partial \mathfrak{h}}{\partial r}(\theta,r) = \mathcal{O}(r^{-\alpha})$$

uniformly in $\theta \in \mathbb{R}_\delta$. Thus using these estimates, we obtain a uniform (in $\varepsilon \in]0,\sigma]$) bound on $\|h(\cdot,\varepsilon)\|_\rho$ and $\|\frac{\partial h}{\partial \varepsilon}(\cdot,\varepsilon)\|_\rho$.

Step 4: Application of Theorem 2.5. Since

$$l(x,0) = \begin{pmatrix} R_1(\theta,\xi,0) \\ R_2(\theta,\xi,0) \end{pmatrix} = \begin{pmatrix} \gamma\xi^{-\alpha} \\ 0 \end{pmatrix},$$

we deduce from (2.1) that

$$E(\theta,\xi) = \frac{\gamma}{1-\alpha}\,\xi^{1-\alpha},$$

and thus in fact $E = E(\xi)$. Then Theorem 2.5 yields the existence of $\hat\sigma \in]0,\sigma]$ and constants $\hat C, \hat D > 0$ such that if $\varepsilon \in [0,\hat\sigma]$ and

$$(x_n)_{0\le n\le N} = (\theta_n,\xi_n)_{0\le n\le N}$$

is a real forward orbit piece of ψ_ε so that $1 < \xi_n < 2$ for all $0 \le n \le N$, then

$$|E(\xi_n) - E(\xi_0)| \le \hat C\varepsilon, \quad 0 \le n \le \min\{N, N_\varepsilon\}, \quad N_\varepsilon = [e^{\hat D/\varepsilon}]. \tag{3.22}$$

Step 5: Going back to the original system. First we fix two numbers $1 < a < 3/2 < b < 2$. Let $(\theta_n, r_n)_{n_1 \le n \le n_2}$ be a real forward orbit piece of f such that $\varepsilon = (2r_{n_1}/3)^{-\alpha} < \hat{\sigma}$, where $\hat{\sigma}$ is from the previous step; by decreasing $\hat{\sigma}$ further, we may assume that in addition

$$\hat{\sigma} \le \frac{\gamma}{(1-\alpha)\hat{C}} \min\left\{ \left(\frac{3}{2}\right)^{1-\alpha} - a^{1-\alpha}, b^{1-\alpha} - \left(\frac{3}{2}\right)^{1-\alpha} \right\} \tag{3.23}$$

as well as

$$\hat{\sigma} \le \min\left\{ \frac{2-b}{m_\alpha C_2}, \frac{a-1}{m_\alpha C_2} \right\} \tag{3.24}$$

are verified. Then we have $\xi_{n_1} = \varepsilon^{1/\alpha} r_{n_1} = 3/2 \in]1,2[$, where in general we let $\xi_n = \varepsilon^{1/\alpha} r_n$. Denote by

$$N_\omega = \max\{m \ge n_1 : 1 < \xi_n < 2 \text{ for } n \text{ with } n_1 \le n \le m\}$$

the longest time such that along the orbit $(\theta_n, \xi_n)_{n_1 \le n \le n_2}$ of ψ_ε it holds that $1 < \xi_n < 2$. From Step 4 and (3.22) it follows that

$$\left| \xi_n^{1-\alpha} - \left(\frac{3}{2}\right)^{1-\alpha} \right| \le \frac{1-\alpha}{\gamma} \hat{C}\, \varepsilon, \quad n_1 \le n \le \min\{N_\omega, n_1 + N_\varepsilon\}, \quad N_\varepsilon = [e^{\hat{D}/\varepsilon}].$$

Thus for $n_1 \le n \le \min\{N_\omega, n_1 + N_\varepsilon\}$ we deduce from $\varepsilon < \hat{\sigma}$ and (3.23) that

$$a^{1-\alpha} < \left(\frac{3}{2}\right)^{1-\alpha} - \frac{1-\alpha}{\gamma} \hat{C}\, \varepsilon < \xi_n^{1-\alpha} \le \left(\frac{3}{2}\right)^{1-\alpha} + \frac{1-\alpha}{\gamma} \hat{C}\, \varepsilon < b^{1-\alpha}. \tag{3.25}$$

We claim that $N_\omega \ge n_1 + N_\varepsilon$. Otherwise (3.25) would be applicable $n = N_\omega$ to imply that $\xi_{N_\omega} \in [a,b]$. We do also know that $\xi_{N_\omega+1} \notin]1,2[$. But then

$$\xi_{N_\omega+1} = \xi_{N_\omega} + \varepsilon R_2(\theta_{N_\omega}, \xi_{N_\omega}, \varepsilon)$$

together with (3.19) and $\varepsilon \le 1$ would lead to $|\xi_{N_\omega+1} - \xi_{N_\omega}| \le m_\alpha\, C_2\, \varepsilon^2 \le m_\alpha\, C_2\, \varepsilon$. This in turn would yield $\xi_{N_\omega+1} \in]1,2[$ by (3.24), which is impossible. This completes the argument for $N_\omega \ge n_1 + N_\varepsilon$, and the previous discussion can be summarized as follows: If $r_{n_1} > (3/2)\, \hat{\sigma}^{-1/\alpha}$, then

$$|r_n^{1-\alpha} - r_{n_1}^{1-\alpha}| \le C_4\, r_{n_1}^{1-2\alpha}, \quad n_1 \le n \le n_1 + [e^{C_5\, r_{n_1}^\alpha}], \tag{3.26}$$

where $C_4 = (\frac{2}{3})^{1-2\alpha}(\frac{1-\alpha}{\gamma})\hat{C}$ and $C_5 = (2/3)^\alpha \hat{D}$.

Step 6: Conclusion. In terms of $s_n = C_5^{\frac{1-\alpha}{\alpha}}\, r_n^{1-\alpha}$ and writing $m = n_1$, (3.26) reads as follows: If $s_m > C_6$, then

$$|s_n - s_m| \le C_7\, s_m^\beta, \quad m \le n \le m + [e^{s_m^\delta}], \tag{3.27}$$

where $\beta = \frac{1-2\alpha}{1-\alpha} < 1$, $\delta = \frac{\alpha}{1-\alpha} > 0$, $C_6 = (3/2)^{1-\alpha}\,(C_5/\hat{\sigma})^{1/\delta}$ and $C_7 = C_4 C_5$. We need to prove that $\limsup_{m\to\infty} \frac{s_m}{(\log m)^{1/\delta}} \le C_*$ for an appropriate constant $C_* > 0$ that will be independent of the initial condition s_0. Suppose now that $\limsup_{n\to\infty} s_n = \infty$, or equivalently $\limsup_{n\to\infty} r_n = \infty$.

We intend to adapt (3.27) to the framework described in Lemma 3.4 and Remark 3.5. The function $g(x) = x + C_7\,x^\beta$ is increasing in $I =]b,\infty[$, where we take $b = 0$ for $\beta \geq 0$ and $b = (C_7|\beta|)^{\frac{1}{1-\beta}}$ for $\beta < 0$. In addition,

$$h(n) = n + [e^{s_n^\delta}]$$

satisfies (3.8), since all the s_n are positive. The numbers b_* and b^* are defined as follows. First we take

$$b_* = \max\{C_6 + 1, 2b, (2C_7)^{\frac{1}{1-\beta}}\}.$$

To obtain b^* so that (3.12) is satisfied for $\gamma_n = s_n$ we need an estimate of the type $s_n \geq C_9\,s_{n+1} - 1$, which is valid for all $n \in \mathbb{N}$. By the definition of the map and by (3.14):

$$r_{n+1} = r_n + r_n^{1-\alpha} F_2(\theta_n, r_n) \leq r_n + C_2\,r_n^{1-2\alpha},$$

which translates into

$$\begin{aligned} s_{n+1} &\leq C_5^{\frac{1-\alpha}{\alpha}}(r_n + C_2\,r_n^{1-2\alpha})^{1-\alpha} \leq C_5^{\frac{1-\alpha}{\alpha}}(r_n^{1-\alpha} + C_2\,r_n^{(1-2\alpha)(1-\alpha)}) \\ &= s_n + C_8\,s_n^{1-2\alpha} \leq (C_8+1)(s_n+1) \end{aligned}$$

for a suitable constant $C_8 > 0$. Expressed differently, we have the bound

$$s_n \geq C_9\,s_{n+1} - 1 \quad \text{for} \quad C_9 = (C_8+1)^{-1}. \tag{3.28}$$

