Five Lectures on

Hausdorff Dimension, Random Trees and Brownian Motion

Peter Mörters

University of Bath

Material zur Winterschule des Graduiertenkollegs

"Stochastische Prozesse und probabilistische Analysis"

Stralsund, April 2003

Contents

Chapter 1

Lecture One: Hausdorff dimension

1.1 Motivation

Dimensions are a tool to measure the size of mathematical objects on a crude scale. For example, in classical geometry dimension is able to tell us that in three-dimensional space a line segment (a one-dimensional object) is smaller than the surface of a ball (a two-dimensional object), but there is no difference between line-segments of different lengths. It may therefore come as a surprise that Hausdorff dimension is able to distinguish the size of so many objects in probability theory, for example

- the ranges of stable processes with different indices,
- the boundaries of supercritical Galton-Watson trees with different mean offspring numbers,
- the n-fold self-intersections of a planar Brownian motion for different n.

In this chapter we introduce the concept of *Hausdorff dimension* and discuss a couple of techniques for the calculation of Hausdorff dimensions. As an example the path of Brownian motion in $d \geq 2$ will be discussed as a project exercise.

1.2 Definition of Hausdorff dimension

How can we capture the dimension of a geometric object? One requirement for a useful definition of dimension is that it should be *intrinsic*. This means that it should be independent of an embedding of the object in an ambient space like $\mathbb{R}^d$. Intrinsic notions of dimension can be defined in arbitrary metric spaces.

The Hausdorff dimension and Hausdorff measure were introduced by Felix Hausdorff in 1919. The Hausdorff measure is based on the notion of a *covering* of the metric space E by sets of finite diameter. A *covering* of E is an at most countable collection of sets $E_1, E_2, E_3, \ldots$ with

$$E \subset \bigcup_{i=1}^{\infty} E_i \,.$$

For every $s \geq 0$ we say that the *s-value* of the covering is

$$\sum_{i=1}^{\infty} |E_i|^s ,$$

where $|E_i|$ denotes the diameter of the set E_i. Informally speaking, the α-value of the most efficient covering by small sets is the α-Hausdorff measure of the set.

Definition 1.1 *For every $\alpha \geq 0$ the* α-Hausdorff measure *of a subset $A \subset E$ of a metric space E is defined as*

$$\mathcal{H}^\alpha(A) = \liminf_{\delta \downarrow 0} \Big\{ \sum_{i=1}^{\infty} |E_i|^\alpha \,:\, (E_i : i \in \mathbb{N}) \text{ is a covering of } A \text{ and } |E_i| < \delta \Big\} .$$

It can be shown that $\mathcal{H}^\alpha$ is a measure on the Borel subsets of E. If $0 \leq s < t$, and $\mathcal{H}^s(E) < \infty$, then $\mathcal{H}^t(E) = 0$. Thus we can define

$$\dim E = \inf \Big\{ s \geq 0 \,:\, \mathcal{H}^s(E) = 0 \Big\} = \inf \Big\{ s \geq 0 \,:\, \mathcal{H}^s(E) < \infty \Big\} = \sup \Big\{ s \geq 0 \,:\, \mathcal{H}^s(E) > 0 \Big\},$$

the Hausdorff dimension *of the set E.*

The Hausdorff dimension has the following crucial properties,

- if $A \subset B$, then $\dim A \leq \dim B$, in particular subsets of $\mathbb{R}^d$ have Hausdorff dimension no larger than d;
- $\dim \bigcup_{k=1}^{\infty} E_k = \sup \{ \dim E_k \,:\, k \geq 1 \}$, this is the *countable stability property*;
- if $f : A \to B$ is Lipschitz, then $\dim f(A) \leq \dim A$.

1.3 Techniques for calculating the Hausdorff dimension

From the definition of the Hausdorff dimension it is plausible that in many cases it is relatively easy to give an upper bound on the dimension: just find an efficient covering of the set. However it looks more difficult to give lower bounds, as we must obtain a lower bound on α-values of *all* coverings of the set. In this section we discuss three important techniques to obtain lower bounds for the Hausdorff dimension.

1.3.1 The mass distribution principle

The basic idea of the *mass distribution principle* is the following: if it is possible to distribute a positive amount of mass on a set E in such a manner that its local concentration is bounded above, then the set must be large in a suitable sense.

For the purpose of this method we call a measure μ on the Borel sets of a metric space E a *mass distribution* on E, if

$$0 < \mu(E) < \infty .$$

The intuition here is that a positive and finite mass is spread over the space E.

Theorem 1.2 (Mass distribution principle) *Suppose E is a metric space and $s \geq 0$. If there is a mass distribution μ on E and constants $C > 0$ and $\delta > 0$ such that*

$$\mu(U) \leq C|U|^s$$

for all closed sets U with diameter $|U| \leq \delta$. Then

$$\mathcal{H}^s(E) \geq \frac{\mu(E)}{C} > 0,$$

and hence $\dim E \geq s$.

Proof: Suppose that $\{U_i : i \geq 1\}$ is a covering of E by arbitrary sets with $0 \leq |U_i| \leq \delta$. Let V_i be the closure of U_i and note that $|U_i| = |V_i|$. We have

$$0 < \mu(E) \leq \mu\Big(\bigcup_{i=1}^{\infty} U_i\Big) \leq \mu\Big(\bigcup_{i=1}^{\infty} V_i\Big) \leq \sum_{i=1}^{\infty} \mu(V_i) \leq C \sum_{i=1}^{\infty} |U_i|^s .$$

Passing to the infimum over all such coverings, and letting $\delta \downarrow 0$ gives the statement. ∎

The following converse of the mass distribution principle is called *Frostman's Lemma*:

Suppose that E is a complete separable metric space and $\mathcal{H}^s(E) > 0$. Then there exists a mass distribution μ on E such that $\mu(B) \leq |B|^s$ for all bounded Borel sets B.

This result was proved by Howroyd (1995) in the present setting, but for closed subsets of $\mathbb{R}^d$ it is older and due to Frostman (1935). The main difficulty arising in the proof is that, if $\mathcal{H}^s(E) = \infty$, one has to find a subset $A \subset E$ with $0 < \mathcal{H}^s(A) < \infty$.

1.3.2 The potential theoretic method

In this section we present a lower bound for the Hausdorff dimension, which is particularly interesting in applications to random fractals, the potential theoretic method. It is based on a localisation of the mass distribution principle, which is almost as easy to prove.

Definition 1.3 *Suppose μ is a mass distribution on E and $s \geq 0$. Then, for every $x \in E$, let $B(x,r)$ be the closed ball centred in x of radius r and define the value*

$$\overline{d}_s(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(B(x,r))}{r^s},$$

the upper s-density *of μ at x.*

Theorem 1.4 (Local mass distribution principle) *If μ is a mass distribution on the metric space E, and $A \subset E$ a Borel set with*

$$\overline{d}_s(\mu, x) < C \text{ for all } x \in A\,,$$

then $\mathcal{H}^s(A) \geq \frac{\mu(A)}{C}$, and, in particular, $\dim A \geq s$ if $\mu(A) > 0$.

We now come to the potential theoretic method and introduce the necessary terminology.

Definition 1.5 *Suppose μ is a mass distribution on a metric space (E, d) and $\alpha \geq 0$. The* α-potential *of a point $x \in E$ with respect to μ is defined as*

$$\phi_\alpha(x) = \int \frac{d\mu(y)}{\mathrm{d}(x,y)^\alpha}\,.$$

In the case $E = \mathbb{R}^3$ and $\alpha = 1$, this is the Newton gravitational potential of the mass μ. The α-energy *of μ is*

$$I_\alpha(\mu) = \int \phi_\alpha(x)\, d\mu(x) = \int\int \frac{d\mu(x)\, d\mu(y)}{\mathrm{d}(x,y)^\alpha}\,.$$

The (Riesz) α-capacity *of E is defined as*

$$C_\alpha(E) = \sup\Big\{ I_\alpha(\mu)^{-1} \,:\, \mu \text{ a mass distribution on } E \text{ with } \mu(E) = 1 \Big\}.$$

The idea of the potential theoretic method is that mass distributions with $I_s(\mu) < \infty$ for large s spread the mass so that at each place the concentration is small. This is only possible on large sets.

Theorem 1.6 (Potential theoretic method) *Let $s \geq 0$ and E a metric space. Suppose there is a mass distribution μ on E with $I_s(\mu) < \infty$. Then $\mathcal{H}^s(E) = \infty$ and, in particular, $\dim E \geq s$.*

Remarks

(a) The theorem states that a set of positive α-capacity has dimension $\geq \alpha$.

(b) In order to get a lower bound on the dimension from this method it suffices to show finiteness of a single integral. This is very convenient for random sets, where it suffices to show that $\mathbb{E} I_s(\mu) < \infty$ for a (random) measure on the random set E in order to get $\dim E \geq s$ almost surely.

