A Birman-Schwinger principle in galactic dynamics
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Abstract

These are the (somewhat extended) lecture notes for four lectures delivered at the spring school during the thematic programme "Mathematical Perspectives of Gravitation beyond the Vacuum Regime" at ESI Vienna in February 2022.

Contents

1 Introduction

These are the (somewhat extended) lecture notes for four lectures delivered at the spring school during the thematic programme "Mathematical Perspectives of Gravitation beyond the Vacuum Regime" at ESI Vienna in February 2022. The main reference for the lectures is [27], which has some overlap with [18], although we wanted to emphasize the action-angle variables approach and put a main focus on the Birman-Schwinger principle, as is done in [27]. Since the lectures have been aimed at newcomers, some parts of them cover basic background material. The author is indepted to the organizers H. Andréasson, D. Fajman, J. Joudioux and T. Oliynyk for making the programme happen despite the Corona virus pandemic, and thanks are due to the ESI for providing a very stimulating working atmosphere. The author is also grateful to the referees for many suggestions that helped to improve these notes.

2 The Birman-Schwinger principle in quantum mechanics

The Birman-Schwinger principle is a widely used and well-established tool in mathematical quantum mechanics. It was introduced through the independent works of Birman [6] and Schwinger [44], with the idea of counting, or at least estimating, the number of eigenvalues of Schrödinger operators on $L^2(\mathbb{R}^n)$. To be more specific, consider (only formal at this point)

$$H = -\Delta + V;$$

to avoid introducing negative parts we will assume that $V \leq 0$.

Theorem 2.1 *The following assertions hold:*

(a) $-e$ *is a (negative) eigenvalue of* H *if and only if* 1 *is an eigenvalue of the Birman-Schwinger operator*

$$B_e = \sqrt{-V}(-\Delta + e)^{-1}\sqrt{-V}. \tag{2.1}$$

Furthermore,

(b) if ϕ *is an eigenfunction of* H *for the eigenvalue* $-e$*, then* $\psi = \sqrt{-V}\phi$ *is an eigenfunction of* B_e *for the eigenvalue* 1*;*

(c) if ψ *is an eigenfunction of* B_e *for the eigenvalue* 1*, then* $\phi = (-\Delta + e)^{-1}(\sqrt{-V}\psi)$ *is an eigenfunction of* H *for the eigenvalue* $-e$*.*

Proof: See [30, Section 4.3.1]. If $H\phi = (-\Delta+V)\phi = (-e)\phi$, then we define $\psi = \sqrt{-V}\phi$ to obtain

$$B_e\psi = \sqrt{-V}(-\Delta+e)^{-1}\sqrt{-V}\sqrt{-V}\phi = \sqrt{-V}(-\Delta+e)^{-1}(-V)\phi = \sqrt{-V}\phi = \psi.$$

Conversely, if $B_e\psi = \psi$ holds and if we put $\phi = (-\Delta+e)^{-1}(\sqrt{-V}\psi)$, it follows that

$$(-\Delta+e)\phi = \sqrt{-V}\psi = \sqrt{-V}\sqrt{-V}(-\Delta+e)^{-1}\sqrt{-V}\psi = (-V)\phi,$$

and hence $H\phi = (-\Delta+V)\phi = (-e)\phi$, which completes the argument. □

The operators B_e enjoy a number of favorable properties. For instance, they are non-negative Hilbert-Schmidt operators (if V decays sufficiently fast and $n \le 3$), and in particular they are compact. Furthermore, their eigenvalues can be ordered: $\lambda_1(e) \ge \lambda_2(e) \ge \ldots \to 0$ and the eigenvalue curves are decreasing in e, in that $\tilde{e} \ge e$ implies that $\lambda_k(e) \le \lambda_k(\tilde{e})$ for all k. Also the number of eigenvalues of H less than or equal to $-e$ agrees with the number of eigenvalues of B_e greater than or equal to 1, counting multiplicities in both cases; cf. [30, Figure 4.1, p. 78] for an illustration. Even more ist true: not only the number of eigenvalues of H can be bounded, but also eigenvalue moments like $\sum_j |-e_j|^\gamma$, where the sum extends over all negative eigenvalues $-e_j$ of H. This fact lies at the heart of many important results in the field. Let us only mention here the Lieb-Thirring bound

$$\sum_j |-e_j| \le L_{1,3} \int_{\mathbb{R}^3} |V(x)|^{5/2}\,dx$$

in three dimensions for an absolute constant $L_{1,3} > 0$ and $V \in L^{5/2}(\mathbb{R}^3)$. It is used in those authors' proof of the stability of matter [31], which has found many generalizations [30], and which is much easier to follow than the original argument by Dyson and Lenard [8].

Good general textbooks that cover the Birman-Schwinger principle are [30, Section 4.3], [41, 45, 46] or [47, Section 7.9].

There is also a large number of further applications of the Birman-Schwinger principle, like to problems including complex-valued potentials; to Dirac operators; to the Bardeen-Cooper-Schrieffer model of superconductivity; to the linearized 2D Euler equations, to list only a few. See [27] for more references.

3 Galactic dynamics: The Vlasov-Poisson system

Galactic dynamics generally refers to the modeling of the time evolution of self-gravitating matter such as galaxies, or on an even larger scale, clusters of galaxies. In mathematical terms, the resulting system is an N-body problem, with N quite large: $N \sim 10^6 - 10^{11}$ for galaxies and $N \sim 10^2 - 10^3$ for clusters of galaxies. This N-body problem consists of coupled Newtonian equations, one for each individual object (the 'objects' in a galaxy are stars, those in a cluster of galaxies are galaxies), and to study the collective behavior of the system. While results may be obtainable numerically in this way, the mathematical complexity of even the three-body problem prevents one from rigorously addressing deeper questions (concerning for instance galaxy formation or stability) for such stellar systems. Therefore, from the early days of the field, a statistical description of the evolution has been proposed, by Vlasov [50] in 1938 for plasmas (in this case a related equation occurs) and by Jeans [24] in 1915 for gravitational systems; see [19] for an interesting historical discussion of the origins of the equation. It is also known as the 'collisionless Boltzmann equation', which refers to the fact that collisions among the stars or galaxies are sufficiently rare to be neglected. A standard source of information on galactic dynamics is [5].

The time evolution of such a system is then governed by a distribution function $f = f(t, x, v)$ that depends on time $t \in \mathbb{R}$, position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The quantity $\int_{\mathcal{X}} dx \int_{\mathcal{V}} dv\, f(t, x, v)$ should be thought of as the number of objects (henceforth called 'particles') at time t, which are located at some point $x \in \mathcal{X} \subset \mathbb{R}^3$ and which have velocities $v \in \mathcal{V} \subset \mathbb{R}^3$. Each individual particle follows a trajectory $(X(s), V(s))$ in phase space $\mathbb{R}^3 \times \mathbb{R}^3$ such that $(X(t), V(t)) = (x, v)$ at time t and

$$\dot{X}(s) = V(s), \quad \dot{V}(s) = -\nabla_x U(s, X(s)), \tag{3.1}$$

where $F = -\nabla_x U$ denotes the Coulomb-type force field that is collectively generated by all particles. The requirement that f be constant along the curves defined by (3.1) then leads to the relation

$$\begin{aligned} 0 &= \frac{d}{ds}\,[f(s, X(s), V(s))] \\ &= \partial_t f(s, X(s), V(s)) + V(s) \cdot \nabla_x f(s, X(s), V(s)) - \nabla_x U(s, X(s)) \cdot \nabla_v f(s, X(s), V(s)) \end{aligned} \tag{3.2}$$

for all s. Evaluated at time t, this yields

$$\partial_t f(t, x, v) + v \cdot \nabla_x f(t, x, v) - \nabla_x U(t, x) \cdot \nabla_v f(t, x, v) = 0 \tag{3.3}$$

for all (t, x, v), which is usually called the Vlasov equation (despite the historic inadequacy of this terminology). The next step is to express the force field F in terms of the distribution function f. Since we are aiming at describing gravitational binding, we need to have $F \sim -\nabla_x V_{\rm C}$ for the Coulomb potential $V_{\rm C}(x) = -\frac{1}{|x|}$ at large distances. This suggests to use the field $F = -\nabla_x U_f$ induced by the Poisson equation

$$\Delta_x U_f(t, x) = 4\pi \rho_f(t, x), \quad \lim_{|x| \to \infty} U_f(t, x) = 0, \quad \text{where} \quad \rho_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v)\, dv \tag{3.4}$$

denotes the charge density induced by f. Observe that $\int_{\mathcal{X}} dx\, \rho_f(t, x)$ represents the number of particles at time t, of any velocity, which are located at some point $x \in \mathcal{X}$. Then

$$U_f(t, x) = -\int_{\mathbb{R}^3} \frac{\rho_f(t, y)}{|x - y|}\, dy \tag{3.5}$$

is Coulomb-like as $|x| \to \infty$.

Initial data $f(0,x,v) = f_0(x,v)$ at time $t = 0$ have to be specified for f only, since then (3.5) determines the initial data $U_f(0,x)$. We will exclusively be interested in classical solutions of (3.3), (3.4), whose global-in-time existence is ensured, under reasonable assumptions on f_0.

Theorem 3.1 ([38, 43] and [32]) *Let f_0 be continuously differentiable and compactly supported. Then the Vlasov-Poisson system (3.3), (3.4), (3.5) has a global and unique solution.*

For a mathematical overview of the system and more background material the reader may wish to consult [12, 36, 42]. Throughout the course we will adopt a dynamical systems viewpoint: Given some initial data f_0, we are interested in what happens to the resulting solution $f(t)$ (that lies in a space of functions depending on (x,v)) as $t \to \infty$?

4 Spherically symmetric solutions

Almost exclusively we will be dealing with spherically symmetric solutions of the Vlasov-Poisson system. A function $g = g(x,v)$ is said to be spherically symmetric, if $g(Ax, Av) = g(x,v)$ for all $A \in \mathrm{SO}(3)$ and $x, v \in \mathbb{R}^3$. Expressed in more sophisticated terms, g needs to be equivariant w.r. to the group action $\mathrm{SO}(3) \times (\mathbb{R}^3 \times \mathbb{R}^3) \to \mathbb{R}^3 \times \mathbb{R}^3$, $(A,x,v) \mapsto (Ax, Av)$. In this case $\rho_g(x) = \rho_g(r)$ and $U_g(x) = U_g(r)$ are radially symmetric; here $r = |x|$. More explicitly,

$$\begin{aligned} U_g(r) &= -\frac{4\pi}{r}\int_0^r s^2\rho_g(s)\,ds - 4\pi\int_r^\infty s\rho_g(s)\,ds, \\ U_g'(r) &= \frac{4\pi}{r^2}\int_0^r s^2\rho_g(s)\,ds = \frac{1}{r^2}\int_{|x|\le r}\rho_g(x)\,dx, \end{aligned} \tag{4.1}$$

where $'$ denotes $\frac{d}{dr}$.

Exercise 4.1 *Prove (4.1) from (3.5).*

It can be shown that a spherically symmetric function $g = g(x,v)$ does in fact only depend upon three variables: $g = \tilde{g}(|x|, |v|, x \cdot v)$.

Exercise 4.2 *Establish this claim.*

In spherical symmetry, the variables

$$r = |x|, \quad p_r = \frac{x \cdot v}{r} \quad \text{and} \quad L = x \wedge v,$$

are most useful. Here $p_r \in \mathbb{R}$ denotes the radial momentum and $L \in \mathbb{R}^3$ is the angular momentum. Since

$$|L|^2 = |x|^2|v|^2 - (x \cdot v)^2 = r^2(|v|^2 - p_r^2), \tag{4.2}$$

we get $|v|^2 = \frac{\ell^2}{r^2} + p_r^2$ for $\ell = |L|$. This implies that a function $g = \tilde{g}(|x|, |v|, x \cdot v)$ can equally well be expressed as a function $g = \hat{g}(r, p_r, \ell)$; of course we are going to identify all versions of g.