Therefore an appropriate choice for $b^* > b_* > b$ is $b^* = (C_8+1)(b_*+1)$. Finally we take $\Gamma_n = (A + Bn)^{\frac{1}{1-\beta}}$ with $B = 2C_7(1-\beta)$ and $A^{\frac{1}{1-\beta}} \geq \max\{s_0, b^*\}$. From the discussion prior to this proof we know that (Γ_n) will be an upper solution to $x_{n+1} = x_n + C_7\,x_n^\beta$, if A is fixed to be sufficiently large (depending on s_0, β, C_6, C_7). Clearly $\Gamma_0 \geq \max\{s_0, b^*\}$ and (Γ_n) is increasing. Lastly, (s_n) satisfies (3.13), the latter due to (3.27): if $s_m \geq b_*$, then $s_m > C_6$ and (3.27) applies. It follows that $s_m - C_7 s_m^\beta \leq s_n \leq g(s_m)$ for $0 \leq m \leq n \leq h(m)$. The lower bound also yields

$$s_n \geq s_m(1 - C_7 s_m^{\beta-1}) \geq \frac{1}{2}\,s_m \geq b$$

for such n. Hence Lemma 3.4 provides us with a non-decreasing function $\sigma : \mathbb{N}_0 \to \mathbb{N}$ such that $s_{\sigma(n)} > \Gamma_n$, $s_m \leq \Gamma_n$ for $m \in \{0,\ldots,\sigma(n)-1\}$ and furthermore

$$\sigma(n+1) > \sigma(n) - 1 + [e^{s_{\sigma(n)-1}^\delta}], \quad n \in \mathbb{N}_0. \tag{3.29}$$

Thus from (3.29), (3.28) and $s_{\sigma(n)} > \Gamma_n$ we deduce

$$\sigma(n+1) > \sigma(n) + e^{s_{\sigma(n)-1}^\delta} - 2 \geq \sigma(n) + e^{(C_9\,s_{\sigma(n)}-1)^\delta} - 2 \geq \sigma(n) + e^{(C_9\Gamma_n - 1)^\delta} - 2.$$

After some straightforward manipulations using the definition of Γ_n and $\frac{\delta}{1-\beta} = 1$, this yields

$$\sigma(n+1) \geq \sigma(n) + C_{10}\,e^{c_{11}n} - 2$$

for constants $C_{10}, c_{11} > 0$ depending upon C_9, δ, A and B. Therefore

$$\sigma(n) = \sigma(0) + \sum_{k=0}^{n-1}(\sigma(k+1) - \sigma(k)) \geq 1 + C_{10} \sum_{k=0}^{n-1} e^{c_{11}k} - 2n \geq C_{10} \frac{e^{c_{11}n} - 1}{e^{c_{11}} - 1} - 2n. \tag{3.30}$$

Thus σ has at least exponential growth, which means that its 'inverse' will remain below a logarithm. More precisely, let

$$\psi(m) = \min\{n \in \mathbb{N}_0 : m < \sigma(n)\}.$$

Then $m \leq \sigma(\psi(m)) - 1$ and hence $s_m \leq \Gamma_{\psi(m)}$. In addition, from (3.30) it follows that $\psi(m) \leq \log(m + C_{12}) + C_{13}$ for suitable constants $C_{12}, C_{13} > 0$. This in turn leads to

$$s_m \leq (A + B\psi(m))^{\frac{1}{1-\beta}} \leq (A + B\log(m + C_{12}) + BC_{13})^{1/\delta}$$

and therefore also

$$\limsup_{m\to\infty} \frac{s_m}{(\log m)^{1/\delta}} \leq C_*$$

for $C_* = B^{1/\delta}$, which completes the proof. Note that C_* is independent of the initial condition s_0, but in general the n_0 from the statement of Theorem 3.1 will depend on s_0. $\square$

4 Application to the ping-pong map

The Fermi-Ulam ping-pong map (see [3]) for the forcing function p is usually expressed in terms of the variables time and velocity at the impacts with one of the rackets. Assuming that this racket is fixed, the equations for the map $(t_0, v_0) \mapsto (t_1, v_1)$ are

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{v_1}, \quad v_1 = v_0 - 2\dot{p}(\tilde{t}),$$

where $\tilde{t} = \tilde{t}(t_0, v_0)$ denotes the hitting time to the other racket, which is obtained from the relation $(\tilde{t} - t_0)v_0 = p(\tilde{t})$. A computation shows that $v_1\, dt_1 \wedge dv_1 = v_0\, dt_0 \wedge dv_0$, and this formula suggests the energy $E = \frac{1}{2}\, v^2$ to be used as the conjugate variable of time. In this way we obtain the symplectic map $\Psi : (t_0, E_0) \mapsto (t_1, E_1)$,

$$t_1 = \tilde{t} + \frac{p(\tilde{t})}{\sqrt{2}(\sqrt{E_0} - \sqrt{2}\,\dot{p}(\tilde{t}))}, \quad E_1 = (\sqrt{E_0} - \sqrt{2}\,\dot{p}(\tilde{t}))^2,$$

where $\tilde{t} = \tilde{t}(t_0, v_0)$ is implicitly defined by means of

$$\tilde{t} = t_0 + \frac{p(\tilde{t})}{\sqrt{2E_0}}.$$

The real domain of the map Ψ contains a half-plane of the type $t_0 \in \mathbb{R}$, $E > R_*$ (see [3]).

As an application of Theorem 3.1 we will obtain the following result, which is an upper bound for the velocities in the analytic case; also see [9, Example 5].

Theorem 4.1 *Let $\delta > 0$ and $p : \mathbb{R}_\delta \to \mathbb{C}$ be holomorphic and such that p maps reals into reals, $|p(z)| \le C$ for $z \in \mathbb{R}_\delta$, and $0 < a \le p(t) \le b$ for $t \in \mathbb{R}$. Then there exist constants $C_*, E_* > 0$, depending only upon the parameters, such that if $(t_n, E_n)_{n\in\mathbb{N}_0}$ is a forward complete real orbit of Ψ with $\liminf_{n\to\infty} E_n \ge E_*$, then there is $n_0 \in \mathbb{N}$ so that*

$$|E_n| \le C_*(\log n)^2, \quad n \ge n_0,$$

and for the velocities $v_n = \sqrt{2E_n}$ this means $|v_n| \le \sqrt{2C_*}\log n$ for $n \ge n_0$.

The idea of the proof is to use Theorem 3.1, not in the coordinates (t, E), but in (w, W) given by $w = \int_0^t \frac{ds}{p(s)^2}$ and $W = p(t)^2 E$. Thus we need to verify that the map $(w_0, W_0) \mapsto (w_1, W_1)$ satisfies the assumptions of Theorem 3.1. This will be accomplished in three steps. First we are going to show that Ψ has a well-defined holomorphic extension. In the second step we will prove that the map $(w_0, W_0) \mapsto (w_1, W_1)$ is exact symplectic, and the function $\mathfrak{h} = \mathfrak{h}(w_0, W_0)$ satisfying

$$W_1\, dw_1 - W_0\, dw_0 = d\mathfrak{h}$$

will be computed. Finally, after applying Theorem 3.1 to this new map, we will go back to the original to obtain the conclusion. Incidentally, we would like to mention that the quantity $W^{1/2}$ appears in [9, Example 5], where it is considered as an adiabatic invariant.

4.1 The complexified map

We start with two lemmas on holomorphic functions.

Lemma 4.2 *Let $g : \mathbb{R}_\delta \to \mathbb{C}$ be holomorphic and such that $\operatorname{Re} g'(z) > 0$ for $z \in \mathbb{R}_\delta$. Then g is one-to-one.*

Proof: This is a particular case of [10, Prop. 1.10]. □

Remark 4.3 Under the assumptions of Lemma 4.2, as g is non-constant and holomorphic, the image $g(\mathbb{R}_\delta) \subset \mathbb{C}$ is open. Thus $g^{-1} : g(\mathbb{R}_\delta) \to \mathbb{R}_\delta$ is well-defined and holomorphic by the inverse function theorem.