Proof: Let μ be a mass distribution on E with $I_s(\mu) < \infty$. Then, clearly, $\mu\{x\} = 0$ for all $x \in E$. Put

$$E_1 = \big\{ x \in E \,:\, \overline{d}_s(\mu, x) > 0 \big\}\,,$$

and note that this is a Borel set. If $x \in E_1$ there exists $\varepsilon > 0$ and $r_i \downarrow 0$ such that

$$\mu\big(B(x, r_i)\big) \geq \varepsilon r_i^s\,.$$

As $\mu\{x\} = 0$ there exist small $0 < q_i < r_i$ such that $B_i = B(x, r_i) \setminus B(x, q_i)$ satisfies

$$\mu(B_i) \geq \frac{1}{4} \varepsilon r_i^s\,.$$

By passing to a subsequence, if necessary, we may assume $r_{i+1} < q_i$ and hence the annuli $B_1, B_2, B_3, \ldots$ are pairwise disjoint. We infer that

$$\phi_s(x) = \int \frac{d\mu(y)}{\mathrm{d}(x,y)^s} \geq \sum_{i=1}^\infty \int_{B_i} \frac{d\mu(y)}{\mathrm{d}(x,y)^s} \geq \frac{1}{4} \sum_{i=1}^\infty \varepsilon r_i^s r_i^{-s} = \infty\,.$$

However, because

$$I_s(\mu) = \int \phi_s(x)\, d\mu(x) < \infty\,,$$

we infer that $\mu(E_1) = 0$. But for all $x \in E \setminus E_1$ we have $\overline{d}_s(\mu, x) = 0$ and thus, for all $C > 0$ by the local mass distribution theorem,

$$\mathcal{H}^s(E) \geq \mathcal{H}^s(E \setminus E_1) \geq \frac{\mu(E \setminus E_1)}{C} = \frac{\mu(E)}{C}\,.$$

Hence, letting $C \downarrow 0$, we get $\mathcal{H}^s(E) = \infty$, as claimed. ∎

1.3.3 Stochastic co-dimension

A different technique to determine the Hausdorff dimension of a set A is based on the following rough idea: take an increasing family of random sets $\{\Gamma[\gamma] : \gamma \geq 0\}$ and determine the probability that $\Gamma[\gamma] \cap A \neq \emptyset$. If the probability is positive we can hope for a lower bound on the dimension of A in terms of the parameter γ. Several families of random sets $\Gamma[\gamma]$ have been suggested, the family of percolation fractals has been particularly successful.

Suppose that $C \subset \mathbb{R}^d$ is a fixed compact unit cube. We denote by $\mathfrak{C}_n$ the collection of compact dyadic subcubes (relative to C) of sidelength 2^{-n}. We also let $\mathfrak{C} = \bigcup_{n=0}^{\infty} \mathfrak{C}_n$.

Given $\gamma \in [0, d]$ we construct a random compact set $\Gamma[\gamma] \subset C$ inductively as follows: We keep each of the 2^d compact cubes in $\mathfrak{C}_1$ independently with probability $p = 2^{-\gamma}$. Let $\mathcal{S}_1$ be the collection of cubes kept in this procedure and $\mathsf{S}(1)$ their union. Pass from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$ by keeping each cube of $\mathfrak{C}_{n+1}$, which is not contained in a previously rejected cube, independently with probability p, and again let $\mathsf{S}(n+1)$ be the union of the surviving cubes. Then the random set

$$\Gamma[\gamma] := \bigcap_{n=1}^{\infty} \mathsf{S}(n)$$

is called a *percolation limit set*. The usefulness of percolation limit sets in fractal geometry comes from the following theorem, see Hawkes (1981) or Peres (1996) for a proof.

Theorem 1.7 *For every $\gamma \in [0, d]$ and every Borel set $A \subset C$ the following properties hold*

(i) if $\dim A < \gamma$, then almost surely, $A \cap \Gamma[\gamma] = \emptyset$,

(ii) if $\dim A > \gamma$, then $A \cap \Gamma[\gamma] \neq \emptyset$ with positive probability,

(iii) if $\dim A > \gamma$, then almost surely $\dim\big(A \cap \Gamma[\gamma]\big) \leq \dim A - \gamma$ and, for all $\varepsilon > 0$, with positive probability $\dim\big(A \cap \Gamma[\gamma]\big) \geq \dim A - \gamma - \varepsilon$.

Remarks:

- Observe that the first part of the theorem gives a *lower* bound γ for the Hausdorff dimension of a set A, if we can show that $A \cap \Gamma[\gamma] \neq \emptyset$ with positive probability.

- As with so many ideas in fractal geometry one of the roots of this method lies in the study of trees, more precisely percolation on trees. See Lyons (1990).
- The proof of Theorem 1.7 exposes a close kinship to the potential theoretic method: a set A is called a *polar set* for the percolation limit set, if $\mathbb{P}\{A \cap \Gamma[\gamma] \neq \emptyset\} = 0$. One can show that a set is polar iff it has γ-capacity zero.
- The experienced reader might notice at this place that the criterion for polarity of a percolation fractal with $\gamma = d-2$ agrees with the criterion for polarity for the path of a Brownian motion. This fact is exploited beautifully in the work of Peres, see e.g. Peres (1996).
- A special case of Theorem 1.7 is that $\dim \Gamma[\gamma] = d - \gamma$ with positive probability. The methods of the next two lectures will enable us to verify this result .

1.4 Exercise: The dimension of the range of Brownian motion

[1] Prove the following theorem:

Theorem 1.8 *Let* $B : [0,1] \to \mathbb{R}^d$ *be a Brownian motion in* $d \geq 2$ *and* $B[0,1] \subset \mathbb{R}^d$ *its path. Then* $\dim B[0,1] = 2$ *almost surely.*

Hints:

For the *upper bound* show that for every $\alpha > 2$, almost surely, there exists $C > 0$ such that $|B(s) - B(t)| \leq C|s-t|^\alpha$ for all $s,t \in [0,1]$. Use this fact to construct an efficient covering explicitly.

For the *lower bound* find an (easy!) measure on $B[0,1]$ and apply the potential theoretic method.

[2] For a Brownian motion $B : [0,1] \to \mathbb{R}$ find the Hausdorff dimension of the *graph*

$$G = \big\{(t, B(t)) \,:\, t \in [0,1]\big\} \subset \mathbb{R}^2 .$$

Chapter 2

Lecture Two: Self-similarity and trees

2.1 Motivation

Much of fractal geometry is devoted to the study of sets featuring some form of self-similarity. As the most important stochastic processes are, in some sense, self-similar it is not surprising that concepts of self-similarity are vital in probability. In this and the next lecture I will stress concepts of self-similarity based on tree structures.

Very roughly speaking, a set is *self-similar* if it can be decomposed into parts which look like scaled copies of the original set. This definition becomes particularly powerful when *look like* is interpreted in a statistical sense. So we would like to call a set *statistically self-similar* if it can be decomposed into parts which have (up to scaling) the same distribution as the whole set. This idea is naturally linked to trees: starting from the root we identify the parts in the decomposition as the offspring of the root. Each part is itself a copy of the whole picture and hence has a decomposition of the same kind as its father, proceeding like this the points of the fractal have a natural address in the tree.

There is no generally accepted definition of a statistically self-similar set, and we do not attempt to give one. In the next two lectures we define a natural class of statistically self-similar sets, the *Galton-Watson fractals*, which are defined using Galton-Watson trees. This class comprises a lot of interesting examples, and at the same time there is a general formula for their Hausdorff dimension. We use the opportunity to explore the relationship between branching processes and self-similarity and introduce basic ideas about probability on trees.

A comprehensive account of the subject of this chapter will appear in the forthcoming book by Russel Lyons and Yuval Peres, "Probability on trees and networks".

2.2 Fractals and trees

We describe a quite general approach to describe fractals by means of *trees* with weights, so called *capacities* associated to the edges, and investigate how the Hausdorff dimension of the fractal can be derived from the tree and the capacities. Later we make the tree as well as the capacities random, which allows to describe a large class of random fractals.

We do not give a formal definition of a tree here, but fix some notation needed later. For any tree we assume that the set V of *vertices* is finite or countable, and let $E \subset V \times V$ be the set of *edges*. For every vertex $v \in V$ the set of *predecessors* $\{w \in V : (w,v) \in E\}$ consists of exactly one element $\overline{v}$, except for the *root* $\rho \in V$, which has no predecessor. For every vertex v there is a unique self-avoiding path from the root to v and the number of edges in this path is the *order* $|v|$ of the vertex $v \in V$. The order $|e|$ of an edge $e = (u,v)$ is the order of its end-vertex v. For every vertex $v \in V$ we also assume that the set of *successors* or *children* $\{w \in V : (v,w) \in E\}$ is finite. Every infinite self-avoiding sequence of vertices (or edges) started in the root is called a *ray*. The set of rays is denoted ∂T.

Definition 2.1 *Let T be an infinite tree. To each vertex $v \in T$ we associate a compact set $\emptyset \neq I_v \subset \mathbb{R}^d$ as follows*

$$I_v = \mathrm{cl}\big(\mathrm{int} I_v\big), \tag{2.1}$$

$$(u,v) \in E \Longrightarrow I_v \subset I_u, \tag{2.2}$$

$$\overline{u} = \overline{v} \textit{ and } u \neq v \Longrightarrow \mathrm{int}\, I_u \cap \mathrm{int}\, I_v = \emptyset, \tag{2.3}$$

$$\textit{for all } \xi = (v_0, v_1, v_2, \dots) \in \partial T \textit{ we have } \lim_{n\to\infty} |I_{v_n}| = 0, \tag{2.4}$$

$$c_1 := \inf_{v \neq \rho} \frac{|I_v|}{|I_{\overline{v}}|} > 0, \tag{2.5}$$

$$c_2 := \inf_{v} \frac{\ell(\mathrm{int} I_v)}{|I_v|^d} > 0. \tag{2.6}$$

Then we associate a set $I(T)$ to the tree T with marks $\{I_v : v \in V\}$, by

$$I(T) := \bigcup_{\xi \in \partial T} \bigcap_{v \in \xi} I_v\,.$$

We call $I(T)$ a tree fractal.

Example 2.2 *The b-adic coding of general sets.* Suppose $C \subset [0,1]$ is a nonempty, compact set and $b \geq 2$ an integer. We can consider the system of closed b-adic subintervals of $[0,1]$. More precisely the intervals in

$$\Big[\frac{k}{b^n}, \frac{k+1}{b^n}\Big] \text{ for } k \in \{0, \dots, b^n - 1\}$$

are the b-adic subintervals of order n. We denote by T the set of those b-adic intervals whose intersection with C is not empty. This set are the vertices of a tree. The tree structure is given as follows: Two such intervals or vertices are connected by an edge if one contains the other one and their orders differ by exactly one. The root of this tree is the only b-adic subinterval of order 0, namely $[0,1]$. We attach the closed b-adic interval corresponding to each vertex. Then it easy to check conditions (2.1) to (2.6) and that C itself is the associated set. We call this the b-adic coding of C. In a similar vein, using b-adic cubes, one can see that every compact subset of $[0,1]^d$ can be coded as a tree fractal.