If we restrict to spherically symmetric solutions, i.e., $f(t) = f(t, \cdot, \cdot)$ is spherically symmetric for all t, then in the new variables (r, p_r, ℓ) the Vlasov-Poisson system can be rewritten as

$$\partial_t f(t, r, p_r, \ell^2) + p_r\, \partial_r f(t, r, p_r, \ell^2) + \Big(\frac{\ell^2}{r^3} - \partial_r U_f(t, r)\Big)\, \partial_{p_r} f(t, r, p_r, \ell^2) = 0$$

and

$$U_f''(t, r) + \frac{2}{r}\, U_f'(t, r) = 4\pi \rho_f(t, r), \quad \lim_{r \to \infty} U_f(t, r) = 0, \quad \rho_f(t, r) = \frac{2\pi}{r^2} \int_0^\infty d\ell\, \ell \int_{\mathbb{R}} dp_r f(t, r, p_r, \ell^2),$$

and moreover

$$U_f(t, r) = -\frac{4\pi}{r} \int_0^r \sigma^2 \rho_f(t, \sigma)\, d\sigma - 4\pi \int_r^\infty \sigma \rho_f(t, \sigma)\, d\sigma \tag{4.3}$$

due to (4.1). From (4.3) it is plain to see that $U_f(t, r) \sim -\frac{M}{r}$ as $r \to \infty$, for

$$M = 4\pi \int_0^\infty \sigma^2 \rho_f(t, \sigma)\, d\sigma = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, v)\, dx\, dv$$

denoting the total mass of the system. It should be noted that M is conserved along solutions of the system, thus in fact $M = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_0(x, v)\, dx\, dv$ is independent of t.

5 Steady state solutions

From a dynamical systems perspective, the easiest solutions of dynamical relevance are steady states, i.e., time-independent solutions. The Vlasov-Poisson system possesses an abundance of such solutions $Q = Q(x, v)$, which we seek to be spherically symmetric. Let $e_Q(x, v) = \frac{1}{2}\,|v|^2 + U_Q(x)$ denote the particle energy and let $\ell^2 = |L|^2$ be as in (4.2).

Lemma 5.1 *Both e_Q and ℓ^2 are conserved along solutions of the characteristic equation $\ddot{X}(t) = -\nabla U_Q(X(t))$ from (3.1).*

Proof: Let us consider ℓ^2 for example. Then for $V = \dot{X}$

$$\begin{aligned} \frac{d}{dt}\,|X(t) \wedge V(t)|^2 &= 2(X(t) \wedge V(t)) \cdot \Big(\dot{X}(t) \wedge V(t) + X(t) \wedge \dot{V}(t)\Big) \\ &= -2(X(t) \wedge V(t)) \cdot \Big(X(t) \wedge \nabla U_Q(X(t))\Big). \end{aligned}$$

From $U_Q(x) = U_Q(|x|) = U_Q(r)$ we deduce $\nabla U_Q(x) = U_Q'(r)\frac{x}{|x|}$, and thus $X(t) \wedge \nabla U_Q(X(t)) = 0$. Note this argument has nothing to do with Vlasov-Poisson, but only relies on the fact that U_Q is what is called a central potential. The calculation for e_Q is similar. $\square$

Lemma 5.1 is the key to obtaining steady state solutions: If we seek a solution in the form $Q(x,v) = Q(e_Q, \ell^2)$ (observe the abuse of notation here), then (3.2), i.e., the Vlasov equation will automatically be satisfied. Thus finding a steady state comes down to solving the semilinear equation

$$\frac{1}{r^2}(r^2 U'(r))' = \Delta U(x) = 4\pi \int_{\mathbb{R}^3} Q\Big(\frac{1}{2}|v|^2 + U(x), |x \wedge v|^2\Big)\, dv, \quad \lim_{r\to\infty} U(r) = 0, \tag{5.1}$$

for $U = U_Q$, if the profile function $Q = Q(e_Q, \ell^2)$ is given; of course we are only interested in nontrivial solutions $U \neq 0$. In fact it is the content of Jeans's theorem (see [4]) that the distribution function Q of every spherically symmetric steady state solution has to be of the form $Q = Q(e_Q, \ell^2)$.

Therefore the question arises for which Q's (5.1) can be solved, and it turns out that there is a variety of possible choices, even if we restrict ourselves to the easier case that $Q = Q(e_Q)$ does not depend on ℓ^2; such steady state solutions are called isotropic. There is a vast literature concerning different classes of ansatz functions (called polytropes, King models, ... etc.), see [5, 36, 42], but for the purpose of this course it will be sufficient to keep in mind the example of the polytropes. They are given by

$$Q(e_Q) = (e_0 - e_Q)_+^k \tag{5.2}$$

for a fixed cut-off energy $e_0 < 0$ and $k \in]-\frac{1}{2}, \frac{7}{2}[$; here $s_+ = \max\{s, 0\}$. Then

$$\rho_Q(r) = c_n (e_0 - U_Q(r))_+^n, \quad n = k + \frac{3}{2} \in]1,5[, \quad c_n = (2\pi)^{3/2} \frac{\Gamma(k+1)}{\Gamma(k+\frac{5}{2})}, \tag{5.3}$$

and $U_Q(0) < e_0$.

Exercise 5.2 *Prove (5.3).*

The potential U_Q is not known explicitly. All the polytropic steady state solutions do have finite radius r_Q (i.e., the density ρ_Q is supported in $[0, r_Q]$ and $r_Q < \infty$) and finite mass $M_Q = \int_{\mathbb{R}^3} \rho_Q(x)\, dx = 4\pi \int_0^{r_Q} r^2 \rho_Q(r)\, dr = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} Q(x,v)\, dx\, dv$; see [4] or [39]. The limiting case $k = 7/2$ is called the Plummer sphere, where M_Q is still finite, but $r_Q = \infty$. It is also important to note that $Q'(e_Q) < 0$ inside the support of the polytropes. In general, this property is very much tied to (linear) stability; also see [5, footnote 10, p. 433].

6 Action angle variables

Action angle variables are particularly well-suited for Hamiltonian systems. We start with a one-degree-of freedom example, a reliable general source on the topic being [51].

Example 6.1 (Action-angle variables) We consider $n = 1$, $q, p \in \mathbb{R}$ and

$$H(q,p) = \frac{1}{2}\,p^2 + V(q),$$

where the potential V should be such that the phase portrait of the resulting system $\ddot{q} = -V'(q)$ contains a fixed point (which we take to be the origin) that is encircled by a family of periodic solutions γ_h, parameterized by their energy h in some interval $h \in]h_0, h_1[$. Then

$$\gamma_h \subset \Big\{ (q,p) \in \mathbb{R} \times \mathbb{R} : \frac{1}{2}\,p^2 + V(q) = h \Big\},$$

but not necessarily $\gamma_h = \{\ldots\}$, since the energy level set $\{\ldots\}$ may consist of several components; for instance this is the case for $V(q) = q^2(q^2 - 1)$, or a version thereof shifted appropriately to place one set of periodic orbits about the origin, where the left and the right interior of the homoclinic orbits both contain solutions of the same period and energy. Let $q_\pm(h)$ denote the intersections of γ_h and the q-axis $\{p = 0\}$, i.e., $V(q_\pm(h)) = h$ is required along with $q_-(h) < 0 < q_+(h)$. If $T(h)$ is the period of γ_h, then

$$T(h) = 2 \int_{q_-(h)}^{q_+(h)} \frac{ds}{\sqrt{2(h - V(s))}}. \tag{6.1}$$

Next denote by $2\pi I(h)$ the area encircled by γ_h. Since the orbit is transversed in the clockwise direction, the Green-Riemann formula (or Stokes's theorem) says that

$$2\pi I(h) = \int_{\gamma_h} p\, dq$$

for the action I. Furthermore, elementary calculus tells us that

$$2\pi I(h) = 2 \int_{q_-(h)}^{q_+(h)} \sqrt{2(h - V(s))}\, ds,$$

noting that the height function at q is just $\pm p = \sqrt{2(h - V(q))}$. Recalling that $V(q_\pm(h)) = h$, we see that in particular

$$I'(h) = \frac{1}{2\pi}\, T(h) > 0$$

holds. Thus the function $h \mapsto I(h)$ (on $]h_0, h_1[$) admits an inverse function that is denoted by $I \mapsto h(I)$. Differentiating the relation $h(I(h)) = h$, we get

$$1 = h'(I(h)) I'(h) = \frac{1}{2\pi}\, h'(I(h))\, T(h).$$

Now we consider the transformation

$$\Phi : (q,p) \mapsto (\theta, I) \tag{6.2}$$

that is obtained from the so–called generating function

$$S(q,I)=\int_{q_-(h(I))}^{q}\sqrt{2(h(I)-V(s))}\,ds.$$

In general, a generating function depends on one "old" variable (here: q) and one "new" variable (here: I). This means that $p=\partial_q S$ and $\theta=\partial_I S$, in the following sense: Given (q,p), where for instance $p\ge 0$, the relation

$$p=\partial_q S(q,I)=\sqrt{2(h(I)-V(q))} \tag{6.3}$$

has to be solved for $I=I(q,p)$, and then the assignment

$$\theta(q,p)=\partial_I S(q,I(q,p))$$

completes the definition of the transformation (6.2).

A key feature of transformations derived from generating functions in this way is that they are canonical (i.e., symplectic). To see this, differentiating $p=\partial_q S(q,I)$ w.r. to p implies that $1=(\partial^2_{qI}S)(\partial_p I)$. Therefore we deduce from $\theta=\partial_I S(q,I)$ that

$$\begin{aligned} d\theta\wedge dI &= \Big[(\partial^2_{qI}S)\,dq+(\partial^2_{II}S)\,dI\Big]\wedge dI=(\partial^2_{qI}S)\,dq\wedge dI\\ &= (\partial^2_{qI}S)\,dq\wedge\Big[(\partial_q I)\,dq+(\partial_p I)\,dp\Big]=(\partial^2_{qI}S)(\partial_p I)\,dq\wedge dp=dq\wedge dp, \end{aligned}$$

which means that Φ from (6.2) is indeed canonical.

The meaning of the angular variable θ is as follows. Denote by

$$\tau(q,p)=\int_{q_-(h)}^{q}\frac{ds}{\sqrt{2(h-V(s))}}$$

the time that it takes the solution, if for instance $p>0$, to pass from $(q_-(h),0)$ to (q,p) on γ_h. Noting that

$$\theta(q,p)=\partial_I S(q,I)=\int_{q_-(h)}^{q}\frac{ds}{\sqrt{2(h(I)-V(s))}}\,h'(I)=\tau(q,p)\,\frac{2\pi}{T(h)},$$

this can be rewritten to read

$$\frac{\theta(q,p)}{2\pi}=\frac{\tau(q,p)}{T(h)}.$$

Hence $\theta(q,p)\in[0,2\pi[$ is the angle of clockwise rotation of the line segment $[(q_-(h),0),(0,0)]$ to the line segment $[(q,p),(0,0)]$. The variables I and θ are called action and angle variables, respectively.