Lemma 4.4 *Let $g : \mathbb{R}_\delta \to \mathbb{C}$ be holomorphic such that g maps reals into reals and there exists $\alpha > 0$ so that $\operatorname{Re} g'(z) > \alpha$ for $z \in \mathbb{R}_\delta$. Then $\mathbb{R}_{\alpha\delta} \subset g(\mathbb{R}_\delta)$.*

Proof: Fix $w = a + ib \in \mathbb{R}_{\alpha\delta}$, i.e., we have $|b| < \alpha\delta$. In particular, we can choose $\sigma \in]0, \delta[$ such that $|b| < \alpha\sigma$ holds. To find a solution $z \in \mathbb{R}_\delta$ of $g(z) = w$, note first that $g(\mathbb{R}) = \mathbb{R}$ by assumption. Hence there is $x \in \mathbb{R}$ satisfying $g(x) = a$. We consider the functions $f_1(z) = g(z) - a$ and $f_2(z) = -ib$ and our intention is to apply Rouché's Theorem on the rectangular region bounded by

$$\Gamma = \{\xi : \operatorname{Re}\xi \in [x - \Delta, x + \Delta], \operatorname{Im}\xi = \pm\sigma\} \cup \{\xi : \operatorname{Re}\xi = x \pm \Delta, \operatorname{Im}\xi \in [-\sigma, \sigma]\},$$

where $\Delta > 0$ will be taken to be large enough (see below). Once we have established that

$$|f_1(\xi)| > |f_2(\xi)|, \quad \xi \in \Gamma, \tag{4.1}$$

the proof will be complete, since then $f_1(z) = g(z) - a$ and $f_1(z) + f_2(z) = g(z) - w$ will have the same numbers of zeros inside of Γ; this number is one, by the choice of x and as g is one-to-one by Lemma 4.2. To check (4.1) on the horizontal parts of Γ take $\xi = t \pm i\sigma$, where $t \in [x - \Delta, x + \Delta]$. Then $g(x) \in \mathbb{R}$ yields

$$|f_1(\xi)| = |g(t \pm i\sigma) - g(x)| \geq |\operatorname{Im} g(t \pm i\sigma)|.$$

As g' is real on $\mathbb{R}$, one has, using the hypothesis,

$$|\operatorname{Im} g(t \pm i\sigma)| = \left|\operatorname{Im} \int_x^{t \pm i\sigma} g'(z)\, dz\right| = \left|\operatorname{Im} \int_t^{t \pm i\sigma} g'(z)\, dz\right| = \left|\operatorname{Re} \int_0^{\sigma} g'(t \pm is)\, ds\right| \geq \alpha|\sigma|.$$

It follows that $|f_1(\xi)| \geq \alpha|\sigma| > |b| = |f_2(\xi)|$. It remains to verify (4.1) on the vertical parts of Γ. For, take $\xi = (x \pm \Delta) + is$, where $s \in [-\sigma, \sigma]$. Define $K = \max\{|g(x + iv)| : v \in [-\sigma, \sigma]\}$. Now observe that, for $t \in \mathbb{R}$,

$$\begin{aligned} |\operatorname{Re} g(t + is)| &= \left|\operatorname{Re} g(x + is) + \operatorname{Re} \int_{x+is}^{t+is} g'(z)\, dz\right| \geq \left|\operatorname{Re} \int_x^t g'(u + is)\, du\right| - K \\ &\geq \alpha|t - x| - K \end{aligned}$$

due to the hypothesis. As a consequence,

$$|f_1(\xi)| = |g(x \pm \Delta + is) - a| \geq \alpha\Delta - K - |a| > |b| = |f_2(\xi)|,$$

provided that we fix $\Delta > \alpha^{-1}(|a| + |b| + K)$. □

Now we return to the ping-pong map and define $\varphi(z) = p(z)^2$ for $z \in \mathbb{R}_\delta$. From the assumptions on p we may assume that $a \leq b \leq C$. Then

$$\delta_1 = \frac{a^2\delta}{4C^2}$$

satisfies $\delta_1 \leq \frac{\delta}{4}$. The expression $\operatorname{Arg} z \in]-\pi, \pi]$ for $z \in \mathbb{C} \setminus \{0\}$ will denote the argument of z.

Lemma 4.5 *The function φ satisfies*

$$\operatorname{Re} \varphi(z) \geq \frac{1}{2}\, a^2 \quad \textit{and} \quad |\varphi(z)| \leq 2b^2, \quad z \in \mathbb{R}_{\delta_1},$$

and moreover

$$|\operatorname{Arg} \varphi(z)| < \frac{\pi}{4}, \quad z \in \mathbb{R}_{\delta_1/2}.$$

Proof: From the Cauchy integral formula we deduce that

$$|\varphi'(z)| \leq \frac{2C^2}{\delta} \quad \text{for} \quad z \in \mathbb{R}_{\delta/2}.$$

Take $z = t + is \in \mathbb{R}_{\delta/2}$. Then $\varphi(z) = \varphi(t) + \int_t^{t+is} \varphi'(\xi)\, d\xi$ implies that

$$\operatorname{Re} \varphi(z) \geq a^2 + \operatorname{Re} \int_t^{t+is} \varphi'(\xi)\, d\xi \geq a^2 - \frac{2C^2}{\delta}\, |s|$$

as well as

$$|\varphi(z)| \le b^2 + \Big|\int_t^{t+is} \varphi'(\xi)\, d\xi\Big| \le b^2 + \frac{2C^2}{\delta}\, |s|$$

and

$$|\mathrm{Im}\, \varphi(z)| \le \frac{2C^2}{\delta}\, |s|.$$

Thus if $z \in \mathbb{R}_{\delta_1}$, then $\mathrm{Re}\, \varphi(z) \ge a^2/2$ and $|\varphi(z)| \le b^2 + 2\frac{C^2}{\delta}\frac{a^2\delta}{4C^2} \le 3b^2/2$. In addition, if $z \in \mathbb{R}_{\delta_1/2}$, then $|\mathrm{Im}\, \varphi(z)| \le a^2/4 < \mathrm{Re}\, \varphi(z)$, which yields the claim on the argument. □

Lemma 4.6 *The function $\tau : \mathbb{R}_{\delta_1} \to \mathbb{C}$, $\tau(z) = \int_0^z \frac{d\zeta}{p(\zeta)^2} = \int_0^z \frac{d\zeta}{\varphi(\zeta)}$, is holomorphic, one-to-one with holomorphic inverse, and satisfies*

$$\mathbb{R}_{\sigma(\Delta)} \subset \tau(\mathbb{R}_\Delta) \quad \textit{for} \quad \Delta \in]0, \delta_1], \tag{4.2}$$

where $\sigma(\Delta) = \frac{a^2}{2C^4}\Delta$.

Proof: By Lemma 4.5 the function $\varphi(z)$ does not vanish on the simply connected domain $\mathbb{R}_{\delta_1}$, and hence $\frac{1}{\varphi(z)}$ has a holomorphic primitive $\tau(z)$. Lemma 4.2 in conjunction with Lemma 4.5 implies that τ is one-to-one. Also τ maps reals into reals and satisfies

$$\mathrm{Re}\, \tau'(z) = \mathrm{Re}\left(\frac{1}{\varphi(z)}\right) = \frac{1}{|\varphi(z)|^2}\, \mathrm{Re}\, \varphi(z) \ge \frac{a^2}{2C^4}.$$

Thus Lemma 4.4 applies with $\alpha = \frac{a^2}{2C^4}$ to prove that (4.2) holds. Concerning the fact that $\tau^{-1} : \tau(\mathbb{R}_{\delta_1}) \to \mathbb{C}$ is holomorphic, cf. Remark 4.3. □

To extend the ping-pong map Ψ as a holomorphic map, we take the complex square root to be $\sqrt{z} = |z|^{1/2} \exp((i/2)\mathrm{Arg}\, z)$, where the complex plane is cut along $]-\infty, 0]$. In particular, $\sqrt{z}$ is holomorphic on $\mathbb{C}\setminus]-\infty, 0]$ and extends the positive square root. Note that $\sqrt{\varphi(z)} = p(z)$ holds for all $z \in \mathbb{R}_{\delta_1/2}$. To establish this identity, it is sufficient to adapt the proof of Lemma 4.5 to the function p to conclude that $|\mathrm{Arg}\, p(z)| < \frac{\pi}{4}$ for $z \in \mathbb{R}_{\delta_1/2}$.

Lemma 4.7 *Let $\underline{e} = \frac{8C^2}{\delta^2}$. Then for every $z \in \mathbb{R}_{\delta/4}$ and $E \in \mathbb{C}\setminus]-\infty, 0]$ such that $|E| > \underline{e}$ the equation*

$$\tilde{z} = z + \frac{p(\tilde{z})}{\sqrt{2E}} \tag{4.3}$$

has a unique solution $\tilde{z} = \tilde{z}(z, E)$ lying in $\mathbb{R}_{\delta/2}$. Moreover, $(z, E) \mapsto \tilde{z}(z, E)$ is holomorphic as a function of two variables. In addition, $\Delta \in]0, \delta/4]$ and $z \in \mathbb{R}_\Delta$ implies that $\tilde{z}(z, E) \in \mathbb{R}_{2\Delta}$.

