Contributions to Fractal Geometry and Stochastic Processes

Peter Mörters

Habilitationsschrift

Universität Kaiserslautern
May 2001

Contents

Preface

There is a lasting trend in modern mathematics away from the concentration on smooth and regular objects towards the intensive study of rough or, in some sense, irregular objects. This trend is driven on the one hand by applications, which require to deal with objects like rough signals, white noise or porous media. On the other hand there is the discovery that irregular objects have a rich mathematical structure, and the insight that the understanding of these objects is the key to a mathematical understanding of many phenomena. A certain rôle in this development is also played by the discovery that, instead of being a danger to the aesthetics of mathematics, irregular or fractal objects have an aesthetic value of their own.

The present *Habilitation Thesis* comprises five papers dealing with the analysis of irregular objects, the first two deal with irregular sets and measures from a very general viewpoint, the last three deal with specific models of random phenomena. The papers are chosen to represent my scientific work in the past five years. This implies that they do not constitute a closed body of work devoted to a single subject, but are aimed to show the diversity and broadness of my mathematical interests. The papers touch three of my main research fields:

- the geometry of fractal sets and measures,
- path properties of stochastic processes,
- stochastic processes in random media.

The mathematical areas involved are analysis and probability, but pursuing these interests leads to problems from a large number of mathematical fields, like singular integrals, ergodic theorems, random measures and point processes, Gibbs measures, combinatorics, large deviation theory, variational calculus, partial differential equations, regular variation, ... I hope that the present thesis enables the reader to get a flavour of this variety.

I have decided to leave the papers —up to typographical details— unchanged and collect a joint bibliography at the end of the thesis. The price for this is that some parts of the notation may vary from paper to paper, but the great advantage is that each chapter is essentially self-contained. To give the reader an overview there is an introductory chapter, which surveys the main results and puts them into context. This overview sometimes presents a different motivation or approach to the results and at some places includes a brief description of the main heuristic ideas behind the proofs. Also some additional background and more examples are included. I have not always resisted the temptation to describe some of my results not included in the main part of the thesis.

As is almost unavoidable in mathematics today, the larger projects are the result of joint efforts. I would like to thank all my coauthors for working with me and especially those coauthors whose work is represented here for permitting to include the joint work in this thesis. I would also like to take the opportunity and express my thanks for external funding of these cooperations. The coauthors and supporters are:

- **Chapter 1: Coauthor** David Preiss, University College London.
  Support: Work on this paper was supported by a travelling grant as part of the ARC-project "Geometry of fractal measures" by British Council and DAAD.

- **Chapter 2:**
  Support: Work on this paper was also supported by a travelling grant as part of the ARC-project "Geometry of fractal measures" by British Council and DAAD.

- **Chapter 3: Coauthor** Wolfgang König, Technische Universität Berlin.
  Support: Large parts of this project were completed during my stay in Berlin in 1998/99, where I was supported by a postdoctoral fellowship of the DFG Graduiertenkolleg "Stochastische Prozesse und probabilistische Analysis", Berlin. Further travelling was supported by the DFG Schwerpunkt "Interagierende stochastische Systeme von hoher Komplexität".

- **Chapter 4:**
  Support: The first part of the project was completed while I was post-doctoral fellow at the DFG Graduiertenkolleg "Stochastische Prozesse und probabilistische Analysis", Berlin, the second part was supported by the DFG Project "Dimensionsspektren für die Super-Brownsche Bewegung".

- **Chapter 5: Coauthors** Donald A. Dawson, Toronto, and
  Klaus Fleischmann, WIAS Berlin.
  Support: The first part of the paper came into existence during my stay in Berlin, I would also like thank the WIAS for support during this time. A visit to Toronto in March/April 2000 was supported by Don Dawson's Max Planck award.

There is a long list of people who took part in this work. First of all I owe thanks to Heinrich v. Weizsäcker who has been supporting me in many ways for more than ten years. I have to thank him for the gentle guidance he provided in these years, for asking and answering innumerable questions, for the great scientific atmosphere in his group and much more. I also owe thanks to David Preiss, not only for being a great PhD-supervisor and for introducing me to geometric measure theory and into the world of mathematical research but also for still taking me seriously afterwards. A big thank you must go to Klaus Fleischmann for introducing me to superprocesses and his great help in making one more step forward — without all this influence this thesis wouldn't have been possible. Also this thesis wouldn't have been possible without the many friends and colleagues who worked next door and always had an open ear for my mathematical and nonmathematical problems: Among them were Helen Joyce, Toby O'Neil, Christoph Lossen, Peter Vieten, Holger Scholl, Andreas Martin, Wolfgang König, Nina Gantert, Jochen Blath, Pascal Vogt — special thanks for the latter two for reading various stages of this thesis. Last but not least I want to thank my parents, who always encouraged me to follow my interests. But most of all I must thank my wife Kirsten — but there are better ways of telling her.

Kaiserslautern, May 2001.
Peter Mörters.

List of publications

W. König and P. Mörters. Brownian intersection local times: Upper tail asymptotics and thick points. *HP Laboratories Technical Report* BRIMS 2001–03, submitted for publication.

D.A. Dawson, K. Fleischmann and P. Mörters. Strong clumping of super-Brownian motion in a stable random catalytic medium. *WIAS Preprint* No. 646, submitted for publication.

P. Mörters and N.R. Shieh. Thin and thick points for branching measure on a Galton-Watson tree. Preprint, submitted for publication.

P. Mörters. How fast are the particles of super-Brownian motion?
To appear in *Probab. Theory rel. Fields.*

P. Mörters. A pathwise version of Spitzer's theorem.
To appear in *Limit Theorems in Probability and Statistics, Balatonlelle, Hungary, 1999. Bolyai Society Math. Studies.*

P. Mörters. The average density of super-Brownian motion.
Ann. Inst. Henri Poincaré, Probabilités et Statistiques **37**, 71–100 (2001).

P. Mörters. Almost sure Kallianpur–Robbins laws for Brownian motion in the plane.
Probab. Theory rel. Fields **118**, 49–64, (2000).

H. Joyce and P. Mörters. A set with finite curvature and projections of zero length.
J. Math. Anal. Appl. **247**, 126–135, (2000).

P. Mörters and N.R. Shieh. Small scale limit theorems for the intersection local time of Brownian motion. *El. Journal Prob.* **4**, Paper 9, 1–23, (1999).

P. Mörters. Average densities, tangent measures and rectifiability.
Periodica Math. Hungarica **37**, 65–79, (1998).

P. Mörters. The average density of the path of planar Brownian motion.
Stoch. Proc. Appl. **74**, 133–149, (1998).

P. Mörters and D. Preiss. Tangent measure distributions of fractal measures.
Math. Annalen **312**, 53–93, (1998).

D. Krieg and P. Mörters. Tangent measure distributions of hyperbolic Cantor sets.
Monatsh. Math. **126**, 313–328, (1998).

P. Mörters. Symmetry properties of average densities and tangent measure distributions of measures on the line. *Adv. Appl. Math.* **21**, 146–179, (1998).

P. Mörters. Average densities and linear rectifiability of measures.
Mathematika **44**, 313–324, (1997).

Overview

In this chapter I want to give a survey over the results of this thesis and embed them into a more general context. The five main chapters of this thesis deal with three principal issues

- the local geometry of irregular sets and measures (Chapters 1 and 2),
- dimension spectra and extremal properties of stochastic processes (Chapters 3 and 4),
- superprocesses in random media (Chapter 5).

In each of the three sections of the overview I start with some general and introductory considerations concerning the subject, state the problems and principal results of the respective chapters and hopefully give some insight into the main mathematical techniques involved. I hope that this overview helps to uncover the red thread in this thesis.

Theorems from the main body of the thesis are announced with their number as they appear in the respective chapter, with the chapter number in front typed in boldface, e.g. Theorem **2.1.1** is Theorem 1.1 in Chapter 2. On the whole I try to stick to a consistent notation. I use the following symbols without further explanation:

$\mathbb{R}$	the real numbers;
ℓ^d, ℓ	the Lebesgue measure, with or without explicit reference to the dimension;
ℓ_I	the Lebesgue measure restricted to the set $I \subset \mathbb{R}^d$;
$U(x,r), B(x,r)$	the open, respectively closed, ball centred in x with radius r;
$\delta_{\{x\}}$ or δ_x	the Dirac measure with unit mass in x,
$\mathcal{M}(U), \mathcal{M}_1(U)$	the space of locally finite (positive) measures, resp. probability measures, on the Borel sets of $U \subset \mathbb{R}^d$ equipped with the vague topology;
o, O	Landau symbols;
$f \sim g$	if $\lim f(x)/g(x) = 1$;
$f \asymp g$	if there are $0 < c \le C < \infty$ with $cf(x) \le g(x) \le Cf(x)$;
$f \approx g$	informally f is approximately equal to g.

1. The local geometry of irregular sets and measures

For many centuries mathematicians were focusing their attention almost exclusively on smooth and geometrically regular objects, on objects which may be locally approximated by linear objects. Only in the last decades rough objects or objects with an, in some sense, irregular geometry have attracted a huge amount of attention in mathematics. Motivation for this development is coming from several sources, among them are

- the discovery that such objects may have a *rich mathematical structure* and their study opens the door to new mathematical ideas and insights,
- the observation that irregular objects come up naturally in the study of *random phenomena*: the most natural requirements on the distribution of random processes we get processes with wild and irregular sample paths,

- the insight that the *influence* of irregular objects (like noise, random media) can change the qualitative behaviour of mathematical systems and may lead to completely different types of behaviour of such systems,
- the fact that *applications* require to deal with mathematical models for irregular objects, like noise, nonsmooth boundaries, or porous media,
- the discovery, supported by computer simulations, that irregular or fractal objects may have a great *aesthetic value* of their own.

One of the pioneers of this development was A.S. Besicovitch, who in the first half of the 20th century studied general (and hence typically nonsmooth) sets in Euclidean spaces by measure theoretic means. He is the founder of the mathematical field of *geometric measure theory.* The principal results of this field allow us to distinguish between regular and irregular sets and thus improve our understanding of both classes, for a closer look at this see Subsection 1.4 below.

The first two chapters of this thesis are part of an attempt to improve our understanding of the local geometry of irregular sets. We assume for the analysis that the irregular sets are equipped with a natural measure which plays the rôle of a *uniform distribution* on the set. This may (but need not) be the α-dimensional Hausdorff measure on the set, if this is positive and finite for some α. The motivating question is whether, under minimal assumptions, the local geometry of such a set can be arbitrarily *bad*, or whether there is some regularity even in almost completely general sets, making arbitrarily bad behaviour impossible. To deal with this question we investigate a general concept from fractal geometry, the *tangent measure distributions.*

Tangent measure distributions are local approximations to irregular sets and one way to understand the main result of Chapter 1 is that the tangent measure distributions at a point x can be seen as the best *stochastically self-similar approximations* to a fractal measure at a point x. This may be compared to tangent spaces being the best *linear approximation* to a smooth manifold at a point x. To explain the concept briefly, we start with a closer look at the problem of defining stochastically self-similar measures.

1.1. Self-similar random measures. To motivate the results of Chapter 1 I describe in this section a natural definition of self-similarity for random elements of $\mathcal{M}(\mathbb{R}^d)$, which is due to Ulrich Zähle [**Zäh88**].

The heuristic idea of (statistical) self-similarity of a random measure is the following: if an observer looks at the measure through a circular window and renormalises size and mass of his observation, he cannot decide from the distribution of this renormalised observation about the original size and position of the window. A first idea to define self-similarity of a random measure therefore suggests that the distribution of the random measure should be invariant with respect to the group of rescaling operators

$$S_\lambda^\alpha : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d) \ , \ S_\lambda^\alpha \nu(E) = \frac{\nu(\lambda E)}{\lambda^\alpha} \, ,$$

where $\alpha \geq 0$ is a fixed renormalisation parameter, which should be linked to the dimension of the random measure, and with respect to the group of shift operators

$$T^u : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d) \ , \ T^u \nu(E) = \nu(u+E) \, .$$

Unfortunately, it turns out that there are no nontrivial examples of probability distributions on $\mathcal{M}(\mathbb{R}^d)$ that are invariant with respect to these operators. To obtain a meaningful definition one has to use non-finite distributions on $\mathcal{M}(\mathbb{R}^d)$.

Definition: A σ-finite distribution $\mathbb{Q}$ on $\mathcal{M}(\mathbb{R}^d)$ with locally finite intensity measure (or expectation) $\Lambda = \int \nu \, \mathbb{Q}(d\nu)$ is called *α-scale invariant* if

- for every vector $u \in \mathbb{R}^d$ we have $\mathbb{Q} = \mathbb{Q} \circ T^u$,
- for every $\lambda > 0$ there is $\gamma(\lambda) > 0$ such that $\mathbb{Q} = \gamma(\lambda)\mathbb{Q} \circ S_\lambda^\alpha$.

Then we automatically have $\gamma(\lambda) = \lambda^{d-\alpha}$.

Example: Let $\{B_t\}$ be a two-sided linear Brownian motion with $B_0 = 0$ and L_t^x a continuous version of the local time at level x. As L_t^x is an increasing function in t it can be interpreted as a locally finite random measure on the real line, supported by the set $\{t : B_t = x\}$. This measure L^x is the local time measure at level x. A 1/2-scale invariant distribution $\mathbb{Q}$ may defined by $\mathbb{Q}(M) = \mathbb{E}\int_{-\infty}^{\infty} 1_M(\mathsf{L}^x)\, dx$ for Borel sets $M \subset \mathcal{M}(\mathbb{R}^d)$. In other words, choose a level x according to Lebesgue measure, independently from the motion, and study the local time measure at the random level x.

The drawback of the definition above is of course the non-finiteness of the distributions $\mathbb{Q}$. One can, however, go back to probability distributions by conditioning: Heuristically, if we choose a measure X according to the distribution $\mathbb{Q}$ and, given X, choose a point x according to X, then x is Λ-distributed and we may ask for the distribution of the measure X given the point x. More formally, disintegration yields the existence of a family $\{\mathbb{P}^x : x \in \mathbb{R}^d\}$ of probability distributions on $\mathcal{M}(\mathbb{R}^d)$ such that for all $M \subset \mathcal{M}(\mathbb{R}^d)$ and $A \subset \mathbb{R}^d$ Borel,

$$\int_M X(A)\, \mathbb{Q}(dX) = \int_A \mathbb{P}^x(M)\, \Lambda(dx)\,.$$

Using the translation invariance of $\mathbb{Q}$ we observe that Λ is a constant multiple of Lebesgue measure and, more importantly, that there exists a probability distribution $\mathbb{P}$ on $\mathcal{M}(\mathbb{R}^d)$ such that

$$\mathbb{P}^x = \mathbb{P} \circ T^x \text{ for Lebesgue-almost all } x\,.$$

$\mathbb{P}$ is called the *Palm distribution associated with* $\mathbb{Q}$. A direct formula for $\mathbb{P}$ is easy to deduce: Choose a Borel set $B \subset \mathbb{R}^d$ of positive and finite Lebesgue measure, then

$$\mathbb{P}(M) = \frac{1}{\Lambda(B)} \int\int_B 1_M(T^x X)\, X(dx)\, \mathbb{Q}(dX) \text{ for } M \subset \mathcal{M}(\mathbb{R}^d) \text{ Borel.}$$

Palm distributions appear in many areas of probability, for example in queuing theory or in the theory of point processes, where they play the rôle of conditional distributions of stationary point processes given a point at the origin. The heuristics in our case is that Palm distributions are probability distributions on $\mathcal{M}(\mathbb{R}^d)$ that have the origin as a typical point of their samples.

It is natural to try and build the theory of statistically self-similar measures on the Palm distribution $\mathbb{P}$ alone, as it is the probabilistic object. More precisely, one defines a probability distribution $\mathbb{P}$ to be the law of an α-self-similar random measure if it is the Palm distribution of an α-scale invariant $\mathbb{Q}$ and hopes to be able to reformulate the definition equivalently without any reference to $\mathbb{Q}$. In order to do this one has to find out how one can see from $\mathbb{P}$ whether it is the Palm distribution of a translation invariant distribution $\mathbb{Q}$, and, if this is the case, how one can seen from $\mathbb{P}$ whether the underlying distribution $\mathbb{Q}$ is α-scale invariant. These questions were settled in papers by Mecke and by U. Zähle.

THEOREM

(i) [**Mec67**] *A probability distribution* $\mathbb{P}$ *on* $\mathcal{M}(\mathbb{R}^d)$ *is the Palm distribution of a translation invariant distribution* $\mathbb{Q}$ *with nonzero intensity measure* $\Lambda \in \mathcal{M}(\mathbb{R}^d)$ *if and only if* $\mathbb{P}$ *does* *not charge the zero measure and the Palm formula*

$$\iint G(\nu,u)\,\nu(du)\,\mathbb{P}(d\nu) = \iint G(T^u\nu,-u)\,\nu(du)\,\mathbb{P}(d\nu) \tag{1}$$

holds for all Borel functions $G:\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d \longrightarrow [0,\infty)$.

(ii) [**Zäh88**] $\mathbb{P}$ *is the Palm distribution of an* α*-scale invariant measure* $\mathbb{Q}$ *if and only if in addition to the above we have* $\mathbb{P}=\mathbb{P}\circ S_\lambda^\alpha$ *for every* $\lambda>0$.

(iii) [**Zäh88**] *If* $\mathbb{P}$ *is the Palm distribution of an* α*-scale invariant measure* $\mathbb{Q}$ *with non-zero intensity measure, then* α *is the carrying dimension of* $\mathbb{P}$*-almost every measure.*

The following definition is now natural, see [**Zäh88**].

Definition: A probability distribution $\mathbb{P}$ on $\mathcal{M}(\mathbb{R}^d)$ is the distribution of an α*-self similar random measure* if

(i) $\mathbb{P}$ satisfies the Palm formula (1) for all Borel functions $G:\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d\longrightarrow[0,\infty)$,
(ii) $\mathbb{P}$ satisfies $\mathbb{P}=\mathbb{P}\circ S_\lambda^\alpha$ for every $\lambda>0$.

In our **example** from above it is easy to check that the Palm distribution $\mathbb{P}$ associated with $\mathbb{Q}$ is just the distribution of the local time measure L^0 at level 0 and hence L^0 is a (1/2)-self-similar random measure.

I now explain, how deterministic self-similar sets may be included in our framework by providing them with a very sparse randomization. Recall the definition of a deterministic self-similar set: Let $S_1,\dots,S_n$ be a family of contracting similitudes on $\mathbb{R}^d$ with contraction ratios $r_1,\dots,r_n$ and define α to be the unique solution of $\sum_{i=1}^n r_i^\alpha=1$. Then there is a unique nonzero compact set K such that

$$K=S_1(K)\cup\ldots\cup S_n(K)\,.$$

We are assuming that there is no overlap of the different parts $S_i(K)$, so that the so-called open-set condition holds. Then α is the Hausdorff dimension of K and a canonical probability measure μ is defined on $K\subset\mathbb{R}^d$ by

$$\mu\big(S_{x_1}\circ\cdots\circ S_{x_k}(K)\big)=r_{x_1}^\alpha\cdots r_{x_k}^\alpha \quad \text{for all } x_1,\dots,x_k\in\{1,\dots,n\}\,.$$

Definition: Let $\mu\in\mathcal{M}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$. Define the family of measures $\{\mu_{x,r}\,:\,r>0\}\subseteq\mathcal{M}(\mathbb{R}^d)$, the *enlargements of* μ *about* x, by

$$\mu_{x,r}(A):=\mu(x+rA):=\mu\big(\{x+ra\,:\,a\in A\}\big) \quad \text{for all Borel sets } A\subset\mathbb{R}^d.$$

For every $x\in\mathbb{R}^d$ and $\alpha\geq 0$ we define probability distributions $\mathbb{P}_\varepsilon^x:=\mathbb{P}_\varepsilon$ on $\mathcal{M}(\mathbb{R}^d)$ by

$$\mathbb{P}_\varepsilon(M):=\frac{1}{\log(1/\varepsilon)}\int_\varepsilon^1 1_M\Big(\frac{\mu_{x,r}}{r^\alpha}\Big)\,\frac{dr}{r} \quad \text{for Borel sets } M\subset\mathcal{M}(\mathbb{R}^d).$$

Note that the choice of the random scales according to the multiplication invariant measure dr/r is well-adapted to the geometry of self-similar sets. The following theorem is due to Graf.

THEOREM [**Gra95**] *Let* μ *be the canonical measure on a deterministic self-similar set as above and* α *its dimension. Then, for* μ*-almost every* x, *the distributions* $\mathbb{P}_\varepsilon$ *converge weakly to the distribution* $\mathbb{P}$ *of an* α*-self-similar random measure, which is the canonical randomization of* μ.

1.2. Tangent measure distributions. The randomization of the self-similar sets in the last paragraph was based on a procedure of zooming into the set around the point x. Hence the self-similar random measures one obtains are local approximations to the set (which, however, in the case of self-similar sets look essentially like the original set). It is worth noting that the definition of the probability distributions $\mathbb{P}_\varepsilon$ makes sense for arbitrary locally finite measures μ and hence the limit points of $\mathbb{P}_\varepsilon$ are in general local approximations to μ around x.

Of course, we cannot generally expect $\varepsilon \mapsto \mathbb{P}_\varepsilon$ to converge as $\varepsilon \downarrow 0$ as in Graf's Theorem, but mild assumptions ensure that for some critical α the family is uniformly tight and nontrivial limit points exist. These limit points are called the α-dimensional *tangent measure distributions* of μ at x. This concept was first introduced by Bandt [**Ban92**], but closely related ideas had been around for some time, see e.g. Furstenberg [**Fur70**]. The word *tangent measure* is justified by an intimate relationship with the measure theoretic concept of tangent measures, see [**Mat95**] for an excellent treatment of this theory, and Subsection 1.4 for some more comments.

Tangent measure distributions generalize the concept of tangent spaces, indeed if μ is the surface measure on a differentiable α-manifold M, the only nontrivial tangent measure distribution at $x \in M$ is the Dirac distribution concentrated at a constant multiple of Lebesgue measure on the tangent space at x. To make such a generalization valuable, it is necessary that for a large class of measures, the tangent measure distributions retain some of the beautiful properties of tangent spaces. In fact, if we want to include the canonical measure on self-similar sets in our class, by Graf's Theorem, the best property we can hope for is α-self similarity. Very surprisingly, it turns out that such an optimal result holds for *every* measure. This is the main result of Chapter 1.

THEOREM **1.2.12** *Let $\alpha \geq 0$ and let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be any measure. Then at μ-almost all points x every α-dimensional tangent measure distribution is the distribution of an α-self similar random measure.*

Remarks:

(i) The statement given in Theorem **1.2.12**, is slightly stronger due to the choice of a weaker type of convergence in the definition of a tangent measure distribution. This convergence ensures that tangent measure distributions always exist.

(ii) In [**Mör98a**] and [**Mör95**] a weaker version of this result was shown. This result is restricted to measures on the real line and makes stronger assumptions on their regularity. The methods of these papers use the order structure of the real line and explicit calculations, which do not extend to the general situation.

In the remainder of this section I give a very rough sketch of parts of the proof of this theorem. The scaling invariance property $\mathbb{P} = \mathbb{P} \circ S_\lambda^\alpha$ follows almost directly from the definition. The surprising fact is that the limit points $\mathbb{P}$ satisfy the Palm formula (1).

Approximate the left hand side in the Palm formula (1) by

$$\iint G(\nu, u)\, \nu(du)\, \mathbb{P}_\varepsilon(d\nu) = \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \int G\Big(\frac{\mu_{x,r}}{r^\alpha}, \frac{y-x}{r}\Big)\, \frac{\mu(dy)}{r^\alpha}\, \frac{dr}{r}$$

and the right hand side by

$$\begin{aligned} &\iint G(T^u\nu, -u)\, \nu(du)\, \mathbb{P}_\varepsilon(d\nu) \\ &= \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \int G\Big(T^{\frac{y-x}{r}}\Big(\frac{\mu_{x,r}}{r^\alpha}\Big), \frac{x-y}{r}\Big)\, \frac{\mu(dy)}{r^\alpha}\, \frac{dr}{r} \\ &= \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \int G\Big(\frac{\mu_{y,r}}{r^\alpha}, \frac{x-y}{r}\Big)\, \frac{\mu(dy)}{r^\alpha}\, \frac{dr}{r}. \end{aligned}$$

Hence, writing $H(x,y,r)=G\big(\frac{\mu_{x,r}}{r^\alpha},\frac{y-x}{r}\big)$, we have to prove that

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\log(1/\varepsilon)}\int_\varepsilon^1\int\frac{H(x,y,r)-H(y,x,r)}{r^\alpha}\,\mu(dy)\,\frac{dr}{r}=0\,.$$

The key step in our proof is an "ergodic theorem", which proves this fact under fairly general conditions on the mapping $H:\mathbb{R}^d\times\mathbb{R}^d\times(0,\infty)\to(0,\infty)$. I hope that this is of some independent interest.

LEMMA **1.3.1** *Whenever a measure $\mu\in\mathcal{M}(\mathbb{R}^d)$, a Borel measurable function $H:\mathbb{R}^d\times\mathbb{R}^d\times(0,\infty)\longrightarrow[0,1]$ and $0<\alpha<d$ satisfy, for some constants $R,A,B\in(0,\infty)$, the conditions*

(i) *$H(x,y;r)=0$, whenever $\|x-y\|\geq Rr$,*
(ii) *$H(x,y;r)=0$, whenever $\mu\big(B(x,5Rr)\big)\geq Ar^\alpha$, and*
(iii) *for all $x,\widetilde{x},y,\widetilde{y}\in\mathbb{R}^d$ and $0<r<\infty$,*

$$|H(x,y;r)-H(\widetilde{x},\widetilde{y};r)|\leq\frac{B}{r}\Big(\|x-\widetilde{x}\|+\|y-\widetilde{y}\|\Big)\,,$$

then, for μ-almost every $x\in\mathbb{R}^d$,

$$\begin{aligned}&\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T\int\frac{H(x,y;e^{-t})-H(y,x;e^{-t})}{e^{-t\alpha}}\,\mu(dy)\,dt\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{\log(1/\varepsilon)}\int_\varepsilon^1\int\frac{H(x,y;r)-H(y,x;r)}{r^\alpha}\,\mu(dy)\,\frac{dr}{r}=0\,.\end{aligned}\tag{2}$$

Remarks:

- The name ergodic theorem is justified by the observation that (2) states that a long time average converges to a spatial average (which in this case is zero).
- The statement of the lemma becomes false if we insert any absolute value signs in our integral: the correctness of the statement depends on some quite subtle cancellation properties.

Outline of the proof: To get some flavour of the proof, we illustrate its general setup. We first discretize the problem in a dyadic grid and show that it suffices to prove that

$$\lim_{n\to\infty}\frac{1}{n}\int\sum_{k=0}^{n-1}H_k(x,y)\,d\mu(y)=0\text{ for }\mu\text{-almost all }x\,,$$

where

$$H_k(x,y)=1_{Q_k^x}(y)\frac{H(x,y;2^{-k})-H(y,x;2^{-k})}{2^{-k\alpha}}$$

and Q_k^x is the dyadic cube of sidelength 2^{-k} containing x. An interesting subtlety here is that, in order to avoid a concentration of μ-mass near the boundary of the cubes, we have to include an averaging procedure with respect to shifts of the measure in the proof of the approximation, we neglect this aspect in this outline.

To prove the convergence we use an L^2-estimate. Writing, for every x,

$$\varphi_x(E)=\sum_{k=0}^{n-1}\frac{\ell^d(E\cap Q_k^x\setminus Q_{k+1}^x))}{\ell^d(Q_k^x\setminus Q_{k+1}^x)}\int_{Q_k^x}H_k(x,y)\,\mu(dy)\,,$$

enables us to express the term in question as

$$\int \Big[\frac{1}{n}\int \sum_{k=0}^{n-1} H_k(x,y)\,\mu(dy)\Big]^2\,\mu(dx) = \frac{1}{n^2}\int \Big(\varphi_x(\mathbb{R}^d)\Big)^2\,\mu(dx)\,.$$

The fact that φ_x is a signed measure allows us to use a suitable partition of $\mathbb{R}^d$ into sets $\{G(I)\,:\,I\in\mathcal{I}\}$ and write

$$\frac{1}{n^2}\int \Big(\varphi_x(\mathbb{R}^d)\Big)^2\,\mu(dx) = \frac{1}{n^2}\sum_{I\in\mathcal{I}}\sum_{J\in\mathcal{I}}\int \varphi_x(G(I))\varphi_x(G(J))\,\mu(dx)\,. \tag{3}$$

The partition is chosen in such a way that the index set $\mathcal{I}$ is a family of good dyadic cubes of sidelength at least $2^{-(n-1)}$, which allow some control on the mass from below. Every set $G(I)$ is the finite union of those dyadic subcubes of I of sidelength $2^{-(n-1)}$, which are not contained in any good strict subcube of I. The proof now proceeds by giving careful individual estimates for the summands on the right hand side of (3), these estimates are the core of the proof.

To convey a little bit of the flavour of these estimates, we give a very rough sketch of the idea behind the main individual estimates. Let q be the smallest integer bigger than $\sqrt{n}$ and let $i\in\{0,\ldots,q-1\}$. Assume $I\in\mathcal{I}$ is a large cube, say of sidelength 2^{-k} for some $iq\le k\le (i+1)q-1$, and denote the collection of these cubes by $\mathcal{I}_i$. We also take a smaller cube K from the class $\mathcal{K}_{i+1}$ consisting of the maximal cubes $K\in\mathcal{I}$ of sidelength 2^{-k} for some $(i+1)q\le k\le n-1$.

Typically, $K\in\mathcal{K}_{i+1}$ is situated far away from I relative to its own size. Now K can be partitioned into sets $\{G(J)\,:\,J\in\mathcal{I}\,,J\subset K\}$. We show that essentially all $\varphi_x(G(J))$ are small if $x\notin K$ and also that typically $\varphi_x(G(I))$ is almost constant for all $x\in K$. Hence

$$\begin{aligned}
&\sum_{I\in\mathcal{I}}\sum_{K\in\mathcal{K}_{i+1}}\sum_{J\subset K}\int \varphi_x(G(I))\varphi_x(G(J))\,\mu(dx)\\
&\quad\approx\ \sum_{I\in\mathcal{I}}\sum_{K\in\mathcal{K}_{i+1}}\sum_{J\subset K}\int_K \varphi_x(G(I))\varphi_x(G(J))\,\mu(dx)\\
&\quad\approx\ \sum_{I\in\mathcal{I}}\varphi_x(G(I))\sum_{K\in\mathcal{K}_{i+1}}\int_K\sum_{J\subset K}\varphi_x(G(J))\,\mu(dx)\\
&\quad=\ \sum_{I\in\mathcal{I}}\varphi_x(G(I))\sum_{K\in\mathcal{K}_{i+1}}\int_K\varphi_x(K)\,\mu(dx).
\end{aligned}$$

But $\int_K\varphi_x(K)\,\mu(dx)=0$ by the anti-symmetry of H_k. The errors made in this approximation are allowed to be of order $\mathsf{o}(n^{3/2})$ as we have to sum over all $i\in\{0,\ldots,q-2\}$ for $q\approx\sqrt{n}$ and we divide by n^2. □

1.3. Examples. In this subsection I want to survey some interesting examples of tangent measure distributions, which are not mentioned in the main body of the thesis. Readers mainly interested in getting an overview over the main part are recommended to skip this subsection and proceed to Subsection 1.4.

1.3.1. *Gibbs and conformal measures on hyperbolic Cantor sets.* I give a brief survey of a result obtained in Krieg and Mörters [**KrM98**], where the tangent measure distributions of two natural measures on hyperbolic Cantor-sets are studied. This example makes use of the *thermodynamic formalism* of Bowen and Ruelle, see [**Fal99**] for an introduction.

Hyperbolic Cantor-sets, or cookie-cutters, are examples of self-similar sets based on *non-linear* contractions. The results relate the concept of tangent measure distributions to the concept of *limit models* of a hyperbolic Cantor set introduced by Bedford and Fisher [**BeF96**] and [**BeF97**]. From our point of view, the interest of the example lies in the fact that in typical cases the tangent measure distributions are natural invariant measures (with respect to a certain semiflow) supported by a nontrivial set of measures with a higher degree of smoothness than the underlying measure.

I first give the definition of a hyperbolic Cantor set. Take $0 < \gamma \le 1$ and an open interval J containing the unit interval $I := [0,1]$. Let $\varphi_0, \varphi_1 : J \to J$ be $\mathcal{C}^{1+\gamma}$-mappings, i.e. the derivatives φ_0', φ_1' are γ-Hölder-mappings, such that

- $\varphi_0(0) = 0$, $\varphi_1(1) = 1$ and $\varphi_0(J) \cap \varphi_1(J) = \emptyset$,
- there are $0 < \alpha, \beta < 1$ and $c > 0$ such that $\alpha^n < (\varphi_{x_0} \circ \cdots \circ \varphi_{x_{n-1}})'(a) < c\beta^n$ for all $a \in J$, $(x_0, \dots, x_{n-1}) \in \{0,1\}^n$ and $n \ge 1$.

Then there is a unique nonempty, compact set $C \subseteq J$ such that

$$C = \varphi_0(C) \cup \varphi_1(C)\,,$$

which is called *hyperbolic (or cookie–cutter) Cantor set* associated to φ_0, φ_1.

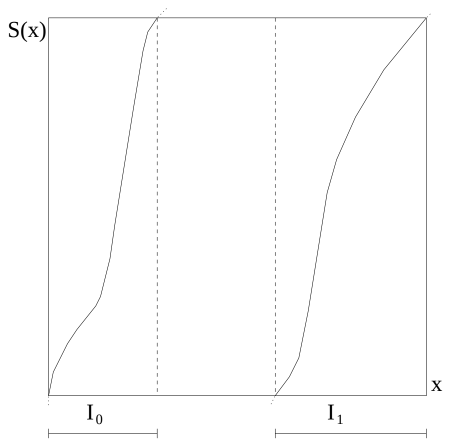

FIGURE 1. Expanding map S for a cookie-cutter.

Abbreviate $I_x^n := I_{x_0 \dots x_{n-1}} := \varphi_{x_0} \circ \cdots \circ \varphi_{x_{n-1}}(I)$ and similarly define J_x^n and $J_{x_0 \dots x_{n-1}}$. On the open set $J_0 \cup J_1$ we can define a $\mathcal{C}^{1+\gamma}$–mapping S by $S(a) = \varphi_i^{-1}(a)$ for $a \in J_i$. C is the maximal invariant set for the expanding map S.

We introduce two canonical measures ν and $\mu \in \mathcal{M}(\mathbb{R})$ supported by the Cantor set C, the *Gibbs* and the *conformal* measure. The idea of the Gibbs measure is to choose the measure of an nth step construction set I_x^n on a logarithmic scale proportionally to the contraction ratio of this set. Then the factor of proportionality is necessarily the dimension of C. For a precise formulation, let $\Sigma^+ = \prod_{i=0}^{\infty}\{0,1\}$ and denote by $\pi : \Sigma^+ \to C$ the canonical address map given by

$$\{\pi(x)\} = \bigcap_{n=1}^{\infty} I_x^n\,.$$

Observe that $\pi^{-1} \circ S \circ \pi$ is the left–shift on $\Sigma^+ = \prod_{i=0}^{\infty}\{0,1\}$. Let $\Sigma^- = \prod_{i=-1}^{-\infty}\{0,1\}$ be the space of left–sided sequences and $\Sigma = \Sigma^- \times \Sigma^+$ the two sided sequences. The thermodynamic formalism shows that there is $\eta > 0$, and a unique shift-invariant and ergodic probability measure $\widetilde{\nu}$ on Σ such that, for all $x \in \Sigma^+$ and $n \geq 1$,

$$\eta < \widetilde{\nu}\{y \in \Sigma \,:\, y_0 = x_0, \dots , y_{n-1} = x_{n-1}\} \left[(S^n)'\big(\pi(x)\big)\right]^s < \eta^{-1},$$

where $s \in (0,1)$ is the unique zero of the *pressure function*

$$P(s) = \lim_{n\to\infty} \frac{1}{n} \log\Big(\sum_{x\in\{0,1\}^n} \big\| \big(\varphi_{x_0} \circ \cdots \circ \varphi_{x_{n-1}}\big)'(y)\big\|^s \Big)\,.$$

Here $\|\cdot\|$ is the supremum norm. The number s is the Hausdorff dimension of C. We can define a measure ν on C by

$$\nu(A) = \widetilde{\nu}(\Sigma^- \times \pi^{-1}(A)) \;\text{ for all } A \subseteq C \text{ Borel.}$$

ν is an S-invariant ergodic probability measure on C and is called the *Gibbs measure on C.*

By μ we shall denote the *s-Hausdorff measure* restricted to C. The conformal transform property of s-Hausdorff measure implies that

$$\mu(S(E)) = \int_E |S'|^s \, d\mu \;\text{ for } E \subseteq J_0 \text{ or } E \subseteq J_1 \text{ Borel.}$$

Therefore μ is called the *conformal measure* on C. It is well–known that the measures μ and ν are mutually absolutely continuous and the density is given by

(4) $$\frac{d\nu}{d\mu}(x) = g(x) := \lim_{n\to\infty} L^n(1)(x)\,,$$

where L^n is the nth iterate of the *Perron-Frobenius Operator* $L : C(X) \to C(X)$ defined by

$$L(f)(x) := \sum_{i=0}^{1} |\varphi_i'(x)|^s f(\varphi_i(x))\,.$$

For $\mathcal{C}^{1+\gamma}$-hyperbolic Cantor sets C and $\widetilde{C}$ with expanding maps S and $\widetilde{S}$ the corresponding dynamical systems (C,S) and $(\widetilde{C},\widetilde{S})$ are called *$\mathcal{C}^{1+\gamma}$-conjugate* (resp. *$\mathcal{C}^1$-conjugate*) if there is an orientation preserving $\mathcal{C}^{1+\gamma}$ (resp. $\mathcal{C}^1$) diffeomorphism $\Phi : I \to I$ such that $\widetilde{S} \circ \Phi = \Phi \circ S$.

We now introduce a complete invariant for $\mathcal{C}^1$-conjugacy of hyperbolic Cantor sets, the scaling function, which is due to Sullivan, see [**Sul87**] and [**BeF96**]. For this purpose define for a positive integer n a mapping $R^{(n)} : \Sigma^- \to \mathbb{R}^2$ by

$$R^{(n)}(y) = \frac{1}{|I_{y_{-n}\cdots y_{-1}}|} \cdot (|I_{y_{-n}\cdots y_{-1}0}|, |I_{y_{-n}\cdots y_{-1}1}|).$$

$R^{(n)}(y)$ is the vector of the relative size of the left and right subparts of $I_{y_{-n}\cdots y_{-1}}$. It is important to note that by increasing n we are *not* zooming down a ray of the tree Σ^+ (from a Cantor set to one of its two next stage subsets) but rather *jump* from one Cantor set to the next one by choosing either in the left or right part of the global Cantor set the subset of the appropriate stage, which is in the same relative position as the previous one in the full tree. In other words we are moving along a ray of the *dual tree* Σ^-. Hence it is plausible, and proved rigorously in [**BeF96**], that, for every $y \in \Sigma^-$, there exists

$$R(y) := \lim_{n\to\infty} R^{(n)}(y).$$

The resulting γ-Hölder continuous function $R : \Sigma^- \to \mathbb{R}^2$ is called the *scaling function* of C.

If (C, S) and $(\widetilde{C}, \widetilde{S})$ are $\mathcal{C}^1$-conjugate, it is easy to see that the corresponding shift-invariant measures $\widetilde{\nu}$ and the dimensions of C and $\widetilde{C}$ coincide. Sullivan showed that for two hyperbolic Cantor sets C and $\tilde{C}$ the dynamical systems (C, S), $(\tilde{C}, \tilde{S})$ are $\mathcal{C}^1$-conjugate if and only if their scaling functions coincide. In other words the scaling function is a complete invariant for $\mathcal{C}^1$-conjugacy. Bedford and Fisher show in [**BeF96**] that this conjugacy automatically has the same degree of smoothness as the Cantor sets themselves.

Following [**BeF96**] we now introduce the limit models. For every $y = (y_i) \in \Sigma^-$ we define an affine mapping $A_n^y : I_{y_{-n} \cdots y_{-1}} \to I$ by

$$A_n^y(a) = \frac{a - \varphi_{y_{-n}} \circ \cdots \circ \varphi_{y_{-1}}(0)}{|I_{y_{-n} \cdots y_{-1}}|},$$

then let $\Phi_n^y = A_n^y \circ \varphi_{y_{-n}} \circ \cdots \circ \varphi_{y_{-1}}$. It turns out that the limit $\Phi^y = \lim_{n \to \infty} \Phi_n^y$ exists in the $\mathcal{C}^1$-norm $\|f\|_{\mathcal{C}^1} = \|f\|_\infty + \|f'\|_\infty$. For $y \in \Sigma^-$ denote $C^y = \Phi^y(C)$. The mappings $\Phi^y : I \to I$ define $\mathcal{C}^{1+\gamma}$–conjugacies between C and C^y and the family $(C^y)_{y \in \Sigma^-}$ is called the family of *ratio Cantor sets* or *limit models* of C. We denote by $\pi^y = \Phi^y \circ \pi$ the address map, by μ^y the s-Hausdorff measure and by ν^y the Gibbs measure on C^y. It should be noted that the mapping $y \mapsto C^y$ is a natural set-valued extension of the concept of the scaling function.

$\mathcal{C}^1$-conjugate Cantor sets have the same limit models. The sets C^y are hyperbolic Cantor sets of the same Hausdorff dimension and (at least) the same degree of smoothness as C. Indeed, we note the important fact that *the limit models of C always have the highest degree of smoothness which occurs in the $\mathcal{C}^1$-conjugacy class of C.*

We may consider the components R_0 and R_1 of the scaling function as functions on the domain Σ by defining $R_i((y, x)) = R_i(y)$, for $y \in \Sigma^-$ and $x \in \Sigma^+$. We define a positive, continuous function on $\Sigma = \Sigma^- \times \Sigma^+$ by

$$f(\underline{x}) = -\log R_{x_0}(y) \text{ for } \underline{x} = (y, x) \in \Sigma.$$

With $\sigma : \Sigma \to \Sigma$ denoting the left shift, the *flow space* Σ_f *under* f is defined as

$$\Sigma_f = \Sigma \times [0, \infty) / \equiv$$

for the equivalence relation $\equiv$ generated by $(\underline{x}, f(\underline{x})) \equiv (\sigma\underline{x}, 0)$. There is a natural semiflow $\Phi = \{\Phi_t : t \geq 0\}$ on Σ_f mapping the class of $(\underline{x}, u)$ onto the class of $(\underline{x}, u + t)$. Φ is called the *flow under the function* f. Recall that we defined $\mu_{x,s}(A) = \mu(x + sA)$. For μ-almost all $a \in C$ there is a continuous mapping

$$\widetilde{M} : \Sigma_f \longrightarrow \mathcal{M}(\mathbb{R})$$

mapping the class of $((y, x), t)$ onto $\nu \in \mathcal{M}(\mathbb{R})$ such that, for all $r > 0$, there exists a constant $\lambda_r > 0$ such that, for all t with $e^{-t} \leq \lambda_r$,

$$\nu(B) = \frac{(\mu^y)_{\pi^y(x), e^{-t}}}{e^{-ts}}(B) \text{ for all } B \subset B(0, r) \text{ Borel.}$$

The main result of [**KrM98**] is a formula for the tangent measure distributions of both Hausdorff and Gibbs measure on a hyperbolic Cantor set in terms of the scaling function and the limit models defined above. Define, for $r > 0$, the σ-field $\mathcal{B}_r$ as generated by the restriction mapping $\mu \mapsto \mu_{B(0,r)}$ with $\mu_{B(0,r)}(A) = \mu(B(0, r) \cap A)$ for $A \subset \mathbb{R}$ Borel.

THEOREM [**KrM98**]

(i) **Tangent measure distributions of the conformal measure:** *At μ-almost every point the measure μ has exactly one s-dimensional tangent measure distribution P, which does not depend on the point. P is the image under $\widetilde{M}$ of the (Φ_t)–invariant measure on Σ_f induced by the shift-invariant measure $\widetilde{\nu}$ on Σ. More explicitly, for $r > 0$ and $\lambda_r > 0$ as above, the tangent measure distribution P is given by*

$$P(A) = \frac{-1}{\int \log R_{x_0}(y)\, d\widetilde{\nu}((y,x))} \int \int_{\lambda_r R_{x_0}(y)}^{\lambda_r} 1_A \left(\frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s} \right) \frac{d\lambda}{\lambda}\, d\widetilde{\nu}((y,x))$$

for all $A \subseteq \mathcal{M}(\mathbb{R})$ in the σ-field $\mathcal{B}_r$.

(ii) **Tangent measure distributions of the Gibbs measure:** *At ν-almost every point $a \in C$ the measure ν has exactly one s-dimensional tangent measure distribution P^a. For $r > 0$ and $\lambda_r > 0$ as above the P^a are given by*

$$P^a(A) = \frac{-1}{\int \log R_{x_0}(y)\, d\widetilde{\nu}((y,x))} \int \int_{\lambda_r R_{x_0}(y)}^{\lambda_r} 1_A \left(g(a) \cdot \frac{(\mu^y)_{\pi^y(x),\lambda}}{\lambda^s} \right) \frac{d\lambda}{\lambda}\, d\widetilde{\nu}((y,x))$$

for all $A \subseteq \mathcal{M}(\mathbb{R})$ in the σ-field $\mathcal{B}_r$. Here g is defined in terms of the Perron-Frobenius operator as in (4).

Heuristically, we sample from the tangent measure distribution of the Hausdorff measure on C at a generic point of the hyperbolic Cantor set by

- first choosing a two-sided sequence (y, x) according to the invariant measure $\widetilde{\nu}$,
- picking the limit model belonging to the ray $y \in \Sigma^-$,
- picking the point $\pi^y(x) \in C^y$ and shifting the limit model such that $\pi^y(x)$ is moved to the origin,
- then rescaling the set with a scale chosen according to the Haar measure on the multiplicative group,
- and finally endowing this set with the Hausdorff measure.

The set thus sampled is an approximation of small neighbourhoods of C around x in a random selection of scales and has (typically) a higher degree of smoothness than the original set C.

1.3.2. *Brownian paths.* Many natural examples of irregular sets arise from random processes, for example the range or level sets of Brownian motion. In this section I briefly discuss the tangent measure distributions of the range of a Brownian motion in $d \geq 2$. It turns out that one can observe a striking difference between the planar (recurrent) and the higher-dimensional (transient) case.

To fix notation let $\{B_t : t \in (-\infty, \infty)\}$ be a two-sided Brownian motion in $\mathbb{R}^d$ for $d \geq 2$ with $B_0 = 0$. Let $I \subset \mathbb{R}$ be an interval. We equip the range of Brownian motion with the *occupation measure* defined by

$$\mu_I(A) = \int_I 1_A(B_s)\, ds\,, \text{ for } A \subset \mathbb{R}^d \text{ Borel.}$$

The tangent measure distributions of μ_I are relatively easy to find in the *transient* case using Brownian scaling. The following result is in the spirit of [**FaX95**] and is made explicit in the diploma thesis [**Wil98**].

THEOREM *Let $d \geq 3$, and $I \subset \mathbb{R}$ be an arbitrary interval. Then, almost surely at μ_I-almost all $x \in \mathbb{R}^d$, the measure μ_I has exactly one 2-dimensional tangent measure distribution P. This distribution P is independent of x and I and coincides with the distribution of the random measure $\mu_{(-\infty,\infty)}$.*

Idea of proof: The **first step** of the proof shows that P is the uniquely determined tangent measure distribution of $\mu := \mu_{(-\infty,\infty)}$ *at the origin.* This can be done quite easily using the well-known fact that the distribution of μ is invariant and ergodic under the (multiplicative) semigroup $\{S^2_\lambda : \lambda > 0\}$ of transformations on $\mathcal{M}(\mathbb{R}^d)$, recall Subsection 1.1. Birkhoff's ergodic theorem shows that, for all $F : \mathcal{M}(\mathbb{R}^d) \to [0,\infty)$ almost surely,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log \varepsilon} \int_\varepsilon^1 F\Big(\frac{\mu_{0,r}}{r^2}\Big)\,\frac{dr}{r} = \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T F \circ S^2_{e^{-t}}(\mu)\, dt = \int F dP,$$

and we get the uniqueness of the tangent measure distributions by selection of a sufficiently rich sequence $F_1, F_2, \ldots$.

Because Brownian motion in $\mathbb{R}^d$, $d \geq 3$, is transient and does not hit points, it is easy to see that the tangent measure distributions of $\mu_{(-\infty,\infty)}$ at the origin and the tangent measure distributions of $\mu_{(-a,b)}$ at the origin coincide for all $a, b > 0$.

To pass in the **second step** from the origin to a typical point one may use several strategies. It can be done by a Markov argument as in [**FaX95**], but I want to present a more flexible approach, based on Palm distributions, which may also be used to study intersections of Brownian paths, see [**MöS99**]. The following lemma can be verified easily by checking (1).

LEMMA [**Mör98b**] *Let $T > 0$ be fixed and pick a random number $X \in [0,T]$ uniformly and independently of the Brownian motion. Then the distribution of the random measure $\mu_{[X-T,X]}$ is a Palm distribution.*

We define the function $G : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \to \{0,1\}$ at (ν, u) as the indicator of the event

the set of tangent measure distributions of ν at u is $\{P\}$.