More examples will be discussed in the exercises and interesting examples of tree fractals arising in a random context will be discussed in depth in the next lecture. We now give a formula for the Hausdorff dimension of tree fractals in terms of the tree and associated capacities.

Definition 2.3 *Suppose* capacities *are assigned to the edges of a tree T, i.e. there is a mapping $C : E \to [0,\infty)$. A* flow *of strength $c > 0$ through a tree with capacities C is a mapping $\theta : E \to [0,c]$ such that*

- *for the root we have $\sum_{w\,:\,\overline{w}=\rho} \theta\big(\rho, w\big) = c$, for every other vertex $v \neq \rho$ we have*
$$\theta\big(\overline{v}, v\big) = \sum_{w\,:\,\overline{w}=v} \theta\big(v, w\big),$$
- *$\theta(e) \le C(e)$, in other words, the flow through the edge e is not bigger than the capacity of this edge.*

Example 2.4 Suppose T is the b-adic tree, i.e. the tree such that every vertex has exactly b successors, and $\lambda > 0$. We associate capacities of $C(e) = \lambda^{-|e|}$ to the edges of the tree. If λ is too big or the tree is too small it may be that no flow is possible.

If $\lambda \le b$, then there is a flow θ through the tree defined by $\theta(u,v) = b^{-|v|}$. Indeed, this is a flow of strength 1, because the flow b^{-n} through every edge is smaller than the capacity λ^{-n} of the edge, and
$$\theta\big(\overline{v}, v\big) = b^{-|v|} = bb^{-(|v|+1)} = \sum_{w\,:\,\overline{w}=v} \theta\big(v, w\big).$$
If $\lambda > b$, then there is no such flow. Suppose there was one, let c be the strength of the flow. One can see by induction that, for each n, one of the b^n edges of order n must have a flow $\theta(e) \ge cb^{-n}$. For sufficiently large n this contradicts the condition $\theta(e) \le \lambda^{-n}$.

A set Π of edges is called a *cutset* if every ray includes an edge from Π.

Theorem 2.5 *If $I(T)$ is the tree fractal associated to the tree $T = (V,E)$ and sets $\{I_v \,:\, v \in V\}$. For every $s \ge 0$ define capacities $C_s(e) = |I_v|^s$ if $e = (\overline{v}, v)$. Then*
$$\dim I(T) = \inf\Big\{ s \,:\, \inf_{\Pi} \sum_{e\in\Pi} C_s(e) = 0 \Big\} = \sup\Big\{ s \,:\, \text{ there is a flow with capacities } C_s \Big\}.$$

I do not want to give the full proof here, but confine myself to mentioning the two fundamental principles on which it rests. The first one is easy to grasp:

A cutset Π of T is a covering of $I(T)$ by sets from the collection $\{I_v \,:\, v \in V\}$ with an s-value $\sum_{e\in\Pi} C_s(e)$.

Hence $\le$ in the first equality is trivial, and $\ge$ amounts to checking that an arbitrary covering of $I(T)$ with small s-value can be replaced by coverings with sets from $\{I_v \,:\, v \in V\}$ without changing the s-value too much.

The second equality follows from a famous result of graph theory, the *max-flow min-cut theorem* of Ford and Fulkerson (1962), which states that

$$\max\Big\{\text{strength}\,(\theta)\ :\ \theta \text{ a flow with capacities } C\Big\} = \inf\Big\{\sum_{e\in\Pi} C(e)\ :\ \Pi \text{ a cutset }\Big\}.$$

Theorem 2.5 is quite similar to the mass distribution principle. A flow corresponds to a mass distribution μ on the rays of the tree, and the capacity conditions correspond to the upper bounds for the mass concentration $\mu(U) \le C|U|^s$.

Example 2.6 *We look at the ternary Cantor set C. The 3-adic coding produces a tree, which is (up to finite subtrees) a binary tree with intervals of length $(1/3)^n$ attached to the edges of order n. Look at the capacities $C_s = 3^{-sn}$ attached to edges of order n. We have seen in Example 2.4 that a flow exists iff $\lambda = 3^s \le 2$, i.e. iff $s \le \log 2/\log 3$. Hence $\dim C = \log 2/\log 3$.*

2.3 Galton-Watson networks

In the previous section we have seen that the Hausdorff dimension of tree fractals depends only on the diameters $|I_v|$ of the sets attached to the vertices. Our aim in the following is to study random tree fractals arising by

- taking a random tree as a basis of the tree fractal,
- assigning random sets to each vertex of the tree.

If we do this in a proper way (i.e. such that conditions (2.1) to (2.6) are satisfied) we get a random fractal and a dimension formula, which relates the Hausdorff dimension to flows on the random tree.

Definition 2.7 Let $X = (N, A_1, \ldots, A_N)$ be a random variable with $N \in \mathbb{N}$ and $0 < A_i \le 1$ almost surely. X is interpreted as a random offspring number N and a *weight* A_i associated to each edge e_i leading to the ith child. We can construct a tree with weighted edges by selecting the offspring of each individual independently according to X. If e is an edge in the tree we let $A(e)$ be the associated weight and define the *capacity* $C(e)$ of the edge e as the product of the weights of all edges in the self-avoiding path from the root to e. The random tree T with these associated capacities is called the *Galton-Watson network* generated by X.

We recall some well-known facts about Galton-Watson networks. The first question must be whether the tree is infinite with positive probability. This event is called *survival.* Only if the tree survives it is suitable for our representation of fractals.

Theorem 2.8 (Survival Theorem) *Suppose the offspring distribution satisfies $p_1 \ne 1$. Then*

$$q := \mathbb{P}\{\ \textit{survival}\ \} > 0 \ \textit{if and only if}\ \mathbb{E}N > 1.$$

We also need a 01-law for Galton-Watson trees. Let A be a set of trees or, equivalently, a property of trees. We say that A is *inherited* if

- every finite tree is in A, and
- if the tree $T \in A$ and v is a vertex of the tree, then the subtree $T(v)$ whose root is v is also in A.

Theorem 2.9 (Galton-Watson 01-Law) *For a Galton-Watson network every inherited property A has either probability q or probability 1. In particular, $\mathbb{P}\{T \in A \mid \text{survival}\} \in \{0,1\}$.*

An example of a an inherited property is the *non*-existence of a flow in the network.

Definition 2.10 Let $X = (N, A_1, \dots, A_N)$ be a random variable with $0 < A_i < 1$ almost surely, which is generating a Galton-Watson network T. Suppose compact sets I_v are randomly assigned to the vertices of the Galton-Watson tree in such a way that the following conditions are satisfied almost surely.

$$I_v = \mathrm{cl}\big(\mathrm{int} I_v\big), \tag{2.7}$$

$$(u,v) \in E \Longrightarrow I_v \subset I_u, \tag{2.8}$$

$$\overline{u} = \overline{v} \text{ and } u \neq v \Longrightarrow \mathrm{int}\, I_u \cap \mathrm{int}\, I_v = \emptyset, \tag{2.9}$$

$$\inf_v \frac{\ell(\mathrm{int} I_v)}{|I_v|^d} \geq c_2 > 0. \tag{2.10}$$

Moreover, assume that the normalized diameters

$$C\big((\overline{v}, v)\big) = \frac{|I_v|}{|I_\rho|}$$

are the capacities of the Galton-Watson network. The the random tree fractal $I(T)$ is called a *Galton-Watson fractal* associated to the network T.

2.4 Exercise: Self-similar sets and iterated function systems

Recall that a mapping $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a *contracting similitude* if there is a number $0 < r < 1$, called the *contraction ratio*, such that

$$|\phi(x) - \phi(y)| = r \cdot |x - y| \text{ for all } x, y \in \mathbb{R}^d.$$

A collection $(\phi_1, \dots, \phi_n)$ of finitely many contracting similitudes is called an *iterated function system.* Denote the contraction ratios by $r_1, \dots r_n$.

A compact set $C \subset X$ is called *attractor of the iterated function system* $(\phi_1, \dots, \phi_n)$ if

$$C = \bigcup_{i=1}^{n} \phi_i(C)\,, \tag{2.11}$$

where $\phi_i(C) := \{\phi_i(x) \,:\, x \in C\}$ is the image of C under ϕ_i. A clever application of Banach's fixed point theorem ensures existence and uniqueness of the attractors of any iterated function system.

An iterated function system $(\phi_1, \ldots, \phi_n)$ satisfies the *open set condition*, if there is a nonempty, bounded open set $V \subset \mathbb{R}^d$ with $\phi_i(V) \cap \phi_j(V) = \emptyset$ for all $i \neq j$, and $V \supset \bigcup_{i=1}^n \phi_i(V)$.

[1] Show that for the attractor C of an iterated function system satisfying the open set condition,

$$\dim C = s ,$$

where s is the unique solution of

$$\sum_{i=1}^n r_i^s = 1 .$$

<u>Hints:</u> For $x = (x_1, x_2, \ldots)$ with $x_i \in \{1, \ldots, n\}$ let $V[x_1, \ldots, x_m] = \phi_{x_1} \circ \cdot \circ \phi_{x_m}(V)$ and show that

$$C = \bigcup_x \bigcap_{m=1}^\infty \mathrm{cl}\, V[x_1, \ldots, x_m].$$

Use this as the basis of the representation of C as a tree fractal and apply Theorem 2.5.

[2] Use this to find the Hausdorff dimension of

- a point,
- the unit interval,
- the Sierpinski gasket,
- the von Koch-curve.

Chapter 3

Lecture Three: The Hausdorff dimension of Galton-Watson fractals

In this lecture we combine Falconer's Theorem on flows in Galton-Watson networks and the Hausdorff dimension formula of Theorem 2.5 to obtain a formula for a large class of random fractals, the Galton-Watson fractals. We discuss several examples including the zero-set of a Brownian bridge.