Proof: Consider the function $g(\tilde{z}, E) = \tilde{z} - \frac{p(\tilde{z})}{\sqrt{2E}}$, so that we need to solve $g(\tilde{z}, E) = z$. For $\tilde{z} \in \mathbb{R}_{\delta/2}$ one has

$$|p'(\tilde{z})| \le \frac{2C}{\delta} \tag{4.4}$$

by the Cauchy integral formula. It follows that

$$\operatorname{Re}\frac{\partial g}{\partial \tilde z}(\tilde z, E) = 1 - \operatorname{Re}\frac{p'(\tilde z)}{\sqrt{2E}} \geq 1 - \frac{|p'(\tilde z)|}{\sqrt{2}|E|^{1/2}} \geq 1 - \frac{2C}{\sqrt{2}\delta \underline{e}^{1/2}} = \frac{1}{2}$$

for $\tilde z \in \mathbb{R}_{\delta/2}$. Thus from Lemma 4.2 we infer that $g(\tilde z, E) = z$ can have at most one solution $\tilde z \in \mathbb{R}_{\delta/2}$. Next let $\Delta \in]0, \delta/4]$. Then $\operatorname{Re}\frac{\partial g}{\partial \tilde z}(\tilde z, E) \geq 1/2$ for $\tilde z \in \mathbb{R}_{2\Delta}$. Therefore we can invoke Lemma 4.4 with $\alpha = 1/2$ to obtain $\mathbb{R}_{\Delta} \subset g(\cdot, E)(\mathbb{R}_{2\Delta})$. Finally we can apply the implicit function theorem to deduce that $\tilde z = \tilde z(z, E)$ is holomorphic on the domain $\mathbb{R}_{\delta/4} \times \{E \in \mathbb{C}\setminus]-\infty, 0] : |E| > \underline{e}\}$. □

Now we are in a position to define the holomorphic extension of the ping-pong map,

$$\Psi : \mathcal{U}_0 \subset \mathbb{C}^2 \to \mathbb{C}^2, \quad (z, E) \mapsto (z_1, E_1),$$

given by

$$z_1 = \tilde z + \frac{p(\tilde z)}{\sqrt{2}(\sqrt{E} - \sqrt{2}\,p'(\tilde z))}, \quad E_1 = (\sqrt{E} - \sqrt{2}\,p'(\tilde z))^2, \tag{4.5}$$

where $\tilde z = \tilde z(z, E)$ is from Lemma 4.7 and

$$\mathcal{U}_0 = \{(z, E) \in \mathbb{R}_{\delta/4} \times (\mathbb{C}\setminus]-\infty, 0]) : |E| > \underline{e}\}.$$

To see that this map is well-defined, we first observe that, according to Lemma 4.7, $\tilde z \in \mathbb{R}_{\delta/2}$. Thus both p and p' can be evaluated at $\tilde z$. Moreover, using (4.4) it follows that the denominator in the equation defining z_1 never vanishes: we have

$$|E|^{1/2} > \underline{e}^{1/2} = \frac{2\sqrt{2}\,C}{\delta} \geq \sqrt{2}\,|p'(\tilde z)|.$$

4.2 The change of variables and the new map

The map Ψ is exact symplectic on the domain

$$\widehat{\mathcal{U}}_0 = \{(z, E) \in \mathcal{U}_0 : z \in \mathbb{R}_{\delta_1/2}\}.$$

More precisely, it satisfies the identity

$$E_1\, dz_1 - E\, dz = dh \tag{4.6}$$

for

$$h(z, E) = -\frac{1}{2}\,p(\tilde z)^2\Big(\frac{1}{z_1 - \tilde z} + \frac{1}{\tilde z - z}\Big). \tag{4.7}$$

The new restriction on the size of $|\operatorname{Im} z|$ guarantees that h is holomorphic on $\widehat{\mathcal{U}}_0$. In fact, both denominators $z_1 - \tilde z$ and $\tilde z - z$ do not vanish. This is a consequence of the definitions of z_1 and $\tilde z$, together with the inequality

$$|p(\tilde z)| \geq \frac{1}{\sqrt{2}}\,a > 0, \quad z \in \mathbb{R}_{\delta_1/2},$$

which in turn follows from Lemmas 4.5 and 4.7.

The generating function for the ping-pong map was computed in [3]. This computation, together with the relationship between the function h and the generating function (see [5]), imply that (4.6), (4.7) holds. Note that all computations in [3] were done on the real domain of Ψ, but once again we rely on the uniqueness of holomorphic extensions.

Later we will need to reformulate (4.6), (4.7) in the new variables (w, W), where $w = \tau(z)$ and $W = p(z)^2 E$. This can be achieved from general principles, without any further computation. For this reason we include a short digression into general maps.

Consider the space $\mathbb{C}^2$ endowed with the 1-form

$$\sigma = p\, dq,$$

where $q, p \in \mathbb{C}$ are the coordinates of a point. Assume that $D, \mathcal{D} \subset \mathbb{C}^2$ are two domains with sub-domains $D_1 \subset D$ and $\mathcal{D}_1 \subset \mathcal{D}$. Let $\chi : \mathcal{D} \to D$ be a holomorphic diffeomorphism such that $\chi(\mathcal{D}_1) \subset D_1$ and

$$\chi^* \sigma = \sigma + dm$$

for some holomorphic function $m : \mathcal{D} \to \mathbb{C}$. In addition, let $T : D_1 \to \mathbb{C}^2$ be a holomorphic map with $T(D_1) \subset D$ and

$$T^* \sigma = \sigma + dh$$

for some holomorphic function $h : D_1 \to \mathbb{C}$. Then $\tilde{T} = \chi^{-1} \circ T \circ \chi : \mathcal{D}_1 \to \mathcal{D}$ is well-defined and a short calculation reveals that

$$\tilde{T}^* \sigma = \sigma + d\mathfrak{h} \tag{4.8}$$

for

$$\mathfrak{h} = h \circ \chi + m - m \circ \tilde{T}. \tag{4.9}$$

In fact, the standard properties of pullbacks of differential forms yield

$$\begin{aligned} d(h \circ \chi) &= \chi^*(dh) = \chi^*(T^*\sigma - \sigma) = (T \circ \chi)^* \sigma - \chi^* \sigma = (\chi \circ \tilde{T})^* \sigma - \chi^* \sigma \\ &= \tilde{T}^*(\chi^* \sigma) - \chi^* \sigma = \tilde{T}^*(\sigma + dm) - \sigma - dm = \tilde{T}^* \sigma - \sigma + d(m \circ \tilde{T} - m), \end{aligned}$$

which proves (4.8).

Now we go back to the ping-pong and introduce the full change of variables $\Gamma : (z, E) \mapsto (w, W)$.

Lemma 4.8 *The map*

$$\Gamma : \mathbb{R}_{\delta_1} \times \mathbb{C} \to \mathbb{C}^2, \quad (z, E) \mapsto (w, W),$$

where $w = \tau(z)$ and $W = p(z)^2 E$, is a holomorphic diffeomorphism between $\mathbb{R}_{\delta_1} \times \mathbb{C}$ and $\Gamma(\mathbb{R}_{\delta_1} \times \mathbb{C})$ that verifies $\Gamma^* \sigma = \sigma$. Moreover, if $\Delta \in]0, \delta_1]$, then $\mathbb{R}_{\sigma(\Delta)} \times \mathbb{C} \subset \Gamma(\mathbb{R}_\Delta \times \mathbb{C})$.

Proof: According to Lemma 4.6, Γ is holomorphic and satisfies $\mathbb{R}_{\sigma(\Delta)} \times \mathbb{C} \subset \Gamma(\mathbb{R}_\Delta \times \mathbb{C})$ for $\Delta \in]0, \delta_1]$, due to (4.2) and the fact that its inverse Γ^{-1} is given by $(w, W) \mapsto (z, p(z)^{-2} W)$, where $z = \tau^{-1}(w)$. This inverse is also holomorphic. To prove that $\Gamma^* \sigma = \sigma$, observe that $dw = \frac{1}{p(z)^2}\, dz$, and hence $W\, dw = E\, dz$ as desired. □

To adjust our situation to the general framework as outlined above, we define

$$\mathcal{D} = \Gamma(\mathbb{R}_{\delta_1} \times \mathbb{C}), \quad D = \mathbb{R}_{\delta_1} \times \mathbb{C}, \quad \chi = \Gamma^{-1}, \quad T = \Psi.$$