Observe that G has the shift-invariance property $G(T^u\mu, -u) = G(\mu, 0)$, for all $u \in \mathbb{R}^d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$. We abbreviate $\mu := \mu_{[X-T,X]}$ and use the Palm formula (1), shift-invariance of G and the first step, to see that,

$$\begin{aligned}\iint G(\mu,u)\,\mu(du)\,\mathbb{P}(d\mu) &= \iint G(T^u\mu,-u)\,\mu(du)\,\mathbb{P}(d\mu)\\ &= \iint G(\mu,0)\,\mu(du)\,\mathbb{P}(d\mu) = \mathbb{E}\big\{\mu(\mathbb{R}^d)\big\} = T.\end{aligned}$$

This implies that $G(\mu, u) = 1$ for μ-almost every x almost surely, which is the statement. □

The case of a planar Brownian motion is very different. Due to recurrence of Brownian motion in the plane we restrict attention to bounded intervals I, as otherwise the occupation measure may not be locally finite. In order to get nontrivial limits one has to use a gauge function, which is not of power type, tangent measures are defined with respect to

$$\varphi(r) = r^2 \log(1/r).$$

It turns out that in this case we do *not* have convergence of $\mathbb{P}_\varepsilon$ to a unique tangent measure distribution as in the previous examples. Unfortunately, I do not know what the set of tangent measure distributions looks like. However, if one changes the definition slightly by moving to a stronger type of averaging, one can get uniqueness and an interesting limit.

The choice of the stronger averaging procedures is motivated in [**Fis87**]. Following the classical theory of Hardy and Riesz one can introduce, for all $n \geq 1$, the *order n average* of a function $f : [0,\infty) \to [0,\infty)$ as

$$\lim_{t\uparrow\infty} \frac{1}{\log^{(n-1)}(t)} \int_{\exp^{(n-1)}(0)}^{t} f(s) \, \frac{ds}{\prod_{j=0}^{n-2} \log^{(j)}(s)} \, ,$$

if it exists. Here $g^{(n)}$ stands for the nth iterate of a function g. The order one average is the usual Cesàro average, the order two average is the logarithmic average. Using Wiener's Tauberian Theorem one can show that there is a hierarchical structure in these averages: existence of the order n average implies existence of the order k average, for all $k \geq n$, with the same value, see [**Fis87**].

If μ is a locally finite measure and $n \geq 2$, we define the φ-dimensional *tangent measure distribution of order n* of μ at x by applying order n averaging to the map $t \mapsto \mu_{x,1/t}(A)/\varphi(1/t)$. More precisely, after an easy substitution, the tangent measure distributions of order n of μ at x are the limit points of

$$\frac{1}{\log^{(n-1)}(1/\varepsilon)} \int_{\varepsilon}^{1/\exp^{(n-1)}(0)} \delta_{\left\{\frac{\mu_{x,s}}{\varphi(s)}\right\}} \, \frac{ds}{s^2 \prod_{j=0}^{n-2} \log^{(j)}(1/s)} \, ,$$

as $\varepsilon \downarrow 0$. Results like our Theorem **1.2.12** can be carried over easily to higher order densities using the hierarchy of the averages. The following result is a variant of Theorem 1.2 in [**Mör00**].

THEOREM [**Mör00**] *Let $\mu = \mu_{[0,T]}$ be the occupation measure of a planar Brownian motion run for a finite time interval of arbitrary length $T > 0$ and define the gauge function $\varphi(r) = r^2 \log(1/r)$.*

(i) *Almost surely, there is exactly one φ–tangent measure distribution of order three of μ at the origin. It is equal to the distribution of a random constant multiple of planar Lebesgue measure. The constant is standard exponentially distributed.*
(ii) *Almost surely, there is exactly one φ–tangent measure distribution P of order three at μ–almost every x. P is equal to the distribution of a random constant multiple of planar Lebesgue measure ℓ^2. The constant is gamma distributed with parameter two. More explicitly,*

$$P = \int_0^\infty \delta_{\{a\cdot\ell^2\}} \, ae^{-a} \, da \, .$$

Idea of proof: To move from statement (i) to (ii) we recall that the gamma distribution with parameter two is the distribution of the sum of two independent standard exponentially distributed random variables. Hence we get P as the unique tangent measure distribution of $\mu_{[-S,T]}$ at the origin for all $S, T > 0$. Now we can use the same method as in the higher dimensional case (Palm distribution) to obtain the result for all typical points.

To obtain the result at the origin we need to work much harder. The proof of [**Mör00**] is based on the approximation of occupation times by crossing numbers. We may assume that T is the first exit time from the unit ball by our Brownian motion W, and let $\mu := \mu[0,T]$.

Fix a number $b > 0$ and define $a_n = e^{-bn}$. For every n we define stopping times $t_0 = 0$, $t_{2k+1} = \inf\{t > t_{2k} : |W_t| > a_{n-1}\}$ and $t_{2k} = \inf\{t > t_{2k-1} : |W_t| < a_n\}$. Define the *crossing numbers* as

$$N_n = \max\{k \geq 0 : t_{2k} < T\},$$

the number of inward crossings of the annulus with radii a_{n-1} and a_n. In [**Mör00**, Theorem 3.1] it is shown that for any Borel set $A \subset U(0,1)$, almost surely,

$$\mu\big(a_n A\big) = a_n^2\, N_n b \left[\frac{\ell^2(A)}{\pi} + \mathsf{o}(1)\right] + \mathsf{o}(na_n^2)\ , \text{ as } n \to \infty. \tag{5}$$

This fact is well-known for $A = B(0,1)$, see [**Ray63**], but for the investigation of the tangent measure distributions we need the asymmetric version (5), which requires a more involved proof. With (5) at our disposal we can reduce the problem to the study of the crossing numbers. This is easy, because the process $\{N_n : n \geq 1\}$ is a Markov chain whose transition probabilities are explicitly known, see [**Ray63**]. The key quantity necessary to finish the proof is the *decay of correlations* of $\{N_n\}$. It turns out that, for $0 < n < m$,

$$\frac{\mathbb{E}\{N_n N_m\} - \mathbb{E}\{N_n\}\mathbb{E}\{N_m\}}{\mathbb{E}\{N_n\}\mathbb{E}\{N_m\}} = \frac{n}{m-1} \sim \frac{1}{m} \text{ as } m \to \infty.$$

This type of decay leads, with a simple L^2-estimate, to the statement

$$\lim_{k\to\infty} \frac{1}{\log k} \sum_{n=1}^{k} \frac{1}{n}\frac{N_n}{n} = 1,$$

which together with (5) leads to the result. It may also be seen that convergence of the tangent measure distributions of order two would require a faster (exponential) decay of the correlation of N_n and N_m and hence does not hold.

Finally, we remark that crossing number approximations are a very versatile tool in the investigation of planar Brownian occupation measures, see [**DPR00b**] for more (deeper) applications. □

The previous theorem is closely related to the *Kallianpur-Robbins law* determining the asymptotic behaviour of integrable additive functionals of planar Brownian motion. To uncover this relationship let $A \subset \mathbb{R}^2$ be a bounded Lebesgue continuity set. The *Kallianpur-Robbins law* states that, as $t \to \infty$, the asymptotic distribution of $\mu_{[0,t]}(A)/\log(\sqrt{t})$ is exponential with expectation $\ell^2(A)/\pi$. By Brownian scaling, with $r = t^{-2}$,

$$\frac{\mu_{[0,t]}(A)}{\log(\sqrt{t})} \stackrel{d}{=} \frac{\mu_{[0,1]}(rA)}{r^2\log(1/r)} = \frac{\mu_{0,r}(A)}{\varphi(r)}, \tag{6}$$

with $\mu = \mu_{[0,1]}$. Hence, as $r \downarrow 0$, the asymptotic distribution of $\frac{\mu_{0,r}(A)}{\varphi(r)}$ is also exponential with expectation $\ell(A)/\pi$.

It is one of the central principles of probability theory that statements about the asymptotic distribution of stochastic processes can be turned into almost sure (or pathwise) statements by suitable averaging. Examples start with the law of large numbers, where Cesàro averaging is sufficient, the pathwise central limit theorem of Brosamler and Fisher [**Bro88**], [**Fis87**], where one needs order two averages, and there are plenty of further examples in the literature, e.g. [**BeD93**], [**FaS98**], [**BCC99**], [**Mör01b**].

Our result shows that order *three* averaging is the type of averaging needed to obtain a pathwise Kallianpur-Robbins law for the scales parameter. Another way to interpret this (and other) pathwise laws is the following: Given any typical sample path, the law of $\mu_{[0,1]}(rA)/[r^2\log(1/r)]$ for a random choice of the scale parameter $r > 0$ is asymptotically exponential with expectation $\ell(A)/\pi$. Here the scale parameters $r > 0$ are chosen according to the laws with density

$$\frac{1}{\log\log(1/\varepsilon)}\ 1_{[\varepsilon,1/e]}(r)\ \frac{dr}{r\ \log(1/r)},$$

and we let $\varepsilon \downarrow 0$, which forces us to look at increasingly smaller random scales.

To close this circle of ideas I would like to point out that the rescaling argument (6) we used in order to turn the *long term* formulation of the Kallianpur-Robbins law into a *small scale* formulation uses a different type of scaling for different one-dimensional marginals. Hence, the pathwise law for the scale parameter does *not* automatically imply a pathwise law for the time parameter. Nevertheless, such a law exists and again the order three average is best possible, see [**Mör00**].

1.3.3. *Further examples.* We finish by mentioning some related publications. Arbeiter and Patzschke [**ArP96**] consider a class of random self-similar fractals endowed with the natural self-similar measure μ and show that a unique tangent measure distribution exists at almost all points, which does not depend on the point. They also use a whole family of alternative measures on the set and show that the tangent measure distribution of μ at almost all points with respect to the alternative measure exists uniquely, but is different from the one above. Very recently, Bandt [**Ban00**] has looked more systematically at tangent measure distributions of one measure with respect to another one and discussed some examples from the realm of self-similar sets. The example of Brownian paths discussed here can be extended to intersections of independent Brownian paths equipped with the intersection local times, see [**MöS99**]. Finally there is a large amount of literature about cruder concepts, most notably the *average densities*, which we discuss in more detail in the next subsection. For a survey see [**Fal99**].

1.4. Average densities and rectifiability of measures. In Chapter 2 of this thesis tangent measure distributions are used as a tool to solve a problem posed by Falconer and Marstrand about the rectifiability of measures, see Theorem **2.1.2**. In this subsection I explain the background of this result and outline the main ideas behind the proof.

We start the subsection by recalling some fundamental facts of geometric measure theory. Consider a nonnegative, nonzero Radon measure μ on $\mathbb{R}^d$ and let $\alpha \geq 0$. The measure μ is called *α-rectifiable* if α is an integer and μ is absolutely continuous with respect to α-Hausdorff measure restricted to a countable family of differentiable α-manifolds. In the case of $\alpha = 1$ we say that μ is *linearly rectifiable*. An important problem in geometric measure theory is to find local criteria for the rectifiability of measures.

Many of the classical criteria for rectifiability are linked to the notion of the density of the measure: We define the *lower α-density* of μ at x as the number

$$\underline{d}^\alpha(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(U(x,r))}{r^\alpha},$$

and the *upper α-density* of μ at x as the number

$$\overline{d}^\alpha(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(U(x,r))}{r^\alpha}.$$

It is not hard to show that for an α-rectifiable measure the lower and upper α-density of μ are positive and coincide at μ-almost every point, see e.g. [**Mat95**, 16.2]. Much harder is the question whether a converse statement holds. A crucial step in this direction was accomplished in the early sixties by Marstrand, who showed that the equality

$$0 < \underline{d}^\alpha(\mu, x) = \overline{d}^\alpha(\mu, x) < \infty \ \mu\text{-almost everywhere} \tag{7}$$

implies that α must be an integer, see [**Mar64**] or [**Mat95**, Chapter 14]. If $\alpha = 1$ the question whether (7) implies linear rectifiability was answered earlier by Besicovitch [**Bes38**] in the case of measures in the plane, and by Moore [**Moo50**] in the case of measures in arbitrary Euclidean spaces. The affirmative answer to this question in the much harder case of arbitrary integers α was given in the eighties in a very deep paper by Preiss [**Pre87**], see also [**Mat95**, Chapter 17] for an accessible account of the proof.

The results of Marstrand and Preiss imply that for measures of fractal nature the lower and upper densities cannot coincide. As it would be very desirable to describe the *lacunarity* of a fractal measure about its points by means of a density parameter, Bedford and Fisher introduce in [**BeF92**] the notion of *average densities* into fractal geometry. Applying classical summation techniques of Hardy and Riesz to the density functions they define the *lower* and *upper average α-densities of order two* of μ at x as

$$\underline{D}^\alpha(\mu,x) = \liminf_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\mu(U(x,r))}{r^\alpha}\frac{dr}{r} \ ,$$

and

$$\overline{D}^\alpha(\mu,x) = \limsup_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\mu(U(x,r))}{r^\alpha}\frac{dr}{r} \ .$$

Observing that the choice of a positive upper bound in the integral is irrelevant to the value of the average density we get the following inequalities:

$$\underline{d}^\alpha(\mu,x) \le \underline{D}^\alpha(\mu,x) \le \overline{D}^\alpha(\mu,x) \le \overline{d}^\alpha(\mu,x).$$

Several authors have shown that for measures with some self-similarity properties the lower and upper average densities in the appropriate dimension coincide and define an interesting fractal parameter. Examples include the natural measures on hyperbolic Cantor sets, see [**BeF92**], random self-similar sets, see [**PZM94**], paths of stable processes, see [**FaX95**], intersections of Brownian paths in $\mathbb{R}^3$, see [**MöS99**], and also the states of super-Brownian motion in $\mathbb{R}^d$ for $d \ge 3$, see [**Mör01a**]. However there are also natural examples of measures where this approach fails to give a density parameter, for example in the case of the paths of Brownian motion in the plane, see [**Mör98b**].

It is natural to ask whether one can get statements about the geometric regularity of the measure μ from weaker inequalities than (7) by using the average densities. The examples mentioned above show that we cannot expect to infer regularity of μ from the equality of the lower and upper average densities, but it is interesting to ask for the consequences of equality of the lower and lower average densities or of the upper and upper average densities. This problem was first studied by Falconer and Springer in [**FaS95**] and their results were later improved by Marstrand in [**Mar96**]. They showed the following theorem:

THEOREM [**FaS95**], [**Mar96**]

Suppose μ is a nonnegative, nonzero Radon measure on $\mathbb{R}^d$ and $\alpha \ge 0$ such that

(i) $0 < \overline{D}^\alpha(\mu,x) = \overline{d}^\alpha(\mu,x) < \infty$ *for μ-almost every x, or*
(ii) $0 < \underline{d}^\alpha(\mu,x) = \underline{D}^\alpha(\mu,x) < \infty$ *for μ-almost every x,*

then α must be an integer.

An elegant proof of this result, which also removes a technical restriction necessary to carry out Marstrand's original proof, can also be found in Chapter 1.

It is natural to ask whether the inequalities above also imply α-rectifiability of μ. The answer is *no* for the first inequality, even in the case $\alpha = 1$. The first example of this type can be found in [**ONe95**]. Let us sketch a simpler example here.

Example: We construct a sequence (I_n) of compact sets $I_n \subseteq \mathbb{R}^2$ as follows: Let $I_0 = B(0,1)$. In the first step inscribe 2 touching, closed balls of radius $1/2$ with centres on the vertical diameter in I_0. Denote the resulting set I_1. In the next step inscribe 4 touching, closed balls of radius 1/8 with centres on the horizontal diameter in each ball of I_1. Call the resulting set I_2. Having constructed I_n, we inscribe 2^{n+1} touching, closed balls of radius $(\prod_{i=1}^{n+1} 2^i)^{-1}$ with centres on the horizontal diameter (if n odd) or on the vertical diameter (if n even) in each ball of I_n and denote the resulting set I_{n+1}, see Figure 2.

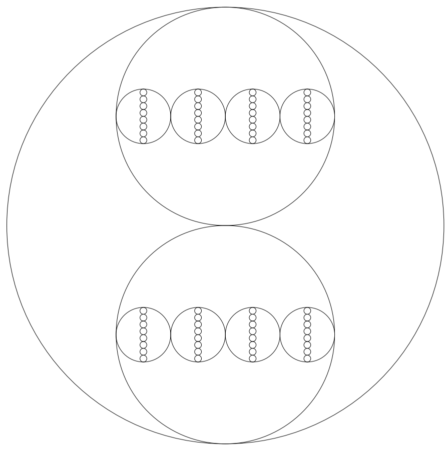

FIGURE 2. The first four steps in the construction of I.

We denote by $\mathcal{B}$ the family of balls appearing in the construction and let

$$I := \bigcap_{n=0}^{\infty} I_n \,.$$

Obviously I is compact. There is a natural measure μ supported by I which is characterised by the property that for every ball $B \in \mathcal{B}$ we have $\mu(B \cap I) = \mathrm{diam}(B)$.

In this example it is not hard to check that, for μ-almost every x, we have

$$\frac{1}{2} \leq \underline{d}^1(\mu, x) < \underline{D}^1(\mu, x) = \overline{D}^1(\mu, x) = \overline{d}^1(\mu, x) = 2 \,.$$

As $\underline{d}^1(\mu, x) < \overline{d}^1(\mu, x)$ the measure μ is not rectifiable, still we have $0 < \overline{D}^1(\mu, x) = \overline{d}^1(\mu, x) < \infty$ at μ-almost every point.

The previous example has shown that for 1-dimensional measure the equality of upper average and upper density does not imply rectifiability. At a first glance surprisingly, it turns out that this is different for the lower densities. This the main result of Chapter 2.

THEOREM **2.1.2** *Suppose μ is a nonnegative Radon measure on $\mathbb{R}^d$. Then*

$$0 < \underline{d}^1(\mu, x) = \underline{D}^1(\mu, x) < \infty \quad \textit{for } \mu\textit{-almost every } x$$

implies that

$$0 < \underline{d}^1(\mu, x) = \overline{d}^1(\mu, x) < \infty \quad \textit{for } \mu\textit{-almost every } x$$

and hence μ is linearly rectifiable.

We sketch the ideas of the proof in the following paragraphs. It is an open problem whether an analogue of this result holds for arbitrary integers α and we try to point out in the sequel which parts of our proof remain valid for integers α different from 1.

To formulate the first step in our proof we introduce the language of tangent measures. This language was introduced by Preiss in his seminal paper [**Pre87**] and has turned out to be an extremely useful tool in geometric measure theory. See [**Mat95**] for an exposition and [**ONe96**], [**Huo97**] for recent progress.

Definition: Let $0 \le \alpha \le d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$. For $r > 0$ recall that $\mu_{x,r} \in \mathcal{M}(\mathbb{R}^d)$ is the *enlargement of* μ *at* x *of factor* $1/r$, i.e. the measure defined by $\mu_{x,r}(A) = \mu(x + rA)$. The set of *$\alpha$-dimensional tangent measures of* μ *at* x is defined as the set

$$\mathrm{Tan}_\alpha(\mu, x) = \Big\{ \lim_{n\to\infty} \mu_{x,r_n}/r_n^\alpha \text{ in the vague topology for some } r_n \downarrow 0 \Big\}.$$

A (tangent) measure ν is called *α-flat* if α is an integer and there is a linear space $V \subseteq \mathbb{R}^d$ of dimension α and some $c > 0$ such that $\nu = c \cdot \mathcal{H}^\alpha|_V$, a multiple of the restriction of α-Hausdorff measure to the space V.

For an α-rectifiable measure at almost all points the set of tangent measures is a singleton consisting of an α-flat measure. This is not hard to prove, see e.g. [**Mat95**, 16.7]. For fractal measures of dimension α however the set of tangent measures is typically very rich.

First step: We show the *existence* of certain flat tangent measures using our hypothesis on μ.

THEOREM **2.2.5** *Let* μ *be a Radon measure on* $\mathbb{R}^d$ *and let* $\alpha \ge 0$.

(a) *Suppose* $0 < \overline{d}^\alpha(\mu, x) = \overline{D}^\alpha(\mu, x) < \infty$ *for μ-almost every x. Then, at μ-almost every x, there is an α-flat measure* $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ *such that*

$$\nu(U(0,1)) = \overline{d}^\alpha(\mu, x)\ .$$

(b) *Suppose* $0 < \underline{d}^\alpha(\mu, x) = \underline{D}^\alpha(\mu, x) < \infty$ *for μ-almost every x. Then, at μ-almost every x, there is an α-flat measure* $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ *such that*

$$\nu(U(0,1)) = \underline{d}^\alpha(\mu, x)\ .$$

We remark that at this stage of the proof we still have symmetry in the statements about lower and upper densities. Also the result applies to arbitrary α, only in the final section we will specialize to the case $\alpha = 1$. Observe that this theorem implies the Theorem of Falconer, Springer and Marstrand.

The proof uses an idea which could be called a *probabilistic method*, which means that we obtain the flat tangent measure by sampling from a suitably chosen probability distribution on $\mathrm{Tan}_\alpha(\mu, x)$.

At this point we note that all α-dimensional tangent measure distributions of μ at x are supported by the set $\mathrm{Tan}_\alpha(\mu, x)$ of tangent measures. We choose $\varepsilon_n \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{1}{\log(1/\varepsilon_n)} \int_{\varepsilon_n}^1 \frac{\mu(U(x,r))}{r^\alpha}\, \frac{dr}{r} = \underline{D}^\alpha(\mu, x)\,.$$

After checking the conditions of Prohorov's Theorem, which is easy, one can choose a convergent subsequence of $(\mathbb{P}^x_{\varepsilon_n})$ and denote the limit $\mathbb{P}$. As the mapping $\nu \mapsto \nu(U(0,1))$ is lower semicontinuous we have

$$\int \nu(U(0,1))\, d\mathbb{P}(\nu) \le \lim_{n\to\infty} \frac{1}{\log(1/\varepsilon_n)} \int_{\varepsilon_n}^1 \frac{\mu(U(x,r))}{r^\alpha}\, \frac{dr}{r} = \underline{D}^\alpha(\mu, x) = \underline{d}^\alpha(\mu, x).$$

But $\nu(U(0,1)) \ge \underline{d}^\alpha(\mu, x)$ for every $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ and therefore we must have

$$\nu(U(0,1)) = c := \underline{d}^\alpha(\mu, x) \text{ for } \mathbb{P}\text{-almost every } \nu.$$

Now the scaling invariance and Palm formula for $\mathbb{P}$ are becoming essential. With their help one can infer that, for $\mathbb{P}$-almost every ν,

$$\nu(U(u,r)) = cr^\alpha \text{ for every } u \in \text{supp } \nu \text{ and every } r > 0.$$

Measures with this property are called *α-uniform.* They are not necessarily α-flat (if $\alpha \geq 3$) but they are α-rectifiable, and hence their tangent measures are flat. Roughly speaking, the tangent measures to tangent measures are again tangent measures at the original point and hence it is quite easy to check that these tangent measures satisfy the requirements of Theorem **2.2.5(b)** Of course, Theorem **2.2.5(a)** can be proved in an analogous same way.

Before passing to the second step of the proof we look at the example from above again.

Example: At a typical point x, the set $\text{Tan}_1(\mu,x)$ contains both 1-flat and non-flat tangent measures. It is shown in [**Mör95**] that for sequences $r_n \downarrow 0$ outside the set Z of vanishing logarithmic density the only possible tangent measures are the 1-flat measures

$$h = \mathcal{H}^1|_{\{(x,0)\,:\,x\in\mathbb{R}\}} \text{ and } v = \mathcal{H}^1|_{\{(0,x)\,:\,x\in\mathbb{R}\}}\,.$$

Moreover, the only tangent measure distribution is

$$P = \frac{1}{2}\cdot\delta_h + \frac{1}{2}\cdot\delta_v\,,$$

where δ_h, δ_v are the Dirac distributions with mass concentrated at h, v, respectively. As the set $\text{Tan}_1(\mu,x)$ is always connected in the vague topology there must be a lot of other tangent measures coming from sequences $r_n \downarrow 0$ in Z. It is easy to see that they are non-flat.

The heuristic message of this example is the following: if an unrectifiable measure μ has flat tangent measures, the flat tangent measures typically come from sequences of scales r_n of large density, i.e.

$$\mu\big(U(x,r_n)\big) \;\approx\; r_n^\alpha\,\overline{d}^1(\mu,x).$$

Therefore we may have flat tangent measures ν with $\nu(U(0,1)) = \overline{d}^1(\mu,x)$. Sequences r_n of scales with low density however correspond to those scales where the "breaking" or fractal nature of the measure becomes apparent and hence they do not lead to flat tangent measures. Information about rectifiability of μ can therefore only be obtained from an analysis of sequences of scales with low density. In other words, only the conclusion of part (b) in Theorem **2.2.5** can lead to rectifiability of μ.

Second step: We show:

THEOREM **2.3.1** *If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is a measure such that at μ-almost every x there is a 1-flat measure $\nu \in \text{Tan}_1(\mu,x)$ with $\nu(U(0,1)) = \underline{d}^1(\mu,x)$, then*

$$0 < \underline{d}^1(\mu,x) = \overline{d}^1(\mu,x) < \infty \quad \textit{for } \mu\textit{-almost every } x,$$

and hence μ is linearly rectifiable.

Remark: It is an open (and probably very hard) problem, whether an analogous statement holds for integers α different from 1.

Note that the assumption $0 < \underline{d}^1(\mu,x) < \infty$ is implicit in our assumption on the tangent measures. Clearly, the Theorems **2.2.5** and **2.3.1** together imply Theorem **2.1.2**.

We indicate the idea in the proof of Theorem **2.3.1**. Assume that x is a generic point and there exists a 1-flat tangent measure ν in x with $\nu(U(0,1)) = \underline{d}^1(\mu,x)$. Roughly speaking, this means that there exist balls $U(x,r)$ of arbitrarily small radius such that

$$\mu\big(U(x,r)\big) \approx \underline{d}^1(\mu,x)\,r$$

and $r^{-1}\mu$ restricted to $U(x,r)$ looks like the 1-Hausdorff measure on $L \cap U(x,r)$ for some line L. Projecting the restriction of μ orthogonally to this line and rescaling, we obtain a measure ν on the unit interval such that

- ν is close in the weak topology to the Lebesgue measure on the interval,
- for all small balls centred in the support of ν the ν-mass is at least equal to the Lebesgue measure.

If we now assume that μ is not rectifiable, we additionally get that the projection of the support of μ in the direction orthogonal to L is small. In other words, the measure ν must be supported by a small compact subset of the interval. As ν is close to the Lebesgue measure on the interval in the weak topology, we know that the complementary intervals of the support cannot be large. In Lemma **2.3.2**, which may be of some independent interest, we show that a measure with these features cannot exist. The proof of this lemma relies on the order structure of $\mathbb{R}$ and unfortunately I have no higher dimensional analogue.

2. Dimension spectra and extremal behaviour of stochastic processes

The central theme of the first two chapters is the *local* geometry of irregular sets around *typical* points. To capture typical points we always equip the irregular set S with a natural measure μ and formulate results which hold for μ-almost every point x.

In the case of random sets we frequently encounter the situation that the set may be parametrized canonically by a deterministic interval I, for example if the set is the sample path of a continuous time process. In this situation we may capture a typical point by fixing a parameter first and looking at the almost sure behaviour at the given parameter. One can fix any finite measure ν on the parametrizing interval and equip the random set $S \subset \mathbb{R}^d$ with the image measure μ of ν under the parametrization, say $\phi : I \to X$. Any local property of the random set, which holds for any given $t \in I$ almost surely at $\phi(t)$, holds almost surely at μ-almost every point. Indeed, if $\chi(S,x) \in \{0,1\}$ is the (measurable) indicator of this property for the random set S at the point x, then Fubini's Theorem gives,

$$\mathbb{E}\Big\{ \int_S \chi(S,x)\, \mu(dx) \Big\} = \int_I \mathbb{E}\Big\{ \chi\big(S,\phi(t)\big) \Big\}\, \nu(dt) = \nu(I) = \mathbb{E}\big\{ \mu(S) \big\},$$

hence $\chi(S,x) = 1$ for μ-almost every x, almost surely. Of course, $\chi(S,x) = 1$ need not hold for *every* x and there is a possibility of *exceptional* or *extremal* behaviour.

We illustrate this by a classical example. *Khinchin's law of the iterated logarithm* describes the size of a small Brownian increment at a typical point. More precisely, let $W = \{W_s : s \geq 0\}$ be a Brownian motion and fix $t \in [0,1]$. Then

$$\limsup_{\varepsilon \downarrow 0} \frac{W_{t+\varepsilon} - W_t}{\sqrt{\varepsilon \log\log(1/\varepsilon)}} = \sqrt{2} \text{ almost surely.}$$

By our observation, the law of the iterated logarithm also holds for almost every point W_t on the Brownian path with respect to the occupation measure, almost surely.

On the other hand, for every $t \in [0,1]$ there is an exceptional set of paths of Wiener measure zero, where Khinchin's law of the iterated logarithm fails, and this set may depend on t. Hence we *cannot* say that the law holds almost surely simultaneously for all $t \in [0,1]$.

If we are allowed to choose t after the Brownian motion W is sampled, we may find considerably larger increments. This is the statement of *Lévy's modulus of continuity*, which may be written in the form,

$$\sup_{0\le t\le 1} \limsup_{\varepsilon\downarrow 0} \frac{W_{t+\varepsilon}-W_t}{\sqrt{\varepsilon\log(1/\varepsilon)}} = \sqrt{2} \text{ almost surely.} \tag{8}$$

Note that the gauge function measuring the size of the increments at the extreme times is considerably larger than the gauge function measuring the size of the increment at fixed (and hence at almost all) times $t\in[0,1]$.

In their seminal paper [**OrT74**] Orey and Taylor further refine (8) by measuring the size of sets of times where extremely large increments occur in terms of a Hausdorff dimension spectrum. Precisely, almost surely, for all $a\in[0,\sqrt{2}]$,

$$\dim\Big\{0\le t\le T \,:\, \limsup_{\varepsilon\downarrow 0} \frac{W_{t+\varepsilon}-W_t}{\sqrt{\varepsilon\log(1/\varepsilon)}} = a\Big\} = 1-\frac{a^2}{2}\,. \tag{9}$$

From this point of view Chapters 3 and 4 deal with the possible behaviour in *untypical* or *extremal* points of irregular sets. Stochastic processes provide a large number of very interesting examples where extremal behaviour can be studied, and the present thesis discusses two important examples in detail. In this section of the overview I would like to focus on the interplay between the *tail behaviour* of certain random variables and *dimension spectra*, which measure the size of sets of points with extremal behaviour.

We discuss three different examples. In Subsection 2.1 we look at extremely thin rays in a supercritical Galton-Watson tree. This example is particularly suited to illustrate our point because variation of the offspring distribution leads to qualitatively different types of dimension spectra. At the same time it has pleasant independence and regularity properties, which make it easier to study than, say, intersections of Brownian paths and it is therefore ideal to demonstrate the essential ingredients behind the lower estimates for the dimension spectra we use. This example is not discussed further in the main body of the thesis.

Perhaps the most interesting recent results in this direction are the results on extremely thick points for the occupation measure of Brownian paths, and more generally for intersection local times. In Subsection 2.2 we briefly review the results of Dembo, Peres, Rosen and Zeitouni [**DPR00a, DPR00b, DPR01**] and discuss the main results of Chapter 3.

The investigation of extremely large Brownian increments is historically the first example, studied already in [**OrT74**]. A recent line of progress here is to look at systems involving a continuum of particles, which generically follow Brownian paths, and investigate the behaviour of extremal particles in this system. These particles frequently exhibit behaviour different from Brownian motion. In Subsection 2.3 we have a closer look at the case of super-Brownian motion, which is the subject of Chapter 4 in this thesis.

2.1. Thin rays in supercritical Galton-Watson trees. We consider a Galton–Watson tree T starting from a single progenitor, the root ρ. Let N be its offspring distribution and suppose it satisfies $m:=EN>1$ and $E[N\log N]<\infty$. We also assume that $P\{N=0\}=0$, hence the tree is infinite.

By ∂T we denote the *boundary* of the tree consisting of all infinite self-avoiding paths in T starting with the root. These paths $\xi\in\partial T$ are called the *rays* of the tree. We equip the boundary ∂T of the tree with the metric d given by $d(u,v)=e^{-n}$, where n is the generation of the last common vertex of $u,v\in\partial T$. With respect to this metric it is well–known that, almost surely on $\partial T\neq\emptyset$, the Hausdorff dimension of ∂T is $a:=\log m$, see [**Haw81**] or [**LyR90**], the exact dimension gauge was obtained in [**Liu96**].

By $\{Z_n : n \geq 0\}$ we denote the *Galton-Watson process* associated with the tree T, i.e. Z_n is the number of vertices of nth generation. By the well-known Kesten-Stigum Theorem under our conditions, almost surely, the *limit variable*

$$W := \lim_{n\to\infty} \frac{Z_n}{m^n}$$

exists in $(0,\infty)$. The limit variable measures, heuristically speaking, the asymptotic size of the tree. Given a vertex $v \in T$ we denote by $W(v)$ the limit variable of the subtree $T(v) \subset T$ consisting of all descendants of the vertex v. As this tree is again a Galton-Watson tree with the same offspring distribution, we may assume that $W(v)$ exists for all vertices $v \in T$. A natural measure μ, called the *branching measure*, can be defined on ∂T such that, for all vertices v of generation n,

$$\mu\big\{\xi \in \partial T \,:\, \xi \text{ is passing through } v\big\} = m^{-n} W(v).$$

For a typical ray $\xi = (v_1, v_2, v_3, \dots)$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log W(v_n) = 0.$$

We call a ray $\xi = (v_1, v_2, v_3, \dots)$ a *thin ray* if

$$\liminf_{n\to\infty} \frac{1}{n} \log W(v_n) < 0. \tag{10}$$

To find a dimension spectrum for the number of thin rays in a Galton-Watson tree we first assume that there is a positive probability $p_1 = P\{N = 1\}$ that a given vertex has only one successor. Intuitively, in this case along thin rays we expect to find long runs of vertices with only one descendant.

THEOREM *Suppose $P\{N = 1\} > 0$ and assume further that N has finite moments of all orders. Define $\tau = -\log P\{N = 1\}/a$.*

(i) [**Bin88**], [**Dub71a**] $\mathbb{P}\{W \leq x\} \asymp x^\tau$ *as* $x \downarrow 0$.
(ii) [**ShT00**] *Almost surely, for* $0 \leq b \leq \frac{a}{\tau}$,

$$\dim \Big\{(v_n) \in \partial T \,:\, \liminf_{n\to\infty} \frac{1}{n} \log W(v_n) = -b\Big\} = a\Big(\frac{a}{b+a}(1+\tau) - \tau\Big).$$

We see that the parameters governing the extremal rays of ∂T correspond naturally to the lower tail behaviour of the random variable W. Fat lower tails of the random variable W make it possible that thin rays occur in the Galton-Watson tree. The behaviour of the lower tail of W can be obtained from parameters of the offspring random variable N. We do not give further insight into the proofs here, as the arguments do not find application in the main body of the thesis.

We now look at the situation that $P\{N = 1\} = 0$. The situation that (at least intuitively) gave rise to thin rays, the occurrence of long sequences of vertices with just one descendant, cannot occur here any more. Indeed, in this case thin rays in the sense of the previous definition do *not* exist. More precisely, we have for *all* rays $\xi = (v_1, v_2, v_3, \dots)$ that

$$\liminf_{n\to\infty} \frac{1}{n} \log W(v_n) \geq 0.$$

In order to capture the fluctuations leading to extremal behaviour, we have to find a maximal increasing gauge sequence $\varphi(n)$ such that there exist rays with

$$\liminf_{n\to\infty} \varphi(n) W(v_n) < \infty.$$

Let $\beta = \operatorname{essinf} N$ and $\lambda = \log m / \log \beta - 1 > 0$. We redefine the notion of a thin ray in the case $P\{N \le 1\} = 0$ as follows: $\xi = (v_1, v_2, v_3, \dots) \in \partial T$ is a *thin ray* if

$$\liminf_{n\to\infty} n^\lambda W(v_n) < \infty.$$

This turns out to be the right way of measuring thin behaviour in the present case.

THEOREM *Suppose $P\{N \le 1\} = 0$. Then, almost surely,*

(i) [**Bin88**], [**Dub71b**] $r := -\limsup_{x\downarrow 0} x^{1/\lambda} \log \mathbb{P}\{W < x\}$ *is positive and finite.*

(ii) [**MöS01**] *Almost surely, for all $\theta \ge \left(\frac{r}{a}\right)^\lambda$, we have*

$$\dim \Big\{ \xi \in \partial T \,:\, \liminf_{n\to\infty} n^\lambda W(v_n) = \theta \Big\} = a - r\theta^{-1/\lambda}.$$

We see that the different lower tail behaviour of the random variable W is reflected in a qualitatively different dimension spectrum. An exponential decay of the lower tails of W only permits thin rays in a much weaker sense than before. Again, the behaviour of the lower tail of W can be obtained from the offspring random variable N.

We now present the proof of (i)⇒(ii) in greater detail, because the underlying method of obtaining a dimension spectrum (in particular lower bounds) from tail asymptotics is coming out particularly clearly here. In Chapter 3 an analogous idea is used, but additional technical details are disturbing the view towards the key ideas.

Sketch of (i)⇒(ii). The *upper bound* is quite easy. It suffices to show that, for any $b > a - r\theta^{-1/\lambda}$, the b-dimensional Hausdorff measure of the set

$$\bigcap_{k=1}^{\infty} \bigcup_{n\ge k} \Big\{ \xi \in \partial T \,:\, n^\lambda W(v_n) < (\theta+\varepsilon) \Big\} \tag{11}$$

equals zero for some small $\varepsilon > 0$. For any positive integer k we obtain a covering of the set in (11) by just taking

$$R(v) = \big\{ \xi \in \partial T \,:\, \xi \text{ is passing through } v \big\}$$

for all those vertices v of generation $n \ge k$ with the property that $n^\lambda W(v_n) < \theta + \varepsilon$. Let I_n be the number of sets $R(v)$ for vertices v of generation n in such a covering, and recall that all these sets have diameter e^{-n}. Then, using (i), and recalling that $a = \log m$, we see that, for sufficiently large k,

$$\begin{aligned} \mathbb{E}\Big[\sum_{n=k}^{\infty} I_n e^{-bn}\Big] &= \sum_{n=k}^{\infty} e^{-bn} m^n \mathbb{P}\Big\{ n^\lambda W(v_n) < \theta + \varepsilon \Big\} = \sum_{n=k}^{\infty} e^{-bn} e^{an} \mathbb{P}\big\{ W < n^{-\lambda}(\theta+\varepsilon) \big\} \\ &\le \sum_{n=k}^{\infty} e^{-n(b-a)} e^{-n(r-\varepsilon)(\theta+\varepsilon)^{-1/\lambda}}. \end{aligned} \tag{12}$$

As $b > a - r\theta^{-1/\lambda}$ we can choose a sufficiently small $\varepsilon > 0$ such that the right hand side in (12) is the remainder of a convergent series. Then $\liminf_{k\uparrow\infty} \sum_{n\ge k} I_n e^{-bn} = 0$ almost surely. We infer that the set in (11) has vanishing b-dimensional Hausdorff measure, which proves the upper bound.

The *lower bound* is more involved. The basic idea is to use the critical exponent of the Bernoulli percolation process on the tree as a measure for the size of the tree. This idea goes back to Lyons [**LyR90**]. If T is any tree and $0 < p < 1$, we attach to each edge e of the tree an independent $\{0,1\}$-valued random variable θ_e with $\mathbb{P}\{\theta_e = 1\} = p$. We denote by T^* the connected component of the root ρ in the graph consisting of all edges e with $\theta_e = 1$. We say that the tree T^* is the result of running *percolation with retention parameter* p on T.

Lyons [**LyR90**] introduced the notion of the *branching number* of a tree, giving the heuristic concept of a mean number of offspring a rigorous foundation. Recall from [**LyR90**] the facts that, for every infinite tree T with branching number b,

- the Hausdorff dimension of the boundary is $\dim \partial T = \log b$.
- Suppose that T^* is the random tree obtained from running percolation with retention parameter p on T.
  - If $p < 1/b$, then $\partial T^* = \emptyset$ almost surely.
  - If $p > 1/b$, then $\partial T^* \neq \emptyset$ with positive probability.

Let us now look at a Galton-Watson tree T. We need the following facts:

- [**Haw81, LyR90**] Let T be a Galton-Watson tree. Then, almost surely on $\partial T \neq \emptyset$, the branching number of T equals the mean offspring number m.
- If we run independent percolation on a Galton-Watson tree T, the unconditional distribution of T^* is again the law of a Galton-Watson tree. If m was the mean offspring number of T, then pm is the mean offspring number of T^*.

Define

$$A(\theta) = \Big\{\xi = (v_n) \in \partial T \,:\, \liminf_{n\to\infty} n^{\lambda} W(v_n) \leq \theta\Big\}. \tag{13}$$

The main step in the proof is the following: For $\theta \geq (r/a)^{\lambda}$ define

$$p = p(\theta) = \exp(-\log m + r\theta^{-1/\lambda}) > 1/a.$$

Then running percolation with this retention parameter yields

$$A(\theta) \cap \partial T^* \neq \emptyset, \qquad \text{almost surely on } \partial T^* \neq \emptyset. \tag{14}$$

Proof of (ii) using (14): We study the subtrees

$$S(\theta) = \Big\{v \in T \,:\, \liminf_{n\to\infty} n^{\lambda} W(v_n) = \theta \text{ for some } \xi \text{ with } v \in \xi\Big\}$$

and

$$\underline{S}(\theta) = \Big\{v \in T \,:\, \liminf_{n\to\infty} n^{\lambda} W(v_n) \leq \theta \text{ for some } \xi \text{ with } v \in \xi\Big\}.$$

The essential idea is that running percolation on T can be seen as a coupling of percolation on all subtrees $S \subset T$, by $S^* = S \cap T^*$. We run percolation with retention parameter p as above on T and hence simultaneously on all $\underline{S}(\delta)$, $\delta > 0$, and on $S(\theta)$. Considering for subtrees $S \subset T$ the boundary ∂S in the natural way as a subset of ∂T, we observe that

$$\partial S(\theta)^* = \partial \underline{S}(\theta)^* \setminus \bigcap_{N=1}^{\infty} \partial \underline{S}(\theta - \tfrac{1}{N})^* \,.$$

Now $\partial \underline{S}(\theta)^* = A(\theta) \cap \partial T^* \neq \emptyset$ almost surely on $\partial T^* \neq \emptyset$, by (14). On the other hand, by the upper bound, we have

$$\dim A(\theta - \tfrac{1}{N}) \leq a - r(\theta - \tfrac{1}{N})^{-1/\lambda} < a - r\theta^{-1/\lambda} = \log(1/p),$$

so that $\partial \underline{S}(\theta - \frac{1}{N})^* = \emptyset$ almost surely. We infer that, with positive probability, $\partial S(\theta)^* \neq \emptyset$. Hence the branching number of $S(\theta)$ is bounded from below by $1/p = \exp(\log m - r\theta^{-1/\lambda})$, which means that the Hausdorff dimension of $\partial S(\theta)$ is bounded from below by $a - r\theta^{-1/\lambda}$. $\square$

Sketch of the proof of (14): An important ingredient in the proof of (14) is the fact that the exponential decay of the probability for extremal behaviour of W remains unchanged if we additionally suppose that the tree survives percolation with the retention parameters p as above, more precisely:

$$(15) \qquad -\limsup_{x\downarrow 0} \frac{\log \mathbb{P}\big(\{W<x\}\cap\{\partial T^*\neq\emptyset\}\big)}{x^{-1/\lambda}} = r.$$

For every positive integer N we let

$$U(N) = \Big\{\xi\in\partial T\ :\ \text{there exists } k\geq N \text{ with } k^\lambda W(v_k) < \theta+\tfrac{1}{N}\Big\}.$$

Our aim is to show that $U(N)\cap\partial T^*$ is dense in ∂T^* almost surely. But we first show how to use this to finish the proof of (14). Note that $U(N)\cap\partial T^*$ is relatively open in ∂T^*, which is a compact, hence complete, metric space. By *Baire's Theorem* we know that

$$\bigcap_{N=1}^{\infty} U(N)\cap\partial T^* \text{ is dense in } \partial T^*.$$

But note that $\bigcap_{N=1}^{\infty} U(N) = A(\theta)$, hence $A(\theta)\cap\partial T^*$ must be dense in ∂T^* almost surely. This implies that $A(\theta)\cap\partial T^*$ is nonempty almost surely conditional on $\partial T^*\neq\emptyset$, which is the required statement.

It remains to show that, for any fixed integer N, the set $U(N)\cap\partial T^*$ is dense in ∂T^* almost surely. To prepare the proof recall (15). One may select an increasing sequence (n_j) of integers with

$$(16) \qquad \mathbb{P}\Big(\big\{W<n_j^{-\lambda}\big(\theta+\tfrac{1}{N}\big)\big\}\cap\big\{\partial T^*\neq\emptyset\big\}\Big) \geq \exp\Big(-n_j\big(r+\tfrac{1}{j}\big)\big(\theta+\tfrac{1}{N}\big)^{-1/\lambda}\Big).$$

To prove that $U(N)\cap\partial T^*$ is dense in ∂T^*, we fix some vertex $v\in T$ and denote by $T(v)\subset T$ the subtree consisting of all vertices which are succeeding v. We need to show that, almost surely conditional on $\partial T(v)^*\neq\emptyset$, we have $U(N)\cap\partial T(v)^*\neq\emptyset$.

Let $\mathcal{G}(n)$ be the σ-field generated by the first n generations of $T(v)$, together with the associated random variables θ_e. Let $\mathcal{K}_n$ be the collection of vertices in $T(v)^*$ of generation n and let K_n be the cardinality of $\mathcal{K}_n$. Note that the random variable K_n is $\mathcal{G}(n)$-measurable.