Since the transformation is canonical, it is sufficient to transform the Hamiltonian function to obtain the equations of motion in the new variables. In this case we obtain

$$\mathcal{H}(\theta,I)=H(\Phi^{-1}(\theta,I))=\frac{1}{2}\,p^2+V(q)=h(I)$$

by (6.3). Here we see the main reason for passing to action-angle variables: the dynamics become very simple, since in the new variables the Hamiltonian is independent of the angular variable. The associated equations of motion are

$$\dot{\theta} = \partial_I \mathcal{H} = h'(I) =: \omega(I), \quad \dot{I} = -\partial_\theta \mathcal{H} = 0,$$

and the corresponding solutions are

$$\theta(t) = \theta_0 + \omega(I_0)t, \quad I(t) = I_0,$$

which is an angular rotation with frequency $\omega(I_0)$. $\diamondsuit$

Exercise 6.2 *Prove the period relation (6.1).*

Exercise 6.3 *Let* $V(q) = \frac{\omega^2}{2} q^2$*, i.e., we consider the harmonic oscillator* $\ddot{q} + \omega q = 0$ *with mass* $m = 1$*. Show the following items:*

(a) The intersection points of the orbit of energy h *with the* q*-axis are* $q_\pm(h) = \pm\frac{\sqrt{2h}}{\omega}$*.*

(b) The period function is $T(h) = \frac{2\pi}{\omega}$*, independently of* h*.*

(c) One has $I(h) = \frac{1}{\omega} h$ *and* $I'(h) = \frac{1}{\omega} = \frac{1}{2\pi} T(h)$*. The inverse function to* $h \mapsto I(h)$ *is* $h(I) = \omega I$ *which yields the constant frequency* $\omega(I) = h'(I) = \omega$*.*

(d) The generating function is

$$S(q, I) = 2I \int_{-1}^{\sqrt{\frac{\omega}{2I}}\, q} \sqrt{1 - \tau^2}\, d\tau$$

(and there is no need to evaluate the integral explicitly).

(e) Calculate that $\partial_I S(q, I(q, p)) = \theta$ *and*

$$\Phi^{-1}(\theta, I) = \left(-\sqrt{\frac{2I}{\omega}} \cos\theta, \sqrt{2I\omega} \sin\theta \right).$$

(f) Establish that

$$q(t) = -\sqrt{\frac{2I_0}{\omega}} \cos(\theta_0 + \omega t), \quad p(t) = \sqrt{2I_0\omega} \sin(\theta_0 + \omega t),$$

is the solution such that $\Phi(q_0, p_0) = (\theta_0, I_0)$*.*

Now we return to the Vlasov-Poisson setting and consider the characteristic equation

$$\ddot{X} = -\nabla U_Q(X(t)) \tag{6.4}$$

for an isotropic steady state solution $Q = Q(e_Q)$; it is (an autonomous) Hamiltonian system. By the spherical symmetry, one can use a canonical change of variables

$$(x, v) \mapsto (p_r, L_3, \ell; r, \varphi, \chi) \tag{6.5}$$

on the support $K = \operatorname{supp} Q$ of Q as described in [5, Ch. 3.5.2] and [49, § 5.3] to simplify matters considerably. Let us first have a look at the variables on the right-hand side of (6.5). Since $L = x \wedge v$, we have $L_3 = x_1 v_2 - x_2 v_1$ for the third component. The angles φ and χ are determined by

$$\sin\varphi = \frac{L_1}{(\ell^2 - L_3^2)^{1/2}}, \quad \cos\varphi = \frac{L_2}{(\ell^2 - L_3^2)^{1/2}},$$
$$\cos\chi = \frac{(e_3 \wedge L)\cdot x}{r(\ell^2 - L_3^2)^{1/2}}, \quad \sin\chi = \frac{\ell\, x_3}{r(\ell^2 - L_3^2)^{1/2}}.$$

From these relations it can be calculated that indeed (6.5) is canonical. The variable pairs $r \leftrightarrow p_r$, $\varphi \leftrightarrow L_3$, and $\chi \leftrightarrow \ell$ are conjugate, their Poisson brackets can be evaluated explicitly; see [49, § 5.3], also for an illustration of how the new coordinates can be read off. The Hamiltonian function for (6.4) is $e_Q(x, v) = \frac{1}{2}|v|^2 + U_Q(x)$. Since the transformation is canonical, we only need to transform the Hamiltonian in order to obtain (6.4) in the new coordinates. Recalling that $|v|^2 = \frac{\ell^2}{r^2} + p_r^2$, it is found that

$$e_Q(r, p_r, \ell) = \frac{1}{2}p_r^2 + U_{\text{eff}}(r, \ell), \quad \text{with} \quad U_{\text{eff}}(r, \ell) = U_Q(r) + \frac{\ell^2}{2r^2}$$

being the effective potential. Now

$$\dot{r} = \frac{\partial e_Q}{\partial p_r} = p_r, \quad \dot{p}_r = -\frac{\partial e_Q}{\partial r} = -U'_{\text{eff}}(r, \ell),$$

thus the resulting equation of motion is

$$\ddot{r} = -U'_{\text{eff}}(r, \ell).$$

This should be viewed as one second order Hamiltonian system in (r, p_r) per each fixed ℓ, where the potential is given by $r \mapsto U_{\text{eff}}(r, \ell)$. The potential has the following shape (where we let $\beta = \ell^2$ and identify functions of ℓ with functions of β):

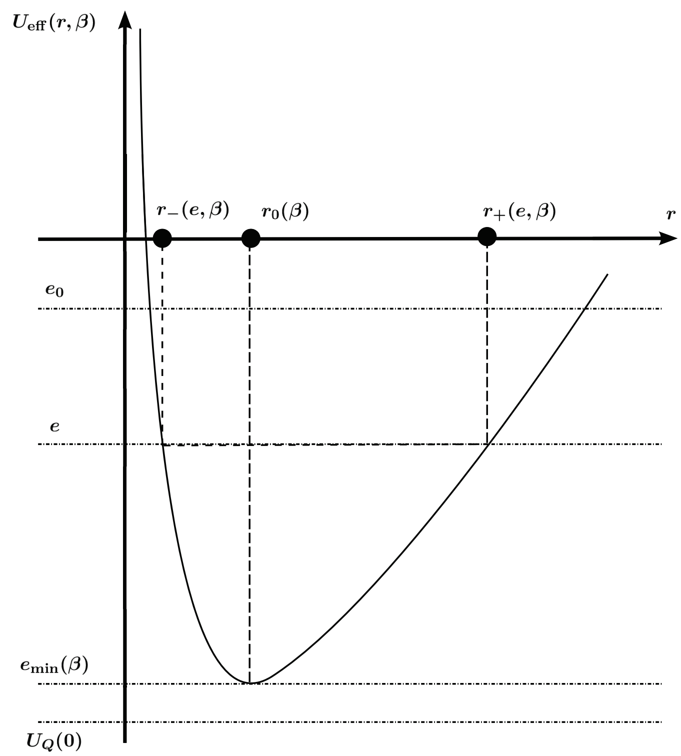

Figure 1: The effective potential $U_{\text{eff}}(r,\ell) = U_{\text{eff}}(r,\beta)$

For $e < 0$ and $\ell > 0$ there are exactly two zeros $0 < r_-(e,\ell) < r_+(e,\ell)$ of $0 = 2(e - U_{\text{eff}}(r,\ell))$ and the minimum is attained at a unique point $r_0(\ell)$.

The new variables $(p_r, L_3, \ell; r, \varphi, \chi)$ in (6.5) are not yet the desired action-angle variables, since $e = e(r, p_r, \ell)$ depends upon r, which plays the role of an angle; remember from the 1D example above that the goal is to get the Hamiltonian independent of the angle(s). Therefore a further canonical transformation

$$(r, p_r) \to (\theta, I) \quad \text{at a fixed } \ell \tag{6.6}$$

will be made. At such a fixed ℓ, we can do this in a region where the orbits of $U_{\text{eff}}(\cdot,\ell)$ are periodic; it is achieved by means of a generating function as above. The angle $\theta \in [0,\pi]$ corresponds to one half-turn of the periodic orbit γ in the potential $U_{\text{eff}}(\cdot,\ell)$, connecting the 'pericenter' r_- to the 'apocenter' r_+; here $\dot r = p_r > 0$ for $r \in]r_-, r_+[$ and $p_r(r_\pm) = 0$. Therefore if $\theta \in [\pi, 2\pi]$, then

$$r(\theta, I, \ell) = r(2\pi - \theta, I, \ell) \quad \text{and} \quad p_r(\theta, I, \ell) = -p_r(2\pi - \theta, I, \ell). \tag{6.7}$$

In other words, we need to determine the (inverse) transformation $(\theta, I) \mapsto (r, p_r)$ only for $\theta \in [0,\pi]$, where we have $p_r \geq 0$. Let $E = E(I,\ell)$ be the solution to

$$I = \frac{1}{2\pi}\int_\gamma p_r\, dr = \frac{1}{\pi}\int_{r_-(E,\ell)}^{r_+(E,\ell)} \sqrt{2(E - U_{\text{eff}}(r,\ell))}\, dr,$$

where γ is as before. Then consider

$$S(r, I, \ell) = \int_{r_-(E(I,\ell),\ell)}^{r} \sqrt{2(E(I,\ell) - U_{\text{eff}}(r',\ell))}\, dr' \tag{6.8}$$

as a generating function for (6.6). The rules for determining the full transformation from S are once again given by

$$\theta = \partial_I S, \quad p_r = \partial_r S.$$

More precisely, the equation

$$\theta = \partial_I S(r, I, \ell) \tag{6.9}$$

has a solution $r = r(\theta, I, \ell)$. In addition, put

$$p_r = p_r(\theta, I, \ell) = \partial_r S(r(\theta, I, \ell), I, \ell).$$

Thus more explicitly

$$p_r(\theta, I, \ell) = \sqrt{2(E(I,\ell) - U_{\text{eff}}(r(\theta, I, \ell), \ell))},$$

which yields

$$E(I, \ell) = \frac{1}{2}\, p_r(\theta, I, \ell)^2 + U_{\text{eff}}(r(\theta, I, \ell), \ell) = e(r(\theta, I, \ell), p_r(\theta, I, \ell), \ell).$$

Now $p_r = \partial_r S$ and (6.8) imply that $e = \frac{p_r^2}{2} + U_{\text{eff}}(r, \ell) = E(I, \ell)$, so E will only depend upon action variables after the transformation (6.6), which leads to the overall transformation

$$(x, v) \mapsto (p_r, L_3, \ell; r, \varphi, \chi) \mapsto (I, L_3, \ell; \theta, \varphi, \chi), \tag{6.10}$$

cf. (6.5). Hence after applying (6.10) the particle energy does only depend upon I and ℓ, both of which are actions. The associated frequencies are

$$\omega_1(I, \ell) = \frac{\partial E(I, \ell)}{\partial I}, \quad \omega_2(I, \ell) = \frac{\partial E(I, \ell)}{\partial L_3} = 0, \quad \omega_3(I, \ell) = \frac{\partial E(I, \ell)}{\partial \ell}, \tag{6.11}$$

and the period functions are

$$T_1(I, \ell) = \frac{2\pi}{\omega_1(I, \ell)}, \quad T_3(I, \ell) = \frac{2\pi}{\omega_3(I, \ell)}.$$

Also (6.9) yields

$$\theta = \partial_I S(r, I, \ell) = \omega_1(I, \ell) \int_{r_-(E(I,\ell),\ell)}^{r} \frac{dr'}{\sqrt{2(E(I,\ell) - U_{\text{eff}}(r',\ell))}}.$$

Since $\theta = 0$ at r_- and $\theta = \pi$ at r_+ (recall that $\dot{r} = p_r > 0$ along this part of the orbit), we obtain

$$\pi = \frac{2\pi}{T_1(I, \ell)} \int_{r_-(E(I,\ell),\ell)}^{r_+(E(I,\ell),\ell)} \frac{dr}{\sqrt{2(E(I,\ell) - U_{\text{eff}}(r,\ell))}},$$

or explicitly

$$T_1(I,\ell) = 2\int_{r_-(E(I,\ell),\ell)}^{r_+(E(I,\ell),\ell)} \frac{dr}{\sqrt{2(E(I,\ell) - U_{\text{eff}}(r,\ell))}} \tag{6.12}$$

for the period function. In particular, $T_1(I,\ell) = T_1(E,\ell)$ by abuse of notation.