Furthermore, we take $m = 0$ and h from (4.7). To introduce $\mathcal{D}_1$ and D_1, let $\Delta = \frac{\delta_1}{4}$ and take $\rho \geq 4$ so large that $2\Delta + \frac{\delta}{2\sqrt{\rho}} < \delta_1$. Then we define

$$\begin{aligned} \mathcal{D}_1 &= \Big\{(w, W) \in \mathbb{R}_{\sigma(\Delta)} \times \mathbb{C} : |\operatorname{Arg} W| < \frac{\pi}{4}, |W| > \rho\, C^2 \underline{e}\Big\}, \\ D_1 &= \{(z, E) \in \mathbb{R}_\Delta \times \mathbb{C} : E \in \mathbb{C} \setminus]-\infty, 0], |E| > \rho \underline{e}\}, \end{aligned}$$

and check all the conditions set out before. First of all, $D_1 \subset D$ and $\mathcal{D}_1 \subset \mathcal{D}$ are immediate, using Lemma 4.8 for the latter. Also $\chi : \mathcal{D} \to D$ is a holomorphic diffeomorphism by definition, and moreover $\chi(\mathcal{D}_1) \subset D_1$. For, let $(z, E) = \Gamma^{-1}(w, W) \in \chi(\mathcal{D}_1)$. Then $w \in \mathbb{R}_{\sigma(\Delta)}$ implies $z \in \mathbb{R}_\Delta$ by Lemma 4.8. Furthermore, since $|p(z)| \leq C$,

$$|E| = \frac{|W|}{|p(z)|^2} > \frac{\rho\, C^2 \underline{e}}{C^2} = \rho \underline{e}$$

and due to Lemma 4.5,

$$|\operatorname{Arg} E| \leq |\operatorname{Arg} W| + \left|\operatorname{Arg} \frac{1}{\varphi(z)}\right| < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2},$$

so that in particular $E \notin]-\infty, 0]$ and accordingly $(z, E) \in D_1$. Next, to see that $T(D_1) \subset D$, we have to make sure that $(z_1, E_1) = T(z, E)$ has $|\operatorname{Im} z_1| < \delta_1$ for $(z, E) \in D_1$. As $z \in \mathbb{R}_\Delta$ and $|E| > \rho \underline{e} > \underline{e}$, we have $\tilde{z} \in \mathbb{R}_{2\Delta}$ by Lemma 4.7. In addition, (4.4) yields $|p'(\tilde{z})| \leq \frac{2C}{\delta}$ and thus by the definition of E_1:

$$|E_1|^{1/2} = |\sqrt{E} - \sqrt{2}\, p'(\tilde{z})| \geq |E|^{1/2} - \frac{2\sqrt{2}\, C}{\delta} \geq \frac{1}{2}\, |E|^{1/2};$$

recall that $\underline{e} = \frac{8C^2}{\delta^2}$ and $\rho \geq 4$. Thus by the definition of z_1:

$$|\operatorname{Im} z_1| \leq |\operatorname{Im} \tilde{z}| + \frac{|p(\tilde{z})|}{\sqrt{2}\, |E_1|^{1/2}} \leq 2\Delta + \frac{\sqrt{2}\, C}{|E|^{1/2}} \leq 2\Delta + \frac{\delta}{2\sqrt{\rho}} < \delta_1,$$

which completes the argument for $T(D_1) \subset D$. Since $D_1 \subset \widehat{\mathcal{U}}_0$, h from (4.7) is well-defined and holomorphic on D_1 and we have $T^*\sigma = \sigma + dh$ due to (4.6).

Now that we have verified the conditions of the general argument, we can conclude from (4.8) and (4.9) that $\Phi = \Gamma \circ \Psi \circ \Gamma^{-1} : \mathcal{D}_1 \to \mathbb{C}^2$, $(w, W) \mapsto (w_1, W_1)$, is well-defined and satisfies

$$W_1\, dw_1 - W\, dw = d\mathfrak{h}, \quad \mathfrak{h} = h \circ \Gamma^{-1}. \tag{4.10}$$

4.3 Application of the main theorem and proof of Theorem 4.1

To summarize, so far we have established that the map $\Phi : \mathcal{D}_1 \to \mathbb{C}^2$ is well-defined and holomorphic. We are going to apply Theorem 3.1 with $f = \Phi$, $\underline{r} = \rho C^2 \underline{e}$, $\eta = \frac{1}{2}$, $\alpha = \frac{1}{2}$ and $\gamma = \sqrt{2}$. For the width of the strip we take δ to be $\sigma(\Delta)$, and in order to write Φ in the required form (3.3), we introduce $F_1 = F$ and $F_2 = G$ for

$$F(w, W) = \sqrt{W}(w_1 - w) - \sqrt{2}, \quad G(w, W) = \frac{W_1 - W}{\sqrt{W}}. \tag{4.11}$$

We also define $\tilde{F}(w, W) = w_1 - w - \frac{\sqrt{2}}{\sqrt{W}}$, so that $F(w, W) = \sqrt{W}\,\tilde{F}(w, W)$. Then the assumptions (a) and (b) of Theorem 3.1 are satisfied. For, recall from (3.2) that Φ needs to be defined on

$$\Omega = \{(w, W) \in \mathbb{R}_\delta \times \mathbb{C} : \operatorname{Re} W > \underline{r}, |\operatorname{Im} W| < \eta |W|\},$$

which is the case due to $\Omega \subset \mathcal{D}_1$.

To derive the needed bounds $|F| = \mathcal{O}(|W|^{-1/2})$ and $|G| = \mathcal{O}(|W|^{-1/2})$ as required by (c), we need to make some preliminary observations. From Lemma 4.7 we know that $\tilde{z} \in \mathbb{R}_{\delta/2}$, so that $|p'(\tilde{z})| \le 2C/\delta$ by (4.4). In general, if $\xi \in \mathbb{C}$ satisfies $\operatorname{Re}\xi > 0$, then $\operatorname{Re}\sqrt{\xi} > \frac{1}{\sqrt{2}}\,|\xi|^{1/2}$. Therefore if $|E| > 2\underline{e}$ and $\operatorname{Re} E > 0$, then

$$\operatorname{Re}\,(\sqrt{E} - \sqrt{2}\,p'(\tilde{z})) > \frac{1}{\sqrt{2}}\,\sqrt{2\underline{e}} - \sqrt{2}\,\frac{2C}{\delta} = 0,$$

and hence $E_1 \in \mathbb{C}\backslash\,]-\infty, 0]$. As a consequence, $\sqrt{E_1}$ can be understood as a single-valued expression and we can write the first equation in (4.5) as

$$z_1 = \tilde{z} + \frac{p(\tilde{z})}{\sqrt{2E_1}}. \tag{4.12}$$

Lastly, if even $z \in \mathbb{R}_\Delta$, $|E| > \rho\underline{e}$ and $\operatorname{Re} E > 0$ holds, then one also has

$$|\sqrt{E} - \sqrt{2}\,p'(\tilde{z})| \ge \sqrt{|E|} - \sqrt{2}\,\frac{2C}{\delta} \ge \frac{1}{2}\sqrt{|E|},$$

and hence

$$|E_1| \ge \frac{1}{4}\,|E|. \tag{4.13}$$

We also have $z \in \mathbb{R}_\Delta \subset \mathbb{R}_{\delta_1}$, and therefore $|\varphi(z)| \le 2b^2$ by Lemma 4.5. It follows that $|E| = |\frac{W}{\varphi(z)}| \ge \frac{1}{2b^2}\,|W|$, and thus $|E_1| \ge \frac{1}{8b^2}\,|W|$ due to (4.13). Then from the definitions of $\tilde{z}$ and z_1, cf. (4.3) and (4.12),

$$|z - \tilde{z}| \le \frac{C}{\sqrt{2}|E|^{1/2}} \le Cb\,|W|^{-1/2}, \quad |z_1 - \tilde{z}| \le \frac{C}{\sqrt{2}|E_1|^{1/2}} \le 2Cb\,|W|^{-1/2}. \tag{4.14}$$

In particular, $|z - z_1| \le 3Cb\,|W|^{-1/2}$. To bound $\tilde{F}$, we first observe that

$$z_1 - z = \tilde{z} + \frac{p(\tilde{z})}{\sqrt{2E_1}} - z = \Big[\frac{1}{\sqrt{2E}} + \frac{1}{\sqrt{2E_1}}\Big]\,p(\tilde{z}). \tag{4.15}$$

Since $|\mathrm{Arg}\, W| < \frac{\pi}{4}$ and $|\mathrm{Arg}\, \frac{1}{\varphi(z)}| < \frac{\pi}{4}$, we have $\sqrt{W} = \sqrt{\varphi(z)}\sqrt{E} = p(z)\sqrt{E}$. Now the expression for $\tilde{F}$ is split up according to

$$\tilde{F}(w,W) = w_1 - w - \frac{\sqrt{2}}{\sqrt{W}} = w_1 - w - \frac{\sqrt{2}}{\sqrt{E}}\,\frac{1}{p(z)} = \tilde{F}_1 + \tilde{F}_2,$$

where

$$\begin{aligned}\tilde{F}_1 &= w_1 - w - \Big[\frac{1}{\sqrt{2E}} + \frac{1}{\sqrt{2E_1}}\Big]\,\frac{1}{p(z)}\\ &= \tau(z_1) - \tau(z) - \frac{z_1 - z}{p(z)p(\tilde{z})}\end{aligned}$$

by (4.15), and

$$\tilde{F}_2 = \Big[\frac{1}{\sqrt{2E_1}} - \frac{1}{\sqrt{2E}}\Big]\,\frac{1}{p(z)}.$$

For the first term,

$$|\tilde{F}_1| = \Big|\int_z^{z_1}\Big(\frac{1}{p(\zeta)^2} - \frac{1}{p(z)p(\tilde{z})}\Big)\,d\zeta\Big|. \tag{4.16}$$