We define a sequence of $\mathcal{G}(n)$-stopping times $N_1, N_2, \dots$ by $N_0 = 1$ and

$$N_k = \min\Big\{n_j > N_{k-1}\ :\ K_{n_j}\geq \exp\Big(n_j\big(r\theta^{-1/\lambda}-\tfrac{1}{k}\big)\Big)\Big\}.$$

By the facts about branching numbers of Galton-Watson trees, almost surely on $\partial T(v)^*\neq\emptyset$, we have $\dim\partial T(v)^* = \log pm = r\theta^{-1/\lambda}$. This implies that the sequence $\{N_j\ :\ j\geq 1\}$ of stopping times is indeed infinite, almost surely on $\partial T(v)^*\neq\emptyset$. For every vertex $w\in T(v)$ of generation N_j define the event

$$E(w) := \Big\{W(w) < N_j^{-\lambda}\big(\theta+\tfrac{1}{N}\big) \text{ and } \partial T(w)^*\neq\emptyset\Big\}.$$

Given $\mathcal{G}(N_j)$ the events $E(w)$, $w\in\mathcal{K}_{N_j}$ are independent and, by (16),

$$\mathbb{P}\Big\{E(w)\,\Big|\,\mathcal{G}(N_j)\Big\} \geq \exp\Big(-N_j\Big(r+\tfrac{1}{j}\Big)\Big(\theta+\tfrac{1}{N}\Big)^{-1/\lambda}\Big).$$

Hence the probability that none of the vertices $w\in\mathcal{K}_{N_j}$ satisfies $E(w)$ is

$$\mathbb{P}\Big\{E(w) \text{ fails for all } w\in\mathcal{K}_{N_j}\,\Big|\,\mathcal{G}(N_j)\Big\} \leq \Big(1-\exp\Big(-N_j\Big(r+\tfrac{1}{j}\Big)\Big(\theta+\tfrac{1}{N}\Big)^{-1/\lambda}\Big)\Big)^{K_{N_j}},$$

which vanishes asymptotically for $j \to \infty$ as $K_{N_j} \geq \exp\big(N_j(r\theta^{-1/\lambda} - \frac{1}{j})\big)$. We infer that, almost surely on $\partial T(v)^* \neq \emptyset$, there exist infinitely many vertices $w \in T(v)^*$ such that $E(w)$ holds. Hence, by a compactness argument, there exists a ray $\xi \in \partial T(v)^* \cap U(N)$ and (14) is shown. $\square$

2.2. Thick points for occupation measures and intersection local times. Let a bunch of p independent Brownian motions $W_1, \dots, W_p$ started in the origin run in $\mathbb{R}^d$ until their first exit times $T_1, \dots, T_p$ from a large ball, or, in the transient case, for infinite time. By classical results of Dvoretzky, Erdös, Kakutani and Taylor the intersection of the paths of these motions, the set

$$S = \bigcap_{i=1}^{p} \big\{ x \in \mathbb{R}^d \,:\, x = W_i(t) \text{ for some } t \in [0, T_i) \big\},$$

contains points different from the starting point if and only if $p < d/(d-2)$ (this includes the case p arbitrary, $d = 2$). In these cases one can define a measure on the set S in three natural ways:

- [**GHR84**] Geman, Horowitz and Rosen define the *confluent Brownian motion* process $W : \prod_{i=1}^{p}[0, T_i) \to \mathbb{R}^{d(p-1)}$ by
$$W(s_1, \dots, s_p) = \big(W_1(s_1) - W_2(s_2), \dots, W_{p-1}(s_{p-1}) - W_p(s_p)\big).$$
They show that this process has a continuous local time at 0, which is a natural measure on $\prod_{i=1}^{p}[0, T_i)$ supported by those $(s_1, \dots, s_p)$ with $W_1(s_1) = \cdots = W_p(s_p)$. The projection of this local time via $(s_1, \dots, s_p) \mapsto W_1(s_1)$ is a natural measure on the intersection of the p Brownian paths.
- [**LeG86**] Le Gall studies the renormalized limit for $\varepsilon \downarrow 0$ of the Lebesgue measure on the intersection of the Wiener sausages
$$S_i[\varepsilon] = \big\{ x \in \mathbb{R}^d \,:\, \big|W_i(s) - x\big| \leq \varepsilon \text{ for some } s \in [0, T_i) \big\}, \quad \text{for } i \in \{1, \dots, p\}.$$
He shows that for a suitable normalization a limit exists in the L^2 sense.
- [**LeG87, LeG89**] Le Gall also defines dimension gauge functions such that the Hausdorff measure on S is a positive and finite measure. The appropriate dimension gauges are
$$\psi_p(r) = \begin{cases} r^2\,[\log(1/r)\log\log\log(1/r)]^p, & \text{if } d = 2, p \geq 1 \text{ arbitrary}, \\ r^{3-p}\,[\log\log(1/r)]^p, & \text{if } d = 3, p \in \{1, 2\}, \\ r^2\,[\log\log(1/r)], & \text{if } d \geq 4, p = 1. \end{cases}$$

It turns out that, with appropriate constants introduced in the gauge and normalization functions, the latter two constructions yield the same measure on S as the first one, as shown in [**LeG86**] and [**LeG87, LeG89**]. We call this measure the *intersection local time* on S and denote it by $\lambda := \lambda[p, d]$. We include the case $p = 1$ of a single Brownian path in all dimensions $d \geq 2$, in which λ is the usual occupation measure. The intersection local times λ can be written symbolically as

$$\lambda(A) = \int_A dy \prod_{j=1}^{p} \int_0^{T_j} ds\, \delta_y(W_j(s)), \text{ for } A \subset \mathbb{R}^d \text{ Borel.} \tag{17}$$

As in the study of thin points for Galton-Watson trees one would first define thick points $x \in S$ in analogy to (10) by the condition

$$\limsup_{r \downarrow 0} \frac{\log \lambda\big(B(x, r)\big)}{-\log r} > \begin{cases} \text{-1} & \text{if } d = 3, p = 2, \\ \text{-2} & \text{if } p = 1, d \geq 2 \text{ or } d = 2, p \geq 1. \end{cases}$$

Can we expect such points to exist? Let us look at the case $d \geq 3$, $p = 1$. After the experience collected in the previous subsection we have to hope for subexponential decay of the upper tails of $\lambda(B(0,r))$. In the present case these tails are known from the famous work of Ciesielski and Taylor [**CiT62**]. They show that, in the case $p = 1$, $d \geq 3$,

$$\mathbb{P}\Big\{\lambda\big(B(0,r)\big) > x\Big\} = \sum_{k=1}^{\infty} \psi_{d,k} \exp\Big(\frac{-p_{d,k}^2}{2r^2} x\Big), \tag{18}$$

where $\{p_{d,k} : k = 1, 2, \dots\}$ are the positive zeroes of the Bessel function J_μ for $\mu = d/2 - 2$ and

$$\psi_{d,k} = \frac{1}{2^{\mu-1}\Gamma(\mu+1)} \frac{p_{d,k}^{\mu-1}}{J_{\mu+1}(p_{d,k})}.$$

For better comparison with later results we give a reformulation of the Ciesielski-Taylor result (18) for the upper tail asymptotics.

Corollary *Let $z : [0, \infty) \to \mathbb{R}$ be the unique solution of the ordinary differential equation*

$$z''(x) + (d-1)\frac{z'(x)}{x} + z(x) = 0 \text{ with } z(0) = 1 \text{ and } z'(0) = 0. \tag{19}$$

Then $p_{d,1}$ is the smallest number $a > 0$ such that $(d-2)z(a) = -az'(a)$ and

$$\lim_{x\uparrow\infty} \frac{1}{x} \log \mathbb{P}\Big\{\lambda\big(B(0,r)\big) > x\Big\} = \frac{-p_{d,1}^2}{2r^2}.$$

Proof. Recall the fact that the Bessel functions J_ν for $\nu = \mu + 1$, solve the *Bessel differential equation* $x^2 J_\nu''(x) + x J_\nu'(x) + (x^2 - \nu^2) J_\nu(x) = 0$. Using this differential equation, the function $z(x) = \Gamma(d/2)(2/x)^\nu J_\nu(x)$ is easily seen to solve (19). Furthermore, using that $J_\nu'(x) = J_\mu(x) - (\nu/x) J_\nu(x)$ one can check that the positive zeroes of J_μ are the solutions of $(d-2)z(a) = -az'(a)$. See [**AbS72**] for facts about Bessel functions. $\square$

As we have an exponential decay we do not expect thick points in the sense above to exist, and have to look for a more modest definition of thick points. To obtain the right definition for thick points of the measure λ, we find a gauge function φ such that

$$0 < \sup_{x\in S} \limsup_{r\downarrow 0} \frac{\lambda\big(B(x,r)\big)}{\varphi(r)} < \infty,$$

and then define a point $x \in S$ to be *thick* if

$$\limsup_{r\downarrow 0} \frac{\lambda\big(B(x,r)\big)}{\varphi(r)} > 0.$$

This program was started by Dembo, Peres, Rosen and Zeitouni [**DPR00a, DPR00b, DPR01**], who analyse thick points for occupation measures and intersection local times in a series of papers. We start our review with the case of Brownian motion in transient dimensions.

Theorem [**DPR00a**] *If $d \geq 3$ and $p = 1$, then $\lambda = \lambda[1, d]$ satisfies, almost surely,*

$$\sup_{x\in S} \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]} = \limsup_{r\downarrow 0} \sup_{x\in S} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]} = \frac{4}{p_{d,1}^2},$$

where $p_{d,1}$ is as above. The corresponding dimension spectrum is

$$\dim\Big\{x \in \mathbb{R}^d : \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]} = a\Big\} = 2 - a\frac{p_{d,1}^2}{2}, \text{ for all } a \in \Big[0, \frac{4}{p_{d,1}^2}\Big].$$

A key fact in the proof is the *localization* of occupation measures: In the *transient* case large occupation measures are accumulated in a relatively *short time range* and the thick points are essentially those points where the time spent in a small ball around the point is relatively large already in a small time horizon. This fact leads to the necessary independence properties to go from tail behaviour to dimension spectra.

THEOREM [**DPR00b, DPR01**] *In the recurrent dimension $d = 2$ the intersection local time measures $\lambda = \lambda[p, 2]$ satisfy, almost surely,*

$$\sup_{x\in S} \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]^{2p}} = \limsup_{r\downarrow 0} \sup_{x\in S} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]^{2p}} = \Big(\frac{2}{p}\Big)^p.$$

and

$$\dim\Big\{x \in \mathbb{R}^d \,:\, \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]^{2p}} = a\Big\} = 2 - pa^{1/p}, \text{ for } a \in \Big[0, \Big(\frac{2}{p}\Big)^p\Big].$$

The case of planar Brownian motion is considerably more difficult, because no localization principle holds. Large occupation measures resp. intersection local times are accumulated in *long time ranges* and the thick points are essentially those points where small neighbourhoods are visited very frequently. The proof of [**DPR00b, DPR01**] investigates extremal behaviour of the crossing numbers (as introduced in Subsection 1.3.2 of the overview) and develops very interesting methods to deal with the dependence problems.

It remains to study the case of the intersections of two Brownian paths in $\mathbb{R}^3$. This is one of the main results of Chapter 3.

THEOREM **3.1.6** *There is a unique continuously differentiable function z on an interval $[0, \xi)$ satisfying $z(0) = 1$, $z'(0) = 0$ and*

$$z''(x) + 2\frac{z'(x)}{x} + z(x)^3 = 0, \text{ for all } x \in [0, \xi), \tag{20}$$

and a smallest number $a \in (0, \xi)$ with $z(a) = -az'(a)$. Let

$$\varrho^2 = \Big(\pi a \int_0^a z^4(s) s^2\, ds\Big)^{-1}.$$

Then, for $\lambda = \lambda[2, 3]$, almost surely,

$$\sup_{x\in S} \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} = \limsup_{r\downarrow 0} \sup_{x\in S} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} = \frac{\varrho^2}{4}.$$

Moreover,

$$\dim\Big\{x \in S \,:\, \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} = a\Big\} = 1 - \sqrt{a}\,\frac{2}{\varrho} \text{ for all } a \in \Big[0, \frac{\varrho^2}{4}\Big].$$

Remark: It seems that the nonlinear equation (20) cannot be solved explicitly. It is nevertheless interesting to compare it with the linear equation (19) arising in the Ciesielski-Taylor result.

In the proof of the latter result a localization principle as in the case $p = 1$ enters, however the calculation of the lower bounds for the dimension spectrum is more involved than in the case $p = 1$, because the set S cannot be naturally parametrized by an interval. Our proof mimicks the proof given for the corresponding problem in the case of thin rays of Galton-Watson trees, see above. We study the intersections of our random set with independent fractal percolation limit sets.

The main part of the proof however is the investigation of the upper tail behaviour of the random variables $\lambda(U(0,r))$ where completely new methods have to be developed.

The following theorem, the main result of Chapter 3, describes the upper tail asymptotics of $\lambda(U)$ for bounded sets $U \subset \mathbb{R}^d$ and all intersection local times $\lambda[p,d]$. It should be noted that

- in $d=2$ it is not directly related to thick points,
- if $p=1$ the result can be obtained from much easier, known methods.

THEOREM **3.1.1** *Suppose that p and d are positive integers satisfying $p < d/(d-2)$. Let U be an open bounded domain in $\mathbb{R}^d$ and $R \in (0,\infty]$ sufficiently large that $B(0,R)$ contains the closure of U. Denote by λ the intersection local time of p independent Brownian motions started in arbitrary fixed points inside U and stopped upon their first exit from the ball $B(0,R)$. Then*

$$\lim_{a\uparrow\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\big\{\lambda(U) > a\big\} = -\frac{p}{\varrho}, \tag{21}$$

where ϱ is defined as

$$\varrho = \sup\Big\{\big\langle g^{2p-1}, \mathfrak{A} g^{2p-1}\big\rangle \;:\; g \in L^{2p}(U) \text{ with } \|g\|_{2p} = 1\Big\}. \tag{22}$$

Here $\mathfrak{A} : L^{\frac{2p}{2p-1}}(U) \to L^{2p}(U)$ is the operator defined by $\mathfrak{A}f(x) = \int_U G(x,y) f(y)\,dy$ and G is the Green's function for Brownian motion stopped upon leaving $B(0,R)$.

Remark: In the special case $d \geq 3$, $U = U(0,1)$ and $R = \infty$ the variational characterization of ϱ can be replaced by the characterization in terms of the ordinary differential equation (19) if $p=1$ and (20) if $p=2$ and $d=3$.

The proof requires several steps, most of them involving analytical arguments. We now give a brief sketch of the main ideas and assume for simplicity that all Brownian motion are starting in the origin.

Step 1: *Tail asymptotics from moment asymptotics.*

On a heuristical level, one can derive a formula for the positive integer moments of $\lambda(U)$ from the symbolical formula (17). Indeed, denoting by $\mathfrak{S}_k$ the group of permutations of k symbols and letting $s_0 = 0$ and $y_{\sigma(0)} = 0$,

$$\begin{aligned}
\mathbb{E}\big[\lambda(U)^k\big] &= \mathbb{E}\Big[\Big(\int_U dy \prod_{j=1}^p \int_0^{T_j} ds\, \delta_y\big(W_j(s)\big)\Big)^k\Big] \\
&= \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^p \mathbb{E}\Big[\sum_{\sigma\in\mathfrak{S}_k} \int\!\cdots\!\int_{0\le s_1\le\cdots\le s_k\le T_j} \prod_{i=1}^k \delta_{y_{\sigma(i)}}(W_j(s_i))\,ds_i\Big] \\
&= \int_U dy_1 \cdots \int_U dy_k \Big[\sum_{\sigma\in\mathfrak{S}_k} \int\!\cdots\!\int_{0\le s_1\le\cdots\le s_k} \prod_{i=1}^k p_{s_i-s_{i-1}}\big(y_{\sigma(i-1)}, y_{\sigma(i)}\big)\,ds_i\Big]^p \\
&= \int_U dy_1 \cdots \int_U dy_k \Big[\sum_{\sigma\in\mathfrak{S}_k} \prod_{i=1}^k G\big(y_{\sigma(i-1)}, y_{\sigma(i)}\big)\Big]^p,
\end{aligned} \tag{23}$$

where $p_s(x,y)$ denotes the transition density of the *stopped* Brownian motion. Le Gall [**LeG87**] has made this argument rigorous. Thanks to this formula it is easier for us to deal with the asymptotic behaviour of $\mathbb{E}[\lambda(U)^k]$ as $k\to\infty$ instead of tail asymptotics. The relationship between the two is given by the following *Tauberian Theorem.*

LEMMA **3.2.3** *Let X be any nonnegative random variable and p a positive integer. Then for any $\kappa \in \mathbb{R}$ the following implication holds:*

$$\lim_{k\uparrow\infty} \frac{1}{k} \log E\Big[\frac{X^k}{(k!)^p}\Big] = -\kappa \quad \Longrightarrow \quad \lim_{a\uparrow\infty} a^{-1/p} \log P\{X > a\} = -pe^{\kappa/p} .$$

Step 2: *Core of the argument.*

Based on Le Gall's moment formula (23) we can show that

$$\lim_{k\uparrow\infty} \frac{1}{k} \log \mathbb{E}\Big[\frac{\lambda(U)^k}{(k!)^p}\Big] = - \inf_{\nu\in\mathcal{M}_1^*(U)} \Big\{ H\big(\overline{\nu}\,|\,\ell\big) + pH\big(\nu\,|\,\overline{\nu}\otimes\overline{\nu}\big) - p\big\langle\overline{\nu}, \log G\big\rangle \Big\}. \tag{24}$$

Here

$$H\big(\mu\,|\,\widetilde{\mu}\big) = \int_U \mu(dx) \log\Big(\frac{\mu(dx)}{\widetilde{\mu}(dx)}\Big)$$

is the *relative entropy* of the measure μ with respect to $\widetilde{\mu}$ on U, we denote by $\mathcal{M}_1^*(U)$ the set of probability measures ν on U^2 with equal marginals called $\overline{\nu}$. We write $\langle\mu, f\rangle$ for $\int_U f\, d\mu$.

We give a *very rough* sketch of the argument, which represents the idea of the proof but neglects almost all of the very considerable technical difficulties.

Given a vector $y = (y_1, \dots, y_k) \in U^k$ we define the *empirical measure*

$$L^1_{y,k} = \frac{1}{k}\sum_{i=1}^k \delta_{y_i}$$

and the *empirical pair measure*

$$L^2_{y,k} = \frac{1}{k}\sum_{i=1}^k \delta_{\{(y_{i-1},y_i)\}}.$$

The main observation is that the integrand on the right hand side of Le Gall's moment formula can be written as

$$\Big[\sum_{\sigma\in\mathfrak{S}_k}\prod_{i=1}^k G\left(y_{\sigma(i-1)}, y_{\sigma(i)}\right)\Big]^p \approx \Big[\sum_{\sigma\in\mathfrak{S}_k} \exp\Big(k\langle L^2_{y_\sigma,k}, \log G\rangle\Big)\Big]^p,$$

where y_σ is the vector y with the entries permuted by $\sigma \in \mathfrak{S}_k$.

We partition the set U into a finite number N of small sets and denote by $\tilde{U}$ the collection of partition sets. We project the empirical measures and pair measures onto $\tilde{U}$ and $\tilde{U}^2$ trusting that the loss due to this cruder point of view turns out to be negligible. From now on all empirical (pair) measures are the projected versions and we denote by $\mathfrak{L}^1_k$ the set of all empirical measure of vectors in U^k. Similarly let $\mathfrak{L}^2_k$ be the set of all empirical pair measures of vectors in U^k. It is important to note that the numbers of elements of both $\mathfrak{L}^1_k$ and $\mathfrak{L}^2_k$ are growing only polynomially in k. We can summarize the permutations that lead to the same empirical pair measure of y_σ and sum over the set $\mathfrak{L}^2_k$ instead.

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_k} \exp\Big(k\langle L^2_{y_\sigma,k}, \log G\rangle\Big) &= \sum_{\substack{\nu\in\mathfrak{L}^2_k\\ \overline{\nu}=L^1_{y,k}}}\sum_{\sigma\in\mathfrak{S}_k} 1_{\{L^2_{y_\sigma,k}=\nu\}} \exp\big(k\langle\nu, \log G\rangle\big)\\ &= \sum_{\substack{\nu\in\mathfrak{L}^2_k\\ \overline{\nu}=L^1_{y,k}}} \#\{\sigma\in\mathfrak{S}_k\,:\, L^2_{y_\sigma,k}=\nu\}\, \exp\big(k\langle\nu, \log G\rangle\big)\,.\end{aligned}$$

By combinatorial arguments one can see that, for any $\widetilde{y} \in U^k$ satisfying $L^2_{\widetilde{y},k} = \nu$ for some ν with $\overline{\nu} = L^1_{y,k}$, we have

$$\frac{1}{k!}\#\{\sigma \in \mathfrak{S}_k \,:\, L^2_{y_\sigma,k} = \nu\} = e^{o(k)}\,\frac{\left[\prod_{n=1}^N \big(k\overline{\nu}(n)\big)!\right]^2}{k!\prod_{n,m=1}^N \big((k\nu(n,m)\big)!}.$$

Stirling's formula yields that this ratio is $e^{o(k)}\exp\big(-kH(\nu\,|\,\overline{\nu}\otimes\overline{\nu})\big)$. Hence, we obtain

$$\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\exp\big(k\langle L^2_{y_\sigma,k},\log G\rangle\big) \approx e^{o(k)}\sum_{\substack{\nu\in\mathfrak{L}^2_k\\ \overline{\nu}=L^1_{y,k}}}\exp\Big(-k\big(H(\nu\,|\,\overline{\nu}\otimes\overline{\nu}) - \langle\nu,\log G\rangle\big)\Big).$$

The exponential rate of growth, as $k\to\infty$, of the sum on the right hand side is given by the maximal rate of the summands, because the number of summands is growing only polynomially. Thus,

$$\frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\exp\big(k\langle L^2_{y_\sigma,k},\log G\rangle\big) \approx e^{o(k)}\exp\Big(-k\inf_{\substack{\nu\in\mathfrak{L}^2_k\\ \overline{\nu}=L^1_{y,k}}}\big\{H(\nu\,|\,\overline{\nu}\otimes\overline{\nu}) - \langle\nu,\log G\rangle\big\}\Big).$$

Now we have to integrate the pth power of this expression. We get,

$$\begin{aligned}\mathbb{E}\Big\{\frac{\lambda(U)^k}{(k!)^p}\Big\} &\approx e^{o(k)}\int_U dy_1\cdots\int_U dy_k\,\exp\Big(-kp\inf_{\substack{\nu\in\mathfrak{L}^2_k\\ \overline{\nu}=L^1_{y,k}}}\big\{H(\nu\,|\,\overline{\nu}\otimes\overline{\nu}) - \langle\nu,\log G\rangle\big\}\Big)\\ &\approx e^{o(k)}\sum_{\mu\in\mathfrak{L}^1_k}\ell\big\{(y_1,\dots,y_k)\in U^k \,:\, L^1_{y,k}\approx\mu\big\}\\ &\qquad\qquad \exp\Big(-kp\inf_{\substack{\nu\in\mathcal{M}_1^*(U)\\ \overline{\nu}=\mu}}\big\{H(\nu\,|\,\overline{\nu}\otimes\overline{\nu}) - \langle\nu,\log G\rangle\big\}\Big).\end{aligned}$$

By Sanov's Theorem we have,

$$\ell\big\{(y_1,\dots,y_k)\in U^k \,:\, L^1_{y,k}\approx\mu\big\} \approx e^{o(k)}\exp\Big(-kH\big(\mu\,|\,\ell\big)\Big).$$

Using this and again the fact that the number of elements in $\mathfrak{L}^1_k$ is growing polynomially in k,

$$\mathbb{E}\Big\{\frac{\lambda(U)^k}{(k!)^p}\Big\} \approx e^{o(k)}\exp\Big(-k\inf_{\mu\in\mathcal{M}_1(U)}\inf_{\substack{\nu\in\mathcal{M}_1^*(U)\\ \overline{\nu}=\mu}}\big\{H(\mu\,|\,\ell) + pH(\nu\,|\,\mu\otimes\mu) - p\langle\nu,\log G\rangle\big\}\Big),$$

which is the same formula as in the claim.

Step 3: *Formulation of the variational problem (22).*

The combinatorial argument of the previous step gives us a description of θ in terms of the very unpleasant variational problem on the right in (24), which is not tractable with standard methods from variational calculus. But it can be replaced by the much more pleasant variational problem (22). To see this we show that, for every $\mu\in\mathcal{M}_1(U)$,

$$\exp\Big(-\frac{1}{p}\inf_{\substack{\nu\in\mathcal{M}_1^*(U)\\ \overline{\nu}=\mu}}\Big\{H\big(\mu\,|\,\ell\big) + pH\big(\nu\,|\,\mu\otimes\mu\big) - p\big\langle\nu,\log G\big\rangle\Big\}\Big) \le \langle g^{2p-1},\mathfrak{A}g^{2p-1}\rangle\,, \tag{25}$$

where g^{2p} is the density of μ. The key here is Jensen's inequality and a variational characterization of the entropy term $H(\nu \,|\, \mu \otimes \mu)$. Equality holds in (25) if and only if there exists some $\varrho > 0$ with

$$\mathfrak{A} g^{2p-1}(x) = \varrho g(x) \quad \text{for } \mu\text{-almost every } x \in U. \tag{26}$$

Then we argue that the maximizers of the right hand side satisfy (26), see below, and obtain

$$\begin{aligned} &\sup\Big\{\big\langle g^{2p-1}, \mathfrak{A} g^{2p-1}\big\rangle \,:\, g \in L^{2p}(U) \text{ with } \|g\|_{2p} = 1\Big\} \\ &\qquad = \exp\Big(-\frac{1}{p} \inf_{\nu \in \mathcal{M}_1^*(U)} \Big\{H\big(\overline{\nu} \,|\, \ell\big) + pH\big(\nu \,|\, \overline{\nu} \otimes \overline{\nu}\big) - p\big\langle \overline{\nu}, \log G\big\rangle\Big\}\Big). \end{aligned}$$

This proves the representation of the limit in Theorem **3.1.1**.

Step 4: *Solution of the variational problem (22).*

Once we have characterized the limit in (21) in terms of the variational problem (22) one can apply methods from variational calculus to solve it. We now restrict our attention to the case of unstopped Brownian motions in $\mathbb{R}^d$ for $d \geq 3$.

The steps in this procedure can be sketched as follows:

- The supremum in (22) is attained. This follows from the fact that
$$\{f \in L^1(U) \,:\, f \geq 0,\, \|f\|_{2p/(2p-1)} \leq 1\}$$
is weakly compact and $f \mapsto \langle f, \mathfrak{A} f\rangle$ is weakly upper semicontinuous on this set.
- The maximizers g of (22) are bounded from 0. This can be checked explicitly.
- A perturbation approach (the Euler-Lagrange equation) shows that the maximizers g satisfy the nonlinear eigenvalue equation (26).
- g is twice continuously differentiable and can be extended to a continuously differentiable function on $\mathbb{R}^d$. It satisfies the differential equation
$$\frac{1}{2}\Delta g(x) = -\frac{1}{\varrho} g^{2p-1}(x) 1_U(x) \text{ for all } x \notin \partial U. \tag{27}$$
with boundary condition $\lim_{x\to\infty} g(x) = 0$. This follows from Itô calculus and potential theory arguments.
- As we do not stop the Brownian motions, the Green's function G depends only on the difference of the arguments. Now we assume $U = B(0,1)$ and recall that this is the situation needed in the proof of Theorem **3.1.6**. By Riesz' rearrangement inequality we can see that all maximizers g of (22) are necessarily rotationally symmetric. In this case (27) can be translated into an ordinary differential equation. If $d = 3$, $p = 2$ this equation can be transformed into (20), which can be easily shown to have (locally) a unique solution. If $d \geq 3$ and $p = 1$ one gets equation (19).

2.3. Large increments of particles in a historical super-Brownian motion. This subsection reviews the results of Chapter 4. We discuss exceptionally large increments of the paths in a system consisting of a continuum of generic Brownian particles, the historical super-Brownian motion. If we study the increments of the particles we may not only look at exceptional times, but additionally at exceptional particles from the (uncountable) population. This may lead to a different quality of exceptional behaviour and we give a thorough investigation of the phenomena that arise.

To put things into context we first look at a different example and recall a well-known result of Fukushima [**Fuk84**]. Let

$$W : [0, \infty) \times [0, \infty) \to \mathbb{R}, (\tau, t) \to W(\tau, t)$$

be a *Brownian sheet*, i.e. a centred Gaussian process with continuous paths and

- $W(\tau, 0) = W(0, t) = 0$ for all $t, \tau \geq 0$, and
- $\mathbb{E}\big\{W(\tau, t)W(\tau', t')\big\} = (\tau \wedge \tau')(t \wedge t')$.

Then, for each fixed $\tau \geq 1$, the process

$$W_\tau : t \mapsto \frac{1}{\sqrt{\tau}} W(\tau, t)$$

is a Brownian motion. We may thus consider the Brownian sheet as a continuum of generic Brownian particles parametrized by the parameter $\tau \geq 1$. Fukushima [**Fuk84**] answered the question about the possibility of larger increments in this continuum of particles in the negative. He showed that, for $T > 0$, almost surely,

$$\sup_{\tau \geq 1} \sup_{0 \leq t \leq T} \limsup_{h \downarrow 0} \frac{|W_\tau(t+h) - W_\tau(t)|}{\sqrt{h \log(1/h)}} = \sqrt{2}.$$

It is worth noting that for a number of related problems the present system of particles *does* show behaviour different from single Brownian particles, e.g. the for the problem of existence of multiple points in various dimensions, see [**LyT86**], [**Pen89**].

In contrast to the example above it turned out in the work of Dawson and Perkins [**DaP91**] that the choice of both exceptional particles *and* exceptional times in a historical super-Brownian motion makes a more extreme behaviour possible. This more extreme behaviour, however, is only visible on the level of a very fine analysis, it is not reflected in a different gauge function compared to the case of a single Brownian particle, but only in different constants appearing in the dimension spectrum.

2.3.1. *Historical super-Brownian motion.* Before formulating the results on increments of super-Brownian motion, we give an intuitive description of the model of historical super-Brownian motion. Super-Brownian motion is the high density limit of a system of branching Brownian motions with a critical, finite variance branching mechanism. It has been an object of intensive research in the last fifteen years, leading to interesting results, for example about its fractal geometry and its potential theoretic properties. Recent deep results emphasize its importance as the natural scaling limit of many interacting particle systems, see e.g. [**CDP99**].

Let N be a positive integer and place N particles in the origin of $\mathbb{R}^d$. Each particle moves independently according to Brownian motion up to time $\frac{1}{4N}$. At this time each particle is replaced by either 0 or 2 particles at its position. This choice is made independently for each particle and both possibilities have probability $1/2$. Every new particle now follows the same rules for the next $\frac{1}{4N}$ time units. Note that, almost surely, after a finite time the particle population has died.

We represent the state of the process at time t by a finite measure

$$\mathfrak{X}^N(t) = \frac{1}{N} \sum \delta_{\big\{W_i : [0,t] \to \mathbb{R}^d\big\}}$$

on the space

$$\mathfrak{P}(t) = \Big\{ w : [0, t] \to \mathbb{R}^d \text{ continuous } \Big\},$$

which we consider as canonically embedded into the space

$$\mathfrak{P} = \Big\{ w : [0,\infty) \to \mathbb{R}^d \text{ continuous with } w(s) = w(\zeta) \text{ for all } s \geq \zeta \geq 0 \Big\}$$

of *stopped paths*. With every path w we associate the minimal possible $\zeta = \zeta(w)$, which is the lifetime of the path. We equip $\mathfrak{P}$ with the metric

$$d\big(w_1, w_2\big) = \|w_1 - w_2\|_\infty + \sqrt{\big|\zeta(w_1) - \zeta(w_2)\big|}\,.$$

Letting $N \to \infty$ at the same time

- increases the number of starting particles,
- decreases the individual mass of a particle,
- increases the branching activity of the particles.

As $N \to \infty$ the finite dimensional distributions of the measure valued process $\mathfrak{X}^N = \{\mathfrak{X}^N(t) : t \geq 0\}$ converge to the finite dimensional distributions of a measure-valued process. This process is a Markov process whose value at each time $t > 0$ is a finite measure on the space $\mathfrak{P}(t)$ of stopped paths w with lifetime $\zeta(w) = t$. It has a continuous version $\{\mathfrak{X}(t) : t \geq 0\}$, the *(historical) super-Brownian motion.* See [**DaP91**] for proofs and more background.

It is important to note that, by passing to the limit, we have moved from finite families of paths to a continuum of infinitesimal paths. Intuitively, each path in $\operatorname{supp} \mathfrak{X}_t$ represents the trail of a particle alive at time t, so that the model describes the individual movements of all the particles. The historical setting also contains the complete description of the genealogical structure of the super-Brownian motion: Any two particles $w_1, w_2 \in \operatorname{supp} \mathfrak{X}_t$ are obtained from the branching of the particle $w : [0,\infty) \to \mathbb{R}^d$ at time $\zeta = \sup\{r : w_1 = w_2 \text{ on } [0,r]\}$ with $w(s) = w_1(s) = w_2(s)$ for all $s \leq \zeta$ and $w(s) = w(\zeta)$ for all $s \geq \zeta$.

2.3.2. *How fast are the particles of super-Brownian motion?* We return to the question of extremal increments of the particles of historical super-Brownian motion. Thanks to a remarkable result of Dawson and Perkins [**DaP91**] we know that in the present model special particles can move faster than a Brownian motion. More precisely, almost surely,

$$\sup_{s>0} \sup_{w \in \operatorname{range} \mathfrak{X}} \limsup_{h \downarrow 0} \frac{|w(s) - w(s-h)|}{\sqrt{h \log(1/h)}} = 2\,, \tag{28}$$

where $\operatorname{range} \mathfrak{X} \subset \mathfrak{P}$ is the closed range of the process $\mathfrak{X}$. Note that taking a supremum over a continuum of particles does *not* lead to a different gauge function, but only to a different constant on the right hand side of (28) when compared to (8).

My aim was to get a deeper understanding of this constant and the relationship to Lévy's modulus (8). We therefore consider a Borel set $E \subset (0,\infty)$ of times and ask for the maximum speed of all particles at all times in E. It turns out that the maximum speed of all particles at all times in E is related to the packing dimension, denoted $\dim_P(E)$, of the set $E \subset \mathbb{R}$. The crucial rôle of packing dimension in hitting problems for a certain class of random fractals is pointed out in [**KPX00**].

The most natural definition of *packing dimension* is based on the family of s-dimensional packing measures. A δ-packing of the set E is a collection of disjoint balls with radii $\leq \delta$ centred in E and s-dimensional packing measure $\mathcal{P}^s$ can be defined in two steps

- $P^s(E) = \lim_{\delta \downarrow 0} \sup \Big\{ \sum_{i=1}^\infty r_i^s \,:\, (B(x_i, r_i)) \text{ a } \delta\text{-packing of } E \Big\}$,

- $\mathcal{P}^s(E) = \inf\Big\{\sum_{i=1}^{\infty} P^s(E_i) \,:\, E = \bigcup_{i=1}^{\infty} E_i\Big\}$.

The packing dimension of E is the critical parameter s where the packing measure switches from infinity to 0,

$$\dim_P E = \inf\{s > 0 \,:\, \mathcal{P}^s(E) = 0\} = \sup\{s > 0 \,:\, \mathcal{P}^s(E) = \infty\}.$$

THEOREM **4.1.1** *Suppose that $\{\mathfrak{X}_t\}$ is a super-Brownian motion in dimension $d \geq 1$ and $E \subset (0,t)$ a Borel set. Then, almost surely on $\{\mathfrak{X}_t \neq 0\}$,*

$$\sup_{s\in E}\sup_{w\in\operatorname{range}\mathfrak{X}}\limsup_{h\downarrow 0}\frac{|w(s)-w(s-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2+2\dim_P(E)}\,.$$

This result includes (28) and also an observation of Verzani [**Ver95**] about the maximal speed of all particles at a particular instant in time. Verzani notes that, at any fixed time $t > 0$, almost surely on $\{\mathfrak{X}_t \neq 0\}$,

$$\sup_{w\in\operatorname{range}\mathfrak{X}}\limsup_{h\downarrow 0}\frac{|w(t)-w(t-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2}, \tag{29}$$

and suggests a proof analogous to the method of [**DaP91**], which is quite different from ours. Our proof of Theorem **4.1.1** sheds some light on his question why the constants in (8) and (29) coincide, see the remark following Lemma **4.2.2**.

The second result of Chapter 4 determines a *Hausdorff-dimension spectrum* for fast paths in the range and support of historical super-Brownian motion, which is analogous to the dimension spectrum (9) of Orey and Taylor. We say that a path $w \in \operatorname{range}\mathfrak{X}$ with lifetime $\zeta = \zeta(w)$ is *a-fast* if

$$\limsup_{h\downarrow 0}\frac{|w(t)-w(t-h)|}{\sqrt{h\log(1/h)}} \geq a\,.$$

Here and throughout this thesis dim stands for the Hausdorff dimension of a metric space.

At a first glance one might wonder whether such a collection of exceptional paths has any chance of having finite dimension as the space $\mathfrak{P}$ of stopped paths clearly has infinite Hausdorff dimension. But the range of a super-Brownian motion is, in a sense, only a small subset of $\mathfrak{P}$. For instance, if one takes all paths of length at least $t > 0$ in the range and restricts these paths to the interval $[0, t-\varepsilon]$, the resulting collection of paths $f : [0, t-\varepsilon] \to \mathbb{R}^d$ is only finite. Indeed, it is known that the range of historical super-Brownian motion has dimension no more than 4.

We consider a closed set $E \subset (0,\infty)$ of times and denote the *range of E* by

$$\mathfrak{R}(E) = \overline{\bigcup_{t\in E}\operatorname{supp}\mathfrak{X}_t}\,.$$

Intuitively this is the collection of particles with total lifetime in the set E. The result of Theorem **4.1.1** can be reformulated as

$$\sup_{w\in\mathfrak{R}(E)}\limsup_{h\downarrow 0}\frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2+2\dim_P(E)}\,.$$

We can give bounds for the dimension of the set of a-fast paths in the range of E, which are sharp for all closed sets E, for which Hausdorff- and packing dimension agree.

Theorem **4.1.3** *Suppose that $E \subset (0,t)$ is a fixed closed set and $\{\mathfrak{X}_t\}$ is a historical super-Brownian motion in dimension $d \geq 2$. Then, almost surely on $\{\mathfrak{X}_t \neq 0\}$, for all $a \in [0, \sqrt{2+2\dim_P(E)}]$,*

$$2+2\dim(E)-a^2 \leq \dim\Big\{w \in \mathfrak{R}(E) \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} \leq 2+2\dim_P(E)-a^2\,.$$

Moreover, if $\dim_P(E) < a^2/2-1$ the set of a-fast paths in $\mathfrak{R}(E)$ is empty. In particular, almost surely for all $a \in [0,2]$,

$$\dim\Big\{w \in \operatorname{range}\mathfrak{X} \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} = 4-a^2\,,$$

and, for all $t>0$, almost surely on $\{\mathfrak{X}_t \neq 0\}$, for all $a \in [0,\sqrt{2}]$,

$$\dim\Big\{w \in \operatorname{supp}\mathfrak{X}_t \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} = 2-a^2\,.$$

The proofs of Theorems **4.1.1** and **4.1.3** are based on three essential techniques.

I. The Brownian snake representation of historical super-Brownian motion.

This representation is due to Le Gall, see e.g. [**LeG93**] or [**LeG99**]. From the point of view of this subsection the *Brownian snake* is a natural parametrization

$$W:[0,\tau] \to \operatorname{range}\mathfrak{X} \subset \mathfrak{P}$$

of the range of (historical) super-Brownian motion by an interval (of positive, but random length). An extension of this representation is also crucial for some parts of Chapter 5, and we give some insight in the final section of this overview.

II. A uniform dimension formula for the Brownian snake.

To use the snake representation for the derivation of dimension spectra we need to understand the relation of the Hausdorff dimension of certain sets $A \subset [0,\tau]$ and their image $W(A) \subset \mathfrak{P}$.

As these sets A typically depend on the random mapping W in a very complicated way, we are looking for *uniform* dimension results: we hope that, in sufficiently high dimensions, the dimension $W(A)$ depends on the dimension of A only, and not on the geometry of A or the random mapping W. A precedent for such nice behaviour is the following classical *dimension doubling theorem* of Kaufman [**Kau69**], which plays a crucial rôle in the study of dimension spectra of Brownian paths.

Theorem [**Kau69**] *Let B be Brownian motion in $\mathbb{R}^d$ for $d \geq 2$. Then, almost surely for all $A \subset [0,\infty)$,*

$$\dim(B(A)) = 2\dim(A)\,.$$

Obviously this result fails to hold in dimension 1, consider the level sets of a Brownian motion.

For our investigation we provide a similar result for the parametrization of historical super-Brownian motion by means of the Brownian snake W. Recall that in the space $\mathfrak{P}$ of stopped paths we associate with every path $w \in \mathfrak{P}$ a (minimal) lifetime $\zeta(w) \geq 0$ and look at the real-valued process $\zeta : t \mapsto \zeta(t) := \zeta(W(t))$.

THEOREM **4.2.3** *Let W be a Brownian snake in dimension $d \geq 2$. Then, almost surely for all $A \subset [0, \tau]$ with $0 \notin \zeta(A)$,*

$$\dim(W(A)) = 4 \dim(A) .$$

Remarks:

- Serlet [**Ser95**] has proved this (in an equivalent formulation) for the case $d \geq 4$. Reducing this to $d \geq 2$ needs a significant refinement of his argument.
- The set of all $t \in [0, \tau]$ with the condition $\zeta(t) = 0$ corresponds heuristically to the initial family of particles placed in the origin. Hence it cannot be included in the regular dimension behaviour of W.

III. The theory of limsup random fractals.

This theory of Khoshnevisan, Peres and Xiao [**KPX00**] is a powerful tool to study Hausdorff dimension for a wide class of random fractals arising in the study of dimension spectra. It may be applied in the study of dimension spectra for random sets which are parametrized by a non-random set.

To define limsup fractals denote, for every $n \geq 1$, by $\mathcal{D}_n$ the collection of all compact dyadic subcubes of $[0,1]^d$ of sidelength 2^{-n}. Suppose further that we have a collection $\{Z_n(I) \, : \, I \in \mathcal{D}_n\}$ of random variables taking values in $\{0,1\}$. The associated *discrete limsup random fractal* is defined as $A := \limsup_{n\to\infty} A_n$, where

$$A_n = \bigcup_{\substack{I \in \mathcal{D}_n \\ Z_n(I)=1}} \operatorname{int}(I) ,$$

and $\operatorname{int}(I)$ denotes the interior of I. The framework of Khoshnevisan, Peres and Xiao is based on two assumptions:

- The *index assumption* states that for each $n \geq 1$ the distribution of $Z_n(I)$ is the same for each $I \in \mathcal{D}_n$ and that
$$\lim_{n\to\infty} \frac{\log_2 \mathbb{E}\{Z_n(I)\}}{n} = -\gamma$$
for some $\gamma > 0$, where $\log_2$ is the logarithm of base 2.
- The *correlation assumption* states that for each fixed n the random variables $Z_n(I)$ and $Z_n(J)$ are only weakly correlated for most pairs $I, J \in \mathcal{D}_n$. More precisely, for each $\varepsilon > 0$ define
$$f(n,\varepsilon) = \max_{I\in\mathcal{D}_n} \#\Big\{ J \in \mathcal{D}_n \, : \, \operatorname{Cov}\{Z_n(I), Z_n(J)\} \geq \varepsilon \, \mathbb{E}\{Z_n(I)\} \, \mathbb{E}\{Z_n(J)\} \Big\}.$$
Then, for all $\varepsilon > 0$,
$$\limsup_{n\to\infty} \frac{\log_2 f(n,\varepsilon)}{n} = 0 .$$

THEOREM [**KPX00**] *Suppose that A is a discrete limsup random fractal satisfying the two assumptions above. Then, for every analytic set $D \subset [0,1]^d$,*

$$\mathbb{P}\Big\{ A \cap D \neq \emptyset \Big\} = \begin{cases} 1 & \text{if } \dim_P(D) > \gamma, \\ 0 & \text{if } \dim_P(D) < \gamma. \end{cases} \tag{30}$$

Moreover, almost surely,

$$\dim(D) - \gamma \leq \dim(A \cap D) \leq \dim_P(D) - \gamma . \tag{31}$$

This result subsumes and refines earlier methods and approaches based on the method of intersecting the set with an independent random set, e.g. the work of Taylor [**Tay66**], Hawkes [**Haw81**], Perkins [**Per83**], Peres [**Per96**] or Khoshnevisan and Shi [**KhS00**].

Remark: Nontrivial limsup fractals are always dense in the unit cube. Hence, for example, subsets of the intersection set of two Brownian paths in $\mathbb{R}^3$ cannot be limsup-fractals. In the case of a single Brownian path or the super-Brownian motion we have a natural parametrization by an interval in such a way that the exceptional *parameters* are approximately limsup fractals.

The setup of the proof of Theorems **4.1.1** and **4.1.3** is now clear:

(i) We use approximations by limsup-fractals to establish hitting estimates (30) and dimension estimates (31) for the set of parameters $s \in [0, \tau]$ such that $W(s)$ is an a-fast path. Whereas the index assumption is very easy to check, the crucial part of the work is to verify the correlation assumption in this context.
(ii) The hitting estimates together with a result about the packing dimension of inverse images under linear Brownian motion, Lemma **4.2.2**, immediately imply Theorem **4.1.1**.
(iii) The dimension spectrum is obtained by applying the uniform dimension result, Theorem **4.2.3**, to the dimension estimates for the exceptional parameter sets.

3. Superprocesses in random catalytic media

The first four chapters are concerned with sets and measures with an *irregular* or *fractal* behaviour. The geometry of these objects is studied locally in the neighbourhood of typical points and, in special probabilistic models, globally by an analysis of the possibilities of extreme behaviour.

A typical situation where such irregular objects may play a rôle is as a medium influencing the behaviour of a process interacting with this medium. Situations that one can consider are: the medium represents soil, the process is the movement of a drop of liquid in this (possibly porous) medium; or the medium is a food supply distributed in space, the process indicates the temporal and spatial evolution of an animal population; or the medium is a catalyst, the process some substance undergoing a chemical reaction in the presence of the catalyst. In many cases of interest both the process and the medium are random or may be modelled as random objects.

There exists a substantial body of research on stochastic systems in random media or random environments. All these systems have in common that there are *two levels of randomness*: the medium is itself a random object and, given the medium, a random process is evolving with a law depending on the medium. The conditional law of the process given the medium is frequently called the *quenched law*, the unconditional law of the process has the name *annealed law*. Although the systems that are rigorously studied at the moment are mostly still toy models, they allow valuable insight into the possible behaviour of more realistic systems and offer case studies for possible effects of irregular or random media on the qualitative behaviour of processes. Two different kinds of observations have been made frequently:

- In many cases, the systems can be described by use of an *averaging principle*. Roughly speaking such a principle predicts that the behaviour does not change much if we replace the randomly varying medium by a constant averaged medium,
- in other cases the random medium leads to completely *new effects*, which do not occur in the case of a regular or deterministic medium.

Examples of new effects occurring due to the influence exerted by the presence of an irregular or random medium are:

- Sinai [**Sin82**] and Kesten [**Kes86**] construct random walks $\{S_n : n \geq 0\}$ in (different) random environments and study the long-term properties. In both cases the random walks exhibit *subdiffusive* behaviour. E.g. in [**Sin82**] it is shown that for some $\varphi(n)$ with
$$\frac{\varphi(n)}{\sqrt{n}} \longrightarrow 0$$
we have that, in probability with respect to the medium,
$$\frac{S_n}{\varphi(n)} \text{ converges in distribution.}$$
- Solutions of evolution equations in a random medium frequently have the property that with increasing time large spatial irregularities develop, a phenomenon sometimes called *intermittency*. Fleischmann and Molchanov [**FlM90**] discuss such a model, which exhibits *complete localization* as an extreme kind of behaviour: In the long term the total mass of the system is approximately concentrated in the record point of the random medium. Similarly, for the random walk of [**Sin82**] the asymptotic distribution of $S_n/\varphi(n)$ is concentrated on a single point depending on the realization of the random environment.

Chapter 5 deals with a model for a super-Brownian motion interacting with a spatially irregular medium. The irregular medium we are dealing with is a stable measure, which is of course considerably simpler than the measures, which were the objects of study in the previous chapters. The process interacting with the medium is a super-Brownian motion and the interaction takes place via the branching activity of the particles. Heuristically, the rate of branching of each infinitesimal particle is given by the amount of contact between the path of the particle and the medium, which in this case acts as a catalyst. We study the long term behaviour and observe considerable qualitative differences in a comparison with the classical super-Brownian motion.

3.1. Super-Brownian motion in a stable catalytic medium. The model under consideration, the super-Brownian motion in a stable catalytic medium, was introduced by Dawson and Fleischmann in [**DaF91**]. In this and the following subsections we use the notation from Chapter 5, namely that various probabilities and corresponding expectations (for example for the random medium, for the process in the given medium) are distinguished by different fonts (for example, P, E and $\mathbb{P}^\Gamma, \mathbb{E}^\Gamma$).

We first have a closer look at the choice of the medium Γ, which under the law P is a *stable measure* of index $0 < \gamma < 1$. Γ is the random measure on $\mathbb{R}$ such that, for all $\varphi : \mathbb{R} \to [0, \infty)$, we have
$$\mathsf{E}\big\{\exp\langle \Gamma, -\varphi\rangle\big\} = \exp\Big(- \int_{\mathbb{R}} \varphi(x)^\gamma \, dx \Big).$$
In other words, the distribution function of Γ is a stable subordinator of index $0 < \gamma < 1$. The main properties of Γ are

- Γ is a purely atomic measure,
- the atoms of Γ are densely located in $\mathbb{R}$,
- for every $a < b$ we have $\Gamma(a, b) < \infty$ but $\mathsf{E}\Gamma(a, b) = \infty$,
- $\Gamma(I)$ and $\Gamma(J)$ are independent if $I \cap J = \emptyset$.

It should be stressed that it is essential that the medium is a singular (hence, in a sense, irregular) measure. An absolutely continuous measure as a catalyst does not change the qualitative behaviour of the catalytic super-Brownian motion, see [**DaF91**].

Heuristically, the random measure Γ enters in the law of the superprocess as follows: Given the sampled Γ, an infinitesimal particle of $X = X[\Gamma]$ moves according to a Brownian path W. The total amount of collisions with the random medium up to time t is recorded by the collision local time

$$L_{[\Gamma,W]}(t) = \int \Gamma(dy)\, \mathsf{L}^y(t)\,,$$

where $\mathsf{L}^y(t)$ is the local time of the Brownian motion W at level y. Moreover, the particles undergo critical branching with quadratic branching mechanism at a varying rate proportional to the collision local time.

In Chapter 5 the rigorous construction of the measure valued Markov process $X = X[\Gamma]$ in a state space $\mathcal{M}_{\text{tem}}$ of tempered measures is described. The state space $\mathcal{M}_{\text{tem}}$ includes all finite measures on $\mathbb{R}$, but also the Lebesgue measure. It suffices here to indicate that the heuristic description above is reflected in the Laplace transition functional in the given medium. Let $\varphi : \mathbb{R} \to [0, \infty)$ be measurable, then given Γ,

$$\mathbb{E}^\Gamma \left\{ \exp\langle X_t\,, -\varphi\rangle \;\middle|\; X_s = \mu \right\} = \exp\langle \mu, -V^\Gamma_{t-s}\varphi\rangle, \quad \text{for } t > s \geq 0,$$

where $\mu \in \mathcal{M}_{\text{tem}}$ is a tempered measure and the function v defined by $v(t,x) = V^\Gamma_t \varphi(x)$ solves the symbolic partial differential equation

$$\frac{\partial}{\partial t} v = \frac{1}{2}\, \frac{\partial^2}{\partial x^2} v - 2\,\Gamma v^2, \quad \text{with initial condition } v\big|_{t=0} = \varphi.$$

Of course, this equation does not make sense rigorously, but in can be turned into a well-defined integral equation and existence and uniqueness of nonnegative solutions of this equation were established in [**DaF92**]. The process X was constructed in [**DaF91**]. Some of its essential properties are the following:

- [**DaF91**] The process is *persistent* in the sense that, if $X_0 = \ell$, for almost every medium Γ,

$$\mathbb{E}^\Gamma \left\{ X_t(A) \right\} = \ell(A) \text{ for all Borel sets } A \subset \mathbb{R}.$$

- [**DaF97**] The process X has a continuous version.
- [**DFR91**] For each $t > 0$ the state X_t is almost surely an absolutely continuous measure.
- [**DLM95**] Started with μ of compact support we have that the global support,

$$\bigcup_{t>0} \text{supp } X_t$$

  of X is compact almost surely.
- [**DFM00**] Started in a finite measure, the process X dies in finite time almost surely.