3.1 The dimension formula for Galton-Watson fractals

The main result of this chapter is the following dimension formula for Galton-Watson fractals found independently by Falconer (1986) and Mauldin and Williams (1986).

Theorem 3.1 (Dimension formula for Galton-Watson fractals) *Suppose $I(T)$ is a Galton-Watson fractal associated with a Galton-Watson network T with offspring variable $X = (N, A_1, \dots, A_N)$. Then, almost surely on survival,*

$$\dim I(T) = \min\Big\{\alpha : \mathbb{E}\Big\{\sum_{i=1}^{N} A_i^{\alpha}\Big\} \le 1\Big\}.$$

Example 3.2 We look at *percolation fractals.* Fix $p \in (0,1)$ and a positive integer n. Divide $[0,1]^d$ into n^d equal subcubes. Keep each independently with probability p. Apply the same procedure to the remaining cubes until infinity. Now the Galton-Watson network has a generating random variable $(N, A_1, \dots, A_N)$ with N binomial with parameters n^d and p and A_i deterministic with $A_i = 1/n$. The probability that the resulting random set is nonempty is positive if and only if $p > 1/n^d$. Moreover,

$$\mathbb{E}\Big\{\sum_{i=1}^{N} A_i^{\alpha}\Big\} = \Big(\frac{1}{n}\Big)^{\alpha}\mathbb{E}N = \frac{n^d p}{n^{\alpha}}.$$

This is ≤ 1 if and only if $\alpha \ge d + \frac{\log p}{\log n}$. Hence, almost surely on survival,

$$\dim I(T) = d + \frac{\log p}{\log n}.$$

The key step in the proof of Theorem 3.1 is to find a criterion —in terms of the offspring variable X alone— for the existence of a flow on the Galton-Watson network generated by X. The problem of survival of a Galton-Watson tree, which we solved in the previous section, is a special case of this: let all $A_i = 1$, then flow is possible if and only if the tree is infinite.

The best possible answer to our question is given by the following theorem of Falconer (1986). Note that the excluded case is trivial.

Theorem 3.3 (Falconer's Theorem) *Suppose that a Galton-Watson network is given by the offspring variable $X = (N, A_1, \dots, A_N)$ and assume that $\sum_{i=1}^N A_i \neq 1$ with positive probability. Let*

$$\gamma := E\Big\{ \sum_{i=1}^N A_i \Big\}.$$

(1) *If $\gamma \le 1$ then almost surely* no *flow is possible.*

(2) *If $\gamma > 1$ then flow is possible almost surely on survival of the tree.*

We give a proof, which is due to Falconer (part (1)) and Lyons and Peres (part (2)). The second part of the proof uses the idea of *percolation.*

Proof of (1): If v is a vertex in the tree let $A(v)$ be the weight of the unique edge ending in v. We equip the subtree with root v with capacities $e \mapsto C_v(e) := C(e)/A(v)$ and let $\theta(v)$ be the maximal strength of a flow in this subtree. Abbreviating $\theta := \theta(\rho)$ we have

$$\theta = \sum_{v:\overline{v}=\rho} \big(A(v) \wedge (A(v)\theta(v))\big) = \sum_{v:\overline{v}=\rho} A(v)\big(1 \wedge \theta(v)\big). \tag{3.1}$$

Now suppose that $\gamma \le 1$ and the random variable $X = (N, A_1, \dots, A_N)$ determines the children of the root and weights attached to the edges connecting the root to its children. Using independence, and the fact that θ and $\theta(v)$ have the same distribution for every edge v,

$$\begin{aligned}
\mathbb{E}\{\theta\} &= \sum_{n=0}^\infty \mathbb{E}\Big\{\theta 1_{\{N=n\}}\Big\} = \sum_{n=0}^\infty \sum_{v=1}^n \mathbb{E}\Big\{A(v)(1 \wedge \theta(v)) 1_{\{N=n\}}\Big\} \\
&= \sum_{n=0}^\infty \sum_{v=1}^n \mathbb{E}\Big\{A(v) 1_{\{N=n\}}\Big\}\mathbb{E}\Big\{1 \wedge \theta(v)\Big\} = \sum_{n=0}^\infty \sum_{v=1}^n \mathbb{E}\Big\{A(v) 1_{\{N=n\}}\Big\}\mathbb{E}\Big\{1 \wedge \theta\Big\} \\
&= \sum_{n=0}^\infty \mathbb{E}\Big\{\sum_{v=1}^N A(v) 1_{\{N=n\}}\Big\}\mathbb{E}\Big\{1 \wedge \theta\Big\} = \gamma \mathbb{E}\{1 \wedge \theta\} \le \mathbb{E}\{1 \wedge \theta\}.
\end{aligned}$$

Hence $\theta \le 1$ almost surely and $\mathbb{P}\{\theta > 0\} > 0$ only if $\gamma = 1$. This already shows that no flow is possible if $\gamma < 1$. In the case $\gamma = 1$ we write, for every nonnegative random variable Y,

$$\|Y\| := \inf\Big\{a : \mathbb{P}\{Y > a\} = 0\Big\}.$$

By independence, we get from (3.1), using that $\theta \le 1$, that

$$\|\theta\| = \Big\| \sum_{v=1}^{N} A(v) \Big\| \, \|\theta\|.$$

hence, if $\|\theta\| > 0$ we have $\| \sum_{v=1}^N A(v)\| = 1$. As $\mathbb{E}\{\sum_{v=1}^N A(v)\} = \gamma = 1$ we must have $\sum_{v=1}^N A(v) = 1$, which is the excluded case. Hence $\theta = 0$ almost surely, which means that no flow is possible. This finishes the proof of the first part. ∎

Proof of (2): We first look at a fixed deterministic tree T with deterministic weights $A(e)$ attached to the edges. We introduce a family of random variables on this tree T as follows. Independently for every edge $e \in E$ we let

$$X(e) = \begin{cases} 1 & \text{with probability } A(e) \\ 0 & \text{with probability } 1 - A(e). \end{cases}$$

The intuition is that an edge e is *open* if $X(e) = 1$ and otherwise closed. We consider the subtree $T^* \subset T$ consisting of all edges which are connected to the root by a path of open edges. Let

$$Q(T) := P\big\{T^* \text{ is infinite } \big\}.$$

For any cutset Π note that $\sum_{e\in\Pi} C(e)$ is the expected number of edges in Π, which are also in T^*. Note that

$$\sum_{e\in\Pi} C(e) = E\Big\{ \sum_{e\in\Pi} 1_{\{e\in T^*\}} \Big\} \ge P\Big\{ e \in T^* \text{ for some } e \in \Pi \Big\} \ge P\Big\{ T^* \text{ is infinite } \Big\}.$$

If $\theta(T)$ is the maximal strength of a flow in T, then the last inequality together with the max-flow min-cut theorem shows that

$$Q(T) > 0 \implies \theta(T) > 0. \tag{3.2}$$

Now we use this result again for a Galton-Watson network T, in this way performing a two-step experiment: first sampling the network T and the reducing it to T^*. As a result of the experiment, T^* is another Galton-Watson tree. Denoting by $e_1, \ldots, e_N$ the edges starting from the root we get for the mean offspring number of T^*,

$$\begin{aligned}
\mathbb{E}\Big\{ \sum_{i=1}^{N} X(e_i) \Big\} &= \mathbb{E}\Big\{ \sum_{n=1}^{\infty}\sum_{i=1}^{n} X(e_i) 1_{\{N=n\}} \Big\} = \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathbb{E}\Big\{ X(e_i) 1_{\{N=n\}} \Big\} \\
&= \sum_{n=1}^{\infty}\sum_{i=1}^{n} \mathbb{E}\Big\{ A(e_i) 1_{\{N=n\}} \Big\} = \sum_{n=1}^{\infty} \mathbb{E}\Big\{ \sum_{i=1}^{n} A(e_i) 1_{\{N=n\}} \Big\} \\
&= \mathbb{E}\Big\{ \sum_{i=1}^{N} A(e_i) \Big\} = \gamma.
\end{aligned}$$

If $\gamma > 1$, by the survival criterion for Galton-Watson trees, we have

$$0 < q = \mathbb{P}\big\{T^* \text{ is infinite } \big\} = \mathbb{E}\big\{Q(T)\big\}.$$

Hence $Q(T) > 0$ with positive probability, and by (3.2) we infer that $\theta(T) > 0$ with positive probability. In other words, $\mathbb{P}\{\theta(T) = 0\} < 1$. As the event $\{\theta(T) = 0\}$ is inherited, we infer from the Galton-Watson 01-law that $\mathbb{P}\big\{\theta(T) = 0 \,|\, \text{ survival } \big\} = 1$. This had to be shown. ■

Modulo some technical steps the proof of Theorem 3.1 follows by combining Falconer's theorem and the dimension formula of Theorem 2.5. An interesting corollary comes from the fact that in the *critical case* $\gamma = 1$ flow is impossible unless we are in the excluded case $\sum_{i=1}^{N} A_i^\alpha = 1$.

Corollary 3.4 *If* $\dim I(T) = \alpha$ *and* $\sum_{i=1}^{N} A_i^\alpha \neq 1$ *with positive probability, then* $\mathcal{H}^\alpha(I(T)) = 0$ *almost surely.*

Example 3.5 Compare the following two random fractals: On the one hand a percolation fractal based on dividing the unit interval $[0, 1]$ into three non-overlapping intervals of length $1/3$ and keeping each with probability $p = 2/3$, on the other hand the random fractal obtained by dividing $[0, 1]$ into three non-overlapping intervals of length $1/3$ and choosing two of the three intervals at random, proceeding like this until infinity.

In both cases we obtain fractals of Hausdorff dimension $\alpha = \log 2/ \log 3$. To see this in the second case just observe that the 3-adic coding tree of the fractal is the dyadic tree, exactly as in the case of the ordinary ternary Cantor set. The previous corollary indicates a significant difference between the two examples. Whereas for the first case, by the corollary, the α-Hausdorff measure is zero, one can show that in the second case the α-Hausdorff measure is strictly positive. This can be seen from the fact that there exists a flow on the coding tree with capacities $C_\alpha(\overline{v}, v) = |I_v|^\alpha$ in the second example, whilst there is none in the first.