From geometric considerations we deduce that

$$\begin{aligned}|\zeta - z| &\le |z_1 - z| \le 3Cb\,|W|^{-1/2},\\ |\zeta - \tilde{z}| &\le \max\{|\tilde{z} - z|, |\tilde{z} - z_1|\} \le 2Cb\,|W|^{-1/2},\end{aligned}$$

for any point ζ on the segment $[z, z_1]$. The upper and lower bounds for p provided by Lemma 4.5 together with the estimate for $|p'|$ allow us to find a constant $K_1 > 0$ such that for each $\zeta \in [z, z_1]$:

$$\Big|\frac{1}{p(\zeta)^2} - \frac{1}{p(z)p(\tilde{z})}\Big| \le K_1 \max\{|\zeta - z|, |\zeta - \tilde{z}|\} \le 3CK_1 b\,|W|^{-1/2}.$$

As a consequence, (4.16) yields

$$|\tilde{F}_1| \le 3CK_1 b\,|W|^{-1/2}\,|z_1 - z| \le 9C^2 K_1 b^2\,|W|^{-1}.$$

To bound $\tilde{F}_2$, we note that by (4.5) and (4.4)

$$|\sqrt{E_1} - \sqrt{E}| \le \sqrt{2}\,|p'(\tilde{z})| \le \frac{2\sqrt{2}\,C}{\delta}.$$

Therefore, due to (4.13),

$$|\tilde{F}_2| = \frac{|\sqrt{E_1} - \sqrt{E}|}{\sqrt{2}\sqrt{|E_1|}\sqrt{|E|}}\,\frac{1}{|p(z)|} \le \frac{2C}{\delta\sqrt{|E_1|}\sqrt{|W|}} \le \frac{4C|p(z)|}{\delta|W|} \le \frac{4C^2}{\delta}\,|W|^{-1},$$

and thus we deduce that altogether

$$|F(w,W)| = \sqrt{|W|}\,|\tilde{F}(w,W)| \le \sqrt{|W|}\,(|\tilde{F}_1| + |\tilde{F}_2|) \le C_1 |W|^{-1/2}$$

holds for an appropriate constant $C_1 > 0$ depending only upon δ, C, a, b. Concerning the bound on $|G|$, according to [7, (5.10)] one has

$$W_1 - W = \frac{1}{2}\,\varphi(\tilde z)\int_0^1 (1-\lambda)\Big[\varphi''((1-\lambda)\tilde z + \lambda z) - \varphi''((1-\lambda)\tilde z + \lambda z_1)\Big]\,d\lambda\,, \tag{4.17}$$

where once again $\varphi(z) = p(z)^2$. In the paper just mentioned this relation was used for a real-valued p, but as all functions involved are holomorphic, it extends to the complex-valued case due to the uniqueness theorem in complex analysis. Now $\Delta \le \delta_1/4$ and $\delta_1 \le \delta/4$ yields $|\mathrm{Im}\,((1-\lambda)\tilde z + \lambda z)| \le |\mathrm{Im}\,\tilde z| + |\mathrm{Im}\,z| < 2\Delta + \Delta = 3\Delta < \delta/2$ and similarly $|\mathrm{Im}\,((1-\lambda)\tilde z + \lambda z_1)| \le |\mathrm{Im}\,\tilde z| + |\mathrm{Im}\,z_1| < 2\Delta + \delta_1 < \delta/2$. Owing to the Cauchy integral formula one has $|\varphi'''(z)| \le (2/\delta)^3 C^2$ for $z \in \mathbb{R}_{\delta/2}$. Therefore (4.17) implies that

$$|W_1 - W| \le 4\,C^4\delta^{-3}|z_1 - z| \le 12\,C^5 b\,\delta^{-3}\,|W|^{-1/2},$$

and hence

$$|G(w,W)| \le 12\,C^5 b\,\delta^{-3}\,|W|^{-1},$$

which is in fact better than $G = \mathcal{O}(|W|^{-1/2})$ what we would have needed in assumption (c) of Theorem 3.1.

Lastly we are going to verify the hypothesis (d) of Theorem 3.1, the function $\mathfrak{h}$ being given by (4.10) with h from (4.7). We also note that $\mathfrak{h}_0(w,W) = -\sqrt{2W}$ for our choice of parameters and we need to establish that $|\mathfrak{h} - \mathfrak{h}_0| = \mathcal{O}(1)$. To simplify the estimates, it is convenient to express h and $\mathfrak{h}$ is a different way, which is based on the definition of the maps Ψ and Φ. More precisely, using the various definitions we write

$$\begin{aligned}
h(z,E) &= -\frac{1}{2}\,p(\tilde z)^2\Big(\frac{1}{z_1 - \tilde z} + \frac{1}{\tilde z - z}\Big)\\
&= -\frac{1}{2}\,p(\tilde z)^2\Big(\frac{\sqrt{2E_1}}{p(\tilde z)} + \frac{\sqrt{2E}}{p(\tilde z)}\Big)\\
&= -\frac{1}{2}\,p(\tilde z)\Big(\sqrt{2}\,(\sqrt{E} - \sqrt{2}\,p'(\tilde z)) + \sqrt{2E}\Big)\\
&= -\sqrt{2E}\,p(\tilde z) + p(\tilde z)p'(\tilde z)
\end{aligned}$$

and

$$\mathfrak{h}(w,W) = -\sqrt{2W}\,\frac{p(\tilde z)}{p(z)} + p(\tilde z)p'(\tilde z),$$

where $(z,E) = \Gamma^{-1}(w,W)$. As a consequence,

$$\mathfrak{h}(w,W) - \mathfrak{h}_0(w,W) = \sqrt{2W}\Big(1 - \frac{p(\tilde z)}{p(z)}\Big) + p(\tilde z)p'(\tilde z).$$

Similarly as before, the lower bound on $|p(z)|$ and the upper bound on $|p'|$ together with (4.14) lead to

$$\Big|1 - \frac{p(\tilde z)}{p(z)}\Big| \le K_2\,|z - \tilde z| \le C K_2\,b\,|W|^{-1/2},$$

which proves that

$$|\mathfrak{h}(w,W)-\mathfrak{h}_0(w,W)|\le C_2.$$

Let us now fix $E_* > \frac{1}{a^2}\,\underline{r}$, where $\underline{r}$ appears in the definition of the domain Ω. Suppose that $(t_n,E_n)_{n\in\mathbb{N}_0}$ is a forward complete real orbit of Ψ with $\liminf_{n\to\infty} E_n \ge E_*$. By assumption there exists $N\in\mathbb{N}_0$ such that $E_n > \frac{1}{a^2}\,\underline{r}$ for $n\ge N$. Then $W_n = p(t_n)^2E_n \ge a^2E_n > \underline{r}$ for $n\ge N$ shows that $(w_n,W_n)_{n\ge N}$ is a forward complete real orbit for Φ, and hence Theorem 3.1 is applicable. □

5 An example

Consider the map $f:(\theta,r)\mapsto(\theta_1,r_1)$ defined as

$$\theta_1=\theta+\sqrt{\frac{2}{r_1}},\quad r_1=r-q(\theta), \tag{5.1}$$

where q is a given function. This map is symplectic, because it can be expressed in the form $r=\frac{\partial g}{\partial\theta}$, $r_1=-\frac{\partial g}{\partial\theta_1}$, for the generating function

$$g(\theta,\theta_1)=\frac{2}{\theta_1-\theta}+Q(\theta)$$

with Q denoting a primitive of q. This is possibly the simplest family of maps in the framework of Section 3. We will analyze the dynamics for the particular case where $q(\theta)=-\frac{2\theta}{1+\theta^2}$.

Assuming that $(\theta_0,r_0)\in\mathbb{R}^2$ is such that $\theta_0>0$ and $r_0>0$, we observe that a forward complete orbit $(\theta_n,r_n)_{n\in\mathbb{N}_0}$ can be produced; the sequences θ_n and r_n are positive and increasing. We are going to prove by direct analysis that

$$0<\liminf_{n\to\infty}\frac{r_n}{(\log n)^2}\le\limsup_{n\to\infty}\frac{r_n}{(\log n)^2}<\infty.$$

However, as we will see, Theorem 3.1 is not applicable in this case, since the bound (3.4) does only hold uniformly for θ in bounded sets, but not for $\theta\in\mathbb{R}_\delta$. This could mean that some assumption of Theorem 3.1 can be relaxed, or for particular examples unbounded orbits could exist.

5.1 Applicability of Theorem 3.1

First note that the map (5.1) can be written in the form (3.3), with $\alpha=1/2$, $\gamma=\sqrt{2}$,

$$F_1(\theta,r)=\sqrt{2}\left(\frac{1}{\sqrt{1-\frac{q(\theta)}{r}}}-1\right),\quad F_2(\theta,r)=-\frac{1}{\sqrt{r}}\,q(\theta), \tag{5.2}$$

where in the following the complex square root will be understood as before.