Finally, it should be noted that the spatial dependence of the branching rate introduces *dependence* between the branching activity of particles. Under the *annealed* law, a high branching activity of a collection of particles in a certain spatial region increases the probability that a high branching activity of other particles in the same region more probable.

However, we always use a *quenched* approach in dealing with the model $X = X[\Gamma]$ in the random medium Γ. We *first* sample the catalyst Γ as a stable random measure and then study the reactant process $X[\Gamma]$ in its conditional law given the catalyst Γ.

3.2. The Brownian snake representation of super-Brownian motion in a catalytic medium. In this subsection we show how the Brownian snake, introduced by Le Gall in [**LeG91, LeG93**] can be used in the study of catalytic super-Brownian motion.

I would like to use the opportunity to give a very brief introduction to the circle of ideas around the Brownian snake construction, starting from some well-known discrete arguments and ending with the new ideas on catalytic super-Brownian motion.

The *key idea* of Le Gall is that the *genealogy*, or family structure, of the offspring of a single progenitor can be coded by a *Brownian excursion*. There is a discrete analogue of this idea, attributed to Harris, which can be described easily.

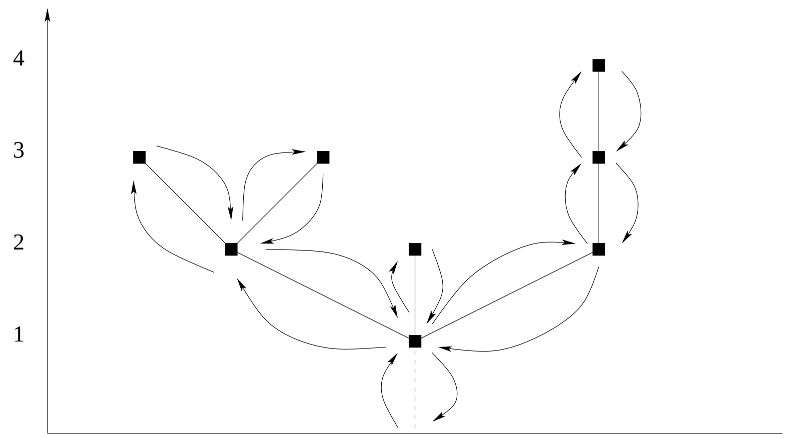

FIGURE 3. Galton-Watson tree with geometric offspring distribution.
The walk through the tree is indicated by arrows.

Suppose that we are dealing with a Galton-Watson branching process with *geometric offspring distribution*, i.e. the offspring variable satisfies $P\{N = k\} = 2^{-k-1}$ for all integers $k \geq 0$. The genealogy of such a branching process can be represented by the corresponding Galton-Watson tree, see Figure 3. This tree is awkward to handle, in particular it seems unclear how the limiting procedure in the construction of the classical super-Brownian motion can be extended to include the convergence, in some sense, of the tree coding the genealogical information.

The idea is to represent the tree as a process by walking through the tree in a particular manner and recording the generation number at each step, with the root being generation number one. We do this in the following way: In the first step we move to the root, then in each of the following steps we move to the leftmost descendant of the current vertex, always recording the generation of the vertex. If a vertex does not have any offspring we move back to its immediate ancestor and then, in the next step, to the leftmost child that has not been visited before. If all children of the current vertex have been visited, we move back to the immediate ancestor, and so forth until we have reached the root and all vertices have been visited.

The score of this procedure in the case of the tree of Figure 3 is shown in Figure 4. The fat squares correspond to first visits of the corresponding vertex, revisited vertices are indicated by grey squares.

It is quite easy to see from the *lack of memory* property of the geometrical distribution that the resulting process is a random walk excursion, if the original tree was a Galton-Watson process

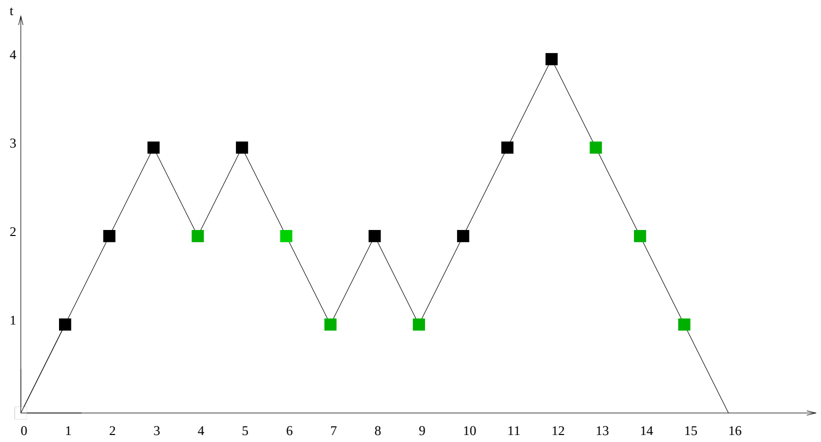

FIGURE 4. Random walk excursion associated to the tree above.

with geometrical offspring distribution. The tree can be recovered easily from the excursion, but also the main features of the tree can be seen directly from it. For example, the total number of particles in generation n is represented by the amount of upcrossings of the interval $(n-1,n)$ and the lifetime of the tree is the maximum of the excursion minus one. A particularly important observation is the following: The generation of the last common ancestor of the two particles visited in the lth and kth step of the excursion, for $l \leq k$, is given by the minimum of the excursion on the time interval $[l,k]$.

It should be pointed out that in the discrete case the nice distributional properties of the "score process" are quite special to the choice of a geometric offspring distribution. However, when we move to the continuous setting, all critical, finite variance offspring laws lead to the same limiting continuous state branching process, the *Feller diffusion*, and it is plausible that this process should inherit the possibility of an excursion representation from the geometric offspring case.

Indeed, the genealogy of a Feller diffusion can be represented by a Brownian excursion. A Brownian excursion is the scaling limit of the random walk excursion, but for its definition one has to leave the realm of probability measures and work with σ-finite measures in the framework of *Itô excursion theory*, the basics of which will be used now. For the moment we denote by $\mathbb{N}$ the *Itô excursion measure*, by ζ a Brownian excursion and by σ its length. We suppose that $\mathbb{N}$ is normalized such that

$$\mathbb{N}\Big\{ \sup_{s\in[0,\sigma]} \zeta_s > \varepsilon \Big\} = \frac{1}{2\varepsilon}.$$

We look at a Poisson point process Π with intensity measure $\mathbb{N}$ and write $\mathsf{L}_s^t = \mathsf{L}_s^t[\zeta]$ for the local time at level t at time $s \in [0,\sigma]$ of the excursion ζ. Combining well-known facts from Itô excursion theory and the first *Ray-Knight Theorem*, see e.g. [**ReY98**], shows that the process $Z = \{Z_t : t \geq 0\}$ with $Z_0 = 1$ and

$$Z_t = \int \mathsf{L}_\sigma^t[\zeta]\, \Pi(d\zeta) \text{ for } t > 0,$$

is a Feller diffusion started in $Z_0 = 1$.

The Brownian excursion however is more than the building block for the representation of the Feller diffusion Z, it also involves a coding of the complete family structure behind Z. Heuristically speaking, every excursion represents the offspring of a single particle. Every $s \in [0, \sigma)$ can be associated to a particle of this family[1], the lifetime of the particle associated to s is ζ_s. The lifetime of the last common ancestor of two particles $s < t$ is, as in the discrete picture, given by $\min_{[s,t]} \zeta$.

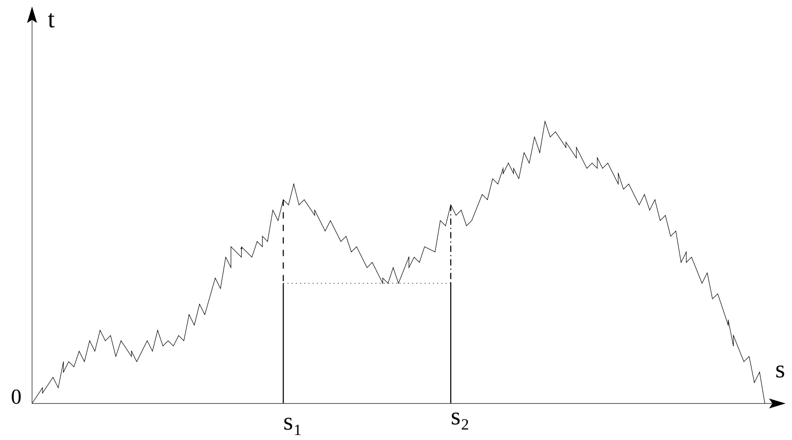

FIGURE 5. Brownian excursion with indicated family relation.

Note that up to now we have only represented the genealogy of the Feller diffusion, neither spatial movements nor catalytic effects have been involved. As a next step, however, we explain Le Gall's *Brownian snake* construction and add a spatial motion component to the picture. This is very natural: Suppose an arbitrary, continuous time-homogeneous Markov-process ξ with state space D is given. Recall that

$$\mathfrak{P} = \Big\{ f \in C([0,\infty), D) : \text{ there exists } \zeta \geq 0 \text{ with } f(r) = f(\zeta) \text{ for all } r \geq \zeta \Big\}$$

is the *space of stopped paths*. Given a Brownian excursion ζ we construct a $\mathfrak{P}$-valued inhomogeneous Markov process $w = \{w_s : s \in [0, \sigma]\}$ called the *Brownian snake with motion process* ξ in such a way that the paths w_s and w_t associated to two particles $s, t \in [0, \sigma]$ agree up to the lifetime of their last common ancestor, and are independent ξ-processes afterwards up to their respective lifetimes ζ_s and ζ_t, when they are stopped. More precisely, given ζ,

- w_s has the law of ξ started in x and stopped at time ζ_s,
- given w_s the path of w_t for $t > s$ is
  - identical to w_s on the interval $[0, m]$ for $m := \min_{[s,t]} \zeta$,
  - distributed like ξ started in $w_s(m)$ on the interval $[m, \zeta_t]$,
  - constant on the interval $[\zeta_t, \infty)$.

Observe that if we consider w_s to be the path of the particle at s, then the construction of the path-valued process is consistent with the genealogical structure coded in the excursion ζ.

Let $\mathbb{N}_x^\xi$ be the (σ-finite) law of the Brownian snake w for the motion process ξ started in x. It is now easy to define a superprocess with homogeneous branching and a quadratic branching

[1]Strictly speaking, there are exceptions to this picture: the beginning and end of each excursion above some level represent the same particle. This ambiguity is inessential for the problems discussed in this thesis.

mechanism (as in the case of a Feller diffusion), with motion process ξ. Indeed, recall that $\{\mathsf{L}_s^t := \mathsf{L}_s^t[\zeta] : t \geq 0, s \in [0,\sigma]\}$ is the family of local times of the Brownian excursion ζ. Denote by Π a Poisson point process with intensity measure

$$\int \mathbb{N}_x^\xi(dw)\,\mu(dx)\,.$$

Then, for $t > 0$ and $\varphi \geq 0$,

$$\langle Z_t, \varphi\rangle = \int\int_0^\sigma \varphi\big(w_s(\zeta_s)\big)\,\mathsf{L}_\bullet^t(ds)\,\Pi(dw)$$

and $Z_0 = \mu$ define the superprocess $Z = \{Z_t : t \geq 0\}$ with motion process ξ. Here the integral is a Stieltjes integral with respect to the nondecreasing function $s \mapsto \mathsf{L}_s^t[\zeta]$. The total mass process of Z is always a Feller diffusion, independently of the motion process.

We now want to use the Brownian snake to represent catalytic branching. Our *idea* is to use a time change to map inhomogeneous branching onto homogeneous branching. A similar idea is used in [**BGJ97**] to represent other (nonquadratic) branching mechanisms.

To describe our approach, let us first look at a generic reactant particle, which moves along a Brownian path $W = \{W(t) : t \geq 0\}$ in $\mathbb{R}$ until its death. The branching of the reactant particle is governed by its *collision local time* $L_{[\Gamma,W]}$ with the medium Γ. $L_{[\Gamma,W]}$ is a continuous additive functional of the Brownian motion W. As the positions of the atoms of Γ are dense in $\mathbb{R}$, it is easy to see that $L_{[\Gamma,W]}$ is strictly increasing.

We use the continuous inverse function $L_{[\Gamma,W]}^{-1}$ of $L_{[\Gamma,W]}$ to introduce a new time scale for the reactant particle on which its collision local time grows linearly. More precisely, given Γ, we define a time-homogeneous continuous Markov process $\xi := \{\xi_r : r \geq 0\}$ with values in $D := [0,\infty) \times \mathbb{R}$ by

$$\xi_r := \big(L_{[\Gamma,W]}^{-1}(r),\, W \circ L_{[\Gamma,W]}^{-1}(r)\big), \ \text{ for } \ r \geq 0,$$

where W is a Brownian motion started in x. The first component of this process can be interpreted as the new individual clock of the Brownian reactant particle, travelling in the medium Γ, and the second component as its position along the new time scale. For all $t > 0$, define the *first exit time*

$$\tau_t := \tau_t(\xi) := \inf\big\{r > 0 : \xi_r \notin [0,t) \times \mathbb{R}\big\}$$

of the path ξ from the domain $D^t := [0,t) \times \mathbb{R}$. At time τ_t the process ξ is in the state $(t, W(t))$ and the reactant particle has accumulated the collision local time $L_{[\Gamma,W]}(t) = \tau_t$ and is placed in $W(t)$. Less formally, a single generic reactant particle of X_t may be represented by a path ξ stopped at the random time τ_t.

We now write $\mathbb{N}_x^\Gamma := \mathbb{N}_{(0,x)}^\xi$ for the Brownian snake associated with the Γ-dependent motion process ξ started in $(0,x)$. Every particle is now represented by a path w_s with the law of ξ, the time change takes care of the correct branching rate. In order to catch exactly those particles with a lifetime t we cannot use the local times at a fixed level of ζ, but have to stop each particle at the individual time $\tau_t(\xi)$. This can be realized using Dynkin's notion of *exit local times.*

We can define, for every $t > 0$, the exit local time at level t of w as the process $L^t[w] := L^t := \big\{L_s^t : s \in [0,\sigma]\big\}$ such that

$$L_s^t := L_s^t[w] := \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^s 1_{\{\tau_t(w_u) < \zeta_u < \tau_t(w_u)+\varepsilon\}}\,du,$$

for every s, $\mathbb{N}_x^\Gamma$–almost surely, where τ_t is the stopping time defined above. Note that the measure associated with the monotone function $s \mapsto L_s^t[w]$ is supported by those s where $\tau_t(w_s) = \zeta_s$ and recall that exactly those paths w_s represent particles of the catalytic super-Brownian motion at time t. The *exit measure at level* $t > 0$ is the measure $Z^t := Z^t[w]$ on $\partial D^t = \{t\} \times \mathbb{R}$ defined by

$$\big\langle Z^t[w], \varphi \big\rangle := \int_0^\sigma \varphi\big(w_s(\zeta_s)\big)\, L_\bullet^t\,(ds), \quad \text{for } \varphi : \partial D^t \to [0,\infty) \text{ measurable.}$$

Slightly abusing notation, for fixed $t > 0$, we identify ∂D^t with $\mathbb{R}$, that is, we consider $Z^t[w]$ and φ as a measure respectively function on $\mathbb{R}$. The measure $Z^t[w]$ can be interpreted as the spatial distribution of the descendants at time t of a reactant particle, which at time 0 is located as x. The quantity $L_\sigma^t[w]$ describes the mass of the totally produced reactant progeny of this particle at time t. Altogether, we have made the following representation of X in terms of the Brownian snake plausible.

THEOREM **5.1.5(i)** *Let* $\mu \in \mathcal{M}_{\text{tem}}$ *be an arbitrary starting measure. Given* Γ*, let* $\Pi = \Pi[\Gamma]$ *be a Poisson point field with intensity measure* $\pi = \pi[\Gamma]$ *defined by*

$$\pi(dw) := \int_{\mathbb{R}} \mathbb{N}_y^\Gamma\,(dw)\,\mu(dy).$$

Then, for almost all Γ*, the superprocess* $X = X[\Gamma]$ *with* $X_0 = \mu$ *can be represented as*

$$\langle X_t\,, \varphi\rangle = \int \big\langle Z^t[w], \varphi \big\rangle\, \Pi(dw),$$

for all $t > 0$ *and* $\varphi : \mathbb{R} \to [0,\infty)$ *measurable.*

The idea of representing catalytic super-Brownian motion by a Brownian snake and using individual time changes for each particle has very recently been taken up and generalized by Klenke [**Kle01**].

3.3. The long term behaviour of super-Brownian motion in a catalytic medium. The Brownian snake gives us the language to describe the long-term behaviour of the catalytic super-Brownian motion in terms of a mass-time-space scaling limit law.

We first look at a classical super-Brownian motion Y without the effect of an irregular random medium. In this case it was shown in [**DaF88**] that a long term clumping effect can be observed. Indeed, Dawson and Fleischmann showed that the rescaled processes $Y^k = \{Y_t^k \,:\, t \geq 0\}$ defined by

$$Y_t^k(\,\cdot\,) = \frac{1}{k} Y_{kt}\big(k\,\cdot\,\big) \text{ for } t \geq 0,$$

converge in distribution, as $k \uparrow \infty$, to a limit process Y^∞, which is a critical measure-valued branching process without a motion component.

THEOREM [**DaF88**] *Suppose* Y *is a classical super-Brownian motion started with Lebesgue measure. Then the rescaled processes* Y^k *converge for* $k \to \infty$ *in law on the Skorokhod space* $\mathfrak{D}\big([0,\infty), \mathcal{M}_{\text{tem}}\big)$ *to a limit process* Y^∞*. For every* $t > 0$ *the random measure* Y_t^∞ *is a compound Poisson measure, i.e.*

$$Y_t = \sum a_i(t)\, \delta_{x_i},$$

where $a_1(t), a_2(t), \ldots$ *are independent, identically distributed and* $x_1, x_2, x_3, \ldots$ *are a homogeneous Poisson point field independent of* $a_i(t)$*. Given the state* $Y_t = \sum a_i(t)\delta_{x_i}$ *at time* $t > 0$*, the state at later times* $s \geq t$ *is*

$$Y_s = \sum a_i(s)\delta_{x_i}$$

with independent Feller diffusions $\{a_i(s) : s \geq t\}$ *started at time* t *in* $a_i(t)$.

In the macroscopic scale the clumps are located at the points of a Poissonian point field and their sizes evolve according to a Feller diffusion. In particular, after a finite lifetime, they are trapped at zero. The spatial irregularities are growing in (macroscopic) time in the sense that clumps disappear in spite of the fact that the overall density remains constant —recall the persistence of the process. Intuitively, one would expect that the influence of the random medium enhances this *intermittency effect*. This is indeed the case and our results show this in several ways.

Suppose that X is a super-Brownian motion in the stable catalytic medium Γ of index $0 < \gamma < 1$, started in the Lebesgue measure. Our aim is to describe the long time behaviour in terms of a *functional limit theorem* under a mass-time-space rescaling.

The scaling is now governed by a new index

$$\eta := \frac{\gamma + 1}{2\gamma} > 1.$$

This index was already identified in [**DaF91**], where a non-functional version of such a limit theorem was obtained. It was shown that, for every $t > 0$, and

$$X_t^k(\,\cdot\,) = \frac{1}{k^\eta} X_{kt}\big(k^\eta\,\cdot\,\big),$$

the random laws of X_t^k converge, as $k \uparrow \infty$, in probability with respect to Γ, to the law of a random measure X_t^∞. However, the crucial question whether this measure is supported by separated atoms could not be settled. If the question can be answered affirmatively, it confirms that the influence of the medium enhances the intermittency effect, as the spatial isolation of the clumps is of a different quality due to the stronger spatial rescaling.

In our setting we can use the Brownian snake construction to give an explicit description of the limit process in a functional limit theorem. Let $X^k[\Gamma] := \{X_t^k : t \geq 0\}$ be the rescaled process and $C\big((0,\infty), \mathcal{M}_{\text{tem}}\big)$ be the space of continuous paths with values in the space of tempered measures, equipped with the topology of uniform convergence on compact intervals.

Theorem **5.1.5(ii)** *In probability with respect to* Γ *the random laws of* $X^k[\Gamma]$ *converge as* $k \uparrow \infty$ *weakly on* $C\big((0,\infty), \mathcal{M}_{\text{tem}}\big)$ *to the law of a process* X^∞. *Let* Π^∞ *be a Poisson point field with intensity measure*

$$\pi^\infty(dw\,dx) := \Big(\int \mathbb{N}_0^\Upsilon(dw)\,\mathsf{P}(d\Upsilon)\Big) \otimes dx.$$

Then the limit process X^∞ *has the representation*

$$\langle X_t^\infty, \varphi\rangle = \int L_\sigma^t[w]\varphi(x)\,\Pi^\infty(dw\,dx), \tag{32}$$

for all $t > 0$ *and* $\varphi : \mathbb{R} \to [0,\infty)$ *measurable.*

Outline of the proof: The proof of this functional limit theorem does *not* split in the typical two steps: convergence of finite dimensional marginals and tightness. Instead, the Brownian snake representations can be used to prove functional convergence directly.

Recall that $X^k[\Gamma]$ is given by $X_t^k(\cdot) = k^{-\eta}X_{kt}(k^\eta\,\cdot)$. As we are aiming for convergence *in probability* with respect to the medium, we may study $X^k = X^k[\Gamma_k]$ in the scaled medium Γ_k defined by

$$\Gamma_k(\,\cdot\,) = k^{\frac{1}{2\gamma}}\,\Gamma\big(k^{-1/2}\,\cdot\big),$$

which has the same law as Γ. We divide the process $\{X_t^k[a,b] : t > 0\}$ into two parts, by splitting the starting mass $X_0^k = \ell$ into the Lebesgue measure on $[a,b]$ and its complement.

Step 1: The part of $X^k[a,b]$ with $X_0^k = \ell_{(-\infty,a)} + \ell_{(b,\infty)}$ vanishes asymptotically.

In order to prove this, we have to verify the so-called *crossing property* of the unscaled process X, which we believe to be of independent interest.

LEMMA **5.1.4** *Suppose that $\{X_t : t \geq 0\}$ is the catalytic super-Brownian motion in the stable medium Γ and $X_0 = \ell_{(0,\infty)}$. Then, for* P*–almost every* Γ,

$$\sup_{t\geq 0} X_t(-\infty,0] < \infty\text{–}\textit{almost surely.}$$

This is proved in Subsection **5.3.3** using a *method of good and bad paths.* A very brief outline of this general method is given in the sketch of the proof of Theorem **5.1.1** below. Step 1 follows easily from the crossing property. Indeed, by translation and, if needed, reflection, it is sufficient to show that, in P–probability, the processes

$$\big\{X_t^k(-\infty,0] \,:\, t>0\big\}, \text{ for } X_0 = \ell_{(0,\infty)},$$

converge in distribution on $C((0,\infty),\mathbb{R})$ to the zero function. Now observe that, for P–almost all Γ,

$$\sup_{t\geq 0} X_t^k(-\infty,0] = k^{-\eta} \sup_{t\geq 0} X_t(-\infty,0] \overset{k\to\infty}{\longrightarrow} 0, \quad \mathbb{P}^\Gamma\text{–almost surely.}$$

Step 2: Almost surely with respect to the medium, the law of the process $X^k[a,b]$ with $X_0^k = \ell_{[a,b]}$ converges to the law of $X^\infty[a,b]$.

It suffices to show convergence of the total mass process $\|X^k\|$ instead of $X^k[a,b]$. Using scaling arguments and the representation of the unscaled process one can express the process $\|X^k\|$ with $X_0^k = \ell_{[a,b]}$ in terms of a snake representation:

$$\|X_t^k\| = \int L_\sigma^t[w]\,\Pi^k(dw)\,,$$

where Π^k is a Poisson field with intensity measure

$$\pi^k[\Gamma] = k^{-\rho}\int_{k^\rho a}^{k^\rho b} \mathbb{N}_0^{T^x\Gamma}\,dx,$$

with $\rho := \eta - 1/2 > 0$. Now the dependence on k is only a dependence of the intensity measures. It thus suffices to show the convergence of the sequence of the intensity measures, which can be based on *Birkhoff's ergodic theorem*: Almost surely with respect to the medium,

$$\pi^k[\Gamma] \overset{k\to\infty}{\longrightarrow} (b-a)\int \mathbb{N}_0^\Upsilon\; P(d\Upsilon).$$

These are the principal arguments of the proof of the functional limit theorem. □

A major result of Chapter 5 is the identification of the Poisson structure of the atoms of X_t^∞.

THEOREM **5.1.1** *We have, for all $t>0$,*

$$\int \mathbb{N}_0^\Upsilon\{w \,:\, L_\sigma^t > 0\}\,\mathsf{P}(d\Upsilon) < \infty. \tag{33}$$

In particular, for all $t>0$, the random measure X_t^∞ is a compound Poisson measure, i.e. the clumps are macroscopically isolated.

Outline of the proof: First, note that the statement of the second sentence is immediate from (33) and (32). The proof of (33) is based on a quantitative refinement of the method of good and bad paths, which in its original form appeared in [**DFM00**]. I only give a very brief and general outline of this method, which is also used in the proof of Lemma **5.1.4**.

To every sample of the random medium Γ we associate a characteristic number $N = N(\Gamma)$. If a sample Γ has a small characteristic number, then large intervals without appropriately large atoms are rare, the medium is not exceptionally thin. Our point of view is that such media samples are more regular than those with a very large characteristic number.

We suppose that an increasing series of Brownian stopping times T_n, adapted to the problem at hand, is fixed and let $T = \sup T_n$. To every such stopping time we can associate a measure Y_{T_n} called *Dynkin's stopped measure.* Roughly speaking, Y_{T_n} represents the population one gets if each individual reactant path is stopped at its stopping time T_n.

A path alive at time T_{n+1} is called a *good path* if the accumulated amount of contact with the medium Γ during the time interval $[T_n, T_{n+1}]$ exceeds a threshold value l_n, formally

$$\int_{T_n}^{T_{n+1}} L_{[W,\Gamma]}(ds) \geq l_n.$$

One can establish upper bounds on the survival probability of the set of good paths in terms of the total mass of Y_{T_n} and the threshold values by means of a comparison with the Feller diffusion, see Lemma **5.3.6**, which is taken from [**DFM00**].

The complementary set of paths alive at time T_{n+1} is called the set of *bad paths.* Bad paths have exceptionally small amount of contact with the medium during the time interval. This may have two reasons

- the medium is exceptionally thin. This effect can be excluded if the characteristic number of the sample medium is sufficiently small.
- the path has an exceptional behaviour. We formalize the requirement that the probability of this event is small in two formal hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$, see Section **5.3.2**.

If these formal requirements hold true, the result of Theorem **5.3.4** gives an explicit upper bound on the survival probability of Y_T given a medium with sufficiently small characteristic number.

In the application of this result, the stopping times T_n have to be chosen carefully such that the formal hypotheses can be verified using large deviation estimates. In the proof of Theorem **5.1.1** the stopping times T_n are deterministic and we write $t = T > 0$. The upper bound obtained from this method is, for $X_0 = \ell_{[0,1]}$, P-almost surely,

$$\mathbb{P}^{\Gamma}\big\{X_t > 0\big\} \leq 1 - \theta^{(2^{N}(\Gamma))}$$

for some constant $\theta > 0$, which depends on $t > 0$, see Lemma **5.3.12**. This bound can be used, together with an upper estimate of the upper tail of the random variable $N(\Gamma)$, see Lemma **5.3.2**, to show finiteness of the integral (33). $\square$

To round out the overview over the main achievements of Chapter 5 we mention that we have identified the major indices related to the limit process X^{∞}.

THEOREM **5.1.10** *Recall that $\eta = (\gamma+1)/(2\gamma)$. The probability that a clump at time s survives for $t > 0$ further macroscopic time units is*

$$\Big(\frac{s}{s+t}\Big)^{\eta}.$$

The tail behaviour of the mass M of a clump at macroscopic time $t > 0$ is governed by

$$\mathbf{P}\big\{M > a\big\} \asymp t^{\eta(\gamma+1)}\, a^{-\gamma-1} \quad \text{as } a \uparrow \infty.$$

Outline of the proof: It is known already from the work of Dawson and Fleischmann [**DaF91**] that the Laplace functional of $X_t^\infty[0,1]$ satisfies

$$\mathbf{E}_\ell\big\{\exp\big(-\theta X_t^\infty[0,1]\big)\big\} = \exp\big(-\mathsf{E}U_t\theta\,(0)\big), \text{ for } \theta \geq 0,$$

where $U_t\theta$ is a solution of

$$U_t\theta\,(y) = \theta - 2\,\mathcal{E}_y\Big\{\int_0^t [U_{t-s}\theta\,(W_s)]^2\; L_{[\Gamma,W]}(ds)\Big\}, \quad \text{for } t \geq 0, \;\; y \in \mathbb{R}.$$

Here $\mathcal{E}_y$ refers to expectation with respect to a Brownian motion W started in y. For the proof we need an expansion of $\mathsf{E}U_t\theta\,(0)$ for $\theta \downarrow 0$. A first idea would be to use the trivial estimate $U_t\theta \leq \theta$ on the right hand side of the equation to get a lower bound for $U_t\theta$ and so forth. This method however very quickly leads to trivial estimates for $\mathsf{E}U_t\theta\,(0)$.

A better idea is to reformulate the equation equivalently in Feynman-Kac form

$$U_t\theta\,(y) = \theta\,\mathcal{E}_y\Big\{\exp\Big(-2\int_0^t U_{t-s}\theta\,(W_s)\,L_{[W,\Gamma]}(ds)\Big)\Big\}, \quad \text{for } t \geq 0, \;\; y \in \mathbb{R}.$$

We may now plug the trivial estimate $U_t\theta \leq \theta$ in the right hand side and obtain a lower bound for $U_t\theta\,(y)$. After taking expectations we get, for a suitable constant $C_2 = C_2(t)$,

$$\mathsf{E}U_t\theta\,(0) \geq \theta - C_2(t)\theta^{\gamma+1} \text{ for all } \theta \in (0,1).$$

This method may be iterated once more, we get a new upper bound for $U_t\theta\,(y)$ and show with considerable effort that also for a suitable constant $C_1 = C_1(t)$,

$$\mathsf{E}U_t\theta\,(0) \leq \theta - C_1(t)\theta^{\gamma+1} \text{ for all } \theta \in (0,1).$$

We may now use that X_t^∞ is a compound Poisson measure and the general Laplace transform of compound Poisson measures yields

$$\mathbf{E}_\ell\big\{-\theta X_t^\infty[0,1]\big\} = \exp\Big(\lambda(t)\big(1-\Lambda^t(\theta)\big)\Big),$$

where Λ^t is the Laplace transform of the mass of an atom and $\lambda(t)$ is the intensity of the Poisson field. This gives the following expansion for $\Lambda^t(\theta)$, for all $t>0$, as $\theta \downarrow 0$,

$$1 - \frac{1}{\lambda(t)}\,\theta + \frac{C_2(t)}{\lambda(t)}\,\theta^{\gamma+1} \;\leq\; \Lambda^t(\theta) \;\leq\; 1 - \frac{1}{\lambda(t)}\,\theta + \frac{C_1(t)}{\lambda(t)}\,\theta^{\gamma+1}.$$

The tail behaviour of the macroscopic clump sizes, i.e. the masses of the atoms of X_t^∞, can be inferred easily from this by means of a Tauberian theorem. □

Summarizing, our description of the long term behaviour of the catalytic super-Brownian motion is given in terms of a functional limit theorem, showing convergence of the process under a mass-space-time rescaling to a limit process. Looking at the limit process at a fixed time we see a Poisson field of isolated mass clumps. As time evolves, mass clumps are vanishing and, as the overall density remains constant, we observe an intermittency effect.

This effect also occurs in the constant medium case, due to the criticality of the branching mechanism, but the random medium enhances this effect considerably. The qualitatively different nature of the clumping effect in the random medium case becomes manifest in two major points:

- the limit theorem needs a *stronger scaling*, but mass clumps are still isolated,
- the mass of a macroscopic clump at a fixed macroscopic time is a *heavy tailed* random variable.

Finally, the temporal evolution of the mass of a clump is independent for different clumps. In contrast to the case of a homogeneous medium this evolution of mass does *not* have the law of a continuous state branching process. Instead, it follows a non-Markovian law, which coincides with the evolution in time of the total mass of the family of a single particle in the unscaled process in the *annealed* medium.

CHAPTER 1

Tangent measure distributions of fractal measures

With David Preiss, University College London
Mathematische Annalen, 312 (1998) 53–93.

Abstract. Tangent measure distributions provide a natural tool to study the local geometry of fractal sets and measures in Euclidean spaces. The idea is, loosely speaking, to attach to every point of the set a family of random measures, called the α-dimensional tangent measure distributions at the point, which describe asymptotically the α-dimensional scenery seen by an observer zooming down towards this point. This tool has been used by Bandt [**Ban92**] and Graf [**Gra95**] to study the regularity of the local geometry of self similar sets, but in this paper we show that its scope goes much beyond this situation and, in fact, it may be used to describe a strong regularity property possessed by *every* measure: We show that, for every measure μ on a Euclidean space and any dimension α, at μ-almost every point, all α-dimensional tangent measure distributions are Palm measures. This means that the local geometry of every dimension of general measures can be described —like the local geometry of self similar sets— by means of a family of statistically self similar random measures. We believe that this result reveals a wealth of new and unexpected information about the structure of such general measures and we illustrate this by pointing out how it can be used to improve or generalize recently proved relations between ordinary and average densities.

Mathematics Subject Classification (2000): *28A75, 28A80, 60G57.*
Key words: *fractal measure, fractal, tangent measure, rectifiability, tangent measure distribution, random measure, statistically self-similar, average density, Palm distribution.*

1. Introduction

The study of the regularity of the local behaviour of a measure μ on a Euclidean space is much simplified by the fact that, under the presence of mild a priori estimates, the weak compactness of bounded sets of measures allows one to define the measures representing the local behaviour of μ at a point x as limit points of enlargements of μ about x. In various disguises, this idea has been used for a long time: Examples include Young measures (see e.g. [**Val89**]), tangent cones to currents (see [**Fed87**, 4.3.16]), Bonnet's recent approach to problems in image segmentation (see [**Bon96**]) or tangent measures (see [**Pre87**]). The applicability of this extremely valuable tool stems from the observation that taking this particular limit of measures often improves their geometric behaviour; this behaviour then can be analyzed and, by approximation, information may be obtained about the original measure.

We illustrate this idea further by looking at the concept of tangent measures, which was introduced in [**Pre87**] for the study of sets and measures of integer dimension and their rectifiability questions. Recall that a measure μ is called α-rectifiable if it is absolutely continuous with respect to α-Hausdorff measure on a countable family of smooth α-manifolds. For a measure μ on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$ we define for every $r > 0$ the $(1/r)$-enlargement $\mu_{x,r}$ by $\mu_{x,r}(A) = \mu(x + rA)$. The *$\alpha$-dimensional tangent measures* of μ at x are the vague limit points of $\mu_{x,r}/r^{\alpha}$ as $r \downarrow 0$. The key to most applications of tangent measures is the fact that on the one hand many weak local regularity properties of μ at x translate into strong almost everywhere regularity properties regularity properties of the tangent measures at x and on the other hand μ is α-rectifiable if and only if at μ-almost every point the set of α-dimensional tangent measures is a singleton consisting of a positive multiple of Lebesgue measure on a linear α-subspace $V \subseteq \mathbb{R}^d$, the approximate tangent space at x. The strength of this method becomes apparent in [**Mat95**] and interesting new results in this direction may be found in [**ONe96**] and [**Huo97**].

The limitations of the tangent measure approach, however, become clear in investigations of non-rectifiable sets and measures. Simple examples show that the set of tangent measures may be rather large and contain measures that do not have much better geometric properties than the original one. Therefore for the investigation of non-rectifiable measures more refined tools seem to be necessary.

One class of tools is based on an averaging idea of Bedford and Fisher in [**BeF92**]. Instead of looking directly at all limit points of the enlargements of μ about x as the enlargement factor goes to infinity, we take a natural family of probability distributions on the set of positive reals whose mass tends to infinity and study the limiting distributions of the induced family of distributions on the set of enlargements. The limiting distributions define "random tangent measures" which are called tangent measure distributions. This concept first appeared in a weaker form in [**BeF92**] and then in its full strength in [**Ban92**] and [**Gra95**]. Bandt and Graf used tangent measure distributions to study the particular case of self similar sets and found that they have unique tangent measure distributions at almost all points, which they described explicitly as scaling invariant Palm measures. Scaling invariant Palm measures were studied and termed self similar random measures by U. Zähle in [**Zäh88**]. We give some more details about these results in the next section.

Starting point of our research was the question to what extent the strong results on connections between rectifiability of a measure and smoothness of its tangent measures have their analogy in the fractal world. Are there analogous connections between self similarity of a measure and statistical self similarity of its tangent measure distributions? More precisely, we were looking for a converse of the results of Bandt and Graf: Is self similarity of a set equivalent to all its tangent measure distributions being Palm measures?

This conjecture has turned out to be wrong. Instead we found a surprising and strong result of opposite nature: We prove here that for *every* measure μ on a Euclidean space and every dimension α at μ-almost every point all α-dimensional tangent measure distributions are Palm measures. As all tangent measure distributions are scaling invariant, this means that for every measure the local geometry of every dimension is, from the statistical point of view, given by a family of self similar random measures and hence not different from the local geometry of a self similar set.

To sum up, one can say that tangent measure distributions are a widely applicable example of the effect described at the beginning of this introduction: The localization process leads to objects of good geometric regularity even if the original objects are highly irregular. We believe that this improves the understanding of the geometry of non-rectifiable sets and measures and we illustrate the strength of this result by means of several examples of its application to questions studied previously in [**BeF92**], [**MöP95**], [**Mör98a**] and [**FaS95**], [**Mar96**].

We also remark that our result is new even on the line. However, in the important case of measures on the line with positive lower and finite upper densities it is possible to deduce the Palm property of tangent measure distributions from very detailed information about the structure of these measures which has been obtained in the thesis [**Mör95**] and will appear in [**Mör98a**].

2. The main result and its applications

In this section we give the full definitions and basic properties of the main geometric measure theoretical notions describing the α-dimensional part of the behaviour of a Radon measure μ on $\mathbb{R}^d$ about a point $x \in \mathbb{R}^d$ and we formulate the precise statement of our main result. We also prove the aforementioned applications, but delay the proof of the main theorem until the Sections 3 and 4. We shall frequently refer to some measure theoretical preliminaries, some of them not belonging to standard knowledge, which we have collected in the appendix of this paper.

Tangent measure distributions, definition and remarks. We denote by $\mathcal{M}(\mathbb{R}^d)$ the Polish space of all nonnegative Radon measures on $\mathbb{R}^d$ endowed with the vague topology. Finite Radon measures on the Polish space $\mathcal{M}(\mathbb{R}^d)$ will be termed *measure distributions* on $\mathbb{R}^d$; depending on the context we may also call them random measures. The measure distributions we are interested in will always be sub-probability measures and often, but not always, they will be probability measures. We say that a sequence $(\mathbb{P}_n)$ of measure distributions *converges vaguely* to a measure distribution $\mathbb{P}$ if the following two conditions hold

(i) $\mathbb{P}(K) \geq \limsup_{k\to\infty} \mathbb{P}_k(K)$ for every compact set $K \subseteq \mathcal{M}(\mathbb{R}^d)$, and
(ii) $\mathbb{P}(L) \leq \liminf_{k\to\infty} \mathbb{P}_k(L)$ for every open set $L \subseteq \mathcal{M}(\mathbb{R}^d)$.

The vague convergence is weaker than the usual weak convergence of probability distributions and this enables us to give a stronger formulation of our main theorem. The relevant properties of the space $\mathcal{M}(\mathbb{R}^d)$ and the vague convergence may be found in the appendix.

DEFINITION 2.1. Suppose μ is a Radon measure on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. We define the family of measures $(\mu_{x,r})_{r>0} \subseteq \mathcal{M}(\mathbb{R}^d)$, the *enlargements of μ about x*, by $\mu_{x,r}(A) = \mu(x + rA)$ for all Borel sets $A \subseteq \mathbb{R}^d$ and probability distributions $\mathbb{P}_\delta^x$ on $\mathcal{M}(\mathbb{R}^d)$ by

$$\mathbb{P}_\delta^x(M) = (|\log\delta|)^{-1} \int_\delta^1 1_M\Big(\frac{\mu_{x,r}}{r^\alpha}\Big)\,\frac{dr}{r} \quad \text{for Borel sets } M \subseteq \mathcal{M}(\mathbb{R}^d).$$

$\mathcal{P}^\alpha(\mu, x)$ is defined as the set of all limit points of $(\mathbb{P}_\delta^x)_{\delta>0}$ as $\delta \downarrow 0$ in the vague convergence, The elements of $\mathcal{P}^\alpha(\mu, x)$ are the *α-dimensional tangent measure distributions* of μ at x.

This approach also allows us to study the local geometry of sets $E \subseteq \mathbb{R}^d$ by means of natural measures on the set, like for example Hausdorff measures.

Remarks:
a) Note that $\mathbb{P}_\delta^x$ is the push-forward under the mapping

$$(1,\infty) \to \mathcal{M}(\mathbb{R}^d)\ ,\ t \mapsto \frac{\mu_{x,1/t}}{(1/t)^\alpha}$$

of the measure p_δ given as the normalized restriction of Haar measure on the group $(0,\infty)$ with multiplication to the interval $(0, 1/\delta)$,

$$p_\delta(A) = (\log(1/\delta))^{-1} \int_1^{1/\delta} 1_A(t)\,\frac{dt}{t} \quad \text{for } A \subseteq \mathbb{R} \text{ Borel}.$$

Observe that, for every bounded set $A \subseteq (1, \infty)$ we have $\lim_{\delta \to 0} p_\delta(A) = 0$ and hence the mass escapes to infinity as $\delta \to 0$. $(p_\delta)_{\delta > 0}$ is a natural choice of probability distributions on the set of scales $(1, \infty)$ because of the scaling invariance property $\lim_{\delta \to 0} p_\delta(sA) = \lim_{\delta \to 0} p_\delta(A)$ for all Borel sets $A \subseteq (1, \infty)$ and $s > 0$. Another motivation for this choice comes from the intimate relationship of the family (p_δ) to the order two averaging method of Hardy and Riesz. Although the behaviour of tangent measure distributions we describe in this paper is sensitive to the choice of (p_δ) our results also hold for tangent measure distributions with respect to the families (p_δ) coming from higher order averaging methods, this follows from the general theory of averaging procedures, see [**BeF92**] and [**Fis87**] for more details.

b) No a priori relation between μ, α and x has been assumed. Observe that, for every μ and all choices of dimension $\alpha \geq 0$ we always have $\mathcal{P}^\alpha(\mu, x) \neq \emptyset$; indeed, Lemma A4 of the appendix shows that for every sequence $\delta_k \downarrow 0$ the sequence $\mathbb{P}^x_{\delta_k}$ has a vaguely convergent subsequence. However, although the $\mathbb{P}^x_\delta$ are probability measures, the $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ need only be sub-probability measures. Hence a tangent measure distribution $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ may be trivial in two ways: Either $\mathbb{P}$ is concentrated on the zero-measure, this case corresponds, roughly speaking, to the situation when the dimension of μ at x is in fact bigger than α, or $\mathbb{P}$ is the zero measure distribution, this case corresponds to the situation when the dimension of μ at x is smaller than α. We give some easy conditions for the nontriviality of tangent measure distributions in Proposition 2.3.

We also point out that no a priori relation between α, μ and x has been assumed; thus, for example, tangent measure distributions of the one dimensional Lebesgue measure ℓ in $\mathbb{R}$ are:

- If $\alpha < 1$ then the unique α-dimensional tangent measure distribution of ℓ at every $x \in \mathbb{R}$ is the Dirac measure concentrated at the zero measure.
- If $\alpha = 1$ then the unique α-dimensional tangent measure distribution of ℓ at every $x \in \mathbb{R}$ is the Dirac measure concentrated at ℓ.
- If $\alpha > 1$ then the unique α-dimensional tangent measure distribution of ℓ at every $x \in \mathbb{R}$ is the zero measure distribution.

c) Several close concepts have been used to describe the behaviour of a Radon measure μ on $\mathbb{R}^d$ about a point. The most classical are the *lower and upper α-dimensional densities* of μ at x defined by

$$\liminf_{r \downarrow 0} \frac{\mu(U(x,r))}{r^\alpha} \quad \text{and} \quad \limsup_{r \downarrow 0} \frac{\mu(U(x,r))}{r^\alpha} ,$$

respectively, where $U(x,r)$ denotes the open Euclidean ball of radius r with centre at x. More recently Bedford and Fisher introduced in [**BeF92**] the notion of *lower and upper average, or order two, α-dimensional densities* of μ at x defined by

$$\liminf_{\delta \downarrow 0} (|\log \delta|)^{-1} \int_\delta^1 \frac{\mu(U(x,r))}{r^\alpha} \frac{dr}{r} \text{ and } \limsup_{\delta \downarrow 0} (|\log \delta|)^{-1} \int_\delta^1 \frac{\mu(U(x,r))}{r^\alpha} \frac{dr}{r} ,$$

respectively. The *α-dimensional tangent measures* of μ at x are defined as elements of the set

$$\operatorname{Tan}_\alpha(\mu, x) = \{\nu = \lim_{n \to \infty} \frac{\mu_{x,r_n}}{r_n^\alpha} \text{ in the vague topology for some } r_n \downarrow 0 \} .$$

Again, no relation between α, μ and x has been assumed. Note also that α-dimensional tangent measures may form a proper subset of the set of tangent measures as considered originally in [**Pre87**]. A further natural concept, which was introduced in [**Gra95**], are the *α-dimensional* *average tangent measures* of μ at x defined as vague limit points of the Radon measures

$$(|\log \delta|)^{-1} \int_\delta^1 \frac{\mu_{x,r}}{r^\alpha} \frac{dr}{r}$$

as $\delta \downarrow 0$. Under certain conditions, formulated in Proposition 2.17, the average tangent measures are the expectations of the tangent measure distributions and the average densities are the values of the average tangent measures at the unit ball. The relation of tangent measure distributions and tangent measures is given in the following proposition.

PROPOSITION 2.2. *Suppose that $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\alpha \geq 0$. For every $x \in \mathbb{R}^d$ all $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ are concentrated on* $\mathrm{Tan}_\alpha(\mu, x)$.

PROOF. If the sequence $\mathbb{P}^x_{\delta_k}$ vaguely converges to $\mathbb{P}$ and $\mathbb{P}(\mathcal{M}(\mathbb{R}^d) \setminus \mathrm{Tan}_\alpha(\mu, x)) > 0$, we find a $\delta > 0$ such that $\mathbb{P}(\mathcal{M}(\mathbb{R}^d) \setminus M_\delta) > 0$, where M_δ is the closure of $\{\mu_{x,r}/r^\alpha, 0 < r \leq \delta\}$. Hence $\liminf \mathbb{P}^x_{\delta_k}(\mathcal{M}(\mathbb{R}^d) \setminus M_\delta) \geq \mathbb{P}(\mathcal{M}(\mathbb{R}^d) \setminus M_\delta) > 0$, which contradicts the definition of $\mathbb{P}^x_\delta$. □

Basic properties of tangent measure distributions. We shall give some statements concerning compactness, localization and scaling invariance properties of tangent measure distributions.

PROPOSITION 2.3. *Suppose that $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\alpha \geq 0$.*

(a) *If μ has finite upper average α-dimensional density at x, then for every sequence $\delta_k \downarrow 0$ the sequence $\mathbb{P}^x_{\delta_k}$ has a weakly convergent subsequence. In particular, all $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ are probability measures and the set $\mathcal{P}^\alpha(\mu, x)$ is compact in the weak topology.*
(b) *If μ has positive lower α-dimensional density at x, then $\mathbb{P}(\{\phi\}) = 0$ for every $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$. (Here ϕ denotes the zero measure.)*

PROOF. If μ has finite upper average α-dimensional density at x, we infer from

$$(|\log \delta|)^{-1} \int_\delta^1 \frac{\mu(U(x, nr))}{r^\alpha} \frac{dr}{r} = n^\alpha (|\log \delta|)^{-1} \int_{n\delta}^n \frac{\mu(U(x, r))}{r^\alpha} \frac{dr}{r}$$

that the numbers

$$D_n = \sup \left\{ (|\log \delta|)^{-1} \int_\delta^1 \frac{\mu(U(x, nr))}{r^\alpha} \frac{dr}{r} : 0 < \delta < 1/2 \right\}$$

are finite. Thus, for any $\eta > 0$, we may choose $c(n) \in (0, \infty)$ such that $\sum_{n=1}^\infty \frac{D_n}{c(n)} < \eta$ and observe that the set

$$K = \{\nu \in \mathcal{M}(\mathbb{R}^d) : \nu U(0, n) \leq c(n) \text{ for } n = 1, 2, \dots\}$$

is compact and that by Chebyshev's inequality $\mathbb{P}^x_\delta(\mathcal{M}(\mathbb{R}^d) \setminus K) < \eta$. Hence the existence of a weakly convergent subsequence and the compactness statement follows from Lemmas A4 and A6 and the fact that $\mathcal{P}^\alpha(\mu, x)$ consist of probability measures follows from Lemma A6. Finally, if μ has positive lower α-dimensional density at x, then $\mathrm{Tan}_\alpha(\mu, x)$ does not contain the zero measure. Hence (b) follows from Proposition 2.2. □

REMARK 2.4. Observe that by Proposition 2.3, whenever μ has positive lower and finite upper α-dimensional densities at x, all tangent measure distributions $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ are nontrivial in the strict sense that $\mathbb{P}(\mathcal{M}(\mathbb{R}^d) \setminus \{\phi\}) = 1$.