To summarize, spherically symmetric functions $g = g(x,v) = g(|x|,|v|,x\cdot v)$ may also be expressed as $g = g(r,p_r,\ell) = g(\theta,I,\ell)$.

7 Function spaces

Next we consider the question of how $K = \operatorname{supp} Q$ can be expressed in terms of the variables $\beta = \ell^2$ and $e = e_Q$. We will stick to the example of the polytropes (5.2), where

$$K = \{e_0 - e_Q \geq 0\}.$$

More precisely, since always $\theta \in [0,2\pi]$ on K for the angular variable θ, we have to exhibit a set D in (e,β) such that

$$K = [0,2\pi] \times D. \tag{7.1}$$

On this domain D we need to have

$$e_0 \geq e \geq U_{\text{eff}}(r,\beta) \geq U_{\text{eff}}(r_0(\beta),\beta) = U_Q(r_0(\beta)) + \frac{\beta}{2r_0(\beta)^2}, \tag{7.2}$$

with $r_0(\beta)$ denoting the unique point where the effective potential $U_{\text{eff}}(r,\beta) = U_Q(r) + \frac{\beta}{2r^2}$ attains its minimum value $e_{\min}(\beta) = U_{\text{eff}}(r_0(\beta),\beta)$. From (7.2) we get

$$2r_0(\beta)^2\,(e_0 - U_Q(r_0(\beta)) \geq \beta.$$

Let

$$J = \{\beta \geq 0 : 2r_0(\beta)^2\,(e_0 - U_Q(r_0(\beta)) \geq \beta\}.$$

Exercise 7.1 *Prove that J is an interval. You may use the general fact that $\beta \mapsto e_{\min}(\beta)$ is increasing.*

Solution : To see this, note that

$$2r^2(e_0 - U_{\text{eff}}(r,\beta)) + \beta = 2r^2\Big(e_0 - U_Q(r) - \frac{\beta}{2r^2}\Big) + \beta = 2r^2(e_0 - U_Q(r)).$$

Therefore

$$2r^2(e_0 - U_Q(r)) \geq \beta \quad \Longleftrightarrow \quad U_{\text{eff}}(r,\beta) \leq e_0,$$

which implies that

$$J = \{\beta \geq 0 : e_{\min}(\beta) \leq e_0\}. \tag{7.3}$$

Now $\beta \mapsto e_{\min}(\beta)$ is increasing by [27, Lemma A.7(c)], and thus J has to be an interval. □

Exercise 7.2 *Prove that* $[0,\varepsilon] \subset J$ *for some* $\varepsilon > 0$ *small enough. You may use the general fact that* $r_0(\beta)^4 = \frac{1}{A(0)}\,\beta + \mathcal{O}(\beta^2)$ *for* $A(0) = U_Q''(0)$ *and* $e_{\min}(\beta) = U_Q(0) + \mathcal{O}(\beta^{1/2})$ *as* $\beta \to 0^+$.

Solution: Due to [27, Lemma A.7(f)] one has

$$r_0(\beta)^4 = \frac{1}{A(0)}\,\beta + \mathcal{O}(\beta^2) \quad \text{and} \quad e_{\min}(\beta) = U_Q(0) + \mathcal{O}(\beta^{1/2})$$

as $\beta \to 0^+$. Since $U_Q(0) < e_0$ (the cut-off energy), the condition $e_{\min}(\beta) \le e_0$ from the characterization of J in (7.3) is satisfied with strict inequality at $\beta = 0$. It follows that $[0,\varepsilon] \subset J$, if $\varepsilon > 0$ is sufficiently small. $\square$

Exercise 7.3 *Prove that* J *is bounded.*

Solution: First, if $\beta \in J$, then $r_0(\beta) \le r_Q$, where $\operatorname{supp}\rho_Q = [0, r_Q]$. Otherwise we would have $r_0(\beta) > r_Q$ for some $\beta \in J \setminus \{0\}$. Since r_Q is characterized by $U_Q(r_Q) = e_0$, this gives $U_Q(r_0(\beta)) > e_0$, and consequently $\beta \le 2r_0(\beta)^2\,(e_0 - U_Q(r_0(\beta)) \le 0$, which is a contradiction. Then $r_0(\beta) \le r_Q$ for $\beta \in J$ in turn leads to the boundedness of J, owing to

$$\beta \le 2r_0(\beta)^2\,(e_0 - U_Q(r_0(\beta)) \le 2r_Q^2\,(e_0 - U_Q(r_0(\beta)) \le 2r_Q^2\,(e_0 - U_Q(0))$$

uniformly for $\beta \in J$. $\square$

Exercise 7.4 *Prove that* $\beta_* = \max J$ *satisfies* $e_{\min}(\beta_*) = e_0$.

Solution: In fact, at β_* we must have $2r_0(\beta_*)^2\,(e_0 - U_Q(r_0(\beta_*)) = \beta_*$. Thus

$$e_{\min}(\beta_*) = U_{\text{eff}}(r_0(\beta_*), \beta_*) = U_Q(r_0(\beta_*)) + \frac{\beta_*}{2r_0(\beta_*)^2} = e_0,$$

which is the claim. $\square$

To summarize, since the condition on e is $e_0 \ge e \ge e_{\min}(\beta)$, we have shown that

$$D = \{(\beta, e) : \beta \in [0, \beta_*], e \in [e_{\min}(\beta), e_0]\} \tag{7.4}$$

and $K = [0, 2\pi] \times D$ for the support K of Q in terms of e and β, and the lower boundary curve $[0, \beta_*] \ni \beta \mapsto e_{\min}(\beta)$ strictly increases from $U_Q(0)$ to e_0.

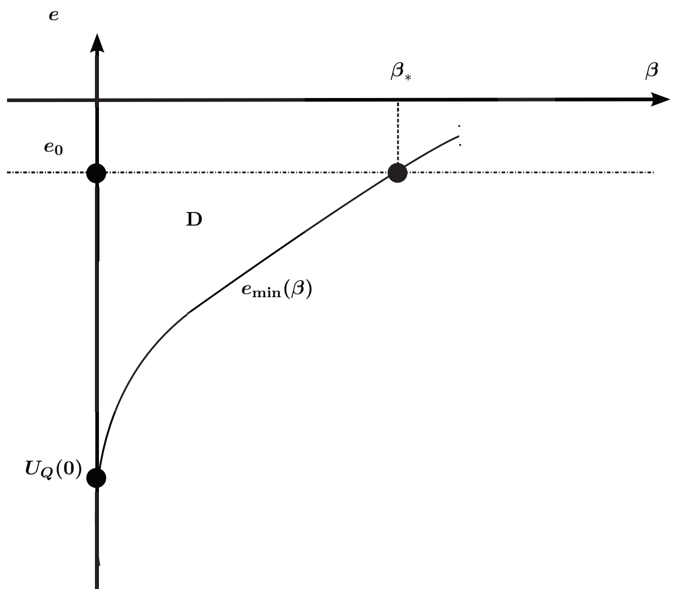

Figure 2: The domain D in coordinates $(e, \beta) = (E, \beta)$

Observe that the reasoning in this section did not depend on the specific form of the polytropic ansatz function (5.2), but only on the general properties of the functions $r_0(\beta)$ and $e_{\min}(\beta)$. It should also be mentioned that $r_0(\beta_*) \in]0, r_Q[$ is verified, see [27, Section 1.7.1].

Going back to (7.1) and using ℓ instead of β, we thus have

$$K = \{(\theta, E, \ell) : \theta \in [0, 2\pi], \ell \in [0, l_*], E \in [e_{\min}(\ell), e_0]\}$$

in the variables (θ, E, ℓ). Since $I = I(E, \ell)$ is the inverse function to $E = E(I, \ell)$ at fixed ℓ, the set K can be equally expressed in the variables (θ, I, ℓ), which is the main observation here. As θ is 2π-periodic, therefore spherically symmetric functions $g(x, v) = g(r, p_r, \ell) = g(\theta, I, \ell)$ of (θ, I, ℓ) that are defined on K, the support of Q, can be expanded into a Fourier series

$$g(\theta, I, \ell) = \sum_{k \in \mathbb{Z}} g_k(I, \ell)\, e^{ik\theta}, \tag{7.5}$$

where $(I, \ell) \simeq (E, \ell) \simeq (E, \beta) \in D$. The Fourier coefficients are

$$g_k(I, \ell) = \frac{1}{2\pi} \int_0^{2\pi} g(\theta, I, \ell)\, e^{-ik\theta}\, d\theta.$$

The series expansion (7.5) is most convenient, since one can easily do calculations on such series, or define Sobolev-type function space.

This motivates the following

Definition 7.5 (**X^α-spaces**) *For $\alpha \geq 0$ denote*

$$X^\alpha = \Big\{ g = \sum_{k\in\mathbb{Z}} g_k(I,\ell)\, e^{ik\theta} : \|g\|^2_{X^\alpha} = 16\pi^3 \sum_{k\in\mathbb{Z}} (1+k^2)^\alpha\, \|g_k\|^2_{L^2_{\frac{1}{|Q'|}}(D)} < \infty \Big\},$$

where

$$D = \{(E,\ell) : \ell \in [0,\ell_*], E \in [e_{\min}(\ell), e_0]\}$$

is from (7.4) and expressed in (I,ℓ), and correspondingly

$$(\phi,\psi)_{L^2_{\frac{1}{|Q'|}}(D)} = \iint_D dI\, d\ell\, \ell\, \frac{1}{|Q'(e)|}\, \overline{\phi(I,\ell)}\, \psi(I,\ell)$$

is a weighted L^2-inner product for suitable functions ϕ,ψ on D; note $e = e(I,\ell)$. The associated scalar product on the Hilbert space X^α is given by

$$(g,h)_{X^\alpha} = 16\pi^3 \sum_{k\in\mathbb{Z}} (1+k^2)^\alpha\, (g_k,h_k)_{L^2_{\frac{1}{|Q'|}}(D)}$$

for $g = \sum_{k\in\mathbb{Z}} g_k\, e^{ik\theta}$ and $h = \sum_{k\in\mathbb{Z}} h_k\, e^{ik\theta}$.

8 Linearization

Now that we have introduced some nice Hilbert spaces, we also need to have a suitable self-adjoint operator in order to come close to a possible Birman-Schwinger setting. Since we are interested in dynamical properties of the system close to an isotropic steady state solution Q, it is natural to consider the linearization about such a steady state.

For, we write $f(t) = Q + g(t)$ with g 'small'. First note that

$$v\cdot\nabla_x f - \nabla_x U_f \cdot \nabla_v f = \{f, e_f\}$$

for $e_f(x,v) = \frac{1}{2}|v|^2 + U_f(x)$, where

$$\{g,h\} = \nabla_x g\cdot\nabla_v h - \nabla_v g \cdot \nabla_x h$$

denotes the standard Poisson bracket of two functions $g = g(x,v)$ and $h = h(x,v)$.