The function q is bounded and holomorphic on any strip $\mathbb{R}_\delta$ with $\delta < 1$; we fix $\delta = 1/2$ for definiteness. It follows that $|q(\theta)| \le C$ for $\theta \in \mathbb{R}_{1/2}$, where $C = 16$. To prove this, consider $\theta \in \mathbb{R}_{1/2}$ and $|\theta| \ge 2$ first. Here we have

$$|q(\theta)| \le \frac{2|\theta|}{||\theta|^2 - 1|} \le \frac{2|\theta|}{(|\theta|^2/2)} \le 2.$$

If $\theta \in \mathbb{R}_{1/2}$ and $|\theta| \le 2$, then $|1 + \theta^2| = |\theta + i||\theta - i| \ge 1/4$ yields

$$|q(\theta)| = \frac{2|\theta|}{|1 + \theta^2|} \le 2 \cdot 4|\theta| \le 16.$$

We now proceed as in the previous section to find an appropriate domain of holomorphy $\Omega \subset \mathbb{C}^2$. Clearly the hypotheses (a)-(c) of Theorem 3.1 are satisfied. The validity of (d) is more delicate. As has been used before, in general the primitive of the form $r_1\, d\theta_1 - r\, d\theta$ is computed from the generating function g via $\mathfrak{h}(\theta, r) = -g(\theta, \theta_1(\theta, r))$. Thus for the map from (5.1) we get

$$\mathfrak{h}(\theta, r) = -\sqrt{2(r - q(\theta))} - Q(\theta).$$

We also note that

$$\mathfrak{h}_0(\theta, r) = -\sqrt{2r},$$

cf. (3.1), and (3.4) says that we should have $|\mathfrak{h}(\theta, r) - \mathfrak{h}_0(\theta, r)|$ bounded, uniformly in $(\theta, r) \in \Omega$, in order that Theorem 3.1 is applicable. The primitive is $Q(\theta) = -\log(1 + \theta^2)$, and the best estimate one can get is $|\mathfrak{h}(\theta, r) - \mathfrak{h}_0(\theta, r)| = \mathcal{O}(1)$ for each $\theta \in \mathbb{R}_{1/2}$, but the bound is not uniform.

5.2 The real dynamics

We start with a useful notion of equivalence for sequences.

Definition 5.1 *Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences of eventually positive numbers. We say that (a_n) is equivalent to (b_n), if there exist constants $C > c > 0$ and $n_0 \in \mathbb{N}$ such that $ca_n \le b_n \le Ca_n$ is verified for $n \ge n_0$; this will be written as $(a_n) \simeq (b_n)$.*

Lemma 5.2 *Let $(\rho_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive numbers such that*

$$(\rho_n) \simeq \Big(\sum_{k=2}^{n-1} \frac{1}{R_k^\sigma}\Big) \tag{5.3}$$

for some $\sigma \ge 1$, where $R_k = \sum_{j=1}^{k-1} \frac{1}{\rho_j^{1/2}}$. If $\sigma > 1$, then (ρ_n) is bounded. If $\sigma = 1$, then $(\rho_n) \simeq ((\log n)^2)$.

The proof is given in the next subsection.

Going back to the map f from (5.1), we consider $(\theta_0, r_0) \in \mathbb{R}^2$ such that $\theta_0 > 0$ and $r_0 > 0$. Let $(\theta_n, r_n)_{n\in\mathbb{N}_0}$ denote the resulting forward complete orbit; we already noted above that θ_n

and r_n are positive and increasing. As a first step we are going to show that $\lim_{n\to\infty}\theta_n = \infty$. Otherwise we would have $0 < \theta_0 \le \theta_n \le C$ for $n \in \mathbb{N}$. Then $r_{n+1} - r_n = -q(\theta_n) = |q(\theta_n)|$ implies that

$$0 < m \le r_{n+1} - r_n \le M < \infty, \quad n \in \mathbb{N},$$

where $m = \inf_{n\in\mathbb{N}} |q(\theta_n)|$ and $M = \sup_{n\in\mathbb{N}} |q(\theta_n)|$. Then $(r_n) \simeq (n)$ and consequently

$$\theta_n = \theta_0 + \sum_{j=1}^{n} \sqrt{\frac{2}{r_j}} \to \infty, \tag{5.4}$$

which is a contradiction.

In terms of $R_n = \sum_{j=1}^{n-1} \frac{1}{\sqrt{r_j}}$ the relation (5.4) can be written as $\theta_n = \theta_0 + \sqrt{2}\, R_{n+1}$, which implies that also $\lim_{n\to\infty} R_n = \infty$ holds, and furthermore $(\theta_n) \simeq (R_{n+1})$. Since $R_{n+1} - R_n \to 0$, we deduce that $(\theta_n) \simeq (R_n)$ is verified. Due to $\lim_{n\to\infty}\theta_n = \infty$ and

$$|q(\theta)|\, \theta = \frac{2\theta^2}{1+\theta^2},$$

there are constants $K > k > 0$ such that

$$\frac{k}{\theta_n} \le |q(\theta_n)| \le \frac{K}{\theta_n}, \quad n \ge 0.$$

If follows that for suitable $n_0 \in \mathbb{N}$ and constants $K^* > k^* > 0$ one has

$$\frac{k^*}{R_n} \le |q(\theta_n)| \le \frac{K^*}{R_n}, \quad n \ge n_0.$$

As a consequence, for $n \ge n_0$ we obtain from $r_n = r_{n_0-1} + \sum_{j=n_0}^{n} |q(\theta_j)|$ that

$$r_{n_0-1} + k^* \sum_{j=n_0}^{n} \frac{1}{R_j} \le r_n \le r_{n_0-1} + K^* \sum_{j=n_0}^{n} \frac{1}{R_j}.$$

From this it is easily deduced that $(r_n) \sim (\sum_{j=2}^{n-1} \frac{1}{R_j})$, and hence Lemma 5.2 applies with $\sigma = 1$. Its conclusion is that $(r_n) \simeq ((\log n)^2)$.

5.3 Some auxiliary results

Lemma 5.3 *For some $a, b > 0$ consider the differential equation $y' = ae^{-by^{1/2}}$, $y > 0$. Then every solution satisfies*

$$\lim_{x\to\infty} \frac{y(x)}{(\ln x)^2} = \frac{1}{b^2}. \tag{5.5}$$

Proof: First we observe that y is increasing and $y'(x) < a$. Thus y is well-defined for $x \to \infty$ and satisfies $y(x) \to \infty$ as $x \to \infty$, since the equation has no equilibrium. By separation of variables,

$$\left(y(x)^{1/2} - \frac{1}{b}\right) e^{by(x)^{1/2}} = \frac{ab}{2}\, x + C,$$

where C is a constant. Taking any $b_1 < b < b_2$ we deduce that, for large x,

$$e^{b_1 y(x)^{1/2}} < \frac{ab}{2}\, x < e^{b_2 y(x)^{1/2}},$$

which yields the claim upon taking the logarithm. $\square$

Lemma 5.4 *Let $\rho : [1,\infty[\to [1,\infty[$ be continuous. Furthermore, suppose that there are constants $0 < \gamma < \Gamma$ and $x_0 > 1$ so that*

$$\gamma\, \rho(x) \le \int_1^x \frac{dy}{R(y)^\sigma} \le \Gamma\, \rho(x), \quad x \ge x_0, \tag{5.6}$$

for some $\sigma \ge 1$, where $R(y) = \int_1^y \frac{d\xi}{\rho(\xi)^{1/2}}$ for $y \ge 1$. If $\sigma > 1$, then ρ is bounded. If $\sigma = 1$, then there are constants $0 < c < C$ such that

$$c \le \frac{\rho(x)}{(\log x)^2} \le C \tag{5.7}$$

for x sufficiently large.