3.2 The dimension of the zero-set of Brownian motion

The problem of this section is to calculate the dimension of the *zero-set* of a Brownian motion $X : [0, 1] \to \mathbb{R}$. This is much more difficult than the problems we were facing in the first exercise, mainly because we do not have a natural measure readily defined on the zero set, as this was the case in the previous two examples. So one way to deal with this problem would be to define a good measure on the zero set. Such a measure exists (the *local time*) but is quite hard to construct.

Alternative ways are based on fine knowledge about distributional properties of Brownian motion and we are going to sketch such a way here, unfortunately I cannot give full details. The idea of the proof is based on Galton-Watson fractals and is due to Graf, Mauldin and Williams (1988).

3.2.1 Brownian motion and Brownian bridge

A first step is to make the problem more symmetric by looking at a *Brownian bridge* instead of a Brownian motion. There are several ways of defining a Brownian bridge B from a Brownian motion X.

- Define the *Brownian bridge* as the process $B(t) := X(t) - tX(1)$, for $t \in [0, 1]$, or

- let $T = \sup\{t < 1 : X(t) = 0\}$. Then the process $C(t) := \sqrt{1/T}\, X(tT)$, for $t \in [0,1]$, is also a Brownian bridge.

Note that for a given sample path X of Brownian motion the two bridges B and C have quite different sample paths. From the second definition it is easy to see that the dimension of the zero set of a Brownian bridge and of a Brownian motion have the same law.

An important property of the Brownian bridge is *symmetry*: if $\{B(t) : 0 \le t \le 1\}$ is a Brownian bridge, then so is $\{\tilde{B}(t) : 0 \le t \le 1\}$ with $\tilde{B}(t) = B(1-t)$.

3.2.2 The zero-set of a Brownian bridge as a Galton-Watson fractal

We now study the dimension of the zero set of a Brownian bridge. We define two random variables

$$T_1 = \sup\{t \le 1/2 : B(t) = 0\} \text{ and } T_2 = \inf\{t \ge 1/2 : B(t) = 0\}.$$

By symmetry the random variables T_1 and $1 - T_2$ have the same distribution (but they are not independent!) The interval (T_1, T_2) does not contain any zeros, and we remove it from $[0,1]$, which leaves us with two random intervals $[0, T_1]$ at the left and $[T_2, 1]$ on the right. Moreover, one can check that the process

$$\big\{\sqrt{1/T_1}\, B(tT_1) : 0 \le t \le 1\big\}$$

is a Brownian bridge, which is independent of $\{B(t) : t \ge T_1\}$.

Now we can represent the zero set of the Brownian bridge as a Galton-Watson fractal: we start with the interval $[0,1]$ and remove a centred random interval $[T_1, 1 - T_2]$. To the left of the removed interval, we have an independent Brownian bridge

$$\big\{\sqrt{1/T_1} B(tT_1) : 0 \le t \le 1\big\}.$$

By symmetry, we also have an independent Brownian bridge

$$\big\{\sqrt{1/(1-T_2)} B(1 - tT_2) : 0 \le t \le 1\big\}.$$

to the right of the removed interval. If we apply the same procedure on each of the remaining bridges, we iteratively construct the zero set of the Brownian bridge by removing all gaps. The essence of all this is the following:

Lemma 3.6 *The zero set of a Brownian bridge B is a Galton Watson fractal with generating random variable $X = (2, T_1, 1 - T_2)$. Hence* $\dim\{t \in [0,1] : B(t) = 0\} = \alpha$, *where α is the unique solution of*

$$\mathbb{E}\{T_1^\alpha + (1 - T_2)^\alpha\} = 1.$$

We can now calculate the dimension by evaluating this expectation for the right value of α.

Lemma 3.7

$$\mathbb{E}\{\sqrt{T_1} + \sqrt{1 - T_2}\} = 1.$$

Proof: By symmetry of the Brownian bridge, T_1 and $1-T_2$ have the same distribution, hence it suffices, to show that $\mathbb{E}\{\sqrt{1-T_2}\} = 1/2$. We have, using the definition of the Brownian bridge and the time inversion property of Brownian motion,

$$\begin{aligned} T_2 &= \inf\big\{1/2 \le t \le 1 \,:\, B(t)=0\big\} = \inf\big\{1/2 \le t \le 1 \,:\, X(t) - tX(1) = 0\big\} \\ &\stackrel{d}{=} \inf\big\{1/2 \le t \le 1 \,:\, tX(1/t) - tX(1) = 0\big\} = \inf\big\{1/2 \le t \le 1 \,:\, X(1/t) - X(1) = 0\big\} \\ &= 1/\sup\big\{1 \le s \le 2 \,:\, X(s) - X(1) = 0\big\}\,. \end{aligned}$$

Because $\{X(s) - X(1) \,:\, s \ge 1\}$ has the same distribution as $\{X(s-1) \,:\, s \ge 1\}$, we have

$$T_2 \stackrel{d}{=} \frac{1}{1 + \sup\big\{0 \le t \le 1 \,:\, X(t) = 0\big\}}$$

and, in particular,

$$\mathbb{E}\sqrt{1-T_2} = \int_0^1 \sqrt{\frac{x}{1+x}}\, f(x)\, dx\,,$$

where f is the density of the random variable $L = \sup\big\{0 \le t \le 1 \,:\, X(t) = 0\big\}$. This random variable has the arcsine-distribution, which can be verified using the *reflection principle* of Brownian motion. We get that

$$\mathbb{E}\sqrt{1-T_2} = \frac{1}{\pi}\int_0^1 \sqrt{\frac{x}{1+x}}\,\frac{dx}{\sqrt{x(1-x)}} = \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{1}{\pi}\arcsin(1) = 1/2.$$

■

We have thus proved the following result.

Theorem 3.8 *Almost surely,* $\dim\{t \in [0,1] \,:\, X(t) = 0\} = \dim\{t \in [0,1] \,:\, B(t) = 0\} = \frac{1}{2}$.

3.3 Exercise: Examples of Galton-Watson fractals

Derive explicitly or implicitly the Hausdorff dimension of the following fractal sets.

(1) Divide $[0,1]$ into three equal parts and keep each independently with probability $p \in (0,1)$. Repeat this independently with the remaining intervals.

(2) Remove from $[0,1]$ a central portion leaving two intervals of random length $A_1, A_2 \in (0,1/2)$. Repeat this, independently, for each remaining interval. Consider in particular the case that A_1, A_2 are independent and uniform on $[0,1/2]$.

(3) Suppose M and N are random integers with $M \ge 2$ and $1 \le N \le M^2$. Divide the unit square of $\mathbb{R}^2$ into M^2 equal squares and keep N of them in some random manner. Repeat this procedure for the remaining squares. Consider in particular the case that $N = M$.

Chapter 4

Lecture Four: Limsup fractals and the Orey-Taylor Theorem

4.1 Motivation

In this lecture we enter into the finer features of Brownian motion and discuss a class of random fractals arising as exceptional points of Brownian motion. We first look at an example, the theorem of Orey and Taylor (1974) about the fast times of a Brownian motion. This motivates the study of limsup fractals, which have been investigated more recently, see Khoshnevisan, Peres and Xiao (2000).

4.2 The Orey-Taylor Theorem

Let $\{B(t) : 0 \leq t \leq 1\}$ be a Brownian motion. The size of the small increments of a Brownian motion is determined by Khinchin's law of the iterated logarithm, which states that, for any *fixed* $t \in [0, 1]$ almost surely,

$$\limsup_{h \downarrow 0} \frac{|B(t) - B(t-h)|}{\sqrt{2h \log\log(1/h)}} = 1\,. \tag{4.1}$$

However, the zero-set of exceptional paths depends on the time t. Taking a maximum over all times $t \in [0, 1]$ shows a different picture, given by Lévy's uniform modulus of continuity, which states that, almost surely,

$$\limsup_{h \downarrow 0} \sup_{t \in [0,1]} \frac{|B(t) - B(t-h)|}{\sqrt{2h \log(1/h)}} = \sup_{t \in [0,1]} \limsup_{h \downarrow 0} \frac{|B(t) - B(t-h)|}{\sqrt{2h \log(1/h)}} = 1\,. \tag{4.2}$$

In other words, there exist exceptional times $t \in [0, 1]$ where the increments are untypical large, i.e. of order $\sqrt{h \log(1/h)}$. Orey and Taylor (1974) studied the natural question, how big – in terms of Hausdorff dimension – the set of exceptional times is.

Theorem 4.1 *Almost surely, for every $a \in [0, \sqrt{2}]$,*

$$\dim \Big\{ t \in [0, 1] \,:\, \limsup_{h \downarrow 0} \frac{|B(t) - B(t+h)|}{\sqrt{h \log(1/h)}} \geq a \Big\} = 1 - \frac{a^2}{2}. \tag{4.3}$$

The sets in the Orey-Taylor theorem can be approximated by limsup fractals. In this lecture we discuss a crude version of these fractals, stressing the basic idea at the expense of technical subtleties.

4.3 Limsup fractals

We only discuss a very basic version of limsup-fractals, applications as the Orey-Taylor theorem need a substantial weakening of the rigid assumptions we make. Nevertheless, I hope to convey the basic ideas.

Let $U = [0,1]^d$ be the d-dimensional unit interval. For any integer $n \geq 0$ let $\mathfrak{D}_n$ be the collection of all open dyadic cubes of sidelength 2^{-n} in U and to every $I \in \mathfrak{D}_n$ we associate a random variable $Z(I)$ taking values in $\{0,1\}$. Assume

- for every $n \in \mathbb{N}$ the random variables $\{Z(I) : I \in \mathfrak{D}_n\}$ are independent,
- for every $n \in \mathbb{N}$ the probabilities $p_n = \mathbb{P}\{Z(I) = 1\}$ do not depend on the choice of $I \in \mathfrak{D}_n$, and for some $\eta > 0$,
$$\lim_{n\to\infty} \frac{1}{n} \log_2 p_n = -\eta.$$

Note that *no independence assumption* is made for $Z(I)$ comparing Is of different generation.