Exercise 8.1 *Prove that if $\phi : \mathbb{R}\to\mathbb{R}$ is a function and $g = g(x,v)$, $h = h(x,v)$, then $\{\phi(g),h\} = \phi'(g)\{g,h\}$.*

Therefore we may write the Vlasov equation (3.3) as

$$\begin{aligned} 0 &= \partial_t f + \{f,e_f\} = \partial_t g + \Big\{ Q+g, \frac{1}{2}|v|^2 + U_Q + U_g \Big\} \\ &= \partial_t g - \nabla_v Q\cdot\nabla_x U_g + v\cdot\nabla_x g - \nabla_v g\cdot\nabla_x U_Q - \nabla_v g\cdot\nabla_x U_g, \end{aligned}$$

which is equivalent to

$$\partial_t g + \mathcal{T} g + \mathcal{K} g = \nabla_v g \cdot \nabla_x U_g, \tag{8.1}$$

where we have introduced the linear operators

$$\begin{aligned} \mathcal{T} g &= v \cdot \nabla_x g - \nabla_v g \cdot \nabla_x U_Q = \{g, e_Q\}, & (8.2) \\ \mathcal{K} g &= -\nabla_v Q \cdot \nabla_x U_g = \{Q, U_g\}; & (8.3) \end{aligned}$$

recall that $U_g = U_g(r)$, whence $\nabla_v U_g = 0$. Since the term on the right-hand side of (8.1) is (formally) quadratic in g, the linearization is found to be

$$\partial_t g + \mathcal{T} g + \mathcal{K} g = 0. \tag{8.4}$$

The next step is to linearize not only the equation itself, but also a suitable Lyapunov-type functional. For this we will closely follow [15] and once again write $f(t) = Q + g(t)$. The total energy

$$\mathcal{H}(f(t)) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^2 \, f(t,x,v) \, dx \, dv - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla U_{f(t)}(t,x)|^2 \, dx$$

is conserved along solutions, so it could be suspected to be a Lyapunov function.

Exercise 8.2 *Prove that $\frac{d}{dt} \mathcal{H}(f(t)) = 0$ for solutions $f(t)$ of the Vlasov-Poisson system.*

The expansion about Q then yields

$$\mathcal{H}(f(t)) = \mathcal{H}(Q) + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e_Q \, g(t) \, dx \, dv - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla U_{g(t)}|^2 \, dx + \mathcal{O}(g^3); \tag{8.5}$$

note that $f \mapsto U_f$ is linear.

Exercise 8.3 *Prove that (8.5) holds (formally).*

The linear term on the right-hand side of (8.5) does not vanish, i.e., Q is not a critical point of $\mathcal{H}$. However, this defect can be remedied by making use of the fact that every 'Casimir functional'

$$\mathcal{C}_\Phi(f(t)) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(f(t,x,v)) \, dx \, dv$$

is also conserved along solutions, provided that Φ is sufficiently well-behaved. Passing from $\mathcal{H}$ to

$$\mathcal{H}_\Phi = \mathcal{H} + \mathcal{C}_\Phi$$

and repeating the expansion, one arrives at

$$\begin{aligned} \mathcal{H}_\Phi(f(t)) &= \mathcal{H}_\Phi(Q) + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (e_Q + \Phi'(Q)) \, g(t) \, dx \, dv \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi''(Q) \, g(t)^2 \, dx \, dv - \frac{1}{8\pi} \int_{\mathbb{R}^3} |\nabla U_{g(t)}|^2 \, dx + \mathcal{O}(g^3). \end{aligned} \tag{8.6}$$

Writing $e = e_Q$, since $Q = Q(e)$, the equation $e + \Phi'(Q(e)) = 0$ can be (formally) solved by taking $\Phi'(\xi) = -Q^{-1}(\xi)$, at least if for instance $Q'(e) < 0$ is verified for the relevant e in the support of Q.

Exercise 8.4 *For the polytropes (5.2), show that* $\Phi(\xi) = \frac{k}{k+1}\,\xi^{\frac{k+1}{k}} - e_0\,\xi$, $\xi \in [0, \infty[$, *is a possible choice.*

Then Q becomes a critical point of this $\mathcal{H}_\Phi$, and due to $1 + \Phi''(Q(e))Q'(e) = 0$ and $Q'(e) < 0$ the expansion (8.6) simplifies to

$$\begin{aligned} \mathcal{H}_\Phi(f(t)) &= \mathcal{H}_\Phi(Q) + \frac{1}{2}\,\mathcal{A}(g(t), g(t)) + \mathcal{O}(g^3), \\ \mathcal{A}(g, g) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,|g|^2 - \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla_x U_g|^2\,dx. \end{aligned}$$

Thus one can expect that the stability of Q will be determined by the properties of the quadratic (second variation) part $\mathcal{A} = 2\,D^2\mathcal{H}_\Phi(Q)$, which we will call the Antonov functional.

Exercise 8.5 *Prove that* $\frac{d}{dt}\,\mathcal{A}(g(t), g(t)) = 0$ *along solutions* $g(t)$ *of the linearized equation (8.4).*

Ideally, to infer stability of Q it would be helpful if $\mathcal{A}$ had some kind of coercivity property. Here one hopes to transfer to our infinite-dimensional problem the following one-degree-of freedom analogue: for $H(q, p) = \frac{1}{2}\,p^2 + V(q)$ the equation of motion is $\ddot{q} = -V'(q)$, and if for instance $V'(0) = 0$ and $V''(0) > 0$, then the equilibrium $q = 0$ is stable.

Now it is the content of the celebrated Antonov stability estimate [2, 3], that

$$\mathcal{A}(\mathcal{T}u, \mathcal{T}u) \geq c\|u\|_Q^2 \tag{8.7}$$

holds for all functions $u = u(x, v)$ that are spherically symmetric and odd in v, i.e., they satisfy $u(x, -v) = -u(x, v)$; the constant $c > 0$ does only depends upon Q. The weighted L^2-inner product is defined as

$$(g, h)_Q = \iint\limits_K \frac{1}{|Q'(e_Q)|}\,\overline{g(x, v)}\,h(x, v)\,dx\,dv \tag{8.8}$$

and it induces the norm $\|\cdot\|_Q$. Perturbations of the form $g = \mathcal{T}u$ are called 'dynamically accessible', for reasons explained in [35], also see [37]. Antonov [2, 3] could prove that the positive definiteness (8.7) is equivalent to the linear stability of Q. Many works followed these pioneering observations, and until to date almost all stability proofs, linear or nonlinear, use the Antonov stability estimate in one way or another. The bound (8.7), or variations thereof, is applied in a number of papers, both in the physics and in the mathematics community, to address a variety of stability issues; see [7, 11, 14, 15, 25, 26, 28, 29, 33, 48] and many further.

In view of (8.7), we first need to obtain a better understanding of spherically symmetric functions g that are odd in v.

Exercise 8.6 *Prove the following facts:*

(a) If $(x, v) \mapsto (p_r, L_3, \ell; r, \varphi, \chi)$ *under the above transformation (6.5), then we have* $(x, -v) \mapsto (-p_r, -L_3, \ell; r, \varphi + \pi, \pi - \chi)$.

(b) If

$$(x, v) \mapsto (p_r, L_3, \ell; r, \varphi, \chi) \mapsto (I, L_3, \ell; \theta, \varphi, \chi)$$

under the transformation (6.10), then

$$(x, -v) \mapsto (-p_r, -L_3, \ell; r, \varphi + \pi, \pi - \chi) \mapsto (I, -L_3, \ell; 2\pi - \theta, \varphi + \pi, \pi - \chi).$$

(c) g *is even in* v *if and only if* $g(r, -p_r, \ell) = g(r, p_r, \ell)$ *if and only if* $g(2\pi - \theta, I, \ell) = g(\theta, I, \ell)$.

(d) g *is odd in* v *if and only if* $g(r, -p_r, \ell) = -g(r, p_r, \ell)$ *if and only if* $g(2\pi - \theta, I, \ell) = -g(\theta, I, \ell)$.

(e) $g(\theta, I, \ell) = \sum_{k \in \mathbb{Z}} g_k(I, \ell)\, e^{ik\theta}$ *is even in* v *if and only if* $g_{-k} = g_k$ *for all* $k \in \mathbb{Z}$. *If* g *is real-valued, then* $g_k(I, \ell) \in \mathbb{R}$.

(f) $g(\theta, I, \ell) = \sum_{k \in \mathbb{Z}} g_k(I, \ell)\, e^{ik\theta}$ *is odd in* v *if and only if* $g_{-k} = -g_k$ *for all* $k \in \mathbb{Z}$, *and in particular* $g_0 = 0$. *If* g *is real-valued, then* $g_k(I, \ell) \in i\mathbb{R}$.

Definition 8.7 (X^α_{odd}-spaces) *For* $\alpha \geq 0$ *denote*

$$X^\alpha_{\mathrm{odd}} = \{g \in X^\alpha : g_{-k} = -g_k \ for \ k \in \mathbb{Z}\}.$$

Now we are in a position to introduce one of our main objects of interest, namely the operator

$$Lu = -\mathcal{T}^2 u - \mathcal{K}\mathcal{T} u. \tag{8.9}$$

The connection to the stability problem outlined above is made in

Lemma 8.8 L *is self-adjoint on the domain* $\mathcal{D}(L) = X^2_{\mathrm{odd}}$ *in* X^0_{odd}. *In addition,*

$$(Lu, u)_Q = \mathcal{A}(\mathcal{T}u, \mathcal{T}u)$$

holds for $u \in X^2_{\mathrm{odd}}$.

See [27, Lemma 1.1] for the proof. Here we only give a somewhat rough argument, why

$$(-\mathcal{T}^2 u, u)_Q = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{dx\, dv}{|Q'(e_Q)|}\, |\mathcal{T}u|^2 \tag{8.10}$$

can be expected to hold; think of $(-\Delta u, u)_{L^2} = \|\nabla u\|^2_{L^2}$ under appropriate hypotheses on u. First we observe that $\mathcal{T}$ from (8.2) can be written as

$$\mathcal{T}g = v \cdot \nabla_x g - \nabla_v g \cdot \nabla_x U_Q = \mathrm{div}_x(vg) - \mathrm{div}_v(g \nabla_x U),$$

since U_Q is independent of v. Therefore

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(\mathcal{T}g)h\,dx\,dv = -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g(\mathcal{T}h)\,dx\,dv$$

through integration by parts, if there are no boundary terms. Thus if u is real-valued and has its support in K, then we have by (8.8)

$$(-\mathcal{T}^2 u, u)_Q = -\iint\limits_K \frac{1}{|Q'(e_Q)|}\,(\mathcal{T}^2 u)\,u\,dx\,dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(\mathcal{T}u)\,\mathcal{T}\Big(\frac{1}{|Q'(e_Q)|}\,u\Big)\,dx\,dv.$$

Clearly $\mathcal{T}$ satisfies the product rule $\mathcal{T}(gh) = (\mathcal{T}g)h + g(\mathcal{T}h)$. Now if we use (8.2) and Exercise 8.1 for $\phi(s) = \frac{1}{|Q'(s)|}$, then we obtain

$$\mathcal{T}\Big(\frac{1}{|Q'(e_Q)|}\Big) = \{\phi(e_Q), e_Q\} = \phi'(e_Q)\{e_Q, e_Q\} = 0,$$

and (8.10) follows, if we proceed in the same way with the second $\mathcal{T}$. Then in order to establish Lemma 8.8, one also needs some further properties of $-\mathcal{T}^2$ and $\mathcal{K}\mathcal{T}$ that are stated in Lemma 9.1(a), (b) below.