Proof: Let $\phi(x) = \int_1^x \frac{dy}{R(y)^\sigma}$. Then $\phi' = R^{-\sigma}$ and $\phi'' = -\sigma R^{-(\sigma+1)} R'$ together with $R' = \rho^{-1/2}$ leads to the differential equation $\phi'' = -\sigma (\phi')^{\frac{\sigma+1}{\sigma}} \rho^{-1/2}$. Then (5.6) yields

$$\gamma\sigma^2\, \frac{\phi'(x)^{\frac{2(\sigma+1)}{\sigma}}}{\phi''(x)^2} \le \phi(x) \le \Gamma\sigma^2\, \frac{\phi'(x)^{\frac{2(\sigma+1)}{\sigma}}}{\phi''(x)^2}, \quad x \ge x_0.$$

Owing to $\phi'' < 0$, this can be rewritten as

$$-\Gamma^{1/2}\sigma\, \frac{\phi'(x)}{\phi(x)^{1/2}} \le \frac{\phi''(x)}{\phi'(x)^{\frac{1}{\sigma}}} \le -\gamma^{1/2}\sigma\, \frac{\phi'(x)}{\phi(x)^{1/2}}, \quad x \ge x_0. \tag{5.8}$$

First we consider the case where $\sigma = 1$. Here we obtain

$$\begin{aligned} \frac{d}{dx}\Big(\log \phi'(x) + 2\Gamma^{1/2}\phi(x)^{1/2} \Big) &\ge 0, \\ \frac{d}{dx}\Big(\log \phi'(x) + 2\gamma^{1/2}\phi(x)^{1/2} \Big) &\le 0, \end{aligned}$$

for $x \ge x_0$. Upon integration and exponentiation one gets

$$\phi'(x)\, e^{2\gamma^{1/2}\phi(x)^{1/2}} \le e^{b_0}, \quad \phi'(x)\, e^{2\Gamma^{1/2}\phi(x)^{1/2}} \ge e^{B_0},$$

for $x \ge x_0$, where $b_0 = \log \phi'(x_0) + 2\gamma^{1/2}\phi(x_0)^{1/2}$ and $B_0 = \log \phi'(x_0) + 2\Gamma^{1/2}\phi(x_0)^{1/2}$. Therefore ϕ is a lower solution of $y' = a e^{-by^{1/2}}$ for $\underline{a} = e^{b_0}$, $\underline{b} = 2\gamma^{1/2}$ and an upper solution for $\overline{a} = e^{B_0}$, $\overline{b} = 2\Gamma^{1/2}$. Let $\underline{y}$ and $\overline{y}$ denote the corresponding solutions with common initial values $\underline{y}(x_0) = \overline{y}(x_0) = \phi(x_0)$. Then

$$\underline{y}(x) \le \phi(x) \le \overline{y}(x), \quad x \ge x_0.$$

According to Lemma 5.3 one has $\lim_{x\to\infty}\frac{y(x)}{(\ln x)^2}=\frac{1}{4\gamma}$ and $\lim_{x\to\infty}\frac{\bar{y}(x)}{(\ln x)^2}=\frac{1}{4\Gamma}$. Recalling (5.6), this leads to (5.7), where we can take for instance $c=\frac{1}{8\gamma\Gamma}$ and $C=\frac{1}{2\gamma\Gamma}$.

In the second case $\sigma>1$, (5.8) can be expressed as

$$-\Gamma^{1/2}\sigma\,\frac{\phi'(x)}{\phi(x)^{1/2}}\le\frac{\phi''(x)}{\phi'(x)^{\frac{1}{\sigma}}}\le-\gamma^{1/2}\sigma\,\frac{\phi'(x)}{\phi(x)^{1/2}}.$$

Upon integration of the inequality on the right-hand side, it is found that

$$\frac{d}{dx}\Big(\frac{\sigma}{\sigma-1}\,\phi'(x)^{\frac{\sigma-1}{\sigma}}+2\gamma^{1/2}\sigma\,\phi(x)^{1/2}\Big)\le 0.$$

Therefore it follows from $\phi'\ge 0$ that

$$2\gamma^{1/2}\sigma\,\phi(x)^{1/2}\le\frac{\sigma}{\sigma-1}\,\phi'(x)^{\frac{\sigma-1}{\sigma}}+2\gamma^{1/2}\sigma\,\phi(x)^{1/2}\le\frac{\sigma}{\sigma-1}\,\phi'(x_0)^{\frac{\sigma-1}{\sigma}}+2\gamma^{1/2}\sigma\,\phi(x_0)^{1/2},$$

which shows that ϕ is bounded. Since $1\le\rho(x)\le\gamma^{-1}\phi(x)$, also ρ is bounded. □

Proof of Lemma 5.2 : First we consider the case where $\sigma=1$.

Step 1: $\rho_n\to\infty$ as $n\to\infty$. Otherwise we would have $\rho_n\to\rho_\infty\in]0,\infty[$ as $n\to\infty$. But then $(R_n)\simeq(n)$, and consequently the series $\sum\frac{1}{R_n}$ is divergent. However, this contradicts (5.3).

Step 2: $R_n\to\infty$ as $n\to\infty$. Otherwise we would have $R_n\to R_\infty\in]0,\infty[$ as $n\to\infty$. Then (5.3) yields

$$(\rho_n)\simeq\Big(\sum_{k=2}^{n-1}\frac{1}{R_k}\Big)\simeq(n),$$

but this in turn leads to $(R_n)\simeq(\sum_{j=1}^{n-1}\frac{1}{j^{1/2}})$, which is divergent as $n\to\infty$.

Step 3: $(\rho_{n+1})_{n\ge1}\simeq(\rho_n)_{n\ge1}$ and $(R_{n+1})_{n\ge1}\simeq(R_n)_{n\ge1}$. To establish these assertions, we first introduce a convenient notion. A sequence (a_n) will be said to have the bounded difference property (BD property, for short), if $a_n\to\infty$ as $n\to\infty$ and the sequence of progressive differences $(a_{n+1}-a_n)$ is bounded. If (a_n) has the BD property, then $(a_{n+1})_{n\ge1}\simeq(a_n)_{n\ge1}$, since $|a_{n+1}/a_n-1|\le C/|a_n|\le 1/2$ for n large enough. The BD property is not invariant under the equivalence of sequences, but if (a_n) has the BD property and $(b_n)\simeq(a_n)$, then $(b_{n+1})\simeq(b_n)$.

Returning to (ρ_n) and (R_n), owing to (5.3) and Step 1 we know that $\sum_{k=2}^{n-1}\frac{1}{R_k}\to\infty$ as $n\to\infty$. Since the differences are bounded (even converging to zero) by Step 2, $(\sum_{k=2}^{n-1}\frac{1}{R_k})$ has the BD property. Invoking (5.3) once more, it follows that $(\rho_{n+1})\simeq(\rho_n)$. Similarly, (R_n) has the BD property, and thus $(R_{n+1})\simeq(R_n)$.

Step 4: To prove that $(\rho_n)\simeq((\log n)^2)$, we may assume that $\rho_n\ge1$ for $n\in\mathbb{N}$. Then the function $\rho:[1,\infty[\to[1,\infty[$ obtained by piecewise linear interpolation from $\rho(n)=\rho_n$ is continuous, increasing and such that $\lim_{x\to\infty}\rho(x)=\infty$. Let $R(y)=\int_1^y\frac{d\xi}{\rho(\xi)^{1/2}}$ for $y\ge1$. According to Lemma 5.4 it is sufficient to establish the estimate (5.6). If $j\le\xi\le j+1$, then $\rho_j\le\rho(\xi)\le\rho_{j+1}$. Since

$$R(n)=\int_1^n\frac{d\xi}{\rho(\xi)^{1/2}}=\sum_{j=1}^{n-1}\int_j^{j+1}\frac{d\xi}{\rho(\xi)^{1/2}}$$

for $n \in \mathbb{N}$, we deduce that $R_{n+1} - \rho_1^{-1/2} \le R(n) \le R_n$. Hence we may employ Step 3 to obtain $(R(n)) \simeq (R_n)$. Finally we observe that if $y \in [j, j+1]$, then $R(j) \le R(y) \le R(j+1)$. For $x \in [N, N+1]$ then

$$\int_1^x \frac{dy}{R(y)} = \sum_{j=1}^{N-1} \int_j^{j+1} \frac{dy}{R(y)} + \int_N^x \frac{dy}{R(y)}$$

yields

$$\sum_{j=1}^{N-1} \frac{1}{R(j+1)} \le \int_1^x \frac{dy}{R(y)} \le \sum_{j=1}^{N} \frac{1}{R(j)}.$$

If we now use (5.3) in conjunction with $\rho_N \le \rho(x) \le \rho_{N+1}$ and $(\rho_{n+1}) \simeq (\rho_n)$, the relation (5.6) follows easily.

In the case where $\sigma > 1$ we need to prove that (ρ_n) is bounded. Assume on the contrary that we would have $\rho_n \to \infty$ as $n \to \infty$ (recall that the sequence is increasing). This would imply $R_n \to \infty$ as $n \to \infty$, as otherwise $R_n \to R_\infty \in]0, \infty[$ as $n \to \infty$ for an appropriate R_∞. Then (5.3) yields

$$(\rho_n) \simeq \Big(\sum_{k=2}^{n-1} \frac{1}{R_k^\sigma} \Big) \simeq (n),$$

but this in turn leads to $(R_n) \simeq (\sum_{j=1}^{n-1} \frac{1}{j^{1/2}})$, which is divergent as $n \to \infty$. Thus we are in the same position as after Steps 1 and 2 in the above argument. An inspection of Steps 3 and 4 shows that they can be straightforwardly adapted to the current setting. In other words, we can apply the case $\sigma > 1$ of Lemma 5.4, and hence the function $\rho(x)$ is found to be bounded. Since $\rho_n = \rho(n)$, the sequence (ρ_n) must be bounded which is a contradiction and completes the proof of Lemma 5.2. □

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