The following lemma is the key to the localization properties of tangent measure distributions.

LEMMA 2.5. *Suppose that $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ are Radon measures, $\alpha \geq 0$ and $x \in \mathbb{R}^d$ is such that*

$$\lim_{r \downarrow 0} \frac{\sup\{|\mu(E) - \nu(E)| : E \subseteq U(x,r)\}}{(\mu+\nu)(U(x,r))} = 0$$

and that $\delta_k \downarrow 0$. Then, if for either of μ or ν the sequence $\mathbb{P}^x_{\delta_k}$ converges vaguely, then both sequences converge and they have the same limit.

PROOF. Denote by $\mathbb{P}_k$ and $\mathbb{Q}_k$ the sequence $\mathbb{P}^x_{\delta_k}$ with respect to μ, respectively ν. Because of the possibility of passing to convergent subsequences, it suffices to assume that $\mathbb{P}_k$ and $\mathbb{Q}_k$ converge vaguely to, say $\mathbb{P}$ and $\mathbb{Q}$, respectively. If $\eta > 0$ and $L \subseteq \mathcal{M}(\mathbb{R}^d)$ is open, we use Lemma A2(b) to find a compact set $K \subseteq L$ such that $\limsup_{k\to\infty} \mathbb{P}_k(K) > \mathbb{P}(L) - \eta$. Since L is a vague neighbourhood of the compact set K, there is a positive integer m such that, with a suitable $0 < \sigma < 1$, L contains every measure γ which satisfies

$$\sup\{|\gamma(E) - \beta(E)| : E \subseteq U(0,m)\} \leq \sigma \text{ for some } \beta \in K. \tag{1}$$

Let $c(n) \in (1,\infty)$ be such that $\beta U(0,n) < c(n)$ for all $\beta \in K$ and let $0 < R < 1$ be such that

$$\frac{\sup\{|\mu(E) - \nu(E)| : E \subseteq U(x,r)\}}{(\mu+\nu)(U(x,r))} < \sigma/(3c(m))$$

for $0 < r < R$; using this with $E = U(x,r)$, we infer that $\nu U(x,r) \leq 2\mu U(x,r)$ and, consequently,

$$\frac{\sup\{|\mu(E) - \nu(E)| : E \subseteq U(x,r)\}}{\mu U(x,r)} < \sigma/c(m)$$

for $0 < r < R$ and $E \subseteq U(x,r)$.

Let K_0 be the set of all measures γ which have the property (1) and, in addition, satisfy

$$\gamma U(0,n) \leq 2c(n) + (n/R)^\alpha \nu U(x,n) \text{ for every } n \geq m\ .$$

Then $K \subseteq K_0 \subseteq L$ and K_0 is compact. Moreover, if $0 < r < R/m$ and $\frac{\mu_{x,r}}{r^\alpha} \in K$, we show that $\frac{\nu_{x,r}}{r^\alpha} \in K_0$. Indeed, if $nr \geq R$, then

$$\frac{\nu_{x,r}}{r^\alpha}(U(0,n)) \leq \frac{\nu U(x,n)}{r^\alpha} \leq (n/R)^\alpha \nu U(x,n)\ ,$$

if $nr < R$, then

$$\frac{\nu_{x,r}}{r^\alpha}(U(0,n)) \leq \frac{\nu U(x,nr)}{\mu U(x,nr)} \frac{\mu_{x,r}}{r^\alpha}(U(0,n)) \leq 2c(n)$$

and, finally,

$$\begin{aligned} &\sup\left\{\left|\frac{\mu_{x,r}}{r^\alpha}(E) - \frac{\nu_{x,r}}{r^\alpha}(E)\right| : E \subseteq U(0,m)\right\} \\ \leq\ & \frac{\sup\{|\mu(E) - \nu(E)| : E \subseteq U(x,mr)\}}{\mu U(x,mr)} \frac{\mu_{x,r}}{r^\alpha}(U(0,m)) < \sigma. \end{aligned}$$

Consequently, $\mathbb{Q}(L) \geq \mathbb{Q}(K_0) \geq \limsup_{k\to\infty} \mathbb{Q}_k(K_0) \geq \limsup_{k\to\infty} \mathbb{P}_k(K) > \mathbb{P}(L) - \eta$, which, as $\eta > 0$ is arbitrary, implies that $\mathbb{Q}(L) \geq \mathbb{P}(L)$ for every open set $L \subseteq \mathcal{M}(\mathbb{R}^d)$. Exchanging the rôles of $\mathbb{P}$ and $\mathbb{Q}$, we finally conclude that $\mathbb{P} = \mathbb{Q}$. ☐

PROPOSITION 2.6. *Suppose that $\alpha \geq 0$ and that $E \subseteq \mathbb{R}^d$ is μ-measurable. Then the restriction of μ to E, which is the Radon measure $\mu|_E$ on $\mathbb{R}^d$ defined by $\mu|_E(A) = \mu(E \cap A)$, satisfies $\mathcal{P}^\alpha(\mu, x) = \mathcal{P}^\alpha(\mu|_E, x)$ for μ-almost every $x \in E$.*

PROOF. For μ-almost every $x \in E$ we infer from the Besicovitch-Morse Density Theorem (see [**Fed87**, 2.9.11]) that

$$\lim_{r \downarrow 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r))} = 0 .$$

For any such $x \in E$ the required statement holds because the measure μ and $\nu = \mu|_E$ satisfy the assumptions of Lemma 2.5. □

REMARK 2.7. Similarly, Lemma 2.5 may be used to describe tangent measure distributions of a measure absolutely continuous with respect to μ: If the Radon-Nikodym derivative of ν with respect to μ is f, and if x is a Lebesgue point of f, i.e. $\lim_{r\to 0} \int_{U(x,r)} |f(x)-f(y)|\, d\mu(y)/\mu U(x,r) = 0$, we can first transform tangent measure distributions of μ to its constant multiple by $f(x)$, and then infer from Lemma 2.5 that these are precisely the tangent measure distributions of ν at x. Since μ-almost every point is a Lebesgue point of f, this applies in μ, hence ν, almost every point of $\mathbb{R}^d$.

In the following proposition we see that tangent measure distributions have an automatic scaling invariance property. This is largely due to the choice of the measures (p_δ) on the set of scales.

PROPOSITION 2.8. *For every $\lambda > 0$ we define the rescaling operator $S_\lambda^\alpha : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ by $S_\lambda^\alpha \nu(E) = (1/\lambda^\alpha) \cdot \nu(\lambda E)$. For all $x \in \mathbb{R}^d$ and $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ we have*

$$\mathbb{P} = \mathbb{P} \circ (S_\lambda^\alpha)^{-1} \quad \textit{for all } \lambda > 0.$$

PROOF. Let $\lambda > 0$ and observe that S_λ^α is a homeomorphism of $\mathcal{M}(\mathbb{R}^d)$ onto itself, hence it preserves compact and open subsets of $\mathcal{M}(\mathbb{R}^d)$. Suppose that $\delta_k \downarrow 0$ and that $\mathbb{P}$ is the vague limit of $\mathbb{P}^x_{\delta_k}$. For every compact set $K \subseteq \mathcal{M}(\mathbb{R}^d)$ and every open set $L \subseteq \mathcal{M}(\mathbb{R}^d)$ we calculate

$$\begin{aligned}
(\mathbb{P} \circ (S_\lambda^\alpha)^{-1})(K) &= \mathbb{P}((S_\lambda^\alpha)^{-1}(K)) \\
&\geq \limsup_{k\to\infty} (|\log \delta_k|)^{-1} \int_{\delta_k}^1 1_{(S_\lambda^\alpha)^{-1}(K)} \left(\frac{\mu_{x,r}}{r^\alpha}\right) \frac{dr}{r} \\
&= \limsup_{k\to\infty} (|\log \delta_k|)^{-1} \int_{\delta_k}^1 1_K \left(\frac{\mu_{x,\lambda r}}{(\lambda r)^\alpha}\right) \frac{dr}{r} \\
&= \limsup_{k\to\infty} (|\log \delta_k|)^{-1} \int_{\delta_k/\lambda}^{1/\lambda} 1_K \left(\frac{\mu_{x,\lambda r}}{(\lambda r)^\alpha}\right) \frac{dr}{r} \\
&= \limsup_{k\to\infty} (|\log \delta_k|)^{-1} \int_{\delta_k}^1 1_K \left(\frac{\mu_{x,r}}{r^\alpha}\right) \frac{dr}{r} \\
&= \limsup_{k\to\infty} \mathbb{P}^x_{\delta_k}(K)
\end{aligned}$$

and, similarly, $(\mathbb{P} \circ (S_\lambda^\alpha)^{-1})(L) \leq \liminf_{k\to\infty} \mathbb{P}^x_{\delta_k}(L)$. Thus $\mathbb{P}^x_{\delta_k}$ converges vaguely to $\mathbb{P} \circ (S_\lambda^\alpha)^{-1}$, which, because of the unique determination of the limit, proves that $\mathbb{P} = \mathbb{P} \circ (S_\lambda^\alpha)^{-1}$. □

Examples of tangent measure distributions. Using Proposition 2.3 we can find the α-dimensional tangent measure distributions in several trivial situations.

PROPOSITION 2.9. *For every measure $\mu \in \mathcal{M}(\mathbb{R}^d)$, at μ-almost every $x \in \mathbb{R}^d$,*

(a) *the only 0-dimensional tangent measure distribution is given by the Dirac measure concentrated at the $\mu(\{x\})$ multiple of the Dirac measure concentrated at the origin (in fact, this holds at every x),*

(b) *the only d-dimensional tangent measure distribution is given either by the zero measure or by the Dirac measure concentrated at a multiple of Lebesgue measure on* $\mathbb{R}^d$ *depending on whether* $\limsup_{r\to 0} \mu U(x,r)/\ell^d U(x,r)$ *is infinite or finite (and, in the latter case, this limit defines the multiple),*

(c) *if* $\alpha > d$*, then the only* α*-dimensional tangent measure distribution is the zero measure distribution.*

PROOF. This is immediate from the fact that the α-dimensional tangent measure distributions are concentrated on the set of α-dimensional tangent measures and the form of these sets, see [**Mat95**, Chap. 14–16]. □

In the case of α-rectifiable sets we see that tangent measure distributions are a natural generalization of the classical concept of a tangent space.

PROPOSITION 2.10. *If* μ *is* α*-rectifiable then, at* μ*-almost every* $x \in \mathbb{R}^d$*, the* α*-dimensional tangent measure distribution at* x *is unique and given by a Dirac measure concentrated at the only tangent measure at this point, which is a constant multiple of Hausdorff measure on the approximate tangent space at the point.*

PROOF. This is immediate from the fact that the α-dimensional tangent measure distributions are concentrated on the set of α-dimensional tangent measures. □

The tangent measure distributions of self similar sets were studied by Graf [**Gra95**], who proved the following statement.

PROPOSITION 2.11. *Suppose* $K \subseteq \mathbb{R}^d$ *is a self similar set in the sense that* K *is the unique compact set such that* $K = \bigcup_{i=1}^n S_i K$ *for a family* $(S_1, \dots, S_n)$ *of contractive similitudes with the property that for the Hausdorff dimension* α *of* K *we have* $\mathcal{H}^\alpha(S_i K \cap S_j K) = 0$ *for all* $i \neq j$*. Then there is a measure distribution* $\mathbb{P}$ *such that for the* α*-Hausdorff measure* μ *on* K *the set of* α*-dimensional tangent measure distributions is* $\mathcal{P}^\alpha(\mu, x) = \{\mathbb{P}\}$ *for* μ*-almost every* x*. Moreover,* $\mathbb{P}$ *is given by its marginals* $\mathbb{P}_B$ *on* $\mathcal{M}(B)$*, for every compact ball* $B \subseteq \mathbb{R}^d$*, by*

$$\mathbb{P}_B(M) = \frac{1}{-\sum_{i=1}^n r_i^\alpha \log r_i} \int_G \int \int_{a(x)}^{b(x)} 1_M\Big(\frac{\mu_{x,r} \circ g}{r^\alpha}\Big) \frac{dr}{r} \, d\mu(x) \, d\nu(g) \,,$$

where $(r_1, \dots, r_n)$ *are the contraction ratios and* G *is the compact group generated by the rotational parts of* $(S_1, \dots, S_n)$*,* ν *is normalized Haar measure on* G *and* $a(x), b(x)$ *are constants depending on* x *and* B*.*

The measure distribution $\mathbb{P}$ in Graf's result has been described before as the natural randomization of μ by Patzschke and U. Zähle [**PZU90**].

Other explicit formulae for tangent measure distributions in self similar situations may be found in [**ArP96**] for the case of random self similar sets, in [**KrM98**] for the case of hyperbolic Cantor sets and in [**Mör00**] for the paths of Brownian motion.

The main result. While it is easy to prove the scaling invariance of tangent measure distributions, it is a more interesting and difficult task to formulate and prove their shift-invariance property. For every $u \in \mathbb{R}^d$ we define the shift operator $T^u : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ by $T^u \nu(E) = \nu(u + E)$. Our main theorem can be seen as a shift-invariance principle for tangent measure distributions.

THEOREM 2.12. *Let $\alpha \geq 0$ and let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be any measure. Then, at μ-almost all points x, every α-dimensional tangent measure distribution $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ satisfies*

$$\iint G(\nu, u)\, d\nu(u)\, d\mathbb{P}(\nu) = \iint G(T^u\nu, -u)\, d\nu(u)\, d\mathbb{P}(\nu) \tag{2}$$

for all Borel functions $G : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow [0, \infty)$.

REMARK 2.13. Because of the simple description of tangent measure distributions of rectifiable measures (see Proposition 2.10), the formula is easy whenever μ is rectifiable. In particular it holds trivially in the special cases $\alpha = 0$ and $\alpha \geq d$.

We point out that there is also a translational invariance property for tangent measures, see [**Pre87**, 2.12], but the invariance property above is of a much more rigid nature and also provides additional information about the local symmetry of measures, which has no analogue for tangent measures. We give more details in the next subsection. Before that, since the formulation of the translational invariance property may not be obvious, we explain some reformulations of Theorem 2.12. The proof of the theorem is deferred to Sections 3 and 4.

Measure distributions satisfying (2) are well known in the theory of random measures and are called Palm measures. The meaning of (2) becomes clearer when we apply it to a function G satisfying the shift invariance condition $G(\nu, u) = G(T^u\nu, 0)$ for all $\nu \in \mathcal{M}(\mathbb{R}^d)$ and $u \in \mathbb{R}^d$. Then we get, for every ball B around the origin,

$$\iint_B G(\nu, u)\, d\nu(u)\, d\mathbb{P}(\nu) = \iint_B G(\nu, 0)\, d\nu(u)\, d\mathbb{P}(\nu)\,,$$

and we can interpret Palm measures as measure distributions which have the origin as a typical point of their realizations. Palm measures appear in many areas of probability, for example in queuing theory or in the theory of point processes, where they play the rôle of conditional distributions of stationary point processes given a point at the origin (see [**Kal83**, Chap.10]).

The original, slightly more restrictive, definition derives the Palm measure from a stationary, i.e. (T^u)-invariant, σ-finite nonzero measure $\mathbb{Q}$ on $\mathcal{M}(\mathbb{R}^d)$. We say such a measure has *finite intensity* if the measure $\int \nu\, d\mathbb{Q}(\nu)$ is locally finite and hence, by shift-invariance, a positive multiple of d-dimensional Lebesgue measure. The multiplicity is called the *intensity* of $\mathbb{Q}$. A measure distribution $\mathbb{P}$ on $\mathcal{M}(\mathbb{R}^d)$ is a *Palm measure* if there is a stationary σ-finite measure $\mathbb{Q}$ on $\mathcal{M}(\mathbb{R}^d)$ with finite intensity and

$$\int_M \nu(B)\, d\mathbb{Q}(\nu) = \int_B \mathbb{P} \circ T^u(M)\, du \quad \text{for all } M \subseteq \mathcal{M}(\mathbb{R}^d),\ B \subseteq \mathbb{R}^d \text{ Borel.}$$

By a theorem of Mecke (see [**Mec67**]) the formula (2) characterizes $\mathbb{P}$ as a Palm measure in this sense.

LEMMA 2.14. *A measure distribution $\mathbb{P}$ on $\mathcal{M}(\mathbb{R}^d)$ is a Palm measure if and only if $\mathbb{P}(\{\phi\}) = 0$, where ϕ is the zero-measure, and the Palm formula (2) holds.*

Clearly $\mathbb{P}(\{\phi\}) = 0$ for all $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ if the lower α-dimensional density of μ at x is positive. Thus our main result implies that

THEOREM 2.15. *Let $\alpha \geq 0$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be any measure. Then, at μ-almost every point x at which the lower α-dimensional density of μ is positive, every tangent measure distribution $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ is a Palm measure.*

This statement establishes an interesting connection to the theory of self similar random measures developed by Ulrich Zähle in [**Zäh88**]. He suggested, heuristically speaking, that a random measure is statistically self similar if it is scaling invariant with respect to its "typical point" and we can interpret Palm measures as those measure distributions which have the origin as a "typical point". More precisely, a measure distribution $\mathbb{P}$ on $\mathcal{M}(\mathbb{R}^d)$ is an *α-self similar random measure* if $\mathbb{P}$ is a Palm measure invariant under the rescaling group $(S_\lambda^\alpha)_{\lambda>0}$.

This concept has been subject of investigations by Patzschke, U. Zähle and M. Zähle, see for example [**PZU90**] and [**PZM94**]. In particular, they have studied the relation of the concept to statistically self similar measures in the constructive sense. We extend the notion of α-self similar random measures to allow, in the natural way, for the possibility that $\mathbb{P}(\{\phi\}) > 0$ and/or that the total mass of $\mathbb{P}$ may be less than one. Of course, no such extension is needed if μ is supposed to have positive lower and finite upper α-dimensional densities. Using the notion of α-self similar random measures, we may reformulate Theorem 2.12 as

THEOREM 2.16. *Let $\alpha \geq 0$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be any measure. Then, at μ-almost all points x, every tangent measure distribution $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ defines an α-self similar random measure.*

Applications to average densities. Without further assumptions on the measure, the connection between average densities and tangent measure distributions is not straightforward, since it involves passing to a limit of an integral of the unbounded function $\nu \mapsto \nu U(0,1)$. For our purposes, the following information will suffice.

PROPOSITION 2.17. *Suppose that $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\alpha \geq 0$, $\delta_n \downarrow 0$ and that $\mathbb{P}_{\delta_n}^x$ converge vaguely to a measure distribution $\mathbb{P}$. If*

$$(3) \qquad \liminf_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu U(x,r)}{r^\alpha} \frac{dr}{r} < \infty,$$

then $\mathbb{P}_{\delta_n}^x$ converge to $\mathbb{P}$ weakly, hence $\mathbb{P}$ is a probability measure and, for every open set $U \subseteq \mathbb{R}^d$,

$$(4) \qquad \int \nu U \, d\mathbb{P}(\nu) \leq \liminf_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu_{x,r}(U)}{r^\alpha} \frac{dr}{r}.$$

If μ has finite upper α-dimensional density at x, then there is a compact subset of $\mathcal{M}(\mathbb{R}^d)$ containing the supports of all measures $\mathbb{P}_\delta^x$, $0 < \delta \leq 1$. In particular, $\mathbb{P}$ is a probability measure and, for every compact set $K \subseteq \mathbb{R}^d$,

$$(5) \qquad \int \nu K \, d\mathbb{P}(\nu) \geq \limsup_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu_{x,r}(K)}{r^\alpha} \frac{dr}{r},$$

and, for every $R > 0$,

$$(6) \qquad \int \nu U(0,R) \, d\mathbb{P}(\nu) = \lim_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu U(x,rR)}{r^\alpha} \frac{dr}{r}.$$

PROOF. As the sequence $\mathbb{P}^x_{\delta_n}$ converges weakly if a subsequence converges weakly, we may assume that the limit in (3) exists. For any $m = 1, 2, \dots$ we infer from

$$\begin{aligned}
\int \nu U(0,m)\, d\mathbb{P}^x_{\delta_n}(\nu) &= (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu U(x,mr)}{r^\alpha}\, \frac{dr}{r} \\
&= m^\alpha (|\log \delta_n|)^{-1} \int_{m\delta_n}^m \frac{\mu U(x,r)}{r^\alpha}\, \frac{dr}{r} \\
&\le m^\alpha (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu U(x,r)}{r^\alpha}\, \frac{dr}{r} \\
&\quad + m^\alpha (|\log \delta_n|)^{-1} \int_1^m \frac{\mu U(x,r)}{r^\alpha}\, \frac{dr}{r}
\end{aligned}$$

that the sequence $\int \nu U(0,m)\, d\mathbb{P}^x_{\delta_n}(\nu)$, $n = 1, 2, \dots$ is bounded by, say, C_m. Thus, for any $\varepsilon > 0$, the measures $\mathbb{P}^x_{\delta_n}$ are concentrated, up to measure ε, on the compact set

$$\bigcap_{m=1}^\infty \{\nu \in \mathcal{M}(\mathbb{R}^d) : \nu U(0,m) \le 2^m C_m/\varepsilon\}\,,$$

which shows that the vaguely convergent sequence $\mathbb{P}^x_{\delta_n}$ converges weakly. The inequality (4) follows, since the function $\nu \mapsto \nu U$ is lower semi-continuous.

If μ has finite upper α-dimensional density at x, then the set $\{\mu_{x,r}/r^\alpha \,:\, 0 < r \le 1\}$ is relatively compact in $\mathcal{M}(\mathbb{R}^d)$; its closure has the required property. Thus the function $\nu \mapsto \nu K$ is upper semi-continuous and bounded on a compact set containing supports of all $\mathbb{P}^x_{\delta_n}$. Hence

$$\int \nu K\, d\mathbb{P}(\nu) \ge \limsup_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu_{x,r} K}{r^\alpha}\, \frac{dr}{r}\,,$$

and

$$\begin{aligned}
\int \nu \overline{U(0,R)}\, d\mathbb{P}(\nu) &\ge \limsup_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu \overline{U(x,Rr)}}{r^\alpha}\, \frac{dr}{r} \\
&\ge \liminf_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu U(x,Rr)}{r^\alpha}\, \frac{dr}{r} \\
&\ge \int \nu U(0,R)\, d\mathbb{P}(\nu) = \int \nu \overline{U(0,R)}\, d\mathbb{P}(\nu)\,,
\end{aligned}$$

using Proposition 2.8 in the last step. This proves Proposition 2.17. □

We may use the formula (2) of Theorem 2.12 with $G(\nu, u) = h(u)$ to infer that, at μ-almost all points x, every tangent measure distribution $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ satisfies

$$\int\!\!\int h(u)\, d\nu(u)\, d\mathbb{P}(\nu) = \int\!\!\int h(-u)\, d\nu(u)\, d\mathbb{P}(\nu)\,, \tag{7}$$

for every Borel function $h : \mathbb{R}^d \longrightarrow [0, \infty)$. Choosing for h characteristic functions only, we get

$$\int \nu(U)\, d\mathbb{P}(\nu) = \int \nu(-U)\, d\mathbb{P}(\nu)\,, \tag{8}$$

for every Borel subset U of $\mathbb{R}^d$. If μ has finite upper α-dimensional density at x, the expectations $\int \nu\, d\mathbb{P}$ of the tangent measure distributions agree with the average tangent measures by Proposition 2.17 and (8) shows that all average tangent measures are symmetric about the origin. A number of unexpected statements concerning average densities, which have no analogy for ordinary densities, also follow from this equality.

PROPOSITION 2.18. *If μ has finite upper α-dimensional density almost everywhere, then at μ-almost all points x the following holds: Whenever $U \subseteq V \subseteq \mathbb{R}^d$ with U closed and V open and $\delta_n \downarrow 0$, we have*

$$\liminf_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu(x+rU)}{r^\alpha}\,\frac{dr}{r} \le \liminf_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu(x-rV)}{r^\alpha}\,\frac{dr}{r}\,.$$

PROOF. We may assume that $\mathbb{P}^x_{\delta_n}$ converges to a measure distribution $\mathbb{P}$. Employing the compactness result of Proposition 2.17 and (8), we infer that

$$\begin{aligned}
&\liminf_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu(x+rU)}{r^\alpha}\,\frac{dr}{r} \\
&\le \int \nu(U)\,d\mathbb{P}(\nu) = \int \nu(-U)\,d\mathbb{P}(\nu) \\
&\le \int \nu(-V)\,d\mathbb{P}(\nu) \\
&\le \liminf_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu(x-rV)}{r^\alpha}\,\frac{dr}{r}\,.
\end{aligned}$$

□

Whenever C is a Borel cone in $\mathbb{R}^d$, one defines the lower average α-dimensional density of a measure μ at a point x with respect to C as

$$\liminf_{\delta\downarrow 0} (|\log \delta|)^{-1} \int_{\delta}^1 \frac{\mu((x+C)\cap U(x,r))}{r^\alpha}\,\frac{dr}{r}\,.$$

If $C = \mathbb{R}^d$, this is the lower average density of the measure μ at a point x, if $d = 1$ and $C = (0,\infty)$ we speak about right lower average density of the measure μ at a point x, etc.

PROPOSITION 2.19. *Suppose $\alpha > 0$, $\mu \in \mathcal{M}(\mathbb{R}^d)$ has finite upper α-dimensional density almost everywhere and C is a closed cone which (except for the origin) is contained in the open cone K. Then, at μ-almost all points x, the lower average α-dimensional density of the measure μ with respect to C does not exceed its lower average α-dimensional density with respect to the cone $-K$.*

PROOF. For every tangent measure distribution $\mathbb{P}$ the scaling invariance property implies that $\int \nu(\overline{U(0,1)} \cap C)\,d\mathbb{P}(\nu) \le \int \nu(U(0,1)\cap K)\,d\mathbb{P}(\nu)$ and we can employ the same argument as in Proposition 2.18. □

As an immediate corollary of these propositions we obtain the following result, which was obtained only under the additional assumption of positive lower α-dimensional density in [**Mör98a**] and, in a weaker form, in [**MöP95**].

THEOREM 2.20. *Let $\alpha > 0$ and $\mu \in \mathcal{M}(\mathbb{R})$ be a measure on the line with finite upper α-dimensional densities μ-almost everywhere. Then, at μ-almost all points x, the right and left lower average α-dimensional densities coincide and each of them is half of the lower average α-dimensional density of μ at x.*

Of course, analogous results can be found for upper average densities.

As our final application we show in Theorem 2.22 how the results of [**FaS95**] and [**Mar96**] follow naturally by using the higher regularity of tangent measure distributions. Our approach removes their unnatural assumption of finiteness of the upper densities for results about lower densities.

LEMMA 2.21. *Suppose that a probability measure $\mathbb{P}$ on $\mathcal{M}(\mathbb{R}^d)$ defines an α-self similar random measure and that, with some $c > 0$, $\mathbb{P}$-almost every ν satisfies $\nu U(0,1) = c$. Then α is an integer.*

PROOF. From the invariance of $\mathbb{P}$ under the rescaling group $(S^\alpha_\lambda)_{\lambda>0}$ we infer that, for every $r > 0$ and $\mathbb{P}$-almost every ν, we have $\nu U(0,r) = cr^\alpha$. Thus, using the Palm formula in the last step,

$$\begin{aligned} 1 &= \mathbb{P}(\{\mu : \mu U(0,r) = cr^\alpha\}) = \frac{1}{cs^\alpha} \int\!\!\int_{U(0,s)} 1_{\{\mu : \mu U(0,r)=cr^\alpha\}}(\nu)\, d\nu(u)\, d\mathbb{P}(\nu) \\ &= \frac{1}{cs^\alpha} \int\!\!\int_{U(0,s)} 1_{\{\mu : \mu U(u,r)=cr^\alpha\}}(\nu)\, d\nu(u)\, d\mathbb{P}(\nu). \end{aligned}$$

As r and s are arbitrary, for $\mathbb{P}$-almost every ν we infer that $\nu U(u,r) = cr^\alpha$ for ν-almost every u and every rational $r > 0$. By continuity, $\mathbb{P}$-almost every ν satisfies that $\nu U(u,r) = cr^\alpha$ for every u in the support of ν and every $r > 0$. Now it suffices to use Marstrand's Theorem ([**Mar64**], see also [**Mat95**, 14.10]) to infer that α is an integer. □

THEOREM 2.22. *Suppose that $\mu \in \mathcal{M}(\mathbb{R}^d)$ and that $\alpha > 0$ is not an integer. Then μ-almost all points x satisfy:*

(a) *The lower average α-dimensional density and the lower α-dimensional density of μ at x are either both zero, or both infinite, or the former is strictly bigger than the latter.*
(b) *The upper average α-dimensional density and the upper α-dimensional density of μ at x are either both zero, or both infinite, or the former is strictly smaller than the latter.*

PROOF. If $d(x)$ and $D(x)$ denote the (ordinary) lower and upper α-dimensional densities of μ, respectively, then, for μ-almost every x, each $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ satisfies $d(x) \le \nu U(0,1) \le D(x)$. To use Lemma 2.21, it suffices to observe that if the average and ordinary lower, respectively upper, densities of μ at x coincide and are finite, then there is a probability measure $\mathbb{P} \in \mathcal{P}^\alpha(\mu,x)$ such that $\mathbb{P}$-almost every ν satisfies that $\nu U(0,1) = d(x)$, respectively $\nu U(0,1) = D(x)$. But if $d(x)$, respectively $D(x)$, is finite, then we can find $\delta_n \downarrow 0$ such that

$$\lim_{n\to\infty} (|\log \delta_n|)^{-1} \int_{\delta_n}^1 \frac{\mu U(x,r)}{r^\alpha} \frac{dr}{r} = d(x) \text{ , respectively } D(x).$$

We can choose $\mathbb{P}$ as the limit of a subsequence of $\mathbb{P}^x_{\delta_n}$. Then, by Proposition 2.3(a), $d(x) \le \nu U(0,1) \le D(x)$ for $\mathbb{P}$-almost every ν, and we may use the inequality (4) of Proposition 2.17 in case of lower densities and the equality (5) of the same proposition in case of upper densities to infer that $\mathbb{P}$-almost every ν satisfies $\nu U(0,1) = d(x)$, respectively $\nu U(0,1) = D(x)$. □

3. The main lemma and its proof

In this section we formulate and prove the main lemma for Theorem 2.12. The proof of the theorem will then be completed in Section 4.

The main lemma. We start by giving an intuitive picture of the rôle of the main lemma without caring too much about technicalities. We approximate the left hand side in the Palm formula (2) by

$$\int\int G(\nu,u)\,d\nu(u)\,\mathbb{P}^x_\delta(\nu) = (|\log\delta|)^{-1}\int_\delta^1 \int G\Big(\frac{\mu_{x,r}}{r^\alpha},\frac{y-x}{r}\Big)\,\frac{d\mu(y)}{r^\alpha}\,\frac{dr}{r}$$

and the right hand side by

$$\begin{aligned}
&\int\int G(T^u\nu,-u)\,d\nu(u)\,\mathbb{P}^x_\delta(\nu)\\
&= (|\log\delta|)^{-1}\int_\delta^1 \int G\Big(T^{\frac{y-x}{r}}\Big(\frac{\mu_{x,r}}{r^\alpha}\Big),\frac{x-y}{r}\Big)\,\frac{d\mu(y)}{r^\alpha}\,\frac{dr}{r}\\
&= (|\log\delta|)^{-1}\int_\delta^1 \int G\Big(\frac{\mu_{y,r}}{r^\alpha},\frac{x-y}{r}\Big)\,\frac{d\mu(y)}{r^\alpha}\,\frac{dr}{r}\,.
\end{aligned}$$

Hence, writing $H(x,y,r) = G\big(\frac{\mu_{x,r}}{r^\alpha},\frac{y-x}{r}\big)$, we have to prove that

$$\lim_{\delta\to 0}(|\log\delta|)^{-1}\int_\delta^1\int \frac{H(x,y,r)-H(y,x,r)}{r^\alpha}\,d\mu(y)\,\frac{dr}{r} = 0\,.$$

Our main lemma shows this for general mappings $H:\mathbb{R}^d\times\mathbb{R}^d\times(0,\infty)\to(0,\infty)$ that fulfill certain conditions. It does not refer explicitly to the notion of tangent measure distributions. Although our results do not depend on the choice of the norm of $\mathbb{R}^d$, it is convenient to equip $\mathbb{R}^d$ with the maximum norm defined, for $u=(u_1,\dots,u_d)$, by

$$\|u\| = \max\{|u_i| \,:\, i=1,\dots,d\}$$

and denote by $B(u,r)$ the closed balls with respect to this norm, i.e., $B(u,r)=\{x\in\mathbb{R}^d : \|u-x\|\le r\}$.

LEMMA 3.1. *Whenever a measure $\mu\in\mathcal{M}(\mathbb{R}^d)$, a Borel measurable function $H:\mathbb{R}^d\times\mathbb{R}^d\times(0,\infty)\longrightarrow[0,1]$ and $0<\alpha<d$ satisfy, for some constants $R,A_1,A_2\in(0,\infty)$, the conditions*

(i) *$H(x,y;r)=0$ whenever $\|x-y\|\ge Rr$,*
(ii) *$H(x,y;r)=0$ whenever $\mu B(x,5Rr)\ge A_1 r^\alpha$, and*
(iii) *for all $x,\widetilde{x},y,\widetilde{y}\in\mathbb{R}^d$ and $0<r<\infty$,*

$$|H(x,y;r)-H(\widetilde{x},\widetilde{y};r)| \le A_2\,\frac{\|x-\widetilde{x}\|+\|y-\widetilde{y}\|}{r}\,,$$

then, for μ-almost every $x\in\mathbb{R}^d$,

$$\lim_{\delta\downarrow 0}\frac{1}{|\log\delta|}\int_\delta^1\int\frac{H(x,y;r)-H(y,x;r)}{r^\alpha}\,d\mu(y)\,\frac{dr}{r}=0. \tag{9}$$

The rest of this section is devoted to the proof of this lemma. But before we give any details we outline the strategy behind our proof.

Outline of the proof. Writing $G(x,y,r)=(1/r^\alpha)(H(x,y;r)-H(y,x;r))$ we have to show that, for large integers n,

$$\frac{1}{n}\int_{2^{-n}}^1\int G(x,y,r)\,d\mu(y)\,\frac{dr}{r} \tag{10}$$

is small. Our first step in the proof is to approximate this situation by discretizing it in a dyadic grid.

Dyadic decomposition of domains is a common tool in analysis, in particular in singular integral theory, see e.g. [**Ste70**]. The tool is useful when combined with Lebesgue integration over the decomposed domain. We will use this idea here by replacing the integration with respect to dr/r in (10) by a sum over cubes with sidelengths comparable to their distances to x. Roughly speaking we approximate (10) by

$$\frac{1}{n}\int\sum_{k=0}^{n-1} H_k(x,y)\,d\mu(y)\,, \tag{11}$$

where $H_k(x,y) \approx \sum_Q 1_Q(x)1_Q(y)G(x,y,2^{-k})$ and the summation is with respect to all dyadic cubes of sidelength 2^{-k}. An interesting subtlety here is that, in order to avoid a concentration of μ-mass near the boundary of the cubes, we have to include an averaging procedure with respect to shifts of the measure in the proof of the approximation.

In order to prove that (11) is small for μ-almost all x we write

$$\varphi_x(E) = \sum_{k=0}^{n-1}\frac{\ell^d(E\cap Q_k\setminus Q_{k+1}))}{\ell^d(Q_k\setminus Q_{k+1})}\cdot\int_{Q_k} H_k(x,y)\,d\mu(y)\,,$$

where Q_k is the unique dyadic cube of sidelength 2^{-k} containing x, and show that

$$\int\Big[\int\sum_{k=0}^{n-1} H_k(x,y)\,d\mu(y)\Big]^2\,d\mu(x) = \int\Big(\varphi_x(\mathbb{R}^d)\Big)^2\,d\mu(x) \tag{12}$$

is small. We can now find a suitable partition of $\mathbb{R}^d$ into sets $\{G(I)\}_{I\in\mathcal{I}}$, which allows us to write

$$\int\Big(\varphi_x(\mathbb{R}^d)\Big)^2\,d\mu(x) = \sum_{I\in\mathcal{I}}\sum_{J\in\mathcal{I}}\int\varphi_x(G(I))\varphi(G(J))\,d\mu(x)\,. \tag{13}$$

Every set $G(I)$ is a finite union of subcubes of a cube I and for all these cubes we have the necessary lower control of the μ-mass. This will be accomplished by means of a stopping time argument. The proof finishes by giving careful individual estimates for the summands on the right hand side of (13).

We now start with the detailed proof. Our plan is first to formulate exactly the dyadic model for the main lemma, then show how this approximation may be used to prove its statement, and only then to embark upon the series of technical estimates which will eventually result in the proof of the dyadic version of our main lemma.

The dyadic model. For the exact formulation of the lemma we define, for $k = 0, 1, \ldots,$ the k-th grid as the family of dyadic cubes

$$\mathcal{Q}_k \quad = \quad \Big\{\prod_{i=1}^{d}\Big[\frac{k_i}{2^k},\frac{k_i+1}{2^k}\Big)\,:\,(k_1,\ldots,k_d)\in\mathbb{Z}^d\Big\}.$$

Any $x\in\mathbb{R}^d$ is contained in a unique cube from $\mathcal{Q}_k$; this cube will be denoted by $Q_k(x)$. The dyadic approximation to our main lemma is given in the following statement.

LEMMA 3.2. *Given $0<\alpha<d$ and $\sigma>0$ and constants $C_1, C_2\in(0,\infty)$, we can find a constant $C_0 = C_0(\sigma,\alpha,d,C_1,C_2)\in(0,\infty)$ having the following property: Whenever a finite measure $\mu\in\mathcal{M}(\mathbb{R}^d)$ and Borel measurable functions $H_k:\mathbb{R}^d\times\mathbb{R}^d\longrightarrow\mathbb{R}$ satisfy the conditions*

(i) $H_k(x,y) = 0$ *whenever* $y\notin Q_k(x)$,

(ii) $H_k(x,y) = 0$ *whenever* $\mu(Q_k(x)) > C_1 2^{-k\alpha}$,
(iii) $|H_k(x,u) - H_k(y,u)| \le C_2 2^{k\alpha+k-i}$ *if* $i \ge k$, $x,y \in I \in \mathcal{Q}_i$ *and* $u \in Q_k(x)$,
(iv) $H_k(y,x) = -H_k(x,y)$ *for all* $k = 0,1,\dots$ *and all* $x,y \in \mathbb{R}^d$,

then for any $n = 1,2,\dots$ *and any non-negative function* $g \in L_2(\mu)$,

$$\int g(x)\Big(\int \sum_{k=0}^{n-1} H_k(x,u)\, d\mu(u) - \sigma n\Big) d\mu(x) \quad \le \quad \mathcal{C}_0 n^{3/4} \mu(\mathbb{R}^d)^{1/2} \|g\|_{L_2(\mu)}.$$

Although we shall not need it, it is interesting to note that our proof allows one to describe the dependence of $\mathcal{C}_0$ on all parameters explicitly; this shows, in particular, that $\mathcal{C}_0$ may be chosen independently of α as long as α is kept away from d.

Proof of Lemma 3.1 using Lemma 3.2. We need only prove Lemma 3.1 under the additional assumption that μ is finite, as the general case immediately follows by using the special one for the restrictions of μ to $B(0,k)$, $k = 1,2,\dots$.

LEMMA 3.3. *Suppose that* $0 < \alpha < d$, $\mu \in \mathcal{M}(\mathbb{R}^d)$ *is a finite measure and* $G : \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty) \longrightarrow \mathbb{R}$ *is a Borel measurable function, and* $\hat{A}_1, \hat{A}_2 \in (0,\infty)$ *are constants such that*

(i) $|G(x,y;r)| \le r^{-\alpha}$ *for all* $x,y \in \mathbb{R}^d$ *and* $r > 0$,
(ii) $G(x,y;r) = 0$ *whenever* $\|x-y\| \ge r$,
(iii) $G(x,y;r) = 0$ *whenever* $\mu B(x,4r) \ge \hat{A}_1 r^\alpha$,
(iv) *for all* $x,y,u \in \mathbb{R}^d$ *and* $0 < r < \infty$,

$$|G(x,u;r) - G(y,u;r)| \le \hat{A}_2 r^{-\alpha-1} \|x-y\| \ ,$$

(v) $G(y,x;r) = -G(x,y;r)$ *for all* $x,y \in \mathbb{R}^d$ *and* $r > 0$.

Then for every $\sigma > 0$ *there is* $\mathcal{C}_1 < \infty$ *such that for any* $n = 1,2,\dots$ *and any non-negative* $g \in L_2(\mu)$,

$$\int g(x)\Big[\int_{2^{-n-1}}^{1/2} \int G(x,y;r)\, d\mu(y) \frac{dr}{r} - \sigma n\Big]\, d\mu(x) \le \mathcal{C}_1 n^{3/4} \|g\|_{L_2(\mu)}. \tag{14}$$

Consequently, for μ*-almost all* $x \in \mathbb{R}^d$, *the equality*

$$\lim_{\delta\downarrow 0} \frac{1}{|\log \delta|} \int_{\delta R}^{R} \int G(x,y;r)\, d\mu(y) \frac{dr}{r} = 0 \tag{15}$$

holds for every $R > 0$.

PROOF. The function

$$\omega(z) = \prod_{i=1}^{d} \big(1 - \min\{|z_i|, 1/2\}\big)^{-1}$$

verifies $1 \le \omega(z) \le 2^d$ and is Lipschitz on $\mathbb{R}^d$; let C_0 be its Lipschitz constant.

We define $C_1 = \hat{A}_1$ and $C_2 = 4^\alpha C_0 + 2^{d+2\alpha+2} \hat{A}_2$, and we prove that, for any $a \in \mathbb{R}^d$ and any $1/4 \le s \le 1/2$, the functions

$$H_k(x,y;a,s) \quad = \quad \sum_{Q \in \mathcal{Q}_k} 1_Q(x) 1_Q(y)\, G(a+x, a+y; 2^{-k}s)\, \omega\big(2^k(y-x)\big)$$

satisfy the assumptions (i)–(iv) of Lemma 3.2 for the measure $T^a\mu$. Clearly, 3.2(i) follows directly from the definition of $H_k(x,y;a,s)$ and 3.2(iv) follows directly from 3.3(v). If $T^a\mu(Q_k(x)) > C_1 2^{-k\alpha}$, then

$$\begin{aligned}\mu(B(a+x,2^{-k+2}s)) &\geq \mu(B(a+x,2^{-k})) \geq \mu(a+Q_k(x)) = T^a\mu(Q_k(x))\\ &> C_1 2^{-k\alpha} \geq \hat{A}_1(2^{-k}s)^\alpha,\end{aligned}$$

which, by 3.3(iii), shows that $G(a+x,a+y;2^{-k}s)=0$ and hence the requirement of 3.2(ii) that $H_k(x,y;a,s)=0$. Finally, the condition 3.2(iii) is also easy, since $i \geq k$, $x,y \in I \in \mathcal{Q}_i$ and $u \in Q_k(x)$ imply, because of 3.3(ii) and 3.3(iv), that

$$\begin{aligned}&|H_k(x,u;a,s) - H_k(y,u;a,s)|\\ &= \big|G(x+a,u+a;2^{-k}s)\,\omega\big(2^k(u-x)\big) - G(y+a,u+a;2^{-k}s)\,\omega\big(2^k(u-y)\big)\big|\\ &\leq \big|\omega\big(2^k(u-x)\big) - \omega\big(2^k(u-y)\big)\big|\,\big|G(x+a,u+a;2^{-k}s)\big|\\ &\quad + \big|\omega\big(2^k(u-y)\big)\big|\,\big|G(x+a,u+a;2^{-k}s) - G(y+a,u+a;2^{-k}s)\big|\\ &\leq (2^{-k}s)^{-\alpha}C_0 2^k\|y-x\| \ + 2^d\hat{A}_2(2^{-k}s)^{-\alpha-1}\|x-y\|\\ &\leq C_2 2^{k\alpha+k-i}\ .\end{aligned}$$

Let $\mathcal{C} = \mathcal{C}_0(\sigma/\log 2,\alpha,d,C_1,C_2)\mu(\mathbb{R}^d)^{1/2}$, where $\mathcal{C}_0$ is the constant introduced in Lemma 3.2 and let $g \in L_2(\mu)$ be non-negative. The statement of Lemma 3.2 then implies that, for every $a \in \mathbb{R}^d$ and every $1/4 \leq s \leq 1/2$,

$$\begin{aligned}\int g(x+a)\Big(\int \sum_{k=0}^{n-1} H_k(x,u;a,s)\ dT^a\mu(u) - \sigma n/\log 2\Big)dT^a\mu(x)\\ \leq \mathcal{C}n^{3/4}\|g\|_{L_2(\mu)}\ .\end{aligned}$$

Hence, denoting

$$\begin{aligned}G_k(x,y;a,s) &= H_k(x-a,y-a;a,s)\\ &= \sum_{Q\in\mathcal{Q}_k} 1_{a+Q}(x)1_{a+Q}(y)G(x,y;2^{-k}s)\,\omega\big(2^k(y-x)\big),\end{aligned}$$

we have that

$$\int g(x)\Big(\int \sum_{k=0}^{n-1} G_k(x,u;a,s)\,d\mu(u) - \sigma n/\log 2\Big)d\mu(x) \leq \mathcal{C}n^{3/4}\|g\|_{L_2(\mu)}. \tag{16}$$

If $\|y-x\| \leq 2^{-k}s$, we use that $s \leq 1/2$ to infer that $\omega\big(2^k(y-x)\big) = \prod_{i=1}^d \big(1-2^k|y_i-x_i|\big)^{-1}$. Thus

$$\begin{aligned}&\int_{[0,1]^d} \sum_{Q\in\mathcal{Q}_k} 1_{a+Q}(x)1_{a+Q}(y)\,d\ell^d(a)\\ &= \ell^d\{a\in[0,1]^d : \sum_{Q\in\mathcal{Q}_k} 1_{a+Q}(x)1_{a+Q}(y) = 1\}\\ &= 1/\omega\big(2^k(y-x)\big)\end{aligned}$$

and we conclude that

$$G(x,y;2^{-k}s) = \int_{[0,1]^d} G_k(x,y;a,s)\,d\ell^d(a). \tag{17}$$

If $\|y-x\| > 2^{-k}s$, then both sides of (17) are zero according to (ii); hence (17) may be used for all x, y. Thus, integrating (16) over $[0,1]^d$ with respect to the Lebesgue measure and over $1/4 \le s \le 1/2$ with respect to the measure $\frac{ds}{s}$ and using that

$$\int_{1/4}^{1/2} G(x,u;2^{-k}s)\frac{ds}{s} = \int_{2^{-k-2}}^{2^{-k-1}} G(x,u;r)\frac{dr}{r}\ ,$$

we get

$$\int g(x)\Big[\int_{2^{-n-1}}^{1/2}\int G(x,y;r)\,d\mu(y)\frac{dr}{r} - \sigma n\Big]\,d\mu(x) \le \mathcal{C}n^{3/4}\|g\|_{L_2(\mu)}\log 2,$$

which shows that (14) holds with $\mathcal{C}_1 = \mathcal{C}\log 2$.