Due to the Antonov bound (8.7) and Lemma 8.8, the spectrum of the second variation (in dynamically accessible directions) is expected to play a major role in all kinds of stability questions.

9 The Birman-Schwinger approach

Therefore the task is to extract as much information on the spectrum of L as possible. To begin with, we recall that the discrete spectrum of a self-adjoint operator L in a Hilbert space, called $\sigma_d(L)$, consists of all eigenvalues of L of finite multiplicity that are isolated points of the spectrum $\sigma(L)$. Its complement $\sigma_{\rm ess}(A) = \sigma(A)\setminus\sigma_d(L)$ is the essential spectrum.

Let us first consider this part of the spectrum for $Lu = -\mathcal{T}^2 u - \mathcal{K}\mathcal{T}u$ from (8.9) on $\mathcal{D}(L) = X^2_{\rm odd}$ in the Hilbert space $X^0_{\rm odd}$.

Lemma 9.1 *The following assertions hold:*

(a) $-\mathcal{T}^2 : X^2_{\rm odd} \to X^0_{\rm odd}$ *is a self-adjoint operator.*

(b) The operator $\mathcal{K}\mathcal{T} : X^0_{\rm odd} \to X^0_{\rm odd}$ *is given by*

$$\mathcal{K}\mathcal{T}g = 4\pi\,|Q'(e_Q)|\,p_r\int_{\mathbb{R}^3} p_r\,g\,dv$$

and it is linear, bounded, symmetric and positive:

$$(\mathcal{K}\mathcal{T}g, g)_{X^0} = \frac{1}{4\pi}\int_{\mathbb{R}^3}|U'_{\mathcal{T}g}(r)|^2\,dx \ge 0 \quad \textit{for} \quad g \in X^0. \tag{9.1}$$

(c) $\mathcal{K}\mathcal{T}$ *is relatively* L*-compact, in that* $\mathcal{D}(\mathcal{K}\mathcal{T}) = X^0 \supset X^2_{\mathrm{odd}} = \mathcal{D}(L)$ *for the domains and* $\mathcal{K}\mathcal{T}(L+i)^{-1} : X^0_{\mathrm{odd}} \to X^0_{\mathrm{odd}}$ *is a compact operator.*

(d) *We have*

$$\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(-\mathcal{T}^2).$$

Proof: See [27, Cor. B.10] for (a), [27, Cor. B.15] for (b) and the proof of [27, Cor. B.19] for (c). Essentially this is due to the fact that $\mathcal{K} : X^0 \to X^0$ is compact, [27, Cor. C.6]. To establish the latter property, one uses that $\mathcal{K}g = |Q'(e_Q)| p_r U_g'(r)$ (see below) and

$$\|\nabla(\nabla U_g)\|_{L^2(\mathbb{R}^3)} \le C\|\nabla(\nabla\Delta^{-1}\rho_g)\|_{L^2(\mathbb{R}^3)} \le C\|\rho_g\|_{L^2(\mathbb{R}^3)}$$

together with the Sobolev embedding theorem; the regularizing property of $\Delta U_g = 4\pi\rho_g$ is central to many stability results for Vlasov-Poisson. (d) This is a consequence of Weyl's Theorem, cf. [20, Thm. 14.6]. □

Thus we need to determine the essential spectrum of $-\mathcal{T}^2$. For this, the variables (θ, I, ℓ) are most convenient, and we are going to use the fact that canonical transformations leave Poisson brackets unaltered. Hence if we write $\Phi = (r, \varphi, \chi)$, $\mathfrak{A} = (p_r, L_3, \ell)$, $\Theta = (\theta, \varphi, \chi)$ and $\mathfrak{I} = (I, L_3, \ell)$ for the coordinates, see (6.10), then

$$\{g, h\}_{xv} = \{g, h\}_{\Phi\mathfrak{A}} = \{g, h\}_{\Theta\mathfrak{I}}.$$

But the functions do depend only upon (r, p_r, ℓ) and (θ, I, ℓ), respectively. Thus

$$\begin{aligned} \{g, h\}_{\Phi\mathfrak{A}} &= (\partial_r g)(\partial_{p_r} h) - (\partial_r h)(\partial_{p_r} g), \\ \{g^*, h^*\}_{\Theta\mathfrak{I}} &= (\partial_\theta g)(\partial_I h) - (\partial_\theta h)(\partial_I g). \end{aligned}$$

Next we recall that $e_Q = e = E(I, \ell)$. Hence due to $\omega_1 = \frac{\partial E}{\partial I}$ and $\partial_\theta E = 0$ we get

$$\mathcal{T}g = \{g, e_Q\} = (\partial_\theta g)(\partial_I E) - (\partial_\theta E)(\partial_I g) = \omega_1 \partial_\theta g,$$

which is appealingly simple in the coordinates (θ, I, ℓ). Since ω_1 is independent of θ, see (6.11), it also follows that

$$-\mathcal{T}^2 g = -\omega_1\partial_\theta(\omega_1\partial_\theta g) = -\omega_1^2\, \partial_\theta^2 g.$$

This relation makes it clear that the properties of the function $\omega_1 = \omega_1(e, \ell) = \omega_1(e, \beta)$, or equivalently of the period function $T_1 = \frac{2\pi}{\omega_1}$, on D will be important.

Lemma 9.2 *The following assertions hold:*

(a) *We have* $\omega_1 \in C^1(\mathring{D}) \cap C(D)$.

(b) *It holds that*

$$\inf \omega_1 = \delta_1 > 0 \quad \textit{and} \quad \sup \omega_1 = \Delta_1 < \infty.$$

Proof: See [27, Thm. 3.6 & Thm. 3.13] for (a) and [27, Thm. 3.2 & Thm. 3.5] for (b). Since ω_1 is continuous on the compact set D and T_1 is non-zero, certainly (b) follows from (a), but it is also possible to give a proof using the explicit period relation (6.12). Similar results have been obtained in [18]. $\square$

There is a result in the spectral theory of self-adjoint operators that asserts that the spectrum of the multiplication operator $M : D(M) = \{u \in L^2(\mathbb{R}^n) : \chi u \in L^2(\mathbb{R}^n)\} \to L^2(\mathbb{R}^n)$, $Mu = \chi u$, for a given real-valued and continuous function χ has the spectrum $\overline{\operatorname{ran}\chi}$, with $\operatorname{ran}\chi$ denoting the range of χ. If we also take into account that on a function $g = \sum_{k\in\mathbb{Z}} g_k\, e^{ik\theta}$ with coefficient $g_k = g_k(I, \ell)$ we have

$$-\mathcal{T}^2 : g \cong (g_k) \mapsto (\omega_1^2\, k^2 g_k),$$

then the following characterization of the essential spectrum of $-\mathcal{T}^2$, and thus of the one of L, is not a big surprise, since $\operatorname{ran}\omega_1 = [\delta_1, \Delta_1]$.

Theorem 9.3 *For the essential spectrum we have*

$$\sigma_{\mathrm{ess}}(L) = \sigma_{\mathrm{ess}}(-\mathcal{T}^2) = \bigcup_{k=1}^{\infty} k^2[\delta_1^2, \Delta_1^2] \quad \textit{and} \quad \delta_1^2 = \min \sigma_{\mathrm{ess}}(L) > 0.$$

If ω_1 is not constant, then there exists $\lambda_c > \delta_1^2$ such that $[\lambda_c, \infty[\subset \sigma_{\mathrm{ess}}(L)$.

Proof: See [27, Cor. B.19]. $\square$

Exercise 9.4 *Prove the last statement of Theorem 9.3.*

Note that due to $\sigma(-\mathcal{T}^2) = \sigma_{\mathrm{ess}}(-\mathcal{T}^2)$, in particular $\sigma(-\mathcal{T}^2) = \bigcup_{k=1}^{\infty} k^2[\delta_1^2, \Delta_1^2]$ holds. What we have done so far is more or less standard, but now we are getting closer to the heart of the matter. As before, we are trying to understand the spectrum of L, but this time, more specifically, possible eigenvalues below the essential spectrum; the following calculation is motivated by [34]. Let $\lambda < \delta_1^2$ and suppose that $Lu = \lambda u$ for some $u \in X^2_{\mathrm{odd}}$ and $u \neq 0$. Then $(-\mathcal{T}^2 - \lambda)u = \mathcal{K}\mathcal{T}u$. Defining $\psi = (-\mathcal{T}^2 - \lambda)u \in X^0_{\mathrm{odd}}$, we get

$$\psi = \mathcal{K}\mathcal{T}(-\mathcal{T}^2 - \lambda)^{-1}\psi. \tag{9.2}$$

Now

$$\mathcal{K}g = -\nabla_v Q \cdot \nabla_x U_g = -Q'(e_Q)v \cdot U_g'(r)\frac{x}{r} = |Q'(e_Q)|\, p_r\, U_g'(r)$$

by the definition of $\mathcal{K}$ in (8.3) and since $Q'(e_Q) < 0$. Thus the image of the operator $\mathcal{K}$ is special. Apart from the factor $|Q'(e_Q)|$, it consists of function $h = h(r, p_r, \ell)$ that factorize as $p_r\tilde{h}(r)$. In particular, due to (9.2), also ψ can be written in this way and we obtain

$$\psi = \mathcal{K}\mathcal{T}u = |Q'(e_Q)|\, p_r\, U_{\mathcal{T}u}'(r). \tag{9.3}$$

To make sense of the following definition, we need to mention that in general for spherically symmetric functions g one has $U_{\mathcal{T}g}'(r) = 4\pi \int_{\mathbb{R}^3} p_r\, g\, dv$.

Exercise 9.5 *Prove this, using that* $\rho_{\mathcal{T}g}(x) = \operatorname{div}_x \int_{\mathbb{R}^3} v\, g\, dv$ *and Gauss's Theorem in* $U_h'(r) = \frac{1}{r^2}\int_{|x|\le r} \rho_h(x)\, dx$.

Definition 9.6 (The Birman-Schwinger operators) *Let*

$$\mathcal{Q}_\lambda \Psi = U'_{\mathcal{T}(-\mathcal{T}^2-\lambda)^{-1}\psi} = 4\pi \int_{\mathbb{R}^3} p_r\, (-\mathcal{T}^2 - \lambda)^{-1}\psi\, dv \tag{9.4}$$

for functions $\Psi = \Psi(r)$, *where we put* $\psi(r, p_r, \ell) = |Q'(e_Q)|\, p_r\, \Psi(r)$ *in terms of a given* Ψ.

Since $\int dv$ is integrated out in (9.4), it turns out that $\mathcal{Q}_\lambda \Psi = (\mathcal{Q}_\lambda \Psi)(r)$ is also a function of r only. Coming back to the spectral problem for L, we started out with an eigenfunction u of L, thereafter put $\psi = (-\mathcal{T}^2 - \lambda)u$, and now let

$$\Psi(r) = U'_{\mathcal{T}(-\mathcal{T}^2-\lambda)^{-1}\psi}(r) = U'_{\mathcal{T}u}(r)$$

to deduce

$$|Q'(e_Q)|\, p_r\, \Psi(r) = |Q'(e_Q)|\, p_r\, U'_{\mathcal{T}u}(r) = \psi$$

from (9.3). Therefore we obtain

$$\mathcal{Q}_\lambda \Psi = U'_{\mathcal{T}(-\mathcal{T}^2-\lambda)^{-1}\psi} = \Psi.$$

In other words, 1 is an eigenvalue of $\mathcal{Q}_\lambda$ with eigenfunction Ψ. Since a converse statement can be verified similarly, we arrive at

Theorem 9.7 *Let* $\lambda < \delta_1^2$. *Then* λ *is an eigenvalue of* L *if and only if* 1 *is an eigenvalue of* $\mathcal{Q}_\lambda$. *More precisely,*

(a) if $u \in X^2_{\mathrm{odd}}$ *is an eigenfunction of* L *for the eigenvalue* λ, *then* $\Psi = U'_{\mathcal{T}u} \in L^2_r$ *is an eigenfunction of* $\mathcal{Q}_\lambda$ *for the eigenvalue* 1*;*

(b) if $\Psi \in L^2_r$ *is an eigenfunction of* $\mathcal{Q}_\lambda$ *for the eigenvalue* 1, *then* $u = (-\mathcal{T}^2 - \lambda)^{-1}(|Q'(e_Q)|\, p_r \Psi) \in X^2_{\mathrm{odd}}$ *is an eigenfunction of* L *for the eigenvalue* λ.