We write $A(n)$ for the union over all $I \in \mathfrak{D}_n$ with $Z(I) = 1$. The *discrete limsup random fractal* associated to the family $Z = \{Z(I) : I \in \mathfrak{D}_n, n \geq 1\}$ is given by

$$A = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A(n).$$

In other words, A is the collection of points which are in infinitely many dyadic intervals I with $Z(I) = 1$. Note that this is very different from a percolation limit set!

Theorem 4.2 (Khoshnevisan, Peres, Xiao) *Almost surely,* $\dim A = d - \eta$.

Proof: For the upper bound we use the obvious explicit covering,

$$A \subset \bigcup_{n=m}^{\infty} \bigcup_{\substack{I\in\mathfrak{D}_n \\ Z(I)=1}} I.$$

Then

$$\mathbb{E}\Big\{ \sum_{n=m}^{\infty} \sum_{\substack{I\in\mathfrak{D}_n \\ Z(I)=1}} |I|^{\alpha} \Big\} = \sum_{n=m}^{\infty} \sum_{I\in\mathfrak{D}_n} \mathbb{P}\{Z(I)=1\} 2^{-n\alpha}$$
$$= \sum_{n=m}^{\infty} \sum_{I\in\mathfrak{D}_n} 2^{-n[\alpha+\eta+o(1)]} = \sum_{n=m}^{\infty} 2^{-n[-d+\alpha+\eta+o(1)]},$$

which converges if $-d+\alpha+\eta>0$. Hence, for any $\alpha > d-\eta$ and $m>0$ we get a covering consisting of sets of diameter $\le \sqrt{d}2^{-m}$ with a.s. finite value. This implies the upper bound. Note that we have not used the independence assumption so far.

For the lower bound we use the percolation method, recall Theorem 1.7, for the proof. Let $\gamma < d-\eta$ and suppose that $\Gamma[\gamma]$ is a percolation limit set independent of Z. We show that, with positive probability, $\Gamma[\gamma]\cap A \neq \emptyset$.

For this purpose we first note that, almost surely, for any open set $V\subset[0,1]^d$, if $V\cap\Gamma[\gamma]\neq\emptyset$, then $\dim(V\cap\Gamma[\gamma]) = d-\gamma$. For the moment we fix $\Gamma[\gamma]$ and consider only the limsup-fractal as random.

Define the open sets $B(m)=\bigcup_{n=m}^\infty A(n)$. We claim that, almost surely, for all $m\ge 1$ the relatively open set $B(m)\cap\Gamma[\gamma]$ is dense in the compact (hence complete) metric space $\Gamma[\gamma]$. If this is true, the set

$$A\cap\Gamma[\gamma]=\bigcap_{m=1}^\infty B(m)\cap\Gamma[\gamma]$$

is dense in $\Gamma[\gamma]$ by the Baire (Category) Theorem. In particular it is nonempty almost surely given $\Gamma[\gamma]\neq\emptyset$.

To prove that $B(m)\cap\Gamma[\gamma]$ is dense in $\Gamma[\gamma]$, we fix an open set V with $V\cap\Gamma[\gamma]\neq\emptyset$. Define $N(n)$ to be the total number of cubes $I\in\mathfrak{D}_n$ with $I\cap\Gamma[\gamma]\cap V\neq\emptyset$. Because $\dim(\Gamma[\gamma]\cap V)=d-\gamma$, we have, for any $0<2\varepsilon<d-\eta-\gamma$,

$$N(n)\ge 2^{n(d-\gamma-\varepsilon)}\ge 2^{n(\eta+\varepsilon)} \text{ for all sufficiently large } n.$$

Let

$$S_n=\sum_{\substack{I\in\mathfrak{D}_n\\ I\cap\Gamma[\gamma]\cap V\neq\emptyset}} Z_n(I),$$

i.e. the total number of cubes $I\in\mathfrak{D}_n$ with $I\cap\Gamma[\gamma]\cap V\cap A(n)\neq\emptyset$. We have, for any $0<\delta<\varepsilon$,

$$\mathbb{P}\{S_n=0\}=\mathbb{P}\{Z_n(I)=0\}^{N(n)}\le[1-2^{-n[\eta+\delta]}]^{2^{n[\eta+\varepsilon]}} \text{ for all sufficiently large } n.$$

This implies that $\lim_{n\to\infty}\mathbb{P}\{S_n=0\}=0$. Hence,

$$\mathbb{P}\{S_n>0 \text{ infinitely often }\}\ge\limsup_{n\to\infty}\mathbb{P}\{S_n>0\}=1.$$

In other words, $B(m)\cap\Gamma[\gamma]\cap V\neq\emptyset$, and as V was an arbitrary open set intersecting $\Gamma[\gamma]$ we infer that $B(m)\cap\Gamma[\gamma]$ is dense in $\Gamma[\gamma]$.

We have thus shown that $A\cap\Gamma[\gamma]\neq\emptyset$ almost surely, given $\Gamma[\gamma]\neq\emptyset$. Let $\check{\Gamma}$ be the union of countably many i.i.d. copies of $\Gamma[\gamma]$. Then $A\cap\check{\Gamma}\neq\emptyset$ almost surely. This implies, by Theorem 1.7 (i), that $\dim A\ge\gamma$ almost surely. For $\gamma\uparrow d-\eta$, we get $\dim A = d-\eta$ with probability one. ■

Remarks:

- A major reason for the success of limsup fractals lies in the fact that the independence assumption between the random variables $\{Z(I) : I\in\mathfrak{D}_n\}$ can be replaced by bounds on the correlation length. The precise condition given in the paper of Khoshnevisan, Peres and Xiao (2002) is that, for

$$f(n,\varepsilon) := \max_{I\in\mathfrak{D}_n} \#\big\{J \in \mathfrak{D}_n \,:\, \mathrm{Cov}(Z(I),Z(J)) \geq \varepsilon\mathbb{E}\{Z_n(I)\}\mathbb{E}\{Z_n(J)\}$$

we have, for all $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{1}{n}\log f(n,\varepsilon) = 0.$$

- Note that limsup fractals are always dense in the base cube. It is however possible to study limsup-subfractals of random sets, by replacing the structure of the underlying 2^d-adic tree by a more general, or even random, tree. This idea is used in König and Mörters (2002).

Let us roughly sketch the connection between the dimension result for limsup fractals and the Orey-Taylor Theorem. Fix a large $n \in \mathbb{N}$ and let $h \approx 2^{-n}$. For $I = [t, t+2^{-n}] \in \mathfrak{D}_n$ we let $Z(I) = 1$ if

$$|B(t) - B(t+h)| \geq a\sqrt{h\log(1/h)}.$$

The limsup fractal A associated with the family $\{Z(I)\}$ can be used as an approximation for the set

$$F[a] := \Big\{t \in [0,1] \,:\, \limsup_{h\downarrow 0} \frac{|B(t)-B(t+h)|}{\sqrt{h\log(1/h)}} \geq a\Big\}.$$

Indeed, for the more difficult lower bound the value h is chosen slightly larger than 2^{-n} by a factor of size n^k, for some $k \geq 0$. This ensures that, by Lévy's uniform modulus of continuity, the set A is a subset of $F[a-\varepsilon]$. Then the independence of $Z(I)$ for various $I \in \mathfrak{D}_n$ only holds if the intervals have a distance of at least $n^k 2^{-n}$ and the assumptions in the definition of limsup-fractals have to be weakened accordingly. The correlation length criterion of Khoshnevisan, Peres and Xiao does the job.

The evaluation of η is very easy here as

$$\begin{aligned}\mathbb{P}\{Z(I)=1\} &= \mathbb{P}\big\{|B(t)-B(t+h)| \geq a\sqrt{h\log(1/h)}\big\}\\ &= 2\int_1^\infty \frac{a\sqrt{\log(1/h)}}{\sqrt{2\pi}} \exp\Big(-(ax\sqrt{h\log(1/h)})^2/2h\Big)\,dx\\ &= \exp\big(-na^2/2\log(2)+o(1)\big).\end{aligned}$$

Hence $\eta = a^2/2$, and thus A has dimension $1-a^2/2$. A tightening of the argument allows to replace $\geq$ by $=$.

4.4 Further examples of limsup-type

Fractals of limsup-type have been studied a lot in the recent literature. A good example involving Brownian motion are the *thick points* for the Brownian occupation measure of Brownian motion in $d \geq 3$, studied by Dembo, Peres, Rosen and Zeitouni.

Theorem 4.3 (Dembo, Peres, Rosen and Zeitouni 2000) *Let $d \geq 3$ and q_d denote the first positive zero of the Bessel function $J_{d/2-2}(x)$. Denote by $T(t,r)$ the time spent by a d-dimensional Brownian motion W inside the ball $B(W(t),r)$. Then, for every $0 < a \leq 4q_d^{-2}$,*

$$\dim\Big\{t \in [0,1]\colon \limsup_{r\downarrow 0} \frac{T(t,r)}{r^2\log(1/r)} = a\Big\} = 1 - a\,\frac{q_d^2}{4} \qquad \textit{almost surely.}$$

The key to this result is the fact that exceptionally large values of $T(t,r)$ are a *local* effect, the behaviour of the paths at times outside the interval $(t - r^{2-\varepsilon}, t + r^{2-\varepsilon})$ does not contribute significantly (in $d \geq 3$). This fact gives the correlation bounds necessary to apply the limsup-fractal machinery. $\diamond$

The following complementary result on the *thin points* for the Brownian occupation measure of Brownian motion in $d \geq 2$, is due to the same authors. It requires a substantial extension of the limsup method to incorporate long-range dependence effects.