To prove (15), we denote

$$\bar{G}(x;r) = \int G(x,y;r)d\mu(y)$$

and note that, by 3.3(ii),3.3(v) and 3.3(iv),

$$\begin{aligned} |G(x,y;r)| &= 1_{B(x,r)}(y)|G(x,y;r) - G(y,y;r)| \le \hat{A}_2 r^{-\alpha-1}\|x-y\|1_{B(x,r)}(y) \\ &\le \hat{A}_2 r^{-\alpha}1_{B(x,r)}(y). \end{aligned}$$

Since, by 3.3(iii), the left side is zero unless $\mu(B(x,r)) \le \hat{A}_1 r^{\alpha}$, we obtain by integrating with respect to $d\mu(y)$ that for all x and r,

$$|\bar{G}(x;r)| \le \hat{A}_2 r^{-\alpha}\mu(B(x,r)) \le \hat{A}_1\hat{A}_2. \tag{18}$$

Next we use (14) for the characteristic function g of the set

$$E_n \;=\; \Big\{x : \int_{2^{-n-1}}^{1/2} \bar{G}(x;r)\,\frac{dr}{r} > 2\sigma n\Big\},$$

to get that

$$\sigma n\mu(E_n) \;\le\; \mathcal{C}_1 n^{3/4}\mu(E_n)^{1/2},$$

hence $\mu(E_n) \le \mathcal{C}_1^2 n^{-1/2}/\sigma^2$, which shows that $\sum_{n=1}^\infty \mu(E_{n^3})$ converges. By the Borel-Cantelli Lemma, for μ-almost every $x \in \mathbb{R}^d$ there is a positive integer m such that $x \notin \bigcup_{n=m}^\infty E_{n^3}$. Consider any such x and the corresponding m; we may clearly also assume that $m \ge 7\hat{A}_1\hat{A}_2/\sigma$. Whenever $0 < \delta < 2^{-m^3}$ we find $n \ge m$ such that $2^{-(n+1)^3} \le \delta < 2^{-n^3}$ and we note that $x \notin E_{n^3}$ implies that

$$\frac{1}{|\log\delta|}\int_{2^{-n^3-1}}^{1/2} \bar{G}(x;r)\,\frac{dr}{r} = \frac{n^3}{|\log\delta|}\frac{1}{n^3}\int_{2^{-n^3-1}}^{1/2} \bar{G}(x;r)\,\frac{dr}{r} \le 2\sigma/\log 2\ .$$

Since (18) gives

$$\frac{1}{|\log\delta|}\int_{1/2}^{R} \bar{G}(x;r)\,\frac{dr}{r} \le \frac{\hat{A}_1\hat{A}_2(|\log R| + \log 2)}{|\log\delta|}$$

and

$$\begin{aligned} \frac{1}{|\log\delta|}\int_{\delta R}^{2^{-n^3-1}} \bar{G}(x;r)\,\frac{dr}{r} &\le \frac{\hat{A}_1\hat{A}_2(|\log R| + ((n+1)^3 - n^3)\log 2)}{|\log\delta|} \\ &\le \frac{\hat{A}_1\hat{A}_2|\log R|}{|\log\delta|} + \frac{7\hat{A}_1\hat{A}_2}{n}, \end{aligned}$$

we infer that

$$\limsup_{\delta\downarrow 0}\frac{1}{|\log\delta|}\int_{\delta R}^{R}\int G(x,y;r)\,d\mu(y)\frac{dr}{r}\le 2\sigma/\log 2$$

for every $R>0$. Since $\sigma>0$ may be arbitrary, it follows that for μ-almost every x the inequality

$$\limsup_{\delta\downarrow 0}\frac{1}{|\log\delta|}\int_{\delta R}^{R}\int G(x,y;r)\,d\mu(y)\frac{dr}{r}\le 0$$

holds for every $R>0$. The statement (15) follows by using this also with G replaced by $-G$. □

PROOF OF LEMMA 3.1. If the assumptions of Lemma 3.1 hold, we let $\hat{A}_1 = A_1R^{-\alpha}$, $\hat{A}_2 = 2A_2R$, and

$$G(x,y;r) = \frac{H(x,y;r/R)-H(y,x;r/R)}{r^\alpha},$$

and note that the assumptions (i), (ii), (iv) and (v) of Lemma 3.3 are clearly satisfied. To establish 3.3(iii), we observe that it follows directly from 3.1(i) if $\|x-y\|\ge r$. If $\|x-y\|<r$ then $B(x,5r)\cap B(y,5r)\supseteq B(x,4r)$, hence $\mu B(x,4r)\ge\hat{A}_1 r^\alpha$ implies that $\mu B(x,5r)\ge A_1(r/R)^\alpha$ and $\mu B(y,5r)\ge A_1(r/R)^\alpha$, hence $H(x,y;r/R)=H(y,x;r/R)=0$ according to 3.1(ii) .

Hence, using (15) of Lemma 3.3, we have

$$\begin{aligned}&\lim_{\delta\downarrow 0}\frac{1}{|\log\delta|}\int_{\delta}^{1}\int\frac{H(x,y;r)-H(y,x;r)}{r^\alpha}\,d\mu(y)\,\frac{dr}{r}\\ &= R^\alpha\lim_{\delta\downarrow 0}\frac{1}{|\log\delta|}\int_{\delta R}^{R}\int G(x,y;r)\,d\mu(y)\frac{dr}{r}=0.\end{aligned}$$

Proof of Lemma 3.2. For the remainder of this section we shall assume that $0<\alpha<d$ and $\sigma, C_1, C_2>0$, the measure μ, the functions H_k and an integer n are fixed. Define the cut-off function

$$\omega_k(x)=\begin{cases}1 & \text{if } C_2^{-1}\sigma 2^{-k\alpha}\le\mu(Q_k(x))\le C_1 2^{-k\alpha},\\ 0 & \text{otherwise,}\end{cases}$$

and let, for $k\in\{0,1,\dots,n-1\}$, $x\in\mathbb{R}^d$ and Borel sets $E\subseteq\mathbb{R}^d$,

$$H_k(x) = \omega_k(x)\int_{Q_k(x)} H_k(x,u)\,d\mu(u)$$

and

$$\varphi_x(E) = (C_1C_2)^{-1}\sum_{k=0}^{n-1}\frac{\ell^d\big(E\cap(Q_k(x)\setminus Q_{k+1}(x))\big)}{\ell^d\big(Q_k(x)\setminus Q_{k+1}(x)\big)}H_k(x).$$

These notions are justified by the following observation.

LEMMA 3.4. *For every positive function $g\in L_2(\mu)$,*

$$\begin{aligned}&\int g(x)\Big(\int\sum_{k=0}^{n-1}H_k(x,u)\,d\mu(u)-\sigma n\Big)d\mu(x)\\ &\le C_1C_2\int g(x)\varphi_x(\mathbb{R}^d)d\mu(x)\\ &\le C_1C_2\|g\|_{L_2(\mu)}\left(\int\big(\varphi_x(\mathbb{R}^d)\big)^2d\mu(x)\right)^{1/2}.\end{aligned}$$

PROOF. By (i) and by the case $i = k$ of (iii), $H_k(x,u) \le C_2 2^{k\alpha} 1_{Q_k(x)}(u)$. Thus, if $\omega_k(x) = 0$, then either $\mu(Q_k(x)) < C_2^{-1}\sigma 2^{-k\alpha}$ and $\int H_k(x,u)\,d\mu(u) \le C_2 2^{k\alpha}\mu(Q_k(x)) < \sigma$, or $\mu(Q_k(x)) > C_1 2^{-k\alpha}$ in which case we infer from (ii) that $\int H_k(x,u)\,d\mu(u) = 0 < \sigma$, and hence

$$\int \sum_{k=0}^{n-1} H_k(x,u)\,d\mu(u) - \sigma n \le \int \sum_{k=0}^{n-1} \omega_k(x) H_k(x,u)\,d\mu(u) = C_1 C_2 \varphi_x(\mathbb{R}^d) .$$

The statement follows by multiplying this inequality by $g(x)$, integrating with respect to $d\mu(x)$, and using Hölder's inequality. □

We therefore devote the rest of this section to the estimate of $\int \big(\varphi_x(\mathbb{R}^d)\big)^2 d\mu(x)$. Since working with the signed measures φ_x may sometimes be slightly awkward, we majorize them by Radon measures ψ_x $(x \in \mathbb{R}^d)$ defined by

$$\psi_x(E) \quad = \quad \sum_{k=0}^{n-1} \frac{\ell^d\big(E \cap (Q_k(x) \setminus Q_{k+1}(x))\big)}{\ell^d\big(Q_k(x) \setminus Q_{k+1}(x)\big)} \omega_k(x).$$

To see the majorization, we first note that $\omega_k(x) \neq 0$ implies that $\omega_k(x) = 1$ and $\mu(Q_k(x)) \le C_1 2^{-k\alpha}$, which proves that

LEMMA 3.5. *For every $x \in \mathbb{R}^d$ and every $k \in \{0,1,\dots,n-1\}$,*

$$\omega_k(x) 2^{k\alpha} \mu(Q_k(x)) \le C_1 .$$

Consequently, $|H_k(x)| \le \omega_k(x) \int_{Q_k(x)} |H_k(x,u)|\,d\mu(u) \le \omega_k(x) 2^{k\alpha} C_2 \mu(Q_k(x)) \le \omega_k(x) C_1 C_2$, and we infer that

LEMMA 3.6. *For every $x \in \mathbb{R}^d$ and every Borel set $E \subseteq \mathbb{R}^d$,*

$$|\varphi_x(E)| \le \psi_x(E) .$$

In addition to the above auxiliary functions and measures, we need to decompose the family of dyadic cubes into several subfamilies. We denote by

$$\mathcal{I} \quad = \quad \mathcal{Q}_0 \cup \bigcup_{k=1}^{n-1} \{Q \in \mathcal{Q}_k : \mu(Q) \ge C_2^{-1}\sigma 2^{-k\alpha}\}$$

the family of those dyadic cubes which have a non-negligible μ measure (or belong to the starting grid).

Noting that $Q_0(x) \in \mathcal{I}$ for every $x \in \mathbb{R}^d$, we define

$$k(x) \quad = \quad \max\{k = 0,1,\dots,n-1 : Q_k(x) \in \mathcal{I}\}$$

and, for $I \in \mathcal{Q}_m$,

$$G(I) \quad = \quad \{x \in I : k(x) = m\}.$$

It will also be convenient to extend the notation $Q_k(x)$ in the following way: If $I \in \mathcal{Q}_m$ and $0 \le k \le m$, then $Q_k(I)$ will denote the unique cube from $\mathcal{Q}_k$ containing I.

Finally, we choose the smallest integer q such that $q^2 \ge n$, define, for $i \in \{0,1,\dots,q-1\}$,

$$\mathcal{I}_i \quad = \quad \mathcal{I} \cap \bigcup_{k=iq}^{(i+1)q-1} \mathcal{Q}_k,$$

and denote by $\mathcal{K}_i$ the set of maximal cubes from the set $\mathcal{I} \cap \bigcup_{k=iq}^{n-1} \mathcal{Q}_k = \bigcup_{j=i}^{q-1} \mathcal{I}_j$, i.e.,

$$\mathcal{K}_i = \left\{ K \in \mathcal{I} \cap \bigcup_{k=iq}^{n-1} \mathcal{Q}_k : K \subseteq I \in \bigcup_{k=iq}^{n-1} \mathcal{Q}_k,\ K \neq I \text{ implies } I \notin \mathcal{I} \right\}.$$

LEMMA 3.7. *If $K \in \mathcal{I}$, the sets $G(I)$ ($I \in \mathcal{I}$, $I \subseteq K$) form a (Borel measurable, finite) partition of K. Consequently, the union of the family $\mathcal{K}_i$ is partitioned into the cubes from $\mathcal{K}_i$ as well as into the sets $G(I)$ ($I \in \bigcup_{k=i}^{q-1} \mathcal{I}_k$). In particular, the sets $G(I)$ ($I \in \mathcal{I}$) form a (Borel measurable, locally finite) partition of $\mathbb{R}^d$.*

PROOF. Suppose that $K \in \mathcal{I} \cap \mathcal{Q}_m$. If $x \in K$, then $k(x) \geq m$, so $I = Q_{k(x)}(x) \subseteq K$, $I \in \mathcal{I}$ and $x \in G(I)$. Hence $K \subseteq \bigcup_{I \in \mathcal{I}, I \subseteq K} G(I)$ which, since $G(I) \subseteq I$, implies that $K = \bigcup_{I \in \mathcal{I}, I \subseteq K} G(I)$. Finally, $x \in G(I) \cap G(J)$ implies that $I = Q_{k(x)}(x) = J$, so $G(I) \cap G(J) = \emptyset$ for $I \neq J$. The bracketed statements about Borel measurability and finiteness are obvious since each $G(I)$ is a finite union of cubes from $\mathcal{Q}_{n-1}$ and the number of those $I \in \mathcal{Q}$ for which $I \subseteq K$ is finite.

Since the cubes from the family $\mathcal{K}_i$ are clearly disjoint, their union is partitioned by them as well as by the sets $G(I)$, where I runs through cubes from $\mathcal{I}$ that are subsets of cubes from $\mathcal{K}_i$. But each such I belongs to $\bigcup_{k=i}^{q-1} \mathcal{I}_k$ and, conversely, each $I \in \bigcup_{k=i}^{q-1} \mathcal{I}_k$ belongs to $\mathcal{I}$, hence, by definition, it is a subset of some cube from $\mathcal{K}_i$. This proves the second statement and, since the union of $\mathcal{K}_0 = \mathcal{Q}_0$ is $\mathbb{R}^d$, also the last one. $\square$

We are now ready to start our main estimates. Note that, according to Lemma 3.7,

$$\varphi_x(\mathbb{R}^d) = \sum_{I \in \mathcal{I}} \varphi_x(G(I))$$

and

$$\sum_{j=i+1}^{q-1} \sum_{J \in \mathcal{I}_j} \varphi_x(G(J)) = \sum_{K \in \mathcal{K}_{i+1}} \varphi_x(K).$$

Hence, using that $|\varphi_x(E)| \leq \psi_x(E)$ by Lemma 3.6, we may estimate

$$\begin{aligned}
&\int \left(\varphi_x(\mathbb{R}^d)\right)^2 d\mu(x) \\
&= \sum_{I,J \in \mathcal{I}} \int \varphi_x(G(I))\, \varphi_x(G(J))\, d\mu(x) \\
&= \sum_{i=0}^{q-1} \sum_{I,J \in \mathcal{I}_i} \int \varphi_x(G(I))\, \varphi_x(G(J))\, d\mu(x) \\
&\quad + 2 \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} \sum_{I \in \mathcal{I}_i} \sum_{J \in \mathcal{I}_j} \int \varphi_x(G(I))\, \varphi_x(G(J))\, d\mu(x) \\
&\leq \sum_{i=0}^{q-1} \sum_{I,J \in \mathcal{I}_i} \int \psi_x(G(I))\, \psi_x(G(J))\, d\mu(x) \qquad (19) \\
&\quad + 2 \sum_{i=0}^{q-2} \sum_{I \in \mathcal{I}_i} \sum_{K \in \mathcal{K}_{i+1}} \int \varphi_x(G(I))\, \varphi_x(K)\, d\mu(x). \qquad (20)
\end{aligned}$$

For $i \geq 1$ we divide the integral $\int \psi_x(G(I))\, \psi_x(G(J))\, d\mu(x)$ in (19) into integrals over $Q_{(i-1)q}(I) \cap Q_{(i-1)q}(J)$ and $\mathbb{R}^d \setminus (Q_{(i-1)q}(I) \cap Q_{(i-1)q}(J))$ and, using that ψ_x is a non-negative measure, estimate the second integral by twice the integral over $\mathbb{R}^d \setminus Q_{(i-1)q}(I)$. Similarly, we divide the integral $\int \varphi_x(G(I))\, \varphi_x(K)\, d\mu(x)$ in (20) into integrals over $\mathbb{R}^d \setminus K$ and K and replace φ_x by ψ_x in the first, but then estimate the second integral in a slightly more sophisticated way by

$$\begin{aligned}
&\int_K \varphi_x(G(I))\, \varphi_x(K)\, d\mu(x) \\
&\quad = \int_K \big(\varphi_x(G(I)) - \varphi_{\zeta(K)}(G(I))\big)\, \varphi_x(K)\, d\mu(x) \\
&\qquad + \int_K \varphi_{\zeta(K)}(G(I))\, \varphi_x(K)\, d\mu(x) \\
&\quad \leq \sup_{z \in K} \big|\varphi_z(G(I)) - \varphi_{\zeta(K)}(G(I))\big| \int_K \psi_x(K)\, d\mu(x) \\
&\qquad + \varphi_{\zeta(K)}(G(I)) \int_K \varphi_x(K)\, d\mu(x),
\end{aligned}$$

where $\zeta(K)$ denotes the centre of K. Thus

$$\int \big(\varphi_x(\mathbb{R}^d)\big)^2\, d\mu(x)$$

$$\leq \sum_{I,J \in \mathcal{I}_0} \int \psi_x(G(I))\, \psi_x(G(J))\, d\mu(x) \tag{21}$$

$$+ \sum_{i=1}^{q-1} \sum_{I,J \in \mathcal{I}_i} \int_{Q_{(i-1)q}(I) \cap Q_{(i-1)q}(J)} \psi_x(G(I))\, \psi_x(G(J))\, d\mu(x) \tag{22}$$

$$+ 2 \sum_{i=1}^{q-1} \sum_{I,J \in \mathcal{I}_i} \int_{\mathbb{R}^d \setminus Q_{(i-1)q}(I)} \psi_x(G(I))\, \psi_x(G(J))\, d\mu(x) \tag{23}$$

$$+ 2 \sum_{i=0}^{q-2} \sum_{I \in \mathcal{I}_i} \sum_{K \in \mathcal{K}_{i+1}} \int_{\mathbb{R}^d \setminus K} \psi_x(G(I))\, \psi_x(K)\, d\mu(x) \tag{24}$$

$$+ 2 \sum_{i=0}^{q-2} \sum_{I \in \mathcal{I}_i} \sum_{K \in \mathcal{K}_{i+1}} \sup_{z \in K} \big|\varphi_z(G(I)) - \varphi_{\zeta(K)}(G(I))\big| \int_K \psi_x(K) d\mu(x) \tag{25}$$

$$+ 2 \sum_{i=0}^{q-2} \sum_{I \in \mathcal{I}_i} \sum_{K \in \mathcal{K}_{i+1}} \varphi_{\zeta(K)}(G(I)) \int_K \varphi_x(K)\, d\mu(x). \tag{26}$$

Taking into account Lemma 3.4, we conclude that to finish the proof of Lemma 3.2 it suffices to estimate each of the terms (21-26) by $\mathcal{C} n^{3/2} \mu(\mathbb{R}^d)$, or by $\mathcal{C} q^3 \mu(\mathbb{R}^d)$, since $n \leq q^2 \leq (\sqrt{n}+1)^2 \leq 4n$ implies $n^{3/2} \leq q^3 \leq 8n^{3/2}$. This will be done, term by term, in a series of lemmas. The first of them follows immediately from the definitions.

LEMMA 3.8. *If $0 \leq i < j \leq n$, then*

$$\psi_x(Q_i(x) \setminus Q_j(x)) = \sum_{k=i}^{j-1} \omega_k(x) \leq j - i.$$

In particular,

$$\psi_x(\mathbb{R}^d) = \psi_x(Q_0(x) \setminus Q_n(x)) \le n \le q^2 \ .$$

LEMMA 3.9. *Whenever $0 \le l < m \le n$ and $x \in I \in \mathcal{Q}_l$, then*

$$\psi_x(Q_m(x) \cap G(I)) = 0 \ .$$

PROOF. Since $\psi_x(Q_m(x) \cap G(I)) \le \sum_{k=m}^{n-1} \omega_k(x)$, the statement is obvious if $m = n$ or $\omega_k(x) = 0$ for all $m \le k \le n-1$. On the other hand, if $p \ge m$ is the least index for which $\omega_p(x) \ne 0$, then $Q_p(z) = Q_p(x) \in \mathcal{I}$ for all $z \in Q_p(x)$, so $k(z) \ge p$ for all such z and we infer that $Q_p(x) \cap G(I) = \emptyset$. Hence $\psi_x(Q_m(x) \cap G(I)) \le \sum_{k=m}^{p-1} \omega_k(x) = 0$, since $\omega_k(x) = 0$ for all $m \le k < p$. □

LEMMA 3.10. *If $0 \le l \le i < j \le n$ and $\mathcal{J} \subseteq \mathcal{I} \cap \bigcup_{k=i}^{j-1} \mathcal{Q}_k$, then*

$$\sum_{I \in \mathcal{J}, x \in Q_l(I)} \psi_x(G(I)) \le 2(j-l) \ .$$

Consequently,

$$\sum_{I,J \in \mathcal{J}} \int_{Q_l(I) \cap Q_l(J)} \psi_x(G(I))\, \psi_x(G(J))\, d\mu(x) \le 4(j-l)^2 \mu(\mathbb{R}^d) \ .$$

PROOF. If $I \in \mathcal{J}$ and $x \in Q_l(I) \setminus I$, then $G(I) \subseteq I \subseteq Q_l(x) \setminus Q_j(x)$. Using also the disjointness of $G(I)$ $(I \in \mathcal{I})$ (Lemma 3.7) and Lemma 3.8, we have

$$\sum_{I \in \mathcal{J}, x \in Q_l(I) \setminus I} \psi_x(G(I)) \le \psi_x(Q_l(x) \setminus Q_j(x)) \le j - l.$$

Similarly, but using first Lemma 3.9, we get

$$\sum_{I \in \mathcal{J}, x \in I} \psi_x(G(I)) = \sum_{I \in \mathcal{J}, x \in I} \psi_x(G(I) \setminus Q_j(x)) \le \psi_x(Q_l(x) \setminus Q_j(x)) \le j - l.$$

Adding these two inequalities gives the first statement. The second statement follows, since

$$\begin{aligned} &\sum_{I,J \in \mathcal{J}} \int_{Q_l(I) \cap Q_l(J)} \psi_x(G(I))\, \psi_x(G(J))\, d\mu(x) \\ &\quad = \int \Big(\sum_{\substack{I \in \mathcal{J} \\ x \in Q_l(I)}} \psi_x(G(I)) \Big)^2 d\mu(x) \le 4(j-l)^2 \mu(\mathbb{R}^d). \end{aligned}$$

□

ESTIMATE OF (21). From Lemma 3.10 with $\mathcal{J} = \mathcal{I}_0$ we infer

$$\sum_{I,J \in \mathcal{I}_0} \int \psi_x(G(I))\, \psi_x(G(J))\, d\mu(x) \le 4q^3 \mu(\mathbb{R}^d) \ .$$

ESTIMATE OF (22). Using Lemma 3.10, we see that

$$\begin{aligned} &\sum_{i=1}^{q-1} \sum_{I,J \in \mathcal{I}_i} \int_{Q_{(i-1)q}(I) \cap Q_{(i-1)q}(J)} \psi_x(G(I)) \psi_x(G(J))\, d\mu(x) \\ &\qquad \le \sum_{i=0}^{q-1} 4(2q)^2 \mu(\mathbb{R}^d) = 16 q^3 \mu(\mathbb{R}^d) \ . \end{aligned}$$

LEMMA 3.11. *If* $0 \le l < k \le n$ *and* $I \in \mathcal{I} \cap \mathcal{Q}_k$, *then*

$$\int_{Q_l(I)\setminus Q_{l+1}(I)} \psi_x(I)\, d\mu(x) \le 2C_1C_2 2^{(l-k)(d-\alpha)}\mu(I)/\sigma\ .$$

PROOF. We first note that $\mu(I) \ge 2^{-k\alpha}C_2^{-1}\sigma$ since $k > 0$ and $I \in \mathcal{I} \cap \mathcal{Q}_k$. If $x \in Q_l(I) \setminus Q_{l+1}(I)$ then $I \cap (Q_i(x) \setminus Q_{i+1}(x)) = \emptyset$ if $i \ne l$ and $I \cap (Q_l(x) \setminus Q_{l+1}(x)) = I$. Hence

$$\begin{aligned}
\psi_x(I) &= \ell^d(I)(2^{-ld} - 2^{-(l+1)d})^{-1}\omega_l(x) \le 2^{-kd}2^{ld+1}\omega_l(x) \\
&\le 2^{-k(d-\alpha)}2^{ld+1}C_2\mu(I)\omega_l(x)/\sigma.
\end{aligned}$$

Since, according to Lemma 3.5,

$$\int_{Q_l(I)\setminus Q_{l+1}(I)} \omega_l(x)\, d\mu(x) \le C_1 2^{-l\alpha}\ ,$$

we infer that

$$\begin{aligned}
&\int_{Q_l(I)\setminus Q_{l+1}(I)} \psi_x(I)\, d\mu(x) \\
&\le 2^{-k(d-\alpha)}2^{ld+1}C_2\mu(I)\int_{Q_l(I)\setminus Q_{l+1}(I)} \omega_l(x)\, d\mu(x)/\sigma \\
&\le 2C_1C_2 2^{(l-k)(d-\alpha)}\mu(I)/\sigma.
\end{aligned}$$

□

LEMMA 3.12. *If* $0 \le m \le k \le n$ *and* $I \in \mathcal{I} \cap \mathcal{Q}_k$, *then*

$$\int_{\mathbb{R}^d\setminus Q_m(I)} \psi_x(I)\, d\mu(x) \le 2C_1C_2 2^{(m-k)(d-\alpha)}\mu(I)/(\sigma(2^{d-\alpha}-1))\ .$$

In particular,

$$\int_{\mathbb{R}^d\setminus I} \psi_x(I)\, d\mu(x) \le 2C_1C_2\mu(I)/(\sigma(2^{d-\alpha}-1))\ .$$

PROOF. Since $\psi_x(\mathbb{R}^d \setminus Q_0(x)) = 0$, the first inequality follows by adding the inequalities from Lemma 3.11 for $l = 0, 1, \dots, m-1$. The second inequality is the case $m = k$ of the first one. □

ESTIMATE OF (23). Using that $\sum_{J\in\mathcal{I}_i}\psi_x(G(J)) \le n \le q^2$ according to Lemma 3.8 and that according to Lemma 3.12, for any $I \in \mathcal{I}_i$,

$$\int_{\mathbb{R}^d\setminus Q_{(i-1)q}(I)} \psi_x(G(I))\, d\mu(x) \le \mathcal{C}_1 2^{-q(d-\alpha)}\mu(I),$$

where $\mathcal{C}_1 = 2C_1C_2/(\sigma(2^{d-\alpha}-1))$, we estimate that

$$\begin{aligned}
&\sum_{i=1}^{q-1}\sum_{I,J\in\mathcal{I}_i}\int_{\mathbb{R}^d\setminus Q_{(i-1)q}(I)}\psi_x(G(I))\,\psi_x(G(J))\,d\mu(x)\\
&\quad\le\; q^2\sum_{i=1}^{q-1}\sum_{I\in\mathcal{I}_i}\int_{\mathbb{R}^d\setminus Q_{(i-1)q}(I)}\psi_x(G(I))\,d\mu(x)\\
&\quad\le\; \mathcal{C}_1 2^{-q(d-\alpha)}q^2\sum_{i=1}^{q-1}\sum_{I\in\mathcal{I}_i}\mu(I)\\
&\quad\le\; \mathcal{C}_1 2^{-q(d-\alpha)}q^4\mu(\mathbb{R}^d)\\
&\quad\le\; \mathcal{C}_2 q^3\mu(\mathbb{R}^d),
\end{aligned}$$

where $\mathcal{C}_2 = \mathcal{C}_1\sup_{m\ge 1}2^{-m(d-\alpha)}m$.

Estimate of (24). We first note that the disjointness of the cubes from $\mathcal{K}_{i+1}$ and the second inequality of Lemma 3.12 imply that

$$\sum_{K\in\mathcal{K}_{i+1}}\int_{\mathbb{R}^d\setminus K}\psi_x(K)\,d\mu(x)\le\mathcal{C}_1\sum_{K\in\mathcal{K}_{i+1}}\mu(K)\le\mathcal{C}_1\mu(\mathbb{R}^d)\,.$$

Thus, using first that $\sum_{I\in\mathcal{I}_i}\psi_x(G(I))\le q^2$, we have

$$\begin{aligned}
&\sum_{i=0}^{q-2}\sum_{I\in\mathcal{I}_i}\sum_{K\in\mathcal{K}_{i+1}}\int_{\mathbb{R}^d\setminus K}\psi_x(G(I))\,\psi_x(K)\,d\mu(x)\\
&\quad\le\; q^2\sum_{i=0}^{q-2}\sum_{K\in\mathcal{K}_{i+1}}\int_{\mathbb{R}^d\setminus K}\psi_x(K)\,d\mu(x)\\
&\quad\le\; q^2\sum_{i=0}^{q-2}\mathcal{C}_1\mu(\mathbb{R}^d)\\
&\quad\le\; \mathcal{C}_1 q^3\mu(\mathbb{R}^d)\,.
\end{aligned}$$

Lemma 3.13. *For any dyadic cube K,*

$$\sum_{I\in\mathcal{I}}\sup_{x,y\in K}\big|\varphi_x(G(I)\setminus K)-\varphi_y(G(I)\setminus K)\big|\le 1.$$

Proof. If $x,y\in K\in\mathcal{Q}_m$ and $0\le k<m$, then $Q_k(x)=Q_k(y)$, $Q_{k+1}(x)=Q_{k+1}(y)$ and $\omega_k(x)=\omega_k(y)$. Using also that $\omega_k(x)2^{k\alpha}\mu(Q_k(x))\le C_1$ by Lemma 3.5 and that, for all $u\in Q_k(x)$, $|H_k(y,u)-H_k(x,u)|\le C_2 2^{k\alpha+k-m}$ by assumption, we infer that

$$\begin{aligned}
|H_k(y)-H_k(x)| &\le \omega_k(x)\int_{Q_k(x)}|H_k(y,u)-H_k(x,u)|\,d\mu(u)\\
&\le C_2\omega_k(x)2^{k\alpha}\mu(Q_k(x))2^{k-m}\\
&\le C_1C_2 2^{k-m}.
\end{aligned}$$

Hence

$$\begin{aligned}
&|\varphi_x(G(I)\setminus K)-\varphi_y(G(I)\setminus K)| \\
&\le (C_1C_2)^{-1}\sum_{k=0}^{m-1}\frac{\ell^d\big(G(I)\cap(Q_k(x)\setminus Q_{k+1}(x))\big)}{\ell^d\big(Q_k(x)\setminus Q_{k+1}(x)\big)}C_1C_2 2^{k-m}
\end{aligned}$$

and, taking supremum over $x,y\in K$, we have

$$\begin{aligned}
&\sup_{x,y\in K}|\varphi_x(G(I)\setminus K)-\varphi_y(G(I)\setminus K)| \\
&\le \sum_{k=0}^{m-1}\frac{\ell^d\big(G(I)\cap(Q_k(\zeta(K))\setminus Q_{k+1}(\zeta(K)))\big)}{\ell^d\big(Q_k(\zeta(K))\setminus Q_{k+1}(\zeta(K))\big)}2^{k-m}.
\end{aligned}$$

Thus, summing over $I\in\mathcal{I}$ and using the disjointness of the sets $G(I)$, we get

$$\sum_{I\in\mathcal{I}}\sup_{x,y\in K}|\varphi_x(G(I)\setminus K)-\varphi_y(G(I)\setminus K)|\le\sum_{k=0}^{m-1}2^{k-m}\le 1.$$

□

Estimate of (25). To use Lemma 3.13, we observe that, whenever $u\in K\in\mathcal{K}_{i+1}$ and $I\in\bigcup_{j=0}^{i}\mathcal{I}_j$, then $k(u)>i$ and hence $u\notin G(I)$. Consequently, $\varphi_x(G(I))=\varphi_x(G(I)\setminus K)$ for every $x\in K$ and, using first that $\int_K\psi_x(K)\,d\mu(x)\le q^2\mu(K)$, since $\psi_x(K)\le\psi_x(\mathbb{R}^d)\le q^2$ by Lemma 3.8, then Lemma 3.13 and finally the disjointness of $\mathcal{K}_{i+1}$, we infer that

$$\begin{aligned}
&\sum_{i=0}^{q-2}\sum_{I\in\mathcal{I}_i}\sum_{K\in\mathcal{K}_{i+1}}\sup_{z\in K}\big|\varphi_z(G(I))-\varphi_{\zeta(K)}(G(I))\big|\int_K\psi_x(K)\,d\mu(x) \\
&\le q^2\sum_{i=0}^{q-2}\sum_{K\in\mathcal{K}_{i+1}}\mu(K)\sum_{j=0}^{i}\sum_{I\in\mathcal{I}_j}\sup_{z\in K}\big|\varphi_z(G(I))-\varphi_{\zeta(K)}(G(I))\big| \\
&= q^2\sum_{i=0}^{q-2}\sum_{K\in\mathcal{K}_{i+1}}\mu(K)\sum_{j=0}^{i}\sum_{I\in\mathcal{I}_j}\sup_{z\in K}\big|\varphi_z(G(I)\setminus K)-\varphi_{\zeta(K)}(G(I)\setminus K)\big| \\
&\le q^2\sum_{i=0}^{q-2}\sum_{K\in\mathcal{K}_{i+1}}\mu(K) \\
&\le q^3\mu(\mathbb{R}^d).
\end{aligned}$$

Lemma 3.14. *For every dyadic cube K,*

$$\int_K\varphi_x(K)\,d\mu(x)=0\,.$$

Proof. If $Q\in\mathcal{Q}_k$, the anti-symmetry of H_k and the fact that $\omega_k(x)=\omega_k(\zeta(Q))$ for every $x\in Q$ imply that

$$\int_Q H_k(x)\,d\mu(x) = \omega_k(\zeta(Q))\int_Q\int_Q H_k(x,u)\,d\mu(x)\,d\mu(u)=0.$$

Hence, if $K\in\mathcal{Q}_m$,

$$\begin{aligned}\int_K \varphi_x(K)\,d\mu(x) &= (C_1C_2)^{-1}\sum_{k=m}^{n-1}\int_K H_k(x)\,d\mu(x)\\ &= (C_1C_2)^{-1}\sum_{k=m}^{n-1}\sum_{Q\in\mathcal{Q}_k, Q\subseteq K}\int_Q H_k(x)\,d\mu(x)\\ &= 0.\end{aligned}$$

□

Estimate of (26). By Lemma 3.14,

$$\sum_{i=0}^{q-2}\sum_{I\in\mathcal{I}_i}\sum_{K\in\mathcal{K}_{i+1}}\varphi_{\zeta(K)}(G(I))\int_K \varphi_x(K)\,d\mu(x) = 0\ .$$

4. Proof of the Palm formula

Noting that trivial cases have been handled in Proposition 2.9, we assume, for the rest of this section, that $0<\alpha<d$ and a measure $\mu\in\mathcal{M}(\mathbb{R}^d)$ are fixed.

We will need to make some, not very difficult but careful, estimates of measures of sets. This will be facilitated by defining the usual distances $F_k(\nu,\tilde\nu)$ of measures ν and $\tilde\nu$ as the supremum of $|\int f(w)\,d\nu(w)-\int f(w)\,d\tilde\nu(w)|$, where f runs through functions $f:\mathbb{R}^d\to\mathbb{R}$ having support in $B(0,k)$ and Lipschitz constant at most one. It is not difficult to see that $(F_k)_{k=1}^\infty$ is a sequence of pseudo-metrics inducing the vague topology of $\mathcal{M}(\mathbb{R}^d)$ (see e.g. [**Mat95**, 14.13]). In addition, we will only need the following simple estimate of the distance of enlargements of measures.

Lemma 4.1. *Whenever $x,\widetilde{x}\in\mathbb{R}^d$ and $r>0$, then*

$$F_k\Big(\frac{\mu_{x,r}}{r^\alpha},\frac{\mu_{\widetilde{x},r}}{r^\alpha}\Big)\le 2\frac{\mu(B(x,\|x-\widetilde{x}\|+kr))}{r^\alpha}\frac{\|x-\widetilde{x}\|}{r}.$$

Proof. If f is as in the definition of F_k then

$$\begin{aligned}&\Big|\int f(w)\,d\mu_{x,r}(w)-\int f(w)\,d\mu_{\widetilde{x},r}(w)\Big|\\ &=\int\Big|f\Big(\frac{w-x}{r}\Big)-f\Big(\frac{w-\widetilde{x}}{r}\Big)\Big|\,d\mu(w)\\ &\le\int_{B(x,kr)}\frac{\|x-\widetilde{x}\|}{r}\,d\mu(w)+\int_{B(\widetilde{x},kr)}\frac{\|x-\widetilde{x}\|}{r}\,d\mu(w)\\ &\le 2\mu(B(x,\|x-\widetilde{x}\|+kr))\frac{\|x-\widetilde{x}\|}{r},\end{aligned}$$

which, divided by r^α, proves the statement. □

Before coming to our last lemma, we need to introduce the distances also in the space $\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d$. We define them by $\varrho_p((\nu,u),(\tilde\nu,\tilde u)) = F_p(\nu,\tilde\nu)+\|u-\tilde u\|$; they form a non-decreasing sequence of pseudo-metrics generating the topology of $\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d$. We will also use the usual notation $\varrho_p((\nu,u),M)=\inf_{(\tilde\nu,\tilde u)\in M}\varrho_p((\nu,u),(\tilde\nu,\tilde u))$ for the ϱ_p distance of $(\nu,u)\in\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d$ from a set $M\subseteq\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d$ and recall that the ϱ_p distance of sets $M,N\subseteq\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d$ is defined as the infimum of $\varrho_p((\nu,u),(\tilde\nu,\tilde u))$, where $(\nu,u)\in M$ and $(\tilde\nu,\tilde u)\in N$.

LEMMA 4.2. *Suppose that $R > 1$ and that $U \subseteq V \subseteq \mathcal{M}(\mathbb{R}^d) \times B(0, R-1)$ are open subsets of $\mathcal{M}(\mathbb{R}^d) \times \mathbb{R}$ such that, for some positive integer p, the ϱ_p distance of U from the complement of V is positive and the values of $\nu(B(0, 5R+p+1))$ for $(\nu, u) \in U$ are bounded. Then, for μ-almost every $x \in \mathbb{R}^d$,*

$$\liminf_{\delta \downarrow 0} \iint \left[1_V(\nu, y) - 1_U(T^y\nu, -y) \right] d\nu(y)\, d\mathbb{P}_\delta^x(\nu) \geq 0.$$

PROOF. We choose $\xi \in (0,1)$ such that $\varrho_p((\nu,u),(\tilde{\nu},\tilde{u})) > \xi$ whenever $(\nu,u) \in U$ and $(\tilde{\nu},\tilde{u}) \notin V$, and we let

$$A_1 = 1 + \sup_{(\nu,u)\in U} \nu(B(0, 5R+p+1)), \qquad A_2 = (2A_1+1)/\xi \,,$$

and observe that our assumptions imply that A_1 is finite. (It may be pointed out that the names of the constants are chosen with the future application of Lemma 3.1 in mind. Also, to avoid trivialities, note that in the case $U = \emptyset$ the statement of the lemma is obvious.)
Define $S : \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty) \to \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$ by

$$S(x,y;r) = \Big(\frac{\mu_{x,r}}{r^\alpha}, \frac{y-x}{r} \Big) \,.$$

Since S is continuous, the set $S^{-1}(U)$ is open, which implies that the function

$$H(x,y;r) \;=\; \max\Big\{ 0, 1 - A_2 \inf_{(\widetilde{x},\widetilde{y};r)\in S^{-1}(U)} \frac{\|x-\widetilde{x}\| + \|y-\widetilde{y}\|}{r} \Big\}.$$

is upper semi-continuous, hence Borel measurable.
We prove that

$$H(x,y;r) \leq 1_V\Big(\frac{\mu_{x,r}}{r^\alpha}, \frac{y-x}{r} \Big) \tag{27}$$

for all $(x,y;r) \in \mathbb{R}^d \times \mathbb{R}^d \times (0,\infty)$. Indeed, otherwise there would exist $(x,y;r)$ and $(\widetilde{x},\widetilde{y};r)$ such that $S(x,y;r) \notin V$, $S(\widetilde{x},\widetilde{y};r) \in U$, and $\|x-\widetilde{x}\| + \|y-\widetilde{y}\| < r/A_2$, and we would infer from $S(\widetilde{x},\widetilde{y};r) \in U$ that

$$\begin{aligned} \mu(B(\widetilde{x}, \|\widetilde{x}-x\| + pr)) &\leq \mu(B(\widetilde{x},(p+1)r)) = \mu_{\widetilde{x},r}(B(0,(p+1))) \\ &\leq \mu_{\widetilde{x},r}(B(0,5R+p+1)) < A_1 r^\alpha. \end{aligned}$$

But then Lemma 4.1 would imply that

$$\begin{aligned} \xi &< \varrho_p\Big(\Big(\frac{\mu_{x,r}}{r^\alpha}, \frac{y-x}{r}\Big), \Big(\frac{\mu_{\widetilde{x},r}}{r^\alpha}, \frac{\widetilde{y}-\widetilde{x}}{r}\Big) \Big) \\ &\leq 2\frac{\mu(B(\widetilde{x}, \|\widetilde{x}-x\|+pr))}{r^\alpha} \frac{\|x-\widetilde{x}\|}{r} + \frac{\|x-\widetilde{x}\| + \|y-\widetilde{y}\|}{r} \\ &\leq 2A_1/A_2 + 1/A_2 \\ &= \xi. \end{aligned}$$

Noting that, for any G,

$$\begin{aligned} \iint G(\nu,y)\, d\nu(y)\, d\mathbb{P}_\delta^x(\nu) &= \frac{1}{|\log\delta|} \int_\delta^1 \int G\Big(\frac{\mu_{x,r}}{r^\alpha}, y\Big) \frac{d\mu_{x,r}(y)}{r^\alpha} \frac{dr}{r} \\ &= \frac{1}{|\log\delta|} \int_\delta^1 \int G\Big(\frac{\mu_{x,r}}{r^\alpha}, \frac{y-x}{r}\Big) d\mu(y) \frac{dr}{r^{1+\alpha}}, \end{aligned}$$

we integrate (27) first with respect to $d\mu(y)$ and then with respect to $dr/r^{1+\alpha}$ to find out that

$$\frac{1}{|\log\delta|}\int_\delta^1\int H(x,y;r)\,d\mu(y)\,\frac{dr}{r^{1+\alpha}} \le \iint 1_V(\nu,y)d\nu(y)\,d\mathbb{P}_\delta^x(\nu) \tag{28}$$

for every $x\in\mathbb{R}^d$ and every $\delta>0$. In a similar way we find an estimate of the integral involving the set U: Since $H(y,x;r)=1$ if $S(y,x;r)\in U$, we have

$$1_U\Big(\frac{\mu_{y,r}}{r^\alpha},\frac{x-y}{r}\Big) \le H(y,x;r). \tag{29}$$

Hence, noting that, for any G,

$$\begin{aligned}\iint G(T^y\nu,-y)\,d\nu(y)\,d\mathbb{P}_\delta^x(\nu) &= \frac{1}{|\log\delta|}\int_\delta^1\int G\Big(\frac{T^y\mu_{x,r}}{r^\alpha},-y\Big)\,\frac{d\mu_{x,r}(y)}{r^\alpha}\,\frac{dr}{r}\\ &= \frac{1}{|\log\delta|}\int_\delta^1\int G\Big(\frac{\mu_{y,r}}{r^\alpha},\frac{x-y}{r}\Big)\,d\mu(y)\,\frac{dr}{r^{1+\alpha}},\end{aligned}$$

we obtain, integrating (29) first with respect to $d\mu(y)$ and then with respect to $dr/r^{1+\alpha}$, that

$$\iint 1_U(T^y\nu,-y)\,d\nu(y)\,d\mathbb{P}_\delta^x(\nu) \le \frac{1}{|\log\delta|}\int_\delta^1\int H(y,x;r)\,d\mu(y)\,\frac{dr}{r^{1+\alpha}} \tag{30}$$

for every $x\in\mathbb{R}^d$ and every $\delta>0$.

We prove that $H(x,y;r)$ satisfies the assumptions of Lemma 3.1. We have already noted that H is Borel measurable, and it is clear that its range is contained in $[0,1]$. If $H(x,y;r)\neq 0$, then we find $(\widetilde{x},\widetilde{y};r)\in S^{-1}(U)$ such that $\|x-\widetilde{x}\|+\|y-\widetilde{y}\|\le r/A_2\le r$ and infer from $S(\widetilde{x},\widetilde{y};r)\in U$ that $\|\widetilde{x}-\widetilde{y}\|<(R-1)r$ and $\mu_{\widetilde{x},r}(B(0,5R+1))<A_1r^\alpha$. Hence

$$\|x-y\|\le\|x-\widetilde{x}\|+\|y-\widetilde{y}\|+\|\widetilde{x}-\widetilde{y}\|<r+(R-1)r=Rr\ ,$$

which proves 3.1(i) , and

$$\mu(B(x,5Rr))\le\mu(B(\widetilde{x},(5R+1)r))=\mu_{\widetilde{x},r}(B(0,5R+1))<A_1r^\alpha\ ,$$

which proves 3.1(ii) . Finally, the requirement 3.1(iii) follows directly from the definition of $H(x,y;r)$.

This finishes the proof of the Lemma, since (30) and (28) and Lemma 3.1 imply that

$$\begin{aligned}&\liminf_{\delta\downarrow 0}\iint\big[1_V(\nu,y)-1_U(T^y\nu,-y)\big]\,d\nu(y)\,d\mathbb{P}_\delta^x(\nu)\\ &\ge\ \liminf_{\delta\to 0}\frac{1}{|\log\delta|}\int_\delta^1\int\frac{H(x,y;r)-H(y,x;r)}{r^\alpha}\,d\mu(y)\,\frac{dr}{r}\\ &=\ 0.\end{aligned}$$

□

Proof of Theorem 2.12. Using that $\mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d$ is a separable metric space, we find a countable basis $\mathcal{U}$ of open sets which is closed under finite unions. Let $\tilde{\mathcal{U}}$ be the set of pairs (U,V) of sets from $\mathcal{U}$ for which there are $R>1$ and a positive integer p such that $U\subseteq V\subseteq\mathcal{M}(\mathbb{R}^d)\times B(0,R-1)$, the ϱ_p distance of U from the complement of V is positive and the values of $\nu(B(0,5R+p+1))$ for $(\nu,u)\in U$ are bounded. Since $\tilde{\mathcal{U}}$ is countable, we infer from Lemma 4.2 that there is a μ null set $N\subseteq\mathbb{R}^d$ such that for every $x\in\mathbb{R}^d\setminus N$,

$$\liminf_{\delta\downarrow 0}\iint\big[1_V(\nu,y)-1_U(T^y\nu,-y)\big]\,d\nu(y)\,d\mathbb{P}_\delta^x(\nu)\ge 0 \tag{31}$$

for every pair $(U,V) \in \tilde{\mathcal{U}}$. We now need one more technical lemma from measure theory.

LEMMA 4.3. *Suppose that X is a locally compact Polish space and that P_k are Radon measures on $\mathcal{M}(X)$ vaguely converging to P_0. Let Λ_k be Borel measures on $\mathcal{M}(X) \times X$ defined by*

$$\Lambda_k(E) = \iint 1_E(\nu, y)\, d\nu(y)\, dP_k(\nu) \ .$$

Then Λ_k are Radon measures converging vaguely to Λ_0.

PROOF. If $K \subseteq \mathcal{M}(X) \times X$ is compact, let $K_1 \subseteq \mathcal{M}(X)$ and $K_2 \subseteq X$ be compact sets such that $K \subseteq K_1 \times K_2$ and let $L_2 \supseteq K_2$ be an open set contained in a compact set $K_3 \subseteq X$. Then there is $0 < c < \infty$ such that $\nu(K_3) < c$ for all $\nu \in K_1$. Given any $\eta > 0$, we use Lemma A3 to find an open set $L_1 \supseteq K_1$ such that

$$\limsup_{k\to\infty} \left| \int \varphi(\nu)\, dP_k(\nu) - \int \varphi(\nu)\, dP_0(\nu) \right| < \eta/c \tag{32}$$

for every continuous function $\varphi : \mathcal{M}(X) \to [0,1]$ such that $\varphi(\nu) = 0$ for $\nu \notin L_1$; intersecting L_1 with the open set $\{\nu : \nu(K_3) < c\}$, if necessary, we may also assume that $\nu(L_2) < c$ for all $\nu \in L_1$. As a byproduct we note that $K_1 \times L_2$ contains K and $\Lambda_k(K_1 \times L_2) < \infty$, so that Λ_k are Radon measures. Denote $L = L_1 \times L_2$. If $\psi : \mathcal{M}(X) \times X \to [0,1]$ is a continuous function such that $\psi(\nu, y) = 0$ if $(\nu, y) \notin L$, we note that the function $\varphi(\nu) = \frac{1}{c} \int \psi(\nu, y)\, d\nu(y)$ is non-negative, continuous, bounded by one, and satisfies $\varphi(\nu) = 0$ for $\nu \notin L_1$. Thus (32) implies that

$$\limsup_{k\to\infty} \left| \int \psi(\nu, y)\, d\Lambda_k(\nu, y) - \int \psi(\nu, y)\, d\Lambda_0(\nu, y) \right| < \eta \ ,$$

which, according to Lemma A3, implies that (Λ_k) vaguely converges to Λ_0. ☐

Let $x \in \mathbb{R}^d \setminus N$ and $\mathbb{P} \in \mathcal{P}^\alpha(\mu, x)$ be fixed. Let $\delta_k \downarrow 0$ be such that $\mathbb{P}^x_{\delta_k}$ converge weakly to $\mathbb{P}$. Then

$$\Lambda_k(E) \quad = \quad \iint 1_E(\nu, y)\, d\nu(y) d\mathbb{P}^x_{\delta_k}(\nu)$$

are, according to the lemma, Radon measures on $\mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$ converging vaguely to the Radon measure

$$\Lambda(E) \quad = \quad \iint 1_E(\nu, y)\, d\nu(y) d\mathbb{P}(\nu).$$

Moreover, since the mapping $\tilde{T} : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$ defined by $\tilde{T}(\nu, y) = (T^y\nu, -y)$ is a homeomorphism, the measures $\widetilde{\Lambda}_k = \Lambda_k \circ \tilde{T}^{-1}$ converge vaguely to $\widetilde{\Lambda} = \Lambda \circ \tilde{T}^{-1}$.