Proof: See [27, Thm. 4.5]. □

Here L^2_r denotes the L^2-Lebesgue space of radially symmetric functions $\Psi(x) = \Psi(r)$ on $\mathbb{R}^3$, where we take

$$\langle \Psi, \Phi \rangle = \int_{\mathbb{R}^3} \overline{\Psi(x)}\, \Phi(x)\, dx = 4\pi \int_0^\infty r^2\, \overline{\Psi(r)}\, \Phi(r)\, dr$$

as the inner product of $\Psi, \Phi \in L^2_r$.

Exercise 9.8 *Prove part (b) of Theorem 9.7.*

If we compare Theorem 9.7 to the quantum mechanics result Theorem 2.1, then we see that also in galactic dynamics there is a Birman-Schwinger principle. Furthermore, it is nice to observe that both are formally identical, if we associate $p_r \sim \sqrt{-V}$ and $-\Delta \sim -\mathcal{T}^2$, and furthermore disregard the velocity average $\int_{\mathbb{R}^3} dv$; the appearance of $|Q'(e_Q)|$ in $|Q'(e_Q)|\, p_r\Psi$ is due to the $(\cdot,\cdot)_Q$ that is used. There is yet another fact that supports the analogy of both approaches. The operator $\mathcal{Q}_\lambda$ from (9.4) can be expressed as

$$\mathcal{Q}_\lambda\Psi = 4\pi\int_{\mathbb{R}^3} p_r\,(-\mathcal{T}^2-\lambda)^{-1}\,(|Q'(e_Q)|\,p_r\Psi)\,dv.$$

Comparing this relation to (2.1), it turns out that both relations do agree, if we apply the same identifications as before.

Theorem 9.7 could only be useful if we are able to gain a better understanding of the operators $\mathcal{Q}_\lambda$, which turn out to have a couple of nice properties. One also notices that $\mathcal{Q}_z$ can not only be defined for $z=\lambda<\delta_1^2$, but for all $z\in\Omega=\mathbb{C}\setminus[\delta_1^2,\infty[$ (or even for all $z\in\mathbb{C}\setminus\sigma_{\mathrm{ess}}(-\mathcal{T}^2)$).

Lemma 9.9 (Properties of $\mathcal{Q}_z$) *The following assertions hold.*

(a) For every $z\in\Omega$ we have $\mathcal{Q}_z\in\mathcal{B}(L_r^2)$, the space of linear and bounded operators on L_r^2. In addition, the map

$$\Omega\ni z\mapsto\mathcal{Q}_z\in\mathcal{B}(L_r^2)$$

is analytic, and we have the representation

$$\begin{aligned}(\mathcal{Q}_z\Psi)(r) &= \frac{16\pi}{r^2}\sum_{k\neq 0}\int_0^\infty d\tilde{r}\,\Psi(\tilde{r})\iint_D d\ell\,\ell\,de\,\mathbf{1}_{\{r_-(e,\ell)\le r,\tilde{r}\le r_+(e,\ell)\}}\,\frac{\omega_1(e,\ell)\,|Q'(e)|}{(k^2\omega_1^2(e,\ell)-z)}\\ &\qquad\times\sin(k\theta(r,e,\ell))\sin(k\theta(\tilde{r},e,\ell))\end{aligned}\tag{9.5}$$

for $\Psi\in L_r^2$.

(b) If $z\in\Omega$, then

$$(\mathcal{Q}_z\Psi)(r)=\langle K_{\bar{z}}(r,\cdot),\Psi\rangle$$

for $\Psi\in L_r^2$. The integral kernel K_z is given by

$$\begin{aligned}&K_z(r,\tilde{r})\\ &=\frac{4}{r^2\tilde{r}^2}\sum_{k\neq 0}\iint_D d\ell\,\ell\,de\,\mathbf{1}_{\{r_-(e,\ell)\le r,\tilde{r}\le r_+(e,\ell)\}}\,\frac{\omega_1(e,\ell)\,|Q'(e)|}{k^2\omega_1^2(e,\ell)-z}\,\sin(k\theta(r,e,\ell))\sin(k\theta(\tilde{r},e,\ell)).\end{aligned}$$

(c) If $z\in\Omega$, then $\mathcal{Q}_z$ is a Hilbert-Schmidt operator on L_r^2.

(d) *If* $\lambda \in]-\infty, \delta_1^2[$*, then* $\mathcal{Q}_\lambda$ *is symmetric and positive. Its spectrum consists of* $\mu_1(\lambda) \geq \mu_2(\lambda) \geq \ldots \to 0$ *(the eigenvalues are listed according to their multiplicities). In addition,*

$$\mu_1(\lambda) = \|\mathcal{Q}_\lambda\| = \sup\{\langle \mathcal{Q}_\lambda \Psi, \Psi\rangle : \|\Psi\|_{L_r^2} \leq 1\}$$

for the largest eigenvalue of $\mathcal{Q}_\lambda$*, where* $\|\cdot\| = \|\cdot\|_{\mathcal{B}(L_r^2)}$.

Proof: See [27, Lemma 4.3]. The representation formula (9.5) is very convenient and obtained from (9.4) by Fourier expanding the functions involved and using the fact that $\psi(r, p_r, \ell) = |Q'(e_Q)|\, p_r\, \Psi(r)$ has

$$\psi_k(I, \ell) = -\frac{i}{\pi}\,|Q'(e)|\,\omega_1(e, \ell)\int_{r_-(e,\ell)}^{r_+(e,\ell)} d\tilde{r}\,\Psi(\tilde{r})\sin(k\theta(\tilde{r}, e, \ell)) \tag{9.6}$$

as its Fourier coefficients. □

Exercise 9.10 *Prove (9.6) from (6.7).*

According to Theorem 9.7, in order to find eigenvalues $\hat{\lambda} < \delta_1^2$, we have to locate such a $\hat{\lambda}$ that additionally satisfies $\mu_1(\hat{\lambda}) = 1$. Therefore we have to study the function $\mu_1 :]-\infty, \delta_1^2[\to]0, \infty[$ in more detail.

Lemma 9.11 *We have* $0 < \mu_1(0) < 1$*, and* μ *is monotone increasing, convex and locally Lipschitz continuous. The limit*

$$\mu_* = \lim_{\lambda \to \delta_1^2 -} \mu_1(\lambda) = \sup\{\mu_1(\lambda) : \lambda \in [0, \delta_1^2[\} \in [\mu_1(0), \infty] \tag{9.7}$$

does exist.

Proof: See [27, Lemma 4.3 & Lemma 4.7(a), (d)]. □

10 An application

It is to be expected that a good understanding of the Birman-Schwinger operators $\mathcal{Q}_z$ and their spectra will lead to new insights into stability-related properties of solutions close to a static solution of the Vlasov-Poisson system.

As an example application, we consider

$$\lambda_* = \inf\{(Lu, u)_Q : u \in X^2_{\mathrm{odd}}, \|u\|_Q = 1\} > 0,$$

which is the 'best constant' in the Antonov stability estimate (8.7); recall Lemma 8.8. In [27] we derived many results related to λ_*, and in particular we were able to characterize the cases where λ_* is attained, in the sense that $\lambda_* = (Lu_*, u_*)_Q$ for some minimizing function $u_* \in X^2_{\rm odd}$ such that $\|u_*\|_Q = 1$. It turns out that then u_* will be an eigenfunction of L corresponding to the eigenvalue λ_*, so that $Lu_* = \lambda_* u_*$. Both u_* and the quantity λ_* will be of fundamental importance for the dynamics of the gravitational Vlasov-Poisson system.

Lemma 10.1 *Let $u_* \in X^2_{\rm odd}$ be a minimizer and define*

$$g_*(t,x,v) = \cos(\sqrt{\lambda_*}t)\, u_*(x,v) - \frac{1}{\sqrt{\lambda_*}} \sin(\sqrt{\lambda_*}t)\, (\mathcal{T} u_*)(x,v).$$

Then g_* is a $\frac{2\pi}{\sqrt{\lambda_*}}$-periodic solution of the equation (8.4) that is obtained by linearizing Vlasov-Poisson about Q.

Proof: Observe that u_* is odd in v. Hence $\rho_{u_*}(x) = \int_{\mathbb{R}^3} u_*(x,v)\,dv = 0$ implies that $U_{u_*} = 4\pi\Delta^{-1}\rho_{u_*} = 0$ and therefore $\mathcal{K}u_* = 0$ by (8.3). Consequently,

$$\begin{aligned}
&\partial_t g_* + \mathcal{T} g_* + \mathcal{K} g_* \\
&= -\sqrt{\lambda_*}\sin(\sqrt{\lambda_*}t)\, u_* - \cos(\sqrt{\lambda_*}t)\,\mathcal{T}u_* + \cos(\sqrt{\lambda_*}t)\,\mathcal{T}u_* - \frac{1}{\sqrt{\lambda_*}}\sin(\sqrt{\lambda_*}t)\,\mathcal{T}^2 u_* \\
&\qquad + \cos(\sqrt{\lambda_*}t)\,\mathcal{K}u_* - \frac{1}{\sqrt{\lambda_*}}\sin(\sqrt{\lambda_*}t)\,\mathcal{K}\mathcal{T}u_* \\
&= -\sqrt{\lambda_*}\sin(\sqrt{\lambda_*}t)\, u_* + \frac{1}{\sqrt{\lambda_*}}\sin(\sqrt{\lambda_*}t)\, Lu_* \\
&= 0,
\end{aligned}$$

as claimed. □

Next we will clarify where λ_* is located as compared to δ_1^2, which is the minimum of the essential spectrum of L; recall Theorem 9.3.

Lemma 10.2 *We have $\lambda_* \le \delta_1^2$.*

Proof: According to [41, Thm. XIII.1] applied to $n = 1$, either

(a) λ_* is an eigenvalue below $\sigma_{\rm ess}(L)$, i.e., $\lambda_* < \inf\sigma_{\rm ess}(L)$, and λ_* is the first eigenvalue, or

(b) $\lambda_* = \inf\sigma_{\rm ess}(L)$ and there are no eigenvalues below λ_*.