Theorem 4.4 (Dembo, Peres, Rosen and Zeitouni 2001) *Let $d \geq 2$ and denote by $T(t,r)$ the time spent by a d-dimensional Brownian motion W inside the ball $B(W(t),r)$. Then, for every $a \geq 1$,*

$$\dim\Big\{t \in [0,1]\ :\ \liminf_{r\downarrow 0} \frac{T(t,r))}{r^2[\log(1/r)]^{-1}} = a\Big\} = 1 - \frac{1}{a}, \ \textit{almost surely.}$$

Chapter 5

Lecture Five: The multifractal spectrum of measures on Brownian paths

5.1 Motivation

An interesting fractal geometric characteristic for discrete and continuous models of statistical physics is the *multifractal spectrum*, which originated in the study of turbulence models. Loosely speaking, the multifractal spectrum evaluates the degree of variation in the intensity of a spatial distribution, or, in other words, it describes quantitatively the irregularities of a fractal measure. More precisely, suppose that μ is a measure on $\mathbb{R}^d$. One can define the *local dimension* of the measure μ in the point x as

$$\lim_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r}, \tag{5.1}$$

which is the rate of decay of the measure μ as the balls zoom into the point x. The value $f(a)$ of the multifractal spectrum is the Hausdorff dimension of the set of points x with local dimension a, i.e.

$$\dim \Big\{ x \in \mathbb{R}^d \, : \, \lim_{r\downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = a \Big\}, \tag{5.2}$$

where $B(x,r)$ denotes the ball of radius r centred in x. In many cases of interest, the limit in (5.1) has to be replaced by liminf or limsup to obtain an interesting nontrivial spectrum.

A measure is called *multifractal* if $f(a) > 0$ for more than one value of $a \in \mathbb{R}$. For example, the occupation measure of a Brownian path in $d \geq 2$ given by

$$\mu(A) = \int_0^1 1_A(B_s)\, ds$$

is *not* multifractal: almost surely, at *every* point $x \in B[0,1]$ the local dimension is exactly two. In this lecture we are looking at two more interesting measures on Brownian paths, the harmonic measure and intersection local times. We focus on dimension three, although analogous results hold true in the case of $d = 2$.

5.2 Intersection exponents

The results of this talk are related to the intersection exponents of Brownian motion. Intersection exponents contain crucial information about the paths of stochastic processes, and they have been studied for Brownian motion and other processes for many years, first by physicists but more recently also by mathematicians. The recent rigorous results of Lawler, Schramm and Werner give us a picture of the values of these exponents in the plane, which is complete in the case of Brownian motion.

Suppose $n, m \geq 1$ are integers and let $W_1, \dots, W_{n+m}$ are independent Brownian motions in $\mathbb{R}^d$, $d = 2, 3$, started uniformly on $\partial B(0,1)$ and running up to the first exit times T^i from a large ball $B(0,r)$. We divide the motions into two packets and look at the union of the paths in each family,

$$\mathfrak{B}_1(r) = W_1[0,T^1] \cup \ldots \cup W_n[0,T^n] \text{ and } \mathfrak{B}_2(r) = W_{n+1}[0,T^{n+1}] \cup \ldots \cup W_{n+m}[0,T^{n+m}].$$

The event that the two packets of Brownian paths fail to intersect has a decreasing probability as $r \uparrow \infty$. Indeed, it is easy, using subadditivity, to show that there exists a constant $\xi_d(n,m)$ such that

$$\mathbb{P}\big\{ \mathfrak{B}_1(r) \cap \mathfrak{B}_2(r) = \emptyset \big\} = r^{-\xi_d(n,m)+o(1)}, \ \textit{as } r \uparrow \infty.$$

The numbers $\xi_d(n,m)$ are called the *intersection exponents.*

The definition can be extended naturally to one real argument. For this purpose define the random variable

$$Z_n(r) = \mathbb{P}\{\mathfrak{B}_1(r) \cap W_{n+1}[0,T^{n+1}] = \emptyset \,\big|\, W_1, \dots, W_n \big\}.$$

Again one can show that, for every $\lambda > 0$, there exists a constant $\xi_d(n,\lambda)$ such that

$$\mathbb{E}\big\{ Z_n(r)^\lambda \big\} = r^{-\xi_d(n,\lambda)+o(1)}, \ \textit{as } r \uparrow \infty.$$

Clearly, this is consistent with the definition of $\xi_d(n,\lambda)$ in the case if an integer λ.

Physicists, e.g. Duplantier and Kwon (1988) have made the striking conjecture that in the plane many of these exponents are rational numbers. Very recently, Lawler, Schramm and Werner, in a seminal series of papers, have been able to verify this rigorously, see Lawler et al (2001) for a survey over the series. They have shown that

$$\xi_2(n,\lambda) = \frac{\big(\sqrt{24n+1} + \sqrt{24\lambda+1} - 2\big)^2 - 4}{48}. \tag{5.3}$$

As the proof of (5.3) is based on conformal invariance, there is no analogue in $d = 3$. The only value known in dimension $d = 3$ is $\xi_3(1,2) = \xi_3(2,1) = 1$, as shown below.

5.3 The multifractal spectrum of harmonic measure

Suppose $A \subset \mathbb{R}^3$ is a compact, nonpolar set. The harmonic measure on the set A is defined as follows. Suppose $B(0,R)$ is a ball such that A is contained inside this ball. Let $W : [0,\infty) \to \mathbb{R}^3$ be a Brownian motion started in a point chosen uniformly from the sphere $\partial B(0,R)$ and let

$$T = \inf\{t > 0 \,:\, W_t \in A\}$$

be the first hitting time of the set A by a Brownian motion. Then the *harmonic measure* h_A on A is given as

$$h_A(B) = \mathbb{P}\{W_T \in B,\, T < \infty\} \text{ for all Borel sets } B \subset A.$$

Note that the support of the harmonic measure is contained in the boundary of the unbounded components of A^c, often called the *frontier* of A.

Now we look at $A = B[0,1] \subset \mathbb{R}^d$, the path of a Brownian motion and study the harmonic measure $h = h_A$ on this set. The following result about the multifractal spectrum of the measure h_A is due to Lawler (1999).

Theorem 5.1 (Lawler 1999) *Define a convex function ξ^* as the Legendre transform of the the strictly concave function $\lambda \mapsto \xi_3(2,\lambda)$, i.e.*

$$\xi^*(a) = \sup_{\lambda>0} \big(\xi_3(2,\lambda) - a\lambda\big).$$

Then, for every a such that $2 - \xi^*(a-1) > 0$, almost surely there exist points $x \in A$ with

$$\lim_{r\downarrow 0} \frac{\log h(B(x,r))}{\log r} = a.$$

Moreover, for these values of a,

$$\dim\Big\{x \in A \,:\, \lim_{r\downarrow 0} \frac{\log h(B(x,r))}{\log r} = a\Big\} = 2 - \xi^*(a-1).$$

Remark: The fact that this spectrum is nontrivial is due to the *non-mean field* nature of Brownian intersections, expressed here in the *strict* concavity of $\xi_3(2,\cdot)$. ◇

In the following I give a heuristic argument for Theorem 5.1, based on the so-called *multifractal formalism.* Let

$$A(j,n) = B\big[\frac{j-1}{e^{2n}}, \frac{j}{e^{2n}}\big].$$

Consider the quantity

$$Q(n,\lambda) = \sum_{j=1}^{e^{2n}} h(A(j,n))^\lambda.$$

This is a random variable, which is intimately related to the multifractal spectrum. Indeed, for large n, we can roughly assume that $A(j,n)$ is the intersection of the Brownian path A with a ball of radius e^{-n} and if the local dimension of the points in this ball is approximately a, then $h(A(j,n)) \approx e^{-an}$. Hence, if we denote by

$$\mathfrak{E}(a) = \Big\{j \,:\, h(A(j,n)) \approx e^{-an}\Big\},$$

we can write

$$Q(n,\lambda) = \sum_a \#\mathfrak{E}(a) e^{-\lambda a n},$$

where the sum extends over an appropriate discretisation of the reals. The Laplace method suggests that, for

$$f(a) = \lim_{n\to\infty} \frac{1}{n} \log\big(\#\mathfrak{E}(a)\big)$$

we have

$$Q(n,\lambda) \approx \exp\big(n \sup_a \{f(a) - \lambda a\}\big).$$

Suppose now that we can show that,

$$Q(n,\lambda) \approx \exp(n g(\lambda)). \tag{5.4}$$

Then we get,

$$\sup_a \big\{f(a) - \lambda a\big\} = g(\lambda).$$

As the sets in $\mathfrak{E}(a)$ roughly cover the set of points of local dimension a, the quantity $f(a)$ is at least an upper bound for the dimension spectrum. If everything is sufficiently regular, this bound is sharp. To identify f note that

$$g(\lambda) = \sup_a \big\{f(a) - \lambda a\big\} = f^*(\lambda).$$

If f is concave, we may invert this Legendre transform and get

$$f(a) = g^*(a) = \inf_{\lambda > 0} \big\{g(\lambda) + \lambda a\big\}. \tag{5.5}$$

Hence it boils down to finding the g in (5.4).

Here the intersection exponents are coming into play. First fix the set $A = B[0,1]$ and look at the harmonic measure of a small ball,

$$h_A(B(x,r)) = P\big\{W \text{ hits } A \text{ first in } B(x,r)\big\} \text{ for } x \in A.$$

The probability that the path W hits $B(x,r)$ is of order $cr^{d-2} = cr$ for some constant $c > 0$. Given that W hits $B(x,r)$, we can look at the time-reversed path starting on the boundary $\partial B(x,r)$ and stopped upon hitting $\partial B(0,R)$, which for our purposes behaves like a Brownian motion. Hence, now looking at the random set A, by scaling and the definition of $\xi_3(2,\lambda)$,

$$\mathbb{E}\big\{h_A(B(x,r))^\lambda\big\} \approx r^{\lambda + \xi_3(2,\lambda) + o(1)}.$$

This gives

$$\mathbb{E}Q(n,\lambda) \approx \exp\big(n(2 - \lambda - \xi_3(2,\lambda) + o(1))\big).$$

Supposing sufficient independence between the different parts of the set A we get that

$$g(\lambda) = \lim_{n\to\infty} \frac{1}{n} \log Q(n,\lambda) \approx 2 - \lambda - \xi_3(2,\lambda),$$

and hence, from (5.5),

$$f(a) = 2 - \sup_{\lambda > 0} \big\{\xi_3(2,\lambda) - \lambda(a-1)\big\}.$$

Remark: As the total harmonic measure is constant, we have $g(1) = 0$ and hence we get $\xi_3(2,1) = 1$.