We intend to use Lemma A5 to show that $\widetilde{\Lambda} \le \Lambda$. Whenever $K \subseteq L$ are subsets of $\mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d$ such that K is compact and L is open, we first choose $R > 2$ such that $K \subseteq \mathcal{M}(\mathbb{R}^d) \times B(0, R-2)$ and find $V \in \mathcal{U}$ such that $K \subseteq V \subseteq L \cap \mathcal{M}(\mathbb{R}^d) \times B(0, R-1)$. Then we choose a positive integer p such that the ϱ_p distance of K from the complement of V is positive. We let ξ denote this distance, define

$$A \quad = \quad 1 + \sup_{(\nu,u)\in K} \nu(B(0, 5R+p+1))$$

and

$$\hat{U} \quad = \quad \{(\nu, u) \in V : \varrho_p((\nu, u), K) < \xi/2, \nu(B(0, 5R+p+1)) < A\},$$

and choose $U \in \mathcal{U}$ such that $K \subseteq U \subseteq \hat{U}$. Then $(U,V) \in \tilde{\mathcal{U}}$, and we conclude from (31) that

$$\begin{aligned}
\liminf_{k\to\infty} \widetilde{\Lambda}_k(U) &= \liminf_{k\to\infty} \iint 1_U(T^y\nu, -y)\, d\nu(y)\, d\mathbb{P}^x_{\delta_k}(\nu) \\
&\le \liminf_{k\to\infty} \iint 1_V(\nu, y)\, d\nu(y)\, d\mathbb{P}^x_{\delta_k}(\nu) \\
&= \liminf_{k\to\infty} \Lambda_k(V) \\
&\le \limsup_{k\to\infty} \Lambda_k(L).
\end{aligned}$$

By Lemma A5, this shows that $\widetilde{\Lambda} \le \Lambda$. But then $\Lambda = \widetilde{\Lambda} \circ \tilde{T}^{-1} \le \Lambda \circ \tilde{T}^{-1} = \widetilde{\Lambda}$, and we conclude that $\widetilde{\Lambda} = \Lambda$, which is the statement of Theorem 2.12.

Appendix: Measure theoretical definitions and preliminaries

In this appendix we recall some notions concerning Radon measures on Polish spaces. Although we use the theory only for the two cases of (possibly infinite) measures on a locally compact space and sub-probability measures on a (non-locally compact) space we think it is worth giving a unified treatment of the two situations.

If X is a separable and completely metrizable topological space, in short a *Polish space*, a non-negative Borel measure ν is termed a *Radon measure* if every compact subset K of X has finite measure and

$$\nu(B) = \sup\{\nu(K) \,:\, K \subseteq B,\ K \text{ compact}\} \text{ for every Borel set } B \subseteq X.$$

We shall often use the facts that Radon measures ν on X also satisfy (see e.g. [**Sch73**, I,§I.2])

$$\nu(B) = \inf\{\nu(U) \,:\, B \subseteq U,\ U \text{ open}\} \text{ for every Borel set } B \subseteq X$$

and that, under the conditions imposed upon X, a Borel measure is Radon if and only if it is locally finite, i.e. every point has a neighbourhood of finite measure (see e.g.[**Sch73**, I,§II.3]).

If ν_k are Radon measures on X, we say that the sequence (ν_k) *converges vaguely* to the Radon measure ν if the following two conditions hold

(i) $\nu(K) \ge \limsup_{k\to\infty} \nu_k(K)$ for every compact set $K \subseteq X$, and
(ii) $\nu(L) \le \liminf_{k\to\infty} \nu_k(L)$ for every open set $L \subseteq X$.

In the following we recall some useful descriptions of the vague convergence of Radon measures. For this purpose the following lemma, which is proved for finite measures in [**Top70**, Lemma 7.7], will turn out to be useful.

LEMMA A1. *Let (ν_k) be a sequence of Radon measures on a Polish space X. Then, for every $\eta > 0$ and every open set $G \subseteq X$ and compact set $K \subseteq X$ with $K \subseteq G$, there are a compact set C with $K \subseteq C \subseteq G$ and an open set H with $K \subseteq H \subseteq G$ such that*

(a) $\eta + \limsup_{k\to\infty} \nu_k(C) \ge \limsup_{k\to\infty} \nu_k(H)$ *and*
(b) $\eta + \liminf_{k\to\infty} \nu_k(C) \ge \liminf_{k\to\infty} \nu_k(H)$.

PROOF. Let G_j be the intersection of G with the open $1/j$-neighbourhoods of K. Pick strictly increasing sequences (a_j) converging to $\inf_i \limsup \nu_k(G_i)$ and (b_j) increasing to $\inf_i \liminf \nu_k(G_i)$. We can find a sequence $k_1 < k_2 < \ldots$ of indices such that $\nu_k(G_j) > b_j$ for all $k \ge k_j$ and $\nu_l(G_j) > a_j$ for some $k_j \le l < k_{j+1}$. For every j we may pick a compact set $K \subseteq C_j \subseteq G_j$ such that $\nu_k(C_j) \ge b_j$ for all $k_j \le k < k_{j+1}$ and $\nu_l(C_j) \ge a_j$ for some

$k_j \le l < k_{j+1}$. Then the set $C = \bigcup C_j$ is compact (see e.g. [**Top70**, Lemma 7.5]) and satisfies $\liminf \nu_k(C) \ge \liminf b_j$ and $\limsup \nu_k(C) \ge \limsup a_j$. Picking j so large that $H = G_j$ satisfies $\limsup \nu_k(H) \le \inf_i \limsup \nu_k(G_i) + \eta$ and $\liminf \nu_k(H) \le \inf_i \liminf \nu_k(G_i) + \eta$ finishes the proof. $\square$

LEMMA A2. *Let (ν_k) be a sequence of Radon measures on a Polish space X converging vaguely to the Radon measure ν.*

(a) *For every compact $K \subseteq X$ and every $\eta > 0$, there is an open set $H \supseteq K$ such that $\limsup_{k\to\infty} \nu_k(H) < \nu(K) + \eta$.*
(b) *For every open set $G \subseteq X$ with $\nu(G) < \infty$ and every $\eta > 0$, there is a compact set $C \subseteq G$ such that $\liminf_{k\to\infty} \nu_k(C) > \nu(G) - \eta$.*

PROOF. To prove (a) we pick an open set $G \supseteq K$ such that $\nu(G \backslash K) < \eta/2$. By Lemma A1(a) we can find a compact set $C \subseteq G$ and an open set $H \supseteq K$ with $\limsup \nu_k(H) \le \eta/2 + \limsup \nu_k(C)$ and, as $\limsup \nu_k(C) \le \nu(C) < \nu(K) + \eta/2$, this proves (a). To prove (b) we pick a compact set $K \subseteq G$ with $\nu(G \setminus K) < \eta/2$ and use Lemma A1(b) to find a compact set $C \subseteq G$ and an open set $H \supseteq K$ with $\eta/2 + \liminf \nu_k(C) \ge \liminf \nu_k(H)$. As $\liminf \nu_k(H) \ge \nu(H) > \nu(G) - \eta/2$, (b) is proved. $\square$

LEMMA A3. *A sequence (ν_k) of Radon measures on a Polish space X converges vaguely to a Radon measure ν if and only if for every compact set $K \subseteq X$ and every $\eta > 0$ there is an open set L with $K \subseteq L \subseteq X$ such that*

$$\limsup_{k\to\infty} \left| \int \varphi(x)\, d\nu_k(x) - \int \varphi(x)\, d\nu(x) \right| < \eta$$

for every continuous function $\varphi : X \to [0,1]$ such that $\varphi(x) = 0$ for $x \notin L$.

PROOF. An easy approximation shows that the given condition implies that (ν_k) converges vaguely to ν. On the other hand, if the convergence is assumed, $\eta > 0$ and $K \subseteq X$ is compact, we can use Lemma A2(a) and (b) to first find an open set $L \supseteq K$ such that $\limsup \nu_k(L) < \nu(K) + \eta/2$ and then a compact set C with $K \subseteq C \subseteq L$ with $\limsup \nu_k(L \setminus C) < \eta$. Observe that $\sup \nu_k(C) < \infty$. If φ is given as in the lemma, we calculate, using Fubini's Theorem and Fatou's Lemma,

$$\begin{aligned}
\int \varphi\, d\nu &\ge \int_C \varphi\, d\nu = \int_0^1 \nu(\{x \in C : \varphi(x) \ge t\})\, dt \\
&\ge \int_0^1 \limsup_{k\to\infty} \nu_k(\{x \in C : \varphi(x) \ge t\})\, dt \\
&\ge \limsup_{k\to\infty} \int_0^1 \nu_k(\{x \in C : \varphi(x) \ge t\})\, dt \\
&= \limsup_{k\to\infty} \int_C \varphi\, d\nu_k \ge \limsup_{k\to\infty} \int \varphi\, d\nu_k - \eta\,,
\end{aligned}$$

and, similarly, we get

$$\begin{aligned}\int \varphi\, d\nu &= \int_0^1 \nu(\{x \in L : \varphi(x) > t\})\, dt \\ &\le \int_0^1 \liminf_{k\to\infty} \nu_k(\{x \in L : \varphi(x) > t\})\, dt \\ &\le \liminf_{k\to\infty} \int_0^1 \nu_k(\{x \in L : \varphi(x) > t\})\, dt = \liminf_{k\to\infty} \int \varphi\, d\nu_k ,\end{aligned}$$

which together implies the condition given in the lemma. □

As an important consequence of the lemma we note that the vague limit of a sequence (ν_k), if it exists, is uniquely determined. Although we have not defined a topology inducing the vague convergence (and, in fact, this may not be possible, see [**Top70**, Chap.7]), the notion of relative sequential compactness with respect to the vague convergence is naturally defined for a set of Radon measures.

LEMMA A4. *A set M of Radon measures on a Polish space X is relatively sequentially compact with respect to the vague convergence if and only if* $\sup\{\nu(K) : \nu \in M\} < \infty$ *for every compact set $K \subseteq X$.*

PROOF. If there is a sequence (ν_k) in M such that, for some compact K, the sequence $\nu_k(K)$ tends to ∞, this sequence clearly has no convergent subsequence and M cannot be relatively sequentially compact. On the other hand suppose (ν_k) is a sequence in M and $(\nu_k(K))$ is bounded for every compact K. We have to find a convergent subsequence. Using Lemma A1(a) we can find a sequence (G_j) of open sets increasing to X with $\sup_k \nu_k(G_j) < \infty$. As any family of finite measures with bounded mass has a vaguely convergent subsequence (see e.g.[**Top70**, Theorem 7.8]), we can pick a subsequence (μ_k) of (ν_k) such that, for every j, the sequence $1_{G_j}\mu_k$ converges, say to $\mu^{(j)}$. For every compact set K there is an index n such that $K \subseteq G_n$ and therefore $\mu^{(i)}|_K = \mu^{(j)}|_K$ for all $i, j \ge n$. Using this, it is easy to see that we can define a Radon measure μ by $\mu(B) = \sup \mu^{(n)}(K)$, where the supremum extends over all compact sets $K \subseteq B$ and n is chosen according to K. It is then straightforward to check that μ is indeed the limit of the sequence (μ_j). □

LEMMA A5. *Suppose that μ_k and ν_k are Radon measures on a Polish space X such that (μ_k) converges vaguely to μ_0 and (ν_k) converges vaguely to ν_0. Suppose further that for every $\eta > 0$ and for every pair $K \subseteq L$ of subsets of X such that K is compact and L is open there is an open set U such that $K \subseteq U \subseteq L$ and*

$$\liminf_{k\to\infty} \mu_k(U) \le \eta + \limsup_{k\to\infty} \nu_k(L) .$$

Then $\mu_0 \le \nu_0$.

PROOF. If $K \subseteq X$ is compact and $\eta > 0$, we use Lemma A3 to find an open set $L \supseteq K$ such that

$$\limsup_{k\to\infty} \left| \int \varphi(x)\, d\mu_k(x) - \int \varphi(x)\, d\mu_0(x) \right| < \eta$$

and

$$\limsup_{k\to\infty} \left| \int \varphi(x)\, d\nu_k(x) - \int \varphi(x)\, d\nu_0(x) \right| < \eta$$

for every continuous function $\varphi : X \to [0,1]$ such that $\varphi(x) = 0$ for $x \notin L$. Let $L \supseteq L_0 \supseteq K$ be an open set such that $\nu_0(L_0 \setminus K) < \eta$ and let $L_0 \supseteq L_1 \supseteq K$ be an open set whose closure is contained in L_0. Using the assumption of the lemma, we find an open set $L_1 \supseteq U \supseteq K$ such that

$$\liminf_{k\to\infty} \mu_k(U) \le \eta + \limsup_{k\to\infty} \nu_k(L_1) \ .$$

Let $\varphi : X \to [0,1]$ be a continuous function such that $\varphi(x) = 1$ for $x \in K$ and $\varphi(x) = 0$ for $x \notin U$ and let $\psi : X \to [0,1]$ be a continuous function such that $\psi(x) = 1$ for $x \in L_1$ and $\psi(x) = 0$ for $x \notin L_0$. Then

$$\begin{aligned} \mu_0(K) &\le \int \varphi(x)\, d\mu_0(x) \le \eta + \liminf_{k\to\infty} \int \varphi(x)\, d\mu_k(x) \le \eta + \liminf_{k\to\infty} \mu_k(U) \\ &\le 2\eta + \limsup_{k\to\infty} \nu_k(L_1) \le 2\eta + \limsup_{k\to\infty} \int \psi(x)\, d\nu_k(x) \\ &\le 3\eta + \int \psi(x)\, d\nu_0(x) \le 3\eta + \nu_0(L_0) \le 4\eta + \nu_0(K). \end{aligned}$$

Since $\eta > 0$ and the compact set K are arbitrary, this shows that $\mu_0 \le \nu_0$. □

A sequence (ν_k) of Radon measures with $\limsup_{k\to\infty} \nu_k(X) < \infty$ is said to *converge weakly* to the Radon measure ν if $\int \varphi\, d\nu_k \to \int \varphi\, d\nu$ for every bounded continuous function φ on X.

LEMMA A6. *Let ν_k be Radon measures on a Polish space X and suppose that $\limsup_{k\to\infty} \nu_k(X) < \infty$ and (ν_k) converges vaguely to a Radon measure ν. Then the following statements are equivalent:*

(a) *(ν_k) converges weakly to ν.*
(b) *$\nu(X) \ge \limsup_{k\to\infty} \nu_k(X)$.*
(c) *For every $\eta > 0$ there is $K \subseteq X$, compact, such that $\limsup \nu_k(X \setminus K) \le \eta$.*

PROOF. As the function 1_X is continuous and bounded, (a) implies (b). If (b) holds, then $\nu(A) \ge \limsup_{k\to\infty} \nu_k(A)$ for every closed set $A \subseteq X$ and we can conclude the weak convergence by means of Fubini's Theorem as exercised in the proof of Lemma A3. Finally, the equivalence of (a) and (c) follows from Prohorov's Theorem (see [**Sch73**, App. §3,Th. 3 and 4]). □

Of course, if the Polish space X is compact, then Radon measures are finite and weak and vague convergence coincide. More generally, if X is a locally compact Polish space, then ν_k converges vaguely to ν if and only if $\int \varphi\, d\nu_k \to \int \varphi\, d\nu$ for every continuous function φ on X with compact support. Then the vague convergence of Radon measures is induced by the vague topology, the smallest topology on the set of Radon measures that makes the mappings $\mu \mapsto \int \varphi\, d\mu$ continuous for all continuous φ with compact support. In the case $X = \mathbb{R}^d$ the set $\mathcal{M}(\mathbb{R}^d)$ of all Radon measures on $\mathbb{R}^d$ equipped with the vague topology is a Polish space, see [**Mat95**, 14.15], and we observe, as a consequence of Lemma A4, that a subset M of $\mathcal{M}(\mathbb{R}^d)$ is relatively compact if and only if for each $n = 1, 2, \dots$ there is $c(n) \in (0,\infty)$ such that $\mu U(0,n) \le c(n)$ for each $\mu \in M$, where $U(x,r)$ denotes the open Euclidean ball of radius r centred in x.

CHAPTER 2

Average densities and linear rectifiability of measures

MATHEMATIKA, 44 (1997) 313–324.

Abstract: We show that a measure on $\mathbb{R}^d$ is linearly rectifiable if and only if the lower 1-density is positive and finite and agrees with the lower average 1-density almost everywhere.

Mathematics Subject Classification (2000): *28A75.*
Key words: *Fractal measures, geometric measure theory, density, average density, regularity of measures, rectifiability, tangent measures, tangent measure distribution, probabilistic method.*

1. Introduction

Let μ be a nonnegative, nonzero Radon measure on $\mathbb{R}^d$ and $\alpha \geq 0$. The *lower α-density* of μ at x is the number

$$\underline{d}^\alpha(\mu, x) = \liminf_{t \downarrow 0} \frac{\mu(U(x,t))}{t^\alpha},$$

where $U(x,t)$ denotes the open Euclidean ball centred in x of radius t, and the *upper α-density* of μ at x is the number

$$\overline{d}^\alpha(\mu, x) = \limsup_{t \downarrow 0} \frac{\mu(U(x,t))}{t^\alpha}.$$

The geometric regularity of the measure μ is intimately related to the behaviour of the densities. We say μ is *α-rectifiable* if μ is absolutely continuous with respect to α-Hausdorff measure restricted to a countable family of smooth α-manifolds and in the case of $\alpha = 1$ we say that μ is *linearly rectifiable.* By Marstrand's Theorem (see [**Mar64**] or [**Mat95**, Chapter 14]) the equality

$$0 < \underline{d}^\alpha(\mu, x) = \overline{d}^\alpha(\mu, x) < \infty \ \mu\text{-almost everywhere} \tag{1}$$

implies that α must be an integer and by Preiss' Regularity Theorem (see [**Pre87**] or [**Mat95**, Chapter 17]) such a measure μ is even α-rectifiable.

A different type of density was introduced by Bedford and Fisher in [**BeF92**], the so called *average density* or *order-two density.* Bedford and Fisher applied a logarithmic average to the density functions and defined the *lower* and *upper average α-density* of μ at x as

$$\underline{D}^\alpha(\mu, x) = \liminf_{\varepsilon \to 0} (|\log \varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu(U(x,t))}{t^\alpha} \frac{dt}{t},$$

and

$$\overline{D}^\alpha(\mu, x) = \limsup_{\varepsilon \to 0} (|\log \varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu(U(x,t))}{t^\alpha} \frac{dt}{t}.$$

We clearly have the following inequalities:

$$\underline{d}^\alpha(\mu,x) \le \underline{D}^\alpha(\mu,x) \le \overline{D}^\alpha(\mu,x) \le \overline{d}^\alpha(\mu,x).$$

It is natural to ask whether one can get statements about the geometric regularity of μ from weaker inequalities than (1), involving the average densities. This program was started by Falconer and Springer in [**FaS95**] and their results were recently improved by Marstrand (see [**Mar96**]), who proved the following theorem:

THEOREM 1.1. *Suppose μ is a nonnegative, nonzero Radon measure on $\mathbb{R}^d$ and $\alpha \ge 0$ such that*

(i) $0 < \overline{D}^\alpha(\mu,x) = \overline{d}^\alpha(\mu,x) < \infty$ *for μ-almost every x, or*
(ii) $0 < \underline{d}^\alpha(\mu,x) = \underline{D}^\alpha(\mu,x) < \infty$ *for μ-almost every x,*

then α must be an integer.

In fact, Marstrand's proof in the case of the second condition needs the additional assumption that $\overline{d}^\alpha(\mu,x) < \infty$ for μ-almost every x. This assumption is removed in [**MöP98**] and a refinement of the argument given there is contained in Section 2 of this paper.

Do the inequalities above also imply α-rectifiability of μ? As Falconer and Springer point out, the answer is clearly no for the first inequality, even in the case $\alpha = 1$, due to an example of O'Neil (see [**ONe95**]). It is the aim of this paper to give the following partial answer to this question in the case of the second inequality.

THEOREM 1.2. *Suppose μ is a nonnegative Radon measure on $\mathbb{R}^d$. Then*

$$0 < \underline{d}^1(\mu,x) = \underline{D}^1(\mu,x) < \infty \quad \textit{for } \mu\textit{-almost every } x \tag{2}$$

if and only if μ is linearly rectifiable.

Of course, if μ is linearly rectifiable it is well known that $0 < \underline{d}^1(\mu,x) = \overline{d}^1(\mu,x) < \infty$ for almost every x and therefore it only remains to prove that (2) implies linear rectifiability of μ. The proof of this statement consists of two parts. In Section 2 we employ the theory of tangent measure distributions to construct, from Theorem 1.1(ii), at almost every point x an α-flat tangent measure ν such that $\nu(U(0,1)) = \underline{d}^\alpha(\mu,x)$ and in Section 3 we finish the proof by showing that in the case $\alpha = 1$ such a tangent measure can only exist if μ is linearly rectifiable.

2. Existence of flat tangent measures

We start by introducing the notion of tangent measures.

DEFINITION 2.1. Let $\mathcal{M}(\mathbb{R}^d)$ be the set of nonnegative Radon measures on $\mathbb{R}^d$. Equipped with the vague topology, which is generated by the mappings $\mu \mapsto \int \varphi\, d\mu$, φ continuous with compact support, $\mathcal{M}(\mathbb{R}^d)$ is a Polish space, see [**Mat95**, Chapter 14].

Let $0 \le \alpha \le d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$. For $r > 0$ define $\mu_{x,r} \in \mathcal{M}(\mathbb{R}^d)$ to be the *enlargement of μ at x of factor* $1/r$, i.e. the measure defined by $\mu_{x,r}(A) = \mu(x + rA)$. The set $\mathrm{Tan}_\alpha(\mu,x)$ of *α-dimensional tangent measures of μ at x* is defined as the set of all limit points of $\mu_{x,r}/r^\alpha$ in the vague topology as $r \downarrow 0$.

A (tangent) measure ν is called *α-uniform* if, for some $c > 0$, $\nu(U(u,r)) = cr^\alpha$ for all $r > 0$ and u in the support of ν. ν is called *α-flat* if α is an integer and there is a linear space $V \subseteq \mathbb{R}^d$ of dimension α and some $c > 0$ such that $\nu = c \cdot \mathcal{H}^\alpha|_V$, a multiple of the restriction of α-Hausdorff measure to the space V.

Preiss introduced the notion of tangent measures in his seminal paper [**Pre87**], he showed that if $0 < \underline{d}^\alpha(\mu,x) = \overline{d}^\alpha(\mu,x) < \infty$ for μ-almost every x, then at μ-almost all x all tangent measures of μ at x are α-flat. This implies that μ is α-rectifiable.

In order to make quantitative statements about the set of tangent measures we introduce a family of probability distributions on $\mathrm{Tan}_\alpha(\mu,x)$, the so-called tangent measure distributions.

DEFINITION 2.2. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $0 \le \alpha \le d$. For $x \in \mathbb{R}^d$ and $\varepsilon > 0$ we define probability distributions $\mathbb{P}_\varepsilon^x$ on $\mathcal{M}(\mathbb{R}^d)$ by

$$\mathbb{P}_\varepsilon^x(M) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 1_M\Big(\frac{\mu_{x,r}}{r^\alpha}\Big)\,\frac{dr}{r} \quad \text{for Borel sets } M \subseteq \mathcal{M}(\mathbb{R}^d).$$

$\mathcal{P}^\alpha(\mu,x)$ is defined as the set of all limit points of $(\mathbb{P}_\varepsilon^x)_{\varepsilon>0}$ as $\varepsilon \downarrow 0$ in the weak topology, i.e. the topology generated by the mappings $P \mapsto \int F\,dP$, F continuous and bounded. The elements of $\mathcal{P}^\alpha(\mu,x)$ are probability distributions on the set $\mathrm{Tan}_\alpha(\mu,x)$, they are the *$\alpha$-dimensional tangent measure distributions* of μ at x.

Tangent measure distributions were introduced by Bandt ([**Ban92**]) and Graf ([**Gra95**]) originally as a tool for the investigation of self-similar sets. They have also turned out to be valuable for the study of more general measures (see [**Mör98a**], [**MöP98**] or the thesis [**Mör95**]), which is due to the invariance properties described in the following theorem. For every $\lambda > 0$ we define the rescaling operator $S_\lambda^\alpha : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ by $S_\lambda^\alpha \nu(E) = (1/\lambda^\alpha)\cdot\nu(\lambda E)$ and for every $u \in \mathbb{R}^d$ we define the shift operator $T^u : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$ by $T^u\nu(E) = \nu(u+E)$.

THEOREM 2.3. *Let $0 \le \alpha \le d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$.*

(i) *At every $x \in \mathbb{R}^d$ every tangent measure distribution $\mathbb{P} \in \mathcal{P}^\alpha(\mu,x)$ fulfills*

$$\mathbb{P} = \mathbb{P} \circ (S_\lambda^\alpha)^{-1} \quad \textit{for all } \lambda > 0. \tag{3}$$

(ii) *At μ-almost every x every tangent measure distribution $\mathbb{P} \in \mathcal{P}^\alpha(\mu,x)$ fulfills*

$$\iint G(\nu,u)\,d\nu(u)\,d\mathbb{P}(\nu) = \iint G(T^u\nu,-u)\,d\nu(u)\,d\mathbb{P}(\nu) \tag{4}$$

for all Borel functions $G : \mathcal{M}(\mathbb{R}^d)\times\mathbb{R}^d \longrightarrow [0,\infty)$.

Whereas the scaling invariance property (3) is easy to check, the shift invariance property (4) is quite difficult. For a proof, an interpretation and a number of applications of the formula (4) see [**MöP98**]. In this paper we shall make use of the properties of tangent measure distributions by means of the following lemma.

LEMMA 2.4. *Suppose that a probability measure $\mathbb{P}$ on $\mathcal{M}(\mathbb{R}^d)$ fulfills the scaling invariance property (3) for some $0 \le \alpha \le d$ and the shift-invariance property (4) and suppose that, with some $c > 0$, $\mathbb{P}$-almost every ν satisfies $\nu(U(0,1)) = c$. Then α is an integer and $\mathbb{P}$-almost every measure ν is α-uniform and, in particular, α-rectifiable.*

PROOF. From (3) we infer that, for every $r > 0$,

$$\begin{aligned}\mathbb{P}(\{\mu : \mu(U(0,r)) = cr^\alpha\}) &= \mathbb{P}\circ(S_r^\alpha)^{-1}(\{\mu : \mu(U(0,1)) = c\})\\ &= \mathbb{P}(\{\mu : \mu(U(0,1)) = c\}) = 1\ .\end{aligned}$$

Thus, using (4) in the last step,

$$\begin{aligned} 1 &= \mathbb{P}(\{\mu : \mu(U(0,r)) = cr^\alpha\}) = \frac{1}{cs^\alpha}\int\int_{U(0,s)} 1_{\{\mu : \mu(U(0,r))=cr^\alpha\}}(\nu)\, d\nu(u)\, d\mathbb{P}(\nu) \\ &= \frac{1}{cs^\alpha}\int\int_{U(0,s)} 1_{\{\mu : \mu(U(u,r))=cr^\alpha\}}(\nu)\, d\nu(u)\, d\mathbb{P}(\nu). \end{aligned}$$

As this holds for arbitrary r and s we infer that, for $\mathbb{P}$-almost every ν, we have that $\nu(U(u,r)) = cr^\alpha$ for ν-almost every u and every rational $r > 0$. By continuity, $\mathbb{P}$-almost every ν satisfies that $\nu(U(u,r)) = cr^\alpha$ for every u in the support of ν and every $r > 0$, i.e. ν is α-uniform. We can now use Marstrand's Theorem to conclude that α is an integer and Preiss' Regularity Theorem to conclude that ν is α-rectifiable, but of course much weaker statements (like in [**Kir88**]) would suffice. ☐

THEOREM 2.5. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a measure such that*

$$0 < \underline{d}^\alpha(\mu,x) = \underline{D}^\alpha(\mu,x) < \infty \quad \text{for } \mu\text{-almost every } x.$$

Then α is an integer and, at μ-almost every x, there is an α-flat measure $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ such that

$$\nu(U(0,1)) = \underline{d}^\alpha(\mu,x)\ .$$

PROOF. To make use of the preceding lemma, we prove that at μ-almost every x there is $\mathbb{P} \in \mathcal{P}^\alpha(\mu,x)$ such that $\mathbb{P}$-almost every ν satisfies $\nu(U(0,1)) = \underline{d}^\alpha(\mu,x)$.

Fix $x \in \mathbb{R}^d$ such that the density condition is fulfilled and choose $\varepsilon_n \downarrow 0$ such that

$$\lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 \frac{\mu(U(x,r))}{r^\alpha}\,\frac{dr}{r} = \underline{D}^\alpha(\mu,x)\,.$$

As, for all $m \ge 1$ and $R > 0$,

$$\begin{aligned} \mathbb{P}^x_{\varepsilon_n}(\{\nu : \nu(U(0,m)) > Rm^\alpha\}) &\le (1/R)\ (|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 \frac{\mu(U(x,mr))}{(mr)^\alpha}\,\frac{dr}{r} \\ &\le (1/R)\ (|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^m \frac{\mu(U(x,r))}{r^\alpha}\,\frac{dr}{r} \end{aligned}$$

we can find, for every $\varepsilon > 0$, a sequence $R_m \uparrow \infty$ such that, for all n,

$$\mathbb{P}^x_{\varepsilon_n}(\{\nu : \nu(U(0,m)) \le R_m \text{ for all } m = 1,2,\dots\ \}) > 1-\varepsilon\,.$$

This set is compact in $\mathcal{M}(\mathbb{R}^d)$ and hence, by Prohorov's Theorem, we can choose a convergent subsequence of $(\mathbb{P}^x_{\varepsilon_n})$ and denote the limit $\mathbb{P}$. As the mapping $\nu \mapsto \nu(U(0,1))$ is lower semicontinuous we have

$$\int \nu(U(0,1))\, d\mathbb{P}(\nu) \le \lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 \frac{\mu(U(x,r))}{r^\alpha}\,\frac{dr}{r} = \underline{D}^\alpha(\mu,x) = \underline{d}^\alpha(\mu,x).$$

But $\nu(U(0,1)) \ge \underline{d}^\alpha(\mu,x)$ for all $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ and therefore we must have $\nu(U(0,1)) = \underline{d}^\alpha(\mu,x)$ for $\mathbb{P}$-almost every ν.

By Lemma 2.4 α is an integer and, for μ-almost every x, the $\mathbb{P} \in \mathcal{P}^\alpha(\mu,x)$ we have constructed has the property that $\mathbb{P}$-almost every $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ is α-rectifiable and fulfills $\nu(U(y,r)) = \underline{d}^\alpha(\mu,x)\cdot r^\alpha$ for all $r > 0$ and y in the support of ν. Fix such a tangent measure $\nu_0 \in \mathrm{Tan}_\alpha(\mu,x)$ and, using that ν_0 is α-rectifiable, fix y_0 in its support such that all tangent measures of ν_0 at y_0 are α-flat. Since, for μ-almost every x, $\mathrm{Tan}_\alpha(\nu,y) \subseteq \mathrm{Tan}_\alpha(\mu,x)$ for all $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ and y

in the support of ν (see e.g. [**Mat95**, Theorem 14.16]), any tangent measure of ν_0 at y_0 fulfills the requirements of the theorem. $\square$

3. Proof of the main theorem

We now look at the case $\alpha = 1$ and show the following theorem.

THEOREM 3.1. *If $\mu \in \mathcal{M}(\mathbb{R}^d)$ is a measure such that at μ-almost every x there is a 1-flat measure $\nu \in \mathrm{Tan}_1(\mu, x)$ with $\nu(U(0,1)) = \underline{d}^1(\mu, x)$, then μ is linearly rectifiable.*

Clearly, Theorems 2.5 and 3.1 together imply Theorem 1.2. The following lemma contains the main ingredient of the proof of Theorem 3.1.

LEMMA 3.2. *For any $0 < \xi < 1/2 - \sqrt{1/5}$ and $0 < p \leq 1$ there is an $0 < \varepsilon_0 = \varepsilon_0(\xi, p) < 1$ such that, for all $0 < \varepsilon < \varepsilon_0$, whenever E is a compact subset of the interval $[a, b]$ such that the Lebesgue measure of $[a, b] \setminus E$ is at least $p \cdot (b - a)$ and every connected component of the set $[a, b] \setminus E$ has length at most $\varepsilon \cdot (b - a)$, and whenever ν is a measure on the line such that $\nu([a, b] \setminus E) \leq \varepsilon \cdot (b - a)$ and*

$$\nu((x - t, x + t)) \geq t \text{ for all } x \in E,\ 0 < t \leq 2\varepsilon \cdot (b - a),$$

then we have

$$\nu([a, b]) \geq (b - a) \cdot \left[\frac{1}{2} + \xi p\right] > \frac{b - a}{2}.$$

PROOF OF LEMMA 3.2. If ξ and p are given, we pick a number ϱ such that $1/2 + \xi < \varrho < 1 - \sqrt{1/5}$ and numbers $\varepsilon_0, \varepsilon_1 > 0$ such that

$$\left(\varrho - \frac{1}{2}\right) \cdot (1 - \varepsilon_1) p - \frac{\varepsilon_1}{2} - 5\varepsilon_0 \geq \xi p. \tag{5}$$

Suppose that E and ν are given as in the formulation of the lemma and $0 < \varepsilon < \varepsilon_0$. We denote by $\mathcal{I}$ the family of connected components of $[a, b] \setminus E$. We can pick a finite subfamily $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ such that

$$\sum_{I \in \tilde{\mathcal{I}}} |I| \geq (1 - \varepsilon_1) \sum_{I \in \mathcal{I}} |I| \geq (1 - \varepsilon_1) p \cdot (b - a). \tag{6}$$

For every $I \in \tilde{\mathcal{I}}$ we denote by $\overline{I}$ the interval consisting of all $x \in [a, b]$ such that the distance of x to I is at most $|I|$. We pick any of the longest $I \in \tilde{\mathcal{I}}$ and remove all $J \in \tilde{\mathcal{I}}$ with $J \subseteq \overline{I}$ from the collection. We can go on with this procedure, always starting with one of the longest remaining intervals which has not been considered and, after a finite number of steps, we have a new collection of intervals, which we order from left to right

$$I_1 < I_2 < \cdots < I_{N-1} < I_N.$$

We now show how the statement of the lemma follows from

$$\nu\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) \geq \varrho \cdot \ell\Big(\bigcup_{i=1}^{N} \overline{I}_i\Big) - 5\varepsilon \cdot (b - a), \tag{7}$$

where ℓ denotes Lebesgue measure. Using Vitali's Covering Theorem (see e.g. [**Mat95**, Theorem 2.2]) we can cover ℓ-almost all of the set $E \setminus \bigcup_{i=1}^{N} \overline{I}_i$ by countably many disjoint intervals

$[x_i - r_i, x_i + r_i]$ centred in $x_i \in E$ with $r_i < \varepsilon(b-a)$, which are contained in $[a,b] \setminus \bigcup_{i=1}^N \overline{I}_i$. Hence

$$\nu\Big([a,b] \setminus \bigcup_{i=1}^N \overline{I}_i\Big) \geq \sum_{i=1}^\infty \nu([x_i - r_i, x_i + r_i]) \geq \sum_{i=1}^\infty r_i \geq \frac{1}{2} \cdot \ell\Big(E \setminus \bigcup_{i=1}^N \overline{I}_i\Big).$$

We also have

$$\ell\Big(([a,b] \setminus E) \setminus \bigcup_{i=1}^N \overline{I}_i\Big) \leq \ell\Big(\bigcup_{I \in \mathcal{I}} I \setminus \bigcup_{I \in \tilde{\mathcal{I}}} I\Big) \leq \varepsilon_1 \cdot (b-a).$$

Altogether we get

$$\begin{aligned} \nu([a,b]) &= \nu\Big([a,b] \setminus \bigcup_{i=1}^N \overline{I}_i\Big) + \nu\Big(\bigcup_{i=1}^N \overline{I}_i\Big) \\ &\geq \frac{1}{2} \cdot \ell\Big(E \setminus \bigcup_{i=1}^N \overline{I}_i\Big) + \varrho \cdot \ell\Big(\bigcup_{i=1}^N \overline{I}_i\Big) - 5\varepsilon(b-a) \\ &\geq \frac{1}{2} \cdot \ell\Big([a,b] \setminus \bigcup_{i=1}^N \overline{I}_i\Big) + \varrho \cdot \ell\Big(\bigcup_{i=1}^N \overline{I}_i\Big) - \varepsilon_1(b-a)/2 - 5\varepsilon(b-a) \\ &= \frac{b-a}{2} + \Big[\varrho - \frac{1}{2}\Big] \cdot \ell\Big(\bigcup_{i=1}^N \overline{I}_i\Big) - \varepsilon_1(b-a)/2 - 5\varepsilon(b-a) \\ &\geq (b-a)\Big[\frac{1}{2} + \Big(\varrho - \frac{1}{2}\Big) \cdot (1-\varepsilon_1)p - \varepsilon_1/2 - 5\varepsilon_0\Big] \\ &\geq (b-a)\Big[\frac{1}{2} + \xi p\Big], \end{aligned}$$

using $\ell\Big(\bigcup_{i=1}^N \overline{I}_i\Big) \geq \sum_{I \in \tilde{\mathcal{I}}} |I| \geq (1-\varepsilon_1)p \cdot (b-a)$ in the penultimate step and (5) in the final step.

It remains to show (7). For this purpose fix some $2 \leq k \leq N$. Denote by Z_k the set of points between the intervals I_{k-1} and I_k, such that the distance to one of the intervals is less than its length. Let

$$C_k = I_{k-1} \cup Z_k \cup I_k\,.$$

We show that there are $\gamma_2, \ldots, \gamma_N$ with $\sum_{i=2}^N \gamma_i \leq \varepsilon(b-a)$ such that

$$\nu(C_k) \geq \varrho\Big[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\Big] - \gamma_k\,. \tag{8}$$

If the distance of I_{k-1} and I_k is larger than the sum of their lengths, then the open interval centred in the right endpoint of I_{k-1} of diameter $2|I_{k-1}|$ and the open interval centred in the left endpoint of I_k of diameter $2|I_k|$ are disjoint and thus, using that the endpoints of these intervals are in E,

$$\nu(C_k) \geq |I_{k-1}| + |I_k| = \frac{2}{3}\Big[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\Big] > \varrho\Big[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\Big].$$

We may thus suppose that Z_k is an interval with $|Z_k| \leq |I_{k-1}| + |I_k|$ and that

$$\nu(C_k) < \varrho\Big[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\Big]. \tag{9}$$

Looking at an open interval centred in the endpoint of the larger of the intervals I_{k-1} and I_k we get $\nu(C_k) \geq \frac{1}{2}\Big(|I_{k-1}| + |I_k|\Big)$. Together with (9) we conclude

$$|Z_k| > \Big(\frac{1-\varrho}{2\varrho}\Big)\Big(|I_{k-1}| + |I_k|\Big)\,. \tag{10}$$

If $|Z_k| \geq \varrho/(2-2\varrho)(|I_{k-1}| + |I_k|)$, say $|Z_k| = \lambda(|I_{k-1}| + |I_k|)$ for some $1 \geq \lambda \geq \varrho/(2-2\varrho)$, then

$$\begin{aligned} \nu(C_k) &\geq \lambda|I_{k-1}| + \lambda|I_k| \\ &= \varrho|Z_k| + (1-\varrho)\cdot\lambda(|I_{k-1}| + |I_k|) \\ &\geq \varrho\Big[\frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\Big]\,, \end{aligned}$$

contradicting (9). Hence

$$|Z_k| < \Big(\frac{\varrho}{2-2\varrho}\Big)\Big(|I_{k-1}| + |I_k|\Big)\,. \tag{11}$$

We know from the construction of the I_k that ζ_k, the centre of C_k, is in Z_k. Let us show that, with $\vartheta = 1/\varrho - 3/2$, the interval $B = (\zeta_k - \vartheta|Z_k|, \zeta_k + \vartheta|Z_k|)$ contains no point of E.
For, if $y \in B \cap E$, let

$$t = \frac{1}{2}\cdot\Big(|I_{k-1}| + |I_k|\Big) + \frac{1-2\vartheta}{2}|Z_k| > 0\,.$$

Then $(y-t, y+t) \subseteq C_k$ and, using (9),

$$1 \leq \frac{\nu((y-t,y+t))}{t} \leq \frac{\nu(C_k)}{t} < \varrho\cdot\frac{|I_{k-1}| + |I_k| + 2|Z_k|}{|I_{k-1}| + |I_k| + (1-2\vartheta)|Z_k|}\,,$$

and hence, using (11),

$$|Z_k| > \frac{1-\varrho}{2\varrho - 1 + 2\vartheta}\Big(|I_{k-1}| + |I_k|\Big) \geq \frac{(1-\varrho)(2-2\varrho)}{(2\varrho - 4 + 2/\varrho)\varrho}\,|Z_k| = |Z_k|\,,$$

a contradiction, which implies $B \cap E = \emptyset$.

Observe that $B \subseteq Z_k$, since $\zeta_k \in B \cap Z_k$ but the boundaries of I_{k-1} and I_k are not. Hence there is a connected component $I \in \mathcal{I}$ of $[a,b] \setminus E$ such that

$$B \subseteq I \subseteq Z_k\,.$$

Define

$$r_k = \min\{|I_{k-1}|, 2\vartheta|Z_k|\} \quad\text{and}\quad s_k = \min\{|I_k|, 2\vartheta|Z_k|\}\,.$$

As $|I| \geq 2\vartheta|Z_k|$ the intersection of the open interval centred in the right endpoint of I_{k-1} of radius r_k and the open interval centred in the left endpoint of I_k of radius s_k is contained in I. Moreover, they are both contained in C_k and therefore

$$\nu(C_k) + \nu(I) \geq r_k + s_k\,. \tag{12}$$

We choose $\gamma_k = \nu(I)$. We look at three cases that might occur in the definition of s_k, r_k.

(1) $r_k = |I_{k-1}|$ and $s_k = |I_k|$.

From (11) and (12) we derive

$$\nu(C_k) + \gamma_k \geq |I_{k-1}| + |I_k| \geq \frac{\varrho}{2}\Big(|I_{k-1}| + |I_k|\Big) + (1-\varrho/2)\,\frac{2-2\varrho}{\varrho}\,|Z_k|\,,$$

and (8) follows since $(1-\varrho/2)\,(2-2\varrho)/\varrho \geq \varrho$.

(2) $r_k = |I_{k-1}|$ and $s_k = 2\vartheta|Z_k|$, or $r_k = 2\vartheta|Z_k|$ and $s_k = |I_k|$.
By symmetry we concentrate on the second case. Since $B \subseteq Z_k$ we have

$$|I_{k-1}| \le \frac{1}{2}\Big(|I_{k-1}| + |I_k| + |Z_k|\Big) - \vartheta|Z_k|$$

and thus

$$|I_k| \ge |I_{k-1}| - (1-2\vartheta)|Z_k|\,.$$

Using this, (12) and (11) we conclude

$$\begin{aligned}\nu(C_k) + \gamma_k &\ge 2\vartheta|Z_k| + \frac{1}{2}|I_k| + \frac{1}{2}|I_{k-1}| - (1/2-\vartheta)|Z_k| \\ &\ge \frac{\varrho}{2}\Big(|I_{k-1}| + |I_k|\Big) + \Big[(1/2-\varrho/2)\frac{2-2\varrho}{\varrho} + (3\vartheta - 1/2)\Big]\,|Z_k|\,,\end{aligned}$$

and (8) follows from $(1/2-\varrho/2)(2-2\varrho)/\varrho + (3\vartheta-1/2) \ge \varrho$.

(3) $r_k = s_k = 2\vartheta|Z_k|$.
From (12) and (10) we derive

$$\nu(C_k) + \gamma_k \ge 4\vartheta|Z_k| \ge \varrho\,|Z_k| + \Big(\frac{4}{\varrho} - 6 - \varrho\Big)\frac{1-\varrho}{2\varrho}\Big(|I_{k-1}| + |I_k|\Big)\,,$$

and (8) follows since $\varrho < 1 - \sqrt{1/5}$ implies $\big(4/\varrho - 6 - \varrho\big)(1-\varrho)/(2\varrho) \ge \varrho/2$.

In all three cases we have verified (8) for a γ_k such that the $\gamma_2, \dots, \gamma_N$ fulfill

$$\sum_{i=2}^N \gamma_i \le \sum_{I\in\mathcal{I}} \nu(I) = \nu([a,b]\setminus E) \le \varepsilon(b-a)\,.$$

To finish the proof we estimate, taking special care of the "boundary" intervals I_1 and I_N,

$$\begin{aligned}\nu\Big(\bigcup_{i=1}^N \overline{I}_i\Big) &\ge \sum_{k=2}^N \nu(C_k) - \sum_{k=1}^N \nu(I_k) \\ &\ge \varrho\cdot\Big[\sum_{k=2}^N \frac{|I_{k-1}|}{2} + \frac{|I_k|}{2} + |Z_k|\Big] - \sum_{k=2}^N \gamma_k - \varepsilon\cdot(b-a) \\ &\ge \varrho\cdot\ell\Big(\bigcup_{k=1}^N \overline{I}_k\Big) - \frac{3}{2}\Big(|I_1| + |I_N|\Big) - 2\varepsilon\cdot(b-a) \\ &\ge \varrho\cdot\ell\Big(\bigcup_{k=1}^N \overline{I}_k\Big) - 5\varepsilon\cdot(b-a)\,,\end{aligned}$$

which is (7) and thus the proof is finished. □

Proof of Theorem 3.1. Denote by $B(x,r)$ the closed Euclidean ball of radius r centred in x. Suppose the statement is false. Then there is some $0<\eta<1$ and $\delta>0$ such that the set

$$\{x\in\mathbb{R}^d\ :\ \delta < \underline{d}^1(\mu,x) < \eta\cdot\overline{d}^1(\mu,x)\}$$

has positive measure. Fix some $0 < \xi < 1/2 - \sqrt{1/5}$ and $1 > 1 - p > \eta$. We can then pick $0 < \varepsilon < \varepsilon_0(\xi, p)$, such that $\varepsilon < \delta$ and $\eta(1 + 5\varepsilon) < 1 - p$ and

$$\frac{1}{2}\Big(1 + \frac{\varepsilon}{\delta - \varepsilon}\Big)\Big(1 + 5\varepsilon\Big) < \frac{1}{2} + \xi p\,, \tag{13}$$

and we can find a compact set $F \subseteq \mathbb{R}^d$ with $\mu(F) > 0$ and numbers $0 < \delta < D < \infty$ with $\delta < \delta < \eta D$ such that there is $R > 0$ such that for all $x \in F$

- $\mu(U(x, r)) > (\delta - \varepsilon)r$ for all $0 < r < R$,
- there is a sequence $r_n \downarrow 0$ such that the tangent measure
$$\widetilde{\mu} = \lim_{n\to\infty} \frac{\mu_{x,r_n}}{r_n}$$
is 1-flat and $\mu(B(x, r_n)) < \delta \cdot r_n$, and
- there is a sequence $s_n \downarrow 0$ such that $\mu(U(x, s_n)) > D \cdot s_n$.

Using the Density Theorem (see e.g. [**Mat95**, Corollary 2.14]) we can fix a density point $y \in F$, i.e. a point $y \in F$ such that $\lim_{r\to 0} \mu(B(y, r) \setminus F)/\mu(B(y, r)) = 0$. Then, in particular,

$$\lim_{n\to\infty} \frac{1}{r_n}\, \mu(B(y, r_n) \setminus F) = 0\ .$$

Denote by L_y the line through y such that $\widetilde{\mu} = c \cdot \mathcal{H}^1|_{L_y - y}$ and by π_y the orthogonal projection onto L_y. Denote by $S(y, r)$ the set of all points $x \in B(y, r)$ such that the distance of x to its projection onto L_y is at most $r\varepsilon/(1 + 5\varepsilon)$ and, similarly, denote by $S(0, 1)$ the set of all $x \in B(0, 1)$ such that the distance of x to its projection onto $L_y - y$ is at most $\varepsilon/(1 + 5\varepsilon)$.Then

$$\limsup_{n\to\infty} \frac{1}{r_n}\, \mu(B(y, r_n) \setminus S(y, r_n)) = \limsup_{n\to\infty} \frac{\mu_{y,r_n}}{r_n}\, (B(0, 1) \setminus S(0, 1)) = 0\ .$$

We may partition $(L_y - y) \cap B(0, 1)$ into finitely many disjoint intervals $J_1, \dots, J_k$ with $0 < |J_i| < \varepsilon/(1+5\varepsilon)$ and for each interval we can choose a nonnegative continuous function f_i on $\mathbb{R}^d$ which is positive in some point of J_i with support contained in the set of those points in $S(0, 1)$ whose projection onto $L_y - y$ hits J_i. We find

$$0 < \int f_i(z)\, d\widetilde{\mu}(z) = \lim_{n\to\infty} \frac{1}{r_n} \int_F f_i\Big(\frac{z - y}{r_n}\Big)\, d\mu(z)$$

and conclude that there is N such that for all $n \geq N$ and every $1 \leq i \leq k$ the set $\pi_y^{-1}(y + r_n J_i) \cap S(y, r_n)$ contains points of F. Putting these facts together we can find $0 < r < R/4$ such that

- every connected component of the set $\pi_y(B(y, r)) \setminus \pi_y(F \cap S(y, r))$ has length less than $r \cdot 2\varepsilon/(1 + 5\varepsilon)$,
- there is an open set $B \supseteq B(y, r)$ such that $\mu(B) \leq d \cdot r$,
- $\mu(B(y, r) \setminus S(y, r)) \leq \dfrac{(\delta - \varepsilon) \cdot \varepsilon}{1 + 5\varepsilon} \cdot r$,
- $\mu(B(y, r) \setminus F) \leq \dfrac{(\delta - \varepsilon) \cdot \varepsilon}{1 + 5\varepsilon} \cdot r$.