Since $\inf\sigma_{\rm ess}(L) = \delta_1^2$, the claim follows. It is also possible to deduce the estimate by direct calculation, see [27, Lemma 3.18]. Here it should only be noted that the result is at least conceivable from the following observation: since $Lu = -\mathcal{T}^2 u - \mathcal{K}\mathcal{T}u$, using (8.10) and (9.1) we get

$$\begin{aligned}(Lu, u)_Q &= (-\mathcal{T}^2 u, u)_Q - (\mathcal{K}\mathcal{T}u, u)_Q \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,|\mathcal{T}u|^2 - \frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla_x U_{\mathcal{T}u}|^2\,dx \\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{dx\,dv}{|Q'(e_Q)|}\,|\mathcal{T}u|^2.\end{aligned}$$

The latter expression equals $(-\mathcal{T}^2 u, u)_Q$, and it is just the quadratic form associated to $-\mathcal{T}^2$. One can construct suitable $u_j \in X^2_{\rm odd}$ such that $\|u_j\|_Q = 1$ and $(-\mathcal{T}^2 u_j, u_j)_Q \to \delta_1^2$ as $j \to \infty$. $\Box$

For the remaining part of these lectures, we will deal with the following result that illustrates the usefulness of the Birman-Schwinger operators.

Theorem 10.3 *We have*

$$\mu_* > 1 \iff \lambda_* < \delta_1^2.$$

In this case $\mu_1(\lambda_*) = 1$ *and* λ_* *is an eigenvalue of* L.

Proof: See [27, Thm. 4.13]. $\Box$

It is not too hard to show that if $\mu_* > 1$, then $\lambda_* = \delta_1^2$ is impossible, so that we must have $\lambda_* < \delta_1^2$. The converse statement is more interesting to prove. Thus let us suppose that $\lambda_* < \delta_1^2$ holds, and assume that we already knew that λ_* is an eigenvalue of L. Let $u_* \in X^2_{\rm odd}$ denote an associated eigenfunction. Using Theorem 9.7(a), it follows that $\Psi_* = U'_{\mathcal{T}u_*} \in L^2_r$ is an eigenfunction of $\mathcal{Q}_{\lambda_*}$ for the eigenvalue 1. Since $\mu_1(\lambda_*)$ is the largest eigenvalue of $\mathcal{Q}_{\lambda_*}$, we get $\mu_1(\lambda_*) \ge 1$. From the Antonov stability estimate $(Lg, g)_Q \ge \lambda_* \|g\|_Q^2$ it can be moreover deduced that $\mu_1(\lambda_*) \le 1$ is verified; see [27, Lemma 4.7(b)]. Hence we obtain $\mu_1(\lambda_*) = 1$ and it remains to show that $\mu_* > 1$. Suppose that on the contrary $\mu_* \le 1$ is satisfied. For $\lambda \in [\lambda_*, \delta_1^2[$ the monotonicity of μ_1 then yields $1 = \mu_1(\lambda_*) \le \mu_1(\lambda) \le \mu_* \le 1$, which means that $\mu_1(\lambda) = 1$ is constant for $\lambda \in [\lambda_*, \delta_1^2[$. Fixing normalized eigenfunctions $\Psi_{\tilde\lambda}$ for $\mu_1(\tilde\lambda)$, where $\lambda_* \le \tilde\lambda < \lambda < \delta_1^2$, we find

$$1 = \mu_1(\tilde\lambda) = \langle \mathcal{Q}_{\tilde\lambda}\Psi_{\tilde\lambda}, \Psi_{\tilde\lambda}\rangle \le \langle \mathcal{Q}_\lambda \Psi_{\tilde\lambda}, \Psi_{\tilde\lambda}\rangle \le \|\mathcal{Q}_\lambda\|\,\|\Psi_{\tilde\lambda}\|^2_{L^2_r} = \mu_1(\lambda) = 1$$

from the general monotonicity of $\lambda \mapsto \langle \mathcal{Q}_\lambda \Psi, \Psi\rangle$, and therefore

$$\langle \mathcal{Q}_\lambda \Psi_{\tilde\lambda}, \Psi_{\tilde\lambda}\rangle = 1, \quad \lambda_* \le \tilde\lambda < \lambda < \delta_1^2.$$

This can be shown to lead to a contradiction upon differentiation w.r. to λ.

To summarize the preceding argument, to establish "$\Longleftarrow$" in Theorem 10.3, we need to prove that $\lambda_* < \delta_1^2$ implies that λ_* is an eigenvalue of L. Since $\inf \sigma_{\rm ess}(L) = \delta_1^2$, this follows from the abstract result mentioned in the proof of Lemma 10.2, as now the stated case (b) does not hold. We also want to indicate a direct 'dynamic' proof of the fact that λ_* is an eigenvalue of L by means of a gradient-flow type argument [27, Appendix C], since this might be useful in future work on the subject. Let

$$\Phi(u) = (Lu, u)_Q = \|\mathcal{T}u\|_Q^2 - (\mathcal{K}\mathcal{T}u, u)_Q$$

be the functional in question. Strictly speaking, one considers Φ to be defined by the expression on the right-hand side, which makes sense for $u \in X^1_{\rm odd}$ only, but we will ignore this fact in what follows. For a given time interval $J = [0, a]$ or $J = [0, \infty[$ and a given continuous function $h : J \to X^1_{\rm odd}$ we introduce the family of operators

$$\begin{aligned} &\mathcal{W}(t, s) : g \mapsto \mathcal{W}(t, s)g, \quad (\mathcal{W}(t, s)g)_k = \mathcal{W}_k(t, s)g_k \ (k \in \mathbb{Z}), \\ &\mathcal{W}_k(t, s)(I, \ell) = \exp\Big(- \int_s^t [k^2\omega_1^2(I, \ell) - \Phi(h(\tau))]\, d\tau\Big), \end{aligned} \tag{10.1}$$

for $t, s \in J$, $t \geq s$; to emphasize the dependence on h, we will at times also write $\mathcal{W}(t, s; h)$. Note the evolution system property

$$\mathcal{W}(t, s) \circ \mathcal{W}(s, \tau) = \mathcal{W}(t, \tau), \quad t, s, \tau \in J,\ t \geq s \geq \tau.$$

We will consider the evolution equation

$$g(t) = \mathcal{W}(t, 0)\psi + \int_0^t \mathcal{W}(t, s)\, \mathcal{K}\mathcal{T}g(s)\, ds \tag{10.2}$$

for $t \geq 0$ and initial data ψ, where $\mathcal{W}(t, s) = \mathcal{W}(t, s; g)$. For this evolution equation one can establish that if $\psi \in X^2_{\rm odd}$ is such that $\|\psi\|_Q = 1$ and $\Phi(\psi) \leq \lambda_* + \varepsilon_*$ (for $\varepsilon_* > 0$ small enough), then there exists a global continuous solution $g : [0, \infty[\to X^1_{\rm odd}$ of (10.2) that satisfies $\|g(t)\|_{X^0} = 1$ for $t \in [0, \infty[$. This result does not rely on $\lambda_* < \delta_1^2$, the condition $\lambda_* \leq \delta_1^2$ is enough. The point about (10.2) is the following. Differentiating (10.1) for $h = g$ w.r. to t, we get

$$\partial_t \mathcal{W}_k(t, s)(I, \ell) = -[k^2\omega_1^2(I, \ell) - \Phi(g(t))]\, \mathcal{W}_k(t, s)(I, \ell)$$

and hence, at least formally,

$$\partial_t(\mathcal{W}(t, s)g) \cong (\partial_t \mathcal{W}_k(t, s)g_k) = (-[k^2\omega_1^2 - \Phi(g(t))]\, \mathcal{W}_k(t, s)g_k) \cong \mathcal{T}^2\mathcal{W}(t, s)g + \Phi(g(t))\, \mathcal{W}(t, s)g.$$

Applying this relation to (10.2), it follows that

$$\begin{aligned} g'(t) &= \mathcal{T}^2\mathcal{W}(t, s)\psi + \Phi(g(t))\psi + \int_0^t [\mathcal{T}^2\mathcal{W}(t, s)\mathcal{K}\mathcal{T}g(s) + \Phi(g(t))\, \mathcal{W}(t, s)\mathcal{K}\mathcal{T}g(s)]\, ds \\ &\quad + \mathcal{K}\mathcal{T}g(t) \\ &= \mathcal{T}^2 g(t) + \Phi(g(t))\, g(t) + \mathcal{K}\mathcal{T}g(t) \\ &= -Lg(t) + \Phi(g(t))\, g(t). \end{aligned} \tag{10.3}$$

This implies that the $\|\cdot\|_Q$-norm is preserved along the solution flow. Since $\Phi(u) = (Lu, u)_Q$ for $u \in X^2_{\rm odd}$ and as the solution $g(t)$ is regular enough, we also deduce from (10.3) that

$$\begin{aligned} \frac{d}{dt}\,\Phi(g(t)) &= \frac{d}{dt}\,(Lg(t), g(t))_Q = 2\,(Lg(t), g'(t))_Q \\ &= 2\,(Lg(t), -Lg(t) + \Phi(g(t))\,g(t))_Q = -2\,(\|Lg(t)\|_Q^2 - \Phi(g(t))^2). \end{aligned}$$

Now if $\|g(0)\|_Q = 1$ initially, then

$$\begin{aligned} \|g'(t)\|_Q^2 &= \|-Lg(t) + \Phi(g(t))g(t)\|_Q^2 \\ &= \|Lg(t)\|_Q^2 - 2\Phi(g(t))\,(Lg(t), g(t))_Q + \Phi(g(t))^2\,\|g(t)\|_Q^2 \\ &= \|Lg(t)\|_Q^2 - \Phi(g(t))^2, \end{aligned}$$

which in turn yields

$$\frac{d}{dt}\,\Phi(g(t)) = -2\,\|g'(t)\|_Q^2 \le 0.$$

Therefore we see that Φ is a Lyapunov function for the evolution. Since $\|g(t)\|_Q = 1$, we also have $\Phi(g(t)) = (Lg(t), g(t))_Q \ge \lambda_*$, and it is a natural question to ask, if we can construct a minimizer of Φ in the following way. Consider a sequence of initial data $(\psi_j) \subset X^2_{\rm odd}$ such that $\Phi(\psi_j) \le \lambda_* + 1/j$ and let g_j denote the corresponding solution to (10.2) so that $g_j(0) = \psi_j$. Then $\lambda_* \le \Phi(g_j(t)) \le \Phi(\psi_j) \le \lambda_* + 1/j$ for all $t \in [0, \infty[$ and $j \in \mathbb{N}$. Hence the key point is to find a sequence of times (t_j) with the properties that $t_j \to \infty$ and $\{g_j(t_j) : j \in \mathbb{N}\} \subset X^0$ is relatively compact. It can be shown that this goal can be accomplished, if the condition $\lambda_* < \delta_1^2$ is imposed; the limiting function u_* will then be the desired eigenfunction of L for the eigenvalue λ_*.

11 Open questions and further topics

In this last section we will outline a few topics for further research.

1. Certainly some more numerical results would provide important insights. For instance, it will be helpful to know the shape of the surface $\mathcal{P} = \{T_1(\beta, e) : \beta \in [0, \beta_*], e \in [e_{\rm min}(\beta), e_0]\}$ for the period function T_1 and for different steady states. It can be expected that the properties of $\mathcal{P}$ will also play a major role for the non-linear dynamics close to steady states; see [40] for some numerically obtained results showing the possible behaviors of such solutions.

2. It should be clarified if it could happen, for some static solution, that $\lambda_* = \delta_1^2$.

3. It should be determined where ω_1 attains its minimum on D. Is it the same point for all "reasonable" static solutions Q?

4. Also it would be interesting to determine the limit μ_* from (9.7) solely in terms of Q.

5. When it comes to relativistic galactic dynamics, the appropriate model is the Einstein-Vlasov system [1]. In the present lectures we have not been dealing with this more general system, but of course it will be tempting to investigate which results could be transferred to Einstein-Vlasov; see [21, 22, 23, 9, 10, 16, 17] for work in this context that is related to the Antonov bound, and the recent paper [13] for some results on a Birman-Schwinger principle for Einstein-Vlasov.

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