5.4 The multifractal spectrum of intersection local time

Let W_1, W_2 be two independent Brownian motions in $\mathbb{R}^3$ with a joint starting point and running for one unit of time. Let $S = W_1[0,1] \cap W_2[0,1]$ be the intersection of the paths, which is a (random) set of Hausdorff dimension 1. This set can be equipped with a finite measure, the intersection local time ℓ, the *(projected) intersection local time*, which can be described symbolically by the formula

$$\ell(A) = \int_A dy \prod_{j=1}^{2} \int_0^1 ds\, \delta_y(W_j(s)), \text{ for } A \subset \mathbb{R}^d \text{ Borel.} \tag{5.6}$$

Rigorous constructions of the random measure ℓ are reviewed in König and Mörters (2002). It is well-known that the local dimension of ℓ-almost all points equals 1, which is also the Hausdorff dimension of the intersection set S.

Points where the local dimension is strictly below the typical value 1 do not exist, in other words there is no nontrivial multifractal spectrum to the left of 1. This follows from recent work of König and Mörters (2002). Indeed, this paper shows how a weaker notion of thick points can be defined in order to obtain a nontrivial spectrum. In a more recent paper, Klenke and Mörters (2003) we are asking whether there are points where the local dimension exceeds 1, these are the *thin points*.

Theorem 5.2 (Klenke and Mörters 2003) *Suppose ℓ is the intersection local time of two Brownian motions in $\mathbb{R}^3$, starting in zero and running for fixed time. Then, almost surely, there exist points $x \in S$ with*

$$\limsup_{r \downarrow 0} \frac{\log \ell(B(x,r))}{\log r} = a,$$

if and only if $1 \le a \le \xi/(\xi - 1)$, for $\xi = \xi_3(2,2)$. Moreover, for these values of a,

$$\dim \Big\{ x \in S \,:\, \limsup_{r \downarrow 0} \frac{\log \ell(B(x,r))}{\log r} = a \Big\} = \frac{\xi}{a} + 1 - \xi.$$

Remark: Here the multifractal spectrum is strictly *convex*, hence it cannot be found by means of the multifractal formalism of the last section, which by its nature always predicts a concave spectrum. ◇

In the following I give a heuristic argument for this result. We start with the problem to find the probability of the event $\ell(B(x,r)) \approx r^a$ given $x \in S$. Intuitively it is clear that in a typical intersection point $x \in S$ two two-sided Brownian motions are attached. This can be made rigorous in terms of the theory of Palm distributions.

As the motions start in the same point they typically have a lot of intersections right at the start. In order to avoid intersections to the extent that $\ell(B(x,r)) \approx r^a$ it is optimal for the Brownian motions to use the following strategy:

(1) Run normally until you hit $\partial B(x, r^a)$. The typical intersection local time until then is $\approx r^a$, and now one has to act.

(2) Do not intersect before leaving $B(x, r^{1-\varepsilon})$. The probability that this happens is, by scaling and the definition of the intersection exponents, $r^{-\xi_3(2,2)(1-\varepsilon-a)+o(1)}$.

(3) Once you reach $\partial B(x, r^{1-\varepsilon})$ never return to $B(x, r)$. The probability that this happens is $\approx 1 - r^{\varepsilon}$.

The total probability (or "cost") of this strategy is $r^{-\xi_3(2,2)(1-\varepsilon-a)+o(1)}$, essentially all of which is incurred in the second step. Note that it leads to an empty annulus $B(x,r) \setminus B(x, r^a)$. It can be shown that this strategy is optimal, i.e. no other strategy to get $\ell(B(x,r)) \approx r^a$ has a significantly higher probability.

We now sketch how the upper bound in our spectrum can be deduced from this. As $\dim S = 1$ we can cover the set S by approximately r^{-a} balls of radius r^a. For each of these balls the probability of containing a point $x \in S$ with $\ell(B(x,r)) \approx r^a$ is approximately $r^{-\xi_3(2,2)(1-a)}$. Choosing a covering of the set

$$\Big\{ x \in S \, : \, \limsup_{r \downarrow 0} \frac{\log \ell(B(x,r))}{\log r} \geq a \Big\}$$

consisting of all these balls for the radii $r = 2^{-n}, n \geq m$ and m fixed, leads to a covering

$$B(x_1, r_1^a), B(x_2, r_2^a), B(x_3, r_3^a), \ldots$$

with s-value

$$\mathbb{E}\Big\{ \sum_{i=1}^{\infty} (r_i^a)^s \Big\} = \sum_{n=m}^{\infty} 2^{-nas} 2^{na} 2^{n\xi_3(2,2)(1-a)},$$

which is finite iff $s > \xi_3(2,2)/a + 1 - \xi_3(2,2)$.

Finally, the actual proof of the (more difficult) lower bound works with the stochastic co-dimension argument of Chapter 1 and uses refinements of the proof of the lower dimension bound for the case of limsup fractals in the case of subfractals of S and in the presence of long-range interactions.

Bibliography

[1] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni. Thick points for spatial Brownian motion: multifractal analysis of occupation measure. *Ann. Probab.* **28(1)**, 1–35 (2000).

[2] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni. Thin points for Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.*, **36**(6), 749–774 (2001).

[3] B. Duplantier and K.-H. Kwon. Conformal invariance and intersections of random walks. *Phys.Rev.Lett.*,**61**, 2514-1517 (1988).

[4] K. Falconer. Random fractals. *Math. Proc. Camb. Philos. Soc.* **100**, 559-582 (1986).

[5] K. Falconer. Cut-set sums and tree processes. *Proc. Am. Math. Soc.* **101**, 337-346 (1987).

[6] K. Falconer. *Fractal geometry: mathematical foundations and applications.* Chichester: John Wiley & Sons. xxii, 288 p. (1997).

[7] K. Falconer. *Techniques in fractal geometry.* Chichester: John Wiley & Sons. xvii, 256 p. (1997).

[8] O. Frostman. Potential d'équilibre et capacité des ensembles avec quelques applications a la théorie des fonctions. *Meddel. Lunds Univ. Math. Sem.* **3**, 1–118, (1935).

[9] S. Graf, R. D. Mauldin, and S. C. Williams The exact Hausdorff dimension in random recursive constructions. *Mem. Am. Math. Soc.* **71**, No.381, (1988).

[10] J. Hawkes. Trees generated by a simple branching process. *J. London Math. Soc.*, **24**, 373–384 (1981).

[11] J.D. Howroyd. On dimension and on the existence of sets of finite positive Hausdorff measure. *Proc. Lond. Math. Soc.*, III. Ser. 70, No.3, 581-604 (1995).

[12] R. Kaufman. Une propriété métrique du mouvement Brownien. *C. R. Acad. Sci. Paris* **268**, 727-728 (1969).

[13] D. Khoshnevisan, Y. Peres and Y. Xiao. Limsup random fractals. *El. J. Probab.* **5**, Paper 4, pp.1–24 (2000).

[14] D. Khoshnevisan and Z. Shi. Fast sets and points for fractional Brownian motion. Séminaire de Probabilités, XXXIV, 393–416, Lecture Notes in Math., 1729, Springer, Berlin, (2000).

[15] A. Klenke and P. Mörters. The multifractal spectrum of Brownian intersection local times. In preparation (2003).

[16] W. König and P. Mörters. Brownian intersection local times: upper tails and thick points. *Ann. Probab.*, **30(4)**, 1605-1656, (2002).

[17] G. F. Lawler. Geometric and fractal properties of Brownian motion and random walk paths in two and three dimensions. Bolyai Math. Soc. Studies, **9**, 219-258 (1999).

[18] G. F. Lawler. The frontier of a Brownian path is multifractal. In preparation (2003)

[19] G. F. Lawler, O. Schramm and W. Werner. The dimension of the Brownian frontier is 4/3. *Math. Res. Lett.* **8**, 401-411 (2001).

[20] R. Lyons. Random walks and percolation on trees. *Ann. Probab.* **18(3)**, 931-958 (1990).

[21] R. Lyons and Y. Peres *Probability on trees and networks.* To appear. Preliminary version currently available at *http://php.indiana.edu/ rdlyons/prbtree/prbtree.html*

[22] P. Mattila *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.* 1st paperback ed. Cambridge Studies in Advanced Mathematics. 44. Cambridge: Cambridge University Press. xii, 343 p. (1999).

[23] S. Orey and S.J. Taylor. How often on a Brownian path does the law of the iterated logarithm fail? *Proc. London Math. Soc.(3)* **28**, 174–192 (1974).

[24] Y. Peres. Remarks on intersection-equivalence and capacity equivalence. *Ann. Inst. H. Poincaré (Physique Théorique)* **64**, 339–347 (1996).

[25] Y. Peres. Intersection-equivalence of Brownian paths and certain branching processes. *Commun. Math. Phys.*.**177**, No.2, 417-434 (1996).

[26] Y. Peres. *Probability on trees: An introductory climb.* In: Bertoin, J. (ed.) et al., Lectures on probability theory and statistics. Ecole d'été de Probabilités de Saint-Flour XXVII-1997. Berlin: Springer. Lect. Notes Math. 1717, 193-280 (1999).

[27] C.A. Rogers. *Hausdorff measures.* With a foreword by K. Falconer. Paperback ed. Cambridge Mathematical Library. Cambridge: Cambridge University Press (1999).