Let $s = r/(1 + 5\varepsilon)$. We define a compact subset of the line L_y by

$$E_y = \pi_y\,(F \cap S(y, r)) \cap B(y, s) \subseteq L_y\ .$$

We now show that the hypotheses of Lemma 3.2 are fulfilled with E_y in the rôle of E and $\pi_y(B(y, s))$ in the rôle of $[a, b]$. By choice of s the connected components of $\pi_y(B(y, s)) \setminus E_y$ have length less than $\varepsilon(2s)$.

By Besicovitch's Covering Theorem (see e.g. [**Mat95**, Theorem 2.8]) we can cover μ-almost all of $F \cap B(y,r)$ with a countable family $(B(x(i),s(i)))$ of disjoint balls centred in F and contained in B such that $\mu(B(x(i),s(i))) \geq D \cdot s(i)$. We infer that

$$\sum_{i=1}^{\infty} s(i) \leq \frac{\mu(B)}{D} \leq \frac{\delta r}{D} < \eta r\,.$$

Hence the set E_y covers at most a proportion of $\eta(1+5\varepsilon) < 1-p$ of the length of $\pi_y(B(y,s))$. In other words,

$$\ell\Big(\pi_y(B(y,s)) \setminus E_y\Big) \geq 2s(1-\eta(1+5\varepsilon)) \geq p(2s).$$

Now define a measure on L_y by

$$\nu = \frac{1}{\delta-\varepsilon} \cdot \mu|_{B(y,r)} \circ \pi_y^{-1}\,.$$

We have

$$\nu(\pi_y(B(y,s)) \setminus E_y) \leq 1/(\delta-\varepsilon) \cdot \Big[\mu(B(y,r) \setminus F) + \mu(B(y,r) \setminus S(y,r))\Big] \leq \varepsilon(2s)\,.$$

Finally, for all $x \in E_y$ and $t \leq 4\varepsilon s < R$, there is $\widetilde{x} \in F \cap S(y,r) \cap \pi_y^{-1}(x)$. As $U(\widetilde{x},t) \subseteq U(x,5\varepsilon s) \subseteq B(y,r)$ we get

$$\nu(U(x,t) \cap L_y) \geq \frac{1}{\delta-\varepsilon} \cdot \mu(U(\widetilde{x},t)) \geq t\,,$$

and thus the hypotheses of Lemma 3.2 are fulfilled. Therefore

$$\nu(\pi_y(B(y,s))) \quad \geq \quad (2s) \cdot \left[\frac{1}{2} + \xi p\right]. \tag{14}$$

On the other hand, from the construction of ν, we get

$$\begin{aligned} \nu(\pi_y(B(y,s))) \quad &\leq \quad \frac{1}{\delta-\varepsilon} \cdot \mu(B(y,r)) \\ &\leq \quad \frac{\delta}{\delta-\varepsilon}(1+5\varepsilon)s\,. \end{aligned} \tag{15}$$

Now (14) and (15) together imply

$$\frac{1}{2}\Big(1+\frac{\varepsilon}{\delta-\varepsilon}\Big)\Big(1+5\varepsilon\Big) \geq \left[\frac{1}{2}+\xi p\right],$$

which contradicts (13) and finishes the proof. □

Remarks:

(i) There are alternative ways to prove Theorem 3.1 from Lemma 3.2. For example one could, instead of showing the existence of densities, use the fact that, if a set F does not contain a linearly rectifiable subset of positive measure, the projections onto almost all lines have Lebesgue measure zero by the Besicovitch-Federer Projection Theorem (see [**Mat95**, Chapter 18]) and then apply Lemma 3.2 with $p=1$.

(ii) Our result naturally raises the question for which dimensions (other than 1) a result like Theorem 3.1 and hence Theorem 1.2 holds. This seems to be a delicate question.

CHAPTER 3

Brownian intersection local times: Upper tails and thick points

WITH WOLFGANG KÖNIG, TU BERLIN AND BRIMS BRISTOL.
HP LABORATORIES TECHNICAL REPORT BRIMS 2001–03.
SUBMITTED FOR PUBLICATION.

Abstract. We equip the intersection of p independent Brownian paths in $\mathbb{R}^d$, $d \geq 2$, with the natural measure λ defined by projecting the intersection local time measure via one of the Brownian motions onto the set of intersection points. Given a bounded domain $U \subset \mathbb{R}^d$ we show that, as $a \uparrow \infty$, the probability of the event $\{\lambda(U) > a\}$ decays with an exponential rate of $a^{1/p}\theta$, where θ is described in terms of a variational problem. In the important special case when U is the unit ball in $\mathbb{R}^3$ and $p = 2$, we characterize θ in terms of an ordinary differential equation. We apply our results to the problem of finding the Hausdorff dimension spectrum for the thick points of the intersection of two independent Brownian paths in $\mathbb{R}^3$.

AMS Subject Classification (2000): *60J65, 60G17, 60J55.*
Key words: *Brownian motion, intersection of Brownian paths, intersection local time, Wiener sausage, upper tail asymptotics, Hausdorff measure, thick points, Hausdorff dimension spectrum, multifractal spectrum.*

1. Introduction and main results

1.1. Aims of the paper. Let a bunch of p independent Brownian motions $W_1, \dots, W_p$ run in $\mathbb{R}^d$ until their first exit times $T_1, \dots, T_p$ from a large ball, or, in the transient case, for infinite time. By classical results of Dvoretzky, Erdös, Kakutani and Taylor the intersection of the paths of these motions,

$$S = \bigcap_{i=1}^{p} \big\{x \in \mathbb{R}^d \,:\, x = W_i(t) \text{ for some } t \in [0, T_i)\big\}, \tag{1}$$

contains points different from the starting point if and only if $p < d/(d-2)$. By work of Geman, Horowitz and Rosen [**GHR84**], in these cases the random set S of intersection points can be equipped with a natural finite measure λ, the *(projected) intersection local time*, which can be described symbolically by the formula

$$\lambda(A) = \int_A dy \prod_{j=1}^{p} \int_0^{T_j} ds\, \delta_y(W_j(s)), \text{ for } A \subset \mathbb{R}^d \text{ Borel.} \tag{2}$$

We review rigorous constructions of the random measure λ in Subsection 2.1 below. In the following we always include the case $p = 1$ of a single Brownian path, in which the rôle of λ is played by the usual occupation measure. However in this case most of our results are known and much easier.

Let $U \subset \mathbb{R}^d$ be a bounded domain in $\mathbb{R}^d$. The first aim of the present paper is to determine asymptotically the upper tails of the random variables $\lambda(U)$. More precisely, we show in Theorem 1.1 that the limit

$$\lim_{a\to\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\{\lambda(U) > a\}$$

exists and describe its value in terms of a natural variational problem associated with p, the stopping rule and the domain $U \subset \mathbb{R}^d$. In the special case that U is the unit ball in $\mathbb{R}^d$ and the Brownian motions run for an infinite amount of time, we can solve this variational problem explicitly in terms of an ordinary differential equation, which is non-linear in the case $p > 1$, see Theorem 1.4.

This result is the key in the solution of a problem posed recently by Dembo, Peres, Rosen and Zeitouni about the *refined multifractal analysis* of intersection local times. Recall that the intersection of two independent Brownian paths in $\mathbb{R}^3$ is a random set of Hausdorff dimension one but length zero. More precisely, by a result of Le Gall [**LeG87, LeG89**], its exact Hausdorff dimension gauge is given by the function $\psi(r) = r[\log\log(1/r)]^2$ and, moreover, the Hausdorff measure for this gauge function coincides up to a constant factor with the projected intersection local time measure on the intersection set S.

The key argument leading to Le Gall's result is the existence of a positive and finite constant c such that, almost surely, for typical x in the intersection set S,

$$\limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r[\log\log(1/r)]^2} = c,$$

where $B(x,r)$ denotes the open ball of radius r with centre in x. Here we are concerned with finer results, which focus on exceptional points in $\mathbb{R}^3$ in a neighbourhood of which the concentration of intersection local time is untypically large. Our first aim is to find an explicit gauge function φ such that

$$0 < \sup_{x\in S} \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{\varphi(r)} < \infty.$$

Having found such a function, we call a point $x \in S$ *thick* if $\limsup_{r\downarrow 0} \lambda(B(x,r))/\varphi(r) > 0$ and ask, how many thick points there are. This is answered in terms of the Hausdorff dimension spectrum for the thick points, which is the function

$$f(a) = \dim\Big\{x \in S \,:\, \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{\varphi(r)} = a\Big\}.$$

Both these problems are intimately related with our upper tail asymptotics and the numerical values featuring the dimension spectrum can be derived from our differential equation. Theorem 1.6 describes just how large the measure λ can be in a neighbourhood of a point and identifies the Hausdorff dimension spectrum of thick points for the intersection local time.

The spectrum of exceptionally thick points is also understood for some other classes of random measures: For example, in the case of the local time measure on the zero set of stable processes [**ShT98**], the branching measure on a supercritical Galton-Watson process [**ShT00**], [**MöS01**], the occupation measure on Brownian paths in dimensions exceeding two [**DPR00a**] and in the plane [**DPR00b**], and also intersections of Brownian paths in the plane [**DPR01**].

The present paper is organized as follows. In Subsections 1.2–1.4, we describe our main results on the upper tails of $\lambda(U)$, the characterization of the variational problem arising, and the dimension spectrum of the thick points, respectively. In Section 2, we survey three constructions of the intersection local time and prove that the upper tail asymptotics of $\lambda(U)$ are completely described by the asymptotics of its high moments. The asymptotic analysis of the high moments is carried out in several steps in Section 3. In Section 4 we analyse the two variational formulae that describe the upper tails resp. the moment asymptotics. Section 5 contains the proof of our results on the thick points of the intersection local time, and in Section 6, we announce further results and applications which will appear elsewhere.

1.2. Upper tail asymptotics. To describe the variational problem featuring the upper tail asymptotics, we first introduce the *Green's function* associated with (possibly stopped) Brownian motion. The Green's function depends on the dimension d of the ambient space and on the way we stop the Brownian motion, but it does *not* depend on the domain U or the number p of motions. Suppose that we stop Brownian motion on its first exit time from a large open ball $B(0,R)$, and, if $d \geq 3$, include the case $R = \infty$ as the case of Brownian motion running until infinity. Denote by $p_s(x,y)$ the transition sub-probability density of the killed motion. For x, y in the open centred ball $B(0,R)$, we let

$$G(x,y) = \int_0^\infty p_s(x,y)\, ds, \tag{3}$$

be the associated *Green's function*. Explicitly, these functions are, see e.g. [**PoS78**, p.114], if $d = 2$,

$$G(x,y) = \frac{1}{\pi} \begin{cases} \log\left|x\frac{R}{|x|} - y\frac{|x|}{R}\right| - \log|x-y| & \text{if } x \neq 0, \\ \log R - \log|y| & \text{if } x = 0, \end{cases} \tag{4}$$

and, if $d \geq 3$ and $0 < R < \infty$, denoting $c_d = \Gamma(d/2-1)/(2\pi^{d/2})$,

$$G(x,y) = c_d \begin{cases} |x-y|^{2-d} - \left|x\frac{R}{|x|} - y\frac{|x|}{R}\right|^{2-d} & \text{if } x \neq 0, \\ |y|^{2-d} - R^{2-d} & \text{if } x = 0. \end{cases} \tag{5}$$

Finally, in the case $d \geq 3$ and $R = \infty$, we have

$$G(x,y) = \frac{c_d}{|x-y|^{d-2}}\,. \tag{6}$$

Note that G is always symmetric, see e.g. [**Bas95**, II(4.5)]. Furthermore, precisely if $p < d/(d-2)$, the function $G^p(0,\cdot)$ is integrable in a neighbourhood of the origin.

Now let $U \subset \mathbb{R}^d$ be an open bounded set whose closure is contained in $B(0,R)$. Define the operator $\mathfrak{A}\colon L^{2p/(2p-1)}(U) \to L^{2p}(U)$, depending on R and $U \subset \mathbb{R}^d$, by

$$\mathfrak{A}f(x) = \int_U G(x,y) f(y)\, dy, \text{ for } x \in U. \tag{7}$$

The operator $\mathfrak{A}$ is symmetric and continuous, and its restriction $\mathfrak{A}\colon L^\infty(U) \to L^\infty(U)$ is moreover symmetric, positive and compact with norm $\int_U \int_U G(x,y)\, dx\, dy$.

We denote by $\mathcal{M}_1(U)$ the set of probability measures on the set $U \subset \mathbb{R}^d$ and, if $\mu \in \mathcal{M}_1(U)$ and f is an integrable or nonnegative function defined on U we use the notation $\langle f, \mu\rangle$ for the integral $\int_U f\, d\mu$. If f, g are both nonnegative functions or their product is Lebesgue integrable on U, we use the notation $\langle f, g\rangle$ to denote $\int_U f(x)g(x)\, dx$. We write $\|\cdot\|_q$ for the norm on $L^q(U)$.

We can now formulate our first main theorem.

THEOREM 1.1 (Upper tail asymptotics). *Suppose that p and d are positive integers satisfying $p < d/(d-2)$. Let U be an open bounded domain in $\mathbb{R}^d$ and $R \in (0,\infty]$ sufficiently large that $B(0,R)$ contains the closure of U. Denote by λ the projected intersection local time of p independent Brownian motions started in fixed points inside U and stopped upon their first exit from the ball $B(0,R)$. Then*

$$\lim_{a\to\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\big\{\lambda(U) > a\big\} = -\frac{p}{\varrho^*}, \tag{8}$$

where ϱ^* is defined as

$$\varrho^* = \sup\Big\{\big\langle g^{2p-1}, \mathfrak{A} g^{2p-1}\big\rangle \,:\, g \in L^{2p}(U) \text{ with } \|g\|_{2p} = 1\Big\}. \tag{9}$$

REMARK 1.2. In the transient case $d \geq 3$ we denote by $\varrho^*(R)$ the supremum in (9) for the Green's function associated with Brownian motion stopped upon leaving $B(0,R)$. Then $\lim_{R\uparrow\infty} \varrho^*(R)$ is equal to the supremum in (9) for the case of unstopped Brownian motions. This follows easily from monotone convergence and the fact that the Green's functions on $B(0,R)$ are increasing to the Green's function in (6).

In the next subsection we turn to an analysis of the variational formula in (9). We already announce that in Proposition 2.1 below we characterize ϱ^* in terms of another variational formula, which is more in the spirit of our actual proof of (8).

1.3. The variational formula in (9)**.** In the special case $p = 1$, the well-known Rayleigh-Ritz formula describes the right hand side in formula (9) as the principal eigenvalue of the compact and symmetric operator $\mathfrak{A}$ on $L^2(U)$. Hence, existence and uniqueness and many more properties of the maximizer are known. For general $p \geq 2$, however, already the uniqueness seems to be an open problem, apart from the special case of the unit ball in $\mathbb{R}^3$. In our next main result, we establish the existence of maximizers and characterize them in terms of the corresponding Euler-Lagrange equations.

THEOREM 1.3. *Suppose that p and d are positive integers satisfying $p < d/(d-2)$. Let U be an open bounded domain in $\mathbb{R}^d$ and $R \in (0,\infty]$ sufficiently large that $B(0,R)$ contains the closure of U. Then the supremum in the variational problem (9) is attained. Every maximizer g in (9) is bounded away from 0 and infinity and has a version that is twice continuously differentiable on U and continuously differentiable on the closure of U. Moreover, g satisfies the following equivalent conditions.*

(*a*) *$\mathfrak{A} g^{2p-1}(x) = \varrho^* g(x)$, for all $x \in U$.*
(*b*) *g is the restriction to U of a continuous function $g\colon B(0,R) \to [0,\infty)$ that vanishes on the sphere $\partial B(0,R)$, or in the case $R = \infty$ converges to 0 as $x \to \infty$. This function g is continuously differentiable inside $B(0,R)$ and is a solution of*

$$\frac{1}{2}\Delta g(x) = -\frac{1}{\varrho^*} g^{2p-1}(x) 1_U(x) \text{ for all } x \in B(0,R) \setminus \partial U. \tag{10}$$

If $p = 1$, the solution (g, ϱ^*) of (a) and hence the maximizer g in (9) is unique.

For $p \geq 2$ and general domains U, there seems to be no reason that, for a solution (g, ϱ) of assertion (a), the number ϱ must coincide with ϱ^* in (9). However, in the important special case that $U = B(0,1)$ is the open unit ball and the Brownian motions run for an infinite time, we can show uniqueness of the minimizer in (9) and describe it more explicitly in terms of an ordinary differential equation. We are able to do this because G is a rotation invariant function of the difference of its arguments, and hence Riesz' rearrangement theorem is applicable.

THEOREM 1.4. *Let $d \geq 3$ and p a positive integer satisfying $p < d/(d-2)$, further let $R = \infty$ and $U = B(0,1)$. Then the maximization problem (9) has a unique solution g. Moreover, g is rotationally symmetric around the origin and can be constructed as follows.*

(i) *There is a unique twice continuously differentiable function z on a maximal interval $[0,\xi)$ satisfying $z(0) = 1$, $z'(0) = 0$ and*

$$z''(x) + (d-1)\frac{z'(x)}{x} + z(x)^{2p-1} = 0, \text{ for all } x \in [0,\xi). \tag{11}$$

There is a smallest number $a \in (0,\xi)$ with $(d-2)z(a) = -az'(a)$. Moreover, z is positive and strictly decreasing on $[0,a]$, and $g(x) = cz(a|x|)$ defines the unique maximizer of (9), if c is the normalization constant such that $\|g\|_{2p} = 1$.

(ii) *We have*

$$\frac{p^2}{\varrho^{*2}} = \begin{cases} \dfrac{a^4}{4}, & \text{for } p = 1, d \geq 3, \\ 4\pi a \displaystyle\int_0^a z^4(s)s^2\,ds, & \text{for } p = 2, d = 3. \end{cases}$$

REMARK 1.5. In the case $p = 1$, one can solve (11) explicitly in terms of Bessel functions. Recall (see e.g., [**AbS72**, 9.1.7]) that the *Bessel function J_ν of the first kind of order $\nu \geq 0$* satisfies the *Bessel differential equation* $x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) = 0$, and define a continuous function $z : [0,\infty) \to \mathbb{R}$ by $z(0) = 1$ and

$$z(x) = \Gamma(d/2)\left(\frac{x}{2}\right)^{\frac{2-d}{2}} J_{\frac{d-2}{2}}(x), \qquad \text{for } x > 0. \tag{12}$$

From Bessel's equation we easily derive that z satisfies (11). Further properties of Bessel functions, see [**AbS72**, 9.1.30], show that $z'(0) = 0$. The continuous function $\zeta(x) = z(x) + (d-2)^{-1}xz'(x)$ takes the positive value $\zeta(0) = 1$ at the origin and is nonpositive at the first zero ξ of z. Hence there is a minimal $a > 0$ with $\zeta(a) = 0$. For example, in dimension $d = 3$ we obtain $z(x) = \frac{1}{x}\sin x$ and $a = \pi/2$ and thus $1/\varrho^* = \pi^2/8$. In dimension $d = 5$ we have $z(x) = 3x^{-3}(\sin x - x\cos x)$ and $a = \pi$, hence $1/\varrho^* = \pi^2/2$.

1.4. The dimension spectrum of thick points. The fine *multifractal structure* of a random measure on $\mathbb{R}^d$ consists of the analysis of its exceptionally thin and thick points. The latter part has been successfully completed in the case of intersection local times of any number of planar Brownian motions and in the case of the occupation measure of a single Brownian motion in transient dimensions $d \geq 3$.

Before stating our new result for the case of intersections of Brownian motions in $\mathbb{R}^3$, let us review the existing results. If λ is the occupation measure of a single Brownian path in $d \geq 3$, it was shown by Dembo et al. in [**DPR00a**] that the correct gauge function is given by $r \mapsto r^2[\log(1/r)]$, in the sense that the number

$$2\varrho^* = \sup_{x\in\mathbb{R}^d} \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]} \in (0,\infty) \tag{13}$$

exists and is non-random. The value of ϱ^* agrees with the value obtained in Theorem 1.4 (ii). The corresponding dimension spectrum is

$$\dim\left\{x \in \mathbb{R}^d \;:\; \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]} = a\right\} = 2 - \frac{a}{\varrho^*}, \text{ for } a \geq 0, \tag{14}$$

where negative values of the dimension mean that the set is empty. However, the assertions in (8) and (9) in this case are known for a long time due to the Ciesielski-Taylor identity [**CiT62**], which is not available in the case of $p > 1$. The proof of (13) and (14) is linked to the upper tail behaviour of Brownian occupation measure in a unit ball via self-similarity: Most of the exceptionally large mass in a small ball around a given point is accumulated in a very short time, hence by scaling the probability of high concentration of occupation measure in a small ball is asymptotically equal to the probability of exceptionally large mass in a unit ball accumulated in a fixed time horizon.

The situation is entirely different in the recurrent dimension $d = 2$. Here, the same authors have shown that the intersection local time measures λ, which in the case $p = 1$ degenerate to the occupation measure, satisfy

$$\left(\frac{2}{p}\right)^p = \sup_{x\in\mathbb{R}^d} \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]^{2p}} \tag{15}$$

and

$$\dim\left\{x \in \mathbb{R}^d \,:\, \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]^{2p}} = a\right\} = 2 - pa^{1/p}, \text{ for } a \geq 0, \tag{16}$$

see [**DPR00b**] in case $p = 1$ and [**DPR01**] in case $p > 1$. The values in the planar case bear no relationship with our result on the upper tail asymptotics in (8) in this case. Heuristically, this is due to the fact that in the planar case, the probability that a small ball has unusually large concentration of mass cannot be related via self-similarity to the probability that the unit ball carries large mass, but depends just on the number of visits of the motions to the ball before the first exit from the large ball $B(0,R)$. In plain words, the asymptotics in (13)–(16) are determined in $d = 2$ by many returns to the small ball in a long time range, and in higher dimensions by relatively large occupation times already in a short time range.

Our next theorem closes the gap in this picture by describing the Hausdorff dimension spectrum of thick points for intersection local time measure on the intersections of two Brownian paths in $\mathbb{R}^3$.

THEOREM 1.6. *Suppose that ϱ^* is as in* (9) *for the case $d = 3$, $p = 2$, $R = \infty$ and U the unit ball, respectively as in Theorem 1.4 (ii) for $p = 2$. Then, almost surely,*

$$\sup_{x\in\mathbb{R}^d} \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} = \limsup_{r\downarrow 0} \sup_{x\in S} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} = \left(\frac{\varrho^*}{2}\right)^2. \tag{17}$$

Moreover,

$$\dim\left\{x \in S \,:\, \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} = a\right\} = 1 - \sqrt{a}\frac{2}{\varrho^*} \text{ for all } a \geq 0\,, \tag{18}$$

where dim *denotes the Hausdorff dimension.*

REMARK 1.7. There is a further difference between the recurrent and transient case, which is worth noting. In the planar case the limsup may be replaced by a liminf without changing the dimension spectrum. This gives the so called spectrum of *consistently thick points.* In higher dimensions this is not the case and the spectrum of consistently thick points requires a different gauge function. This was observed in [**DPR00a**, Theorem 1.6]. The precise Hausdorff dimension spectrum for the consistently thick points of intersection local time measures in $\mathbb{R}^d$, $d \geq 3$, is an *open* problem even in the case $p = 1$ of a single Brownian path.

To complete the picture, we review the results of [**DPR00c**] for *thin* points of the occupation measure λ on a single Brownian path $S = \{W(t) : t \in [0,T]\}$, T the first exit time from a large open ball. It turns out that here the ambient dimension does not play a major rôle. In all $d \geq 2$, we have

$$1 = \inf_{x\in S} \liminf_{r\to 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]^{-1}}\,,$$

and a dimension spectrum given by

$$\dim\Big\{x \in S\,:\, \liminf_{r\downarrow 0} \frac{\lambda(B(x,r))}{r^2[\log(1/r)]^{-1}} = a\Big\} = 2 - \frac{2}{a}.$$

The spectrum of thin points for intersections of Brownian paths seems to be an *open* problem.

2. Preliminaries

In this section, we collect some material to prepare the proofs. First, we survey three rigorous constructions of the intersection local time measures, afterwards we introduce another important variational problem which is strongly linked to (9), and finally we show how to reduce the problem of upper tail asymptotics to the problem of moment asymptotics in our case.

2.1. Brownian intersection local times. Recall that T_i is the first time the motion W_i reaches a sphere of fixed radius $R > 0$, which may be infinity in the transient case. Also recall that we kill W_i at time T_i for $i = 1, \dots, p$. We denote the Euclidean ball in $\mathbb{R}^d$ centred at x with radius $r > 0$ by $B(x,r)$. For every $\varepsilon > 0$ define the *Wiener sausage* around W_i by

$$S^i_\varepsilon = \Big\{x \in \mathbb{R}^d \,:\, \text{there is } t \in [0,T_i) \text{ with } |x - W_i(t)| < \varepsilon\Big\}, \text{ for } i = 1, \dots, p\,, \tag{19}$$

and their intersection $S_\varepsilon = \bigcap_{i=1}^p S^i_\varepsilon$. Recall (1) and observe that

$$S = \bigcap_{\varepsilon>0} S_\varepsilon = \Big\{z \in \mathbb{R}^d \,:\, z = W_1(t_1) = \cdots = W_p(t_p) \text{ for some } t_i \in [0,T_i)\Big\}$$

is the intersection of the p independent Brownian paths. This set is our object of interest. There are three ways to equip S with a natural measure:

- by projecting the local time at 0 for the confluent Brownian motion, see [**GHR84**];
- as renormalized limit for $\varepsilon \downarrow 0$ of the Lebesgue measure on S_ε, see [**LeG86**];
- as a Hausdorff measure on S with an appropriate dimension gauge function, see [**LeG87, LeG89**].

Luckily, all three approaches lead to the same object, the *intersection local time measure* λ on S. Let us first briefly review the approach of Geman, Horowitz and Rosen [**GHR84**], based on local times for the confluent Brownian motion process $W\colon \mathbb{R}^p \to \mathbb{R}^{d(p-1)}$ defined by

$$W(s_1, \dots, s_p) = \big(W_1(s_1) - W_2(s_2), \dots, W_{p-1}(s_{p-1}) - W_p(s_p)\big)\,.$$

With probability one there is a family $\{\mu_y : y \in (\mathbb{R}^d)^{p-1}\}$ of finite measures on $\prod_{i=1}^p [0,T_i)$ with the following two properties:

(i) the mapping $y \mapsto \mu_y$ is continuous with respect to the vague topology on the space $\mathcal{M}(\mathbb{R}^p)$ of locally finite measures on $\mathbb{R}^p$,
(ii) for all Borel functions $g\colon (\mathbb{R}^d)^{p-1} \to [0,\infty]$ and $f\colon \prod_{i=1}^p [0,T_i) \to [0,\infty]$,

$$\int g(y)\langle f, d\mu_y\rangle\, dy = \int_{\prod_{i=1}^p [0,T_i)} f \cdot g \circ W \, ds_p \dots ds_1. \tag{20}$$

These properties imply that μ_y is supported by the level set at y of the confluent Brownian motion, i.e., by

$$\Big\{(s_1,\dots,s_p)\in\prod_{i=1}^p[0,T_i)\ :\ W(s_1,\dots,s_p)=y\Big\}.$$

The family $\{\mu_y\ :\ y\in(\mathbb{R}^d)^{p-1}\}$ is called the family of local times of the confluent Brownian motion process. The local time μ_0 at level 0 is of special interest to us, because it is supported by the set of p-tuples of times where the p motions $W_1,\dots,W_p$ coincide. The image measure λ of μ_0 under the mapping $(t_1,\dots,t_p)\mapsto W_1(t_1)$ is a natural finite measure on the intersections of the Brownian paths, which we hence call the *intersection local time* of the p Brownian motions.

Le Gall [**LeG86**] carried out the second approach. Define

$$s_d(\varepsilon)=\begin{cases}\pi^{-p}\log^p(1/\varepsilon), & \text{if } d=2,\\ (2\pi\varepsilon)^{-2}, & \text{if } d=3 \text{ and } p=2,\\ \frac{2}{\omega_d(d-2)}\varepsilon^{2-d}, & \text{if } d\ge 3 \text{ and } p=1,\end{cases}\tag{21}$$

where ω_d is the volume of the d-dimensional unit ball. By [**LeG86**, Theorème 3.1], the renormalized restriction of the Lebesgue measure on S_ε converges to λ as $\varepsilon\downarrow 0$, in the sense that, for every set $A\subset\mathbb{R}^d$ that is almost surely an λ-continuity set,

$$\lim_{\varepsilon\downarrow 0}s_d(\varepsilon)\ell(S_\varepsilon\cap A)=\lambda(A).\tag{22}$$

The convergence holds in the $L^q(\mathbb{P})$-sense for all $q\in[1,\infty)$ and, in particular, in probability.

Finally, the most natural way to equip S with a uniform distribution is by means of a suitable Hausdorff measure. Le Gall showed in [**LeG87, LeG89**] that λ may be defined intrinsically as a constant multiple of the ψ-Hausdorff measure on the random set S in the case of p spatial Brownian motions for the gauge function

$$\psi_p(r)=\begin{cases}r^2[\log(1/r)\log\log\log(1/r)]^p, & \text{if } d=2, p\in\mathbb{N},\\ r^{3-p}[\log\log(1/r)]^p, & \text{if } d=3, p\in\{1,2\},\\ r^2[\log\log(1/r)], & \text{if } d\ge 4, p=1.\end{cases}\tag{23}$$

Here we have included the results of Ciesielski and Taylor [**CiT62**] and Ray [**Ray63**] in the case $p=1$ for completeness.

2.2. A related variational problem. As announced, we do not directly derive the upper tails of $\lambda(U)$ in (8), but we describe the moment asymptotics and derive (8) from this result. In this subsection, we introduce the variational formula that turns out to feature the moment asymptotics of $\lambda(U)$. Furthermore, we describe the relationship of the two problems of upper tail and moment asymptotics on the stage of the variational formulas. We have to introduce more notation.

For any probability measure μ and finite measure $\widetilde{\mu}$ on the same measurable space the *relative entropy* or *Kullback-Leibler distance* of μ with respect to $\widetilde{\mu}$ is defined as

$$H(\mu\mid\widetilde{\mu})=\begin{cases}\displaystyle\int\mu(dx)\,\log\frac{\mu(dx)}{\widetilde{\mu}(dx)}, & \text{if } \mu\ll\widetilde{\mu},\\ \infty, & \text{otherwise.}\end{cases}\tag{24}$$

If $\widetilde{\mu}$ is also a probability measure, then by Jensen's inequality we always have $H(\mu\,|\,\widetilde{\mu})\ge 0$ and equality holds if and only if $\mu=\widetilde{\mu}$.

More specifically, let $U \subset \mathbb{R}^d$ be an open bounded set and denote by $\mathcal{M}_1(U)$ the space of probability measures on U, equipped with the weak topology. For any $\mu \in \mathcal{M}_1(U)$ we define $I(\mu) = H(\mu \mid \ell)$, the relative entropy of μ with respect to the Lebesgue measure ℓ on U. Note that $I(\mu) \geq -\log \ell(U)$ with equality if and only if μ is the normalized Lebesgue measure on U. Moreover, I is a convex and lower semicontinuous function.

We denote by

$$\mathcal{M}_1^*(U) = \Big\{ \nu \in \mathcal{M}_1(U^2) \ : \ \text{for all Borel sets } A \subset U \text{ we have } \nu(A \times U) = \nu(U \times A) \Big\} \tag{25}$$

the set of probability measures ν on U^2 with equal marginals $\nu_1(A) = \nu(A \times U)$ and $\nu_2(A) = \nu(U \times A)$. For $\nu \in \mathcal{M}_1(U^2)$ we define

$$I_\mu^2(\nu) = \begin{cases} H(\nu \mid \nu_1 \otimes \mu)\,, & \text{if } \nu \in \mathcal{M}_1^*(U), \\ \infty\,, & \text{otherwise.} \end{cases} \tag{26}$$

Note that I_μ^2 is the rate function for the empirical pair measures of an i.i.d. sequence with marginal distribution μ. In particular, I_μ^2 is lower semicontinuous and convex.

Define a function $\mathcal{G} \colon \mathcal{M}_1(U) \longrightarrow \mathbb{R}$ by

$$\mathcal{G}(\mu) = \inf_{\substack{\nu \in \mathcal{M}_1^*(U) \\ \nu_1 = \mu}} \big\{ I_\mu^2(\nu) - \langle \nu, \log G \rangle \big\}\,, \tag{27}$$

where we extend the notation $\langle \cdot, \cdot \rangle$ to integrals on U^2. Observe that it suffices to take the infimum over measures ν satisfying $\nu \ll \mu \otimes \mu$. We can replace $I_\mu^2(\nu)$ in the definition of $\mathcal{G}$ by either the relative entropy $H(\nu \,|\, \mu \otimes \mu)$ or the *mutual information* $H(\nu \,|\nu_1 \otimes \nu_2)$.

For the further development, the variational formula

$$\kappa^* = \inf_{\mu \in \mathcal{M}_1(U)} \Big\{ I(\mu) + p\mathcal{G}(\mu) \Big\} \tag{28}$$

is of fundamental interest. We state its link to the formula (9) as follows.

PROPOSITION 2.1. *For every positive integer $p < d/(d-2)$ we have*

(29)

$$\sup \Big\{ \langle g^{2p-1}, \mathfrak{A} g^{2p-1} \rangle \ : \ g \in L^{2p}(U) \ \textit{with} \ \|g\|_{2p} = 1 \Big\} = \exp\Big(-\frac{1}{p} \inf_{\mu \in \mathcal{M}_1(U)} \Big\{ I(\mu) + p\mathcal{G}(\mu) \Big\} \Big).$$

Moreover, g is a maximizer of the left hand side if and only if the measure $\mu(dx) = g^{2p}(x)\,dx$ is a minimizer on the right hand side of (29). Every minimizing sequence of the variational problem on the right hand side of (29) has a subsequence converging weakly to a minimizer.

Proposition 2.1 is proved in Section 4 together with the results of Subsection 1.3.

2.3. Tail asymptotics from moment asymptotics. For reasons which are explained at the beginning of Section 3, we are able to deal with integer moments of $\lambda(U)$ much better than with the probability of the event $\{\lambda(U) > a\}$. In this section, we show how to derive the upper tail asymptotics of $\lambda(U)$ from moment asymptotics. Let us first state our main result for the moment asymptotics.

PROPOSITION 2.2.

$$\lim_{k \uparrow \infty} \frac{1}{k} \log \mathbb{E}\Big[\frac{\lambda(U)^k}{(k!)^p} \Big] = - \inf_{\mu \in \mathcal{M}_1(U)} \big\{ I(\mu) + p\mathcal{G}(\mu) \big\}.$$

Section 3 is devoted to the proof of Proposition 2.2. In order to derive the upper tail asymptotics from Proposition 2.2 we need the following Tauberian theorem.

LEMMA 2.3. *Let X be any nonnegative random variable and fix $p \in \mathbb{N}$. Then for any $\kappa \in \mathbb{R}$ the following two implications hold:*

(i)
$$\limsup_{k\uparrow\infty} \frac{1}{k} \log E\Big[\frac{X^k}{(k!)^p}\Big] \le -\kappa \quad \Longrightarrow \quad \limsup_{a\uparrow\infty} a^{-1/p} \log P\{X > a\} \le -pe^{\kappa/p}.$$

(ii)
$$\lim_{k\uparrow\infty} \frac{1}{k} \log E\Big[\frac{X^k}{(k!)^p}\Big] = -\kappa \quad \Longrightarrow \quad \lim_{a\uparrow\infty} a^{-1/p} \log P\{X > a\} = -pe^{\kappa/p}.$$

PROOF. The proof of (i) is easy and based on the substitution $a_k = e^{-\kappa} k^p$, Markov's inequality, and Stirling's formula as follows.

$$\begin{aligned}
\limsup_{k\uparrow\infty} a_k^{-1/p} \log P\{X > a_k\} &= e^{\kappa/p} \limsup_{k\uparrow\infty} \frac{1}{k} \log P\Big\{X^k > e^{-k\kappa} k^{kp}\Big\} \\
&\le e^{\kappa/p} \limsup_{k\uparrow\infty} \frac{1}{k} \log E\Big[\frac{X^k}{e^{-k\kappa} k^{kp}}\Big] = e^{\kappa/p} \limsup_{k\uparrow\infty} \Big(\frac{1}{k} \log E\Big[\frac{X^k}{(k!)^p}\Big] + \kappa - p\Big) \\
&\le -pe^{\kappa/p}.
\end{aligned}$$

Since $a_{k+1}^{-1/p}/a_k^{-1/p} \to 1$ as $k \uparrow \infty$, we see that it is sufficient to consider the subsequence a_k rather than an arbitrary sequence tending to infinity.

The proof of (ii) is based on the construction of the transformed measure

$$d\widehat{P}^k(X) = \frac{X^k}{E[X^k]}\, dP(X), \qquad \text{for } k \in \mathbb{N},$$

and the fact that the random variable

$$Y_k = \log\Big(\frac{X}{e^{-\kappa} k^p}\Big)$$

satisfies, for every $\varepsilon > 0$,

$$\lim_{k\uparrow\infty} \widehat{P}^k\{|Y_k| \le \varepsilon\} = 1. \tag{30}$$

To prove this, fix an arbitrary $\varepsilon > 0$ and pick some small number $\alpha > 0$. Then, by the Markov inequality, we may estimate

$$\widehat{P}^k\{Y_k \ge \varepsilon\} = \widehat{P}^k\{X^{k\alpha} \ge e^{(\varepsilon-\kappa)k\alpha} k^{pk\alpha}\} \le e^{-\varepsilon k\alpha} e^{\kappa k\alpha} k^{-pk\alpha} \widehat{E}^k[X^{k\alpha}],$$

where $\widehat{E}^k$ denotes expectation with respect to $\widehat{P}^k$. Note that $\widehat{E}^k[X^{k\alpha}] = E[X^{k(1+\alpha)}]/E[X^k]$. Using our assumption and Stirling's formula, we see that the quotient has the asymptotic behaviour $e^{-(\kappa+p)k\alpha} k^{pk\alpha} (1+\alpha)^{kp(1+\alpha)} e^{\mathsf{o}(k)}$ as $k \uparrow \infty$. Inserting this in the right hand side above, we get

$$\widehat{P}^k\{Y_k \ge \varepsilon\} \le \exp\Big(pk\alpha\Big(-\frac{\varepsilon}{p} - 1 + \frac{1+\alpha}{\alpha}\log(1+\alpha) + \mathsf{o}(1)\Big)\Big), \qquad \text{as } k \uparrow \infty.$$

If $\alpha > 0$ is chosen small enough, then the expression between the inner brackets is negative and bounded away from zero, such that we obtain that $\lim_{k\uparrow\infty} \widehat{P}^k\{Y_k \ge \varepsilon\} = 0$. Analogously one shows that $\lim_{k\uparrow\infty} \widehat{P}^k\{Y_k \le -\varepsilon\} = 0$, and this implies (30).

In order to finish the proof of the lower bound, we keep $\varepsilon > 0$ arbitrarily fixed and substitute this time $a = e^{-\kappa} k^p e^{-\varepsilon}$. It is again clear that the consideration of this subsequence suffices. Note that $\{X > a\} = \{Y_k > -\varepsilon\} \supset \{|Y_k| \leq \varepsilon\}$. This implies that

$$a^{-1/p} \log P\{X > a\} \geq e^{\kappa/p} e^{\varepsilon/p} \frac{1}{k} \log P\{|Y_k| \leq \varepsilon\}.$$

Note that $P\{|Y_k| \leq \varepsilon\} = \widehat{E}^k[X^{-k} 1_{\{|Y_k| \leq \varepsilon\}}] E[X^k]$ and that we may estimate $X^{-k} \geq e^{k(-\varepsilon+\kappa)} k^{-pk}$ on $\{|Y_k| \leq \varepsilon\}$. Using this estimate, our assumption on the asymptotics of $E[X^k]$ and Stirling's formula, we obtain

$$\liminf_{a \uparrow \infty} a^{-1/p} \log P\{X > a\} \geq e^{\kappa/p} e^{\varepsilon/p} \left(-\varepsilon - p + \liminf_{k \uparrow \infty} \frac{1}{k} \log \widehat{P}^k\{|Y_k| \leq \varepsilon\} \right).$$

Because of (30), the latter limit inferior is equal to zero. After letting $\varepsilon \downarrow 0$, we get the assertion. □

Proof of Theorem 1.1. The existence of the limit (8) follows from Proposition 2.2 and Lemma 2.3 (ii), the characterization (9) of the limit is then immediate from Proposition 2.1. □

3. Large moment asymptotics

In this section we prove the large moment asymptotics in Proposition 2.2. In Subsection 3.1, we introduce a main tool: a formula for the integer moments of $\lambda(U)$, which is due to Le Gall. The two main technical steps are carried out in Subsections 3.2 (cutting off the Green's function) and 3.3 (finite partitioning of U). The combinatorial core argument of the proof appears in Subsection 3.4. The ingredients are put together in the final Subsection 3.5, where the proof of Proposition 2.2 is finished.

3.1. Le Gall's moment formula.

Starting point for our analysis of high moments of $\lambda(U)$ is the following formula for the moments of the intersection local times. For $k \in \mathbb{N}$, we denote by $\mathfrak{S}_k$ the set of permutations $\sigma = (\sigma(1), \dots, \sigma(k))$ of the numbers $1, \dots, k$. Denote by $x^1, \dots, x^p \in U$ the starting points of the Brownian motions $W_1, \dots, W_p$.

Lemma 3.1. *For any $k \in \mathbb{N}$,*

$$\mathbb{E}\Big[\lambda(U)^k\Big] = \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^p \sum_{\sigma \in \mathfrak{S}_k} G\big(x^j, y_{\sigma(1)}\big) \prod_{i=2}^k G\big(y_{\sigma(i-1)}, y_{\sigma(i)}\big). \tag{31}$$

Proof. This follows from the results of [**LeG86**], see [**LeG87**, (2.c)]. □

On a heuristical level, Le Gall's formula can be derived from the symbolical formula (2). Indeed,

$$\begin{aligned}
&\mathbb{E}\big[\lambda(U)^k\big] \\
&= \mathbb{E}\Big[\Big(\int_U dy \prod_{j=1}^p \int_0^{T_j} ds\, \delta_y\big(W_j(s)\big)\Big)^k\Big] \\
&= \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^p \mathbb{E}_{x^j}\Big[\sum_{\sigma\in\mathfrak{S}_k} \int_{0\le s_1\le\cdots\le s_k\le T_j}\!\!\cdots\int \prod_{i=1}^k \delta_{y_{\sigma(i)}}(W_j(s_i))\, ds_i\Big] \\
&= \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^p \sum_{\sigma\in\mathfrak{S}_k} \int_{0\le s_1\le\cdots\le s_k}\!\!\cdots\int p_{s_1}\big(x^j, y_{\sigma(1)}\big) \prod_{i=2}^k p_{s_i - s_{i-1}}\big(y_{\sigma(i-1)}, y_{\sigma(i)}\big)\, ds_i\, ds_1 \\
&= \int_U dy_1 \cdots \int_U dy_k \prod_{j=1}^p \sum_{\sigma\in\mathfrak{S}_k} G\big(x^j, y_{\sigma(1)}\big) \prod_{i=2}^k G\big(y_{\sigma(i-1)}, y_{\sigma(i)}\big),
\end{aligned}$$

where $p_s(x,y)$, as before, denotes the transition density of the stopped Brownian motion. For every $j \in \{1,\dots,p\}$ define $\Phi_k^j\colon U^k \to \mathbb{R}$ by

$$\Phi_k^j(y) = \frac{1}{k!} \sum_{\sigma\in\mathfrak{S}_k} G\big(x^j, y_{\sigma(1)}\big) \prod_{i=2}^k G\left(y_{\sigma(i-1)}, y_{\sigma(i)}\right), \qquad \text{for } y = (y_1,\dots,y_k) \in U^k. \tag{32}$$

Note that $\Phi_k^j(y) = \infty$ if $x^j, y_1, \dots, y_k$ are not pairwise distinct. Moreover, $\Phi_k^j(y)$ does not really depend on the vector $y = (y_1,\dots,y_k)$, but only on the set $\{y_1,\dots,y_k\}$, i.e., $\Phi_k^j(y) = \Phi_k^j(y_\sigma)$ for any $\sigma \in \mathfrak{S}_k$, where we put $y_\sigma = (y_{\sigma(1)},\dots,y_{\sigma(k)})$.

Hence, to prove Proposition 2.2 it suffices to show

$$\lim_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \prod_{j=1}^p \Phi_k^j(y) = -\inf_{\mu\in\mathcal{M}_1(U)} \big\{I(\mu) + p\mathcal{G}(\mu)\big\}. \tag{33}$$

3.2. Cutting. In order to derive the upper bound in (33), it is necessary to replace the Green's function by some bounded function. We achieve this by cutting off the Green's function at a large level and show that we do not change the exponential rate of $\Phi_k^j(y)$ asymptotically as the cut-off level gets large. Introduce, for $M \ge 0$, the cut-off Green's function $G_M = G \wedge M$ and denote, for $j = 1,\dots,p$,

$$\Phi_{k,M}^j(y) = \frac{1}{k!} \sum_{\sigma\in\mathfrak{S}_k} G_M\left(x^j, y_{\sigma(1)}\right) \prod_{i=2}^k G_M\left(y_{\sigma(i-1)}, y_{\sigma(i)}\right), \quad \text{for } y = (y_1,\dots,y_k) \in U^k. \tag{34}$$

First, we need some technical preparation.

LEMMA 3.2. *There exists a constant $K_1 > 1$, depending only on U, p and d, such that, for all $a_1,\dots,a_{2p} \in U$ and $q \in \{0,\dots,p\}$,*

$$\int_U \prod_{i=1}^q G(a_{2i-1}, z) G(z, a_{2i}) \prod_{i=q+1}^p G(a_{2i-1}, z)\, dz \le K_1 \prod_{i=1}^q G(a_{2i-1}, a_{2i})\,. \tag{35}$$

PROOF. First note that, as $p < d/(d-2)$, we have $\sup_{a\in U}\int_U G^p(a,z)\,dz < \infty$ and therefore $\sup_{a_1,\dots,a_p\in U}\int_U \prod_{i=1}^p G(a_i,z)\,dz < \infty$.

Now we turn to the proof of our assertion. The case $d=2$ is simple, because the supremum over $a_1,\dots,a_{2p}\in U$ of the left hand side of (35) is finite, and the infimum over $a_1,\dots,a_{2p}\in U$ of the product on the right is positive. The same argument applies in the case $d\geq 3$ when $p=1$, $q=0$ and in the case $d=3$ when $p=2$, $q=0$.

If $d\geq 3$ and $p=q=1$, we use that $U\subset U_1\cup U_2$ where

$$U_1=\left\{z\in U\,:\,|z-a_1|\geq \tfrac14|a_2-a_1|\right\} \text{ and } U_2=\left\{z\in U\,:\,|z-a_2|\geq \tfrac14|a_2-a_1|\right\}.$$

Recall from (5) that $G(x,y)\leq c_d|x-y|^{2-d}$, and pick some $c>0$ such that $G(x,y)\geq c|x-y|^{2-d}$ for any distinct $x,y\in U$. On the first domain we estimate, for some $K>1$,

$$\int_{U_1} G(a_1,z)G(z,a_2)\,dz \leq \frac{c_d}{c}4^{d-2}G(a_1,a_2)\int_U G(z,a_2)\,dz \leq KG(a_1,a_2)\,. \tag{36}$$

The second domain is treated analogously, proving the claim if $p=1$. Now let $d=3$ and $p=2$. If $q=2$ we cover U by four domains

$$U_{ij}=\left\{z\in U\,:\,|z-a_i|\geq \tfrac14|a_2-a_1| \text{ and } |z-a_j|\geq\tfrac14|a_4-a_3|\right\}, i=1,2 \text{ and } j=3,4.$$

On the first domain U_{13} we estimate

$$\begin{aligned}\int_{U_{ij}} G(a_1,z)G(z,a_2)G(a_3,z)G(z,a_4)\,dz &\leq 16\frac{c_d^2}{c^2}G(a_1,a_2)G(a_3,a_4)\int_U G(z,a_2)G(z,a_4)\,dz\\ &\leq KG(a_1,a_2)G(a_3,a_4)\,.\end{aligned} \tag{37}$$

Again the other domains can be treated analogously, which finishes the proof in the case $q=2$. The case $q=1$ is similar. ☐

LEMMA 3.3. *There is $C_0>0$ and, for all sufficiently large $M>1$ and small $\eta\in(0,1)$, there are constants $C_M>0$ and $\varepsilon_\eta>0$ such that, for all $x^1,\dots,x^p\in U$ and for any $k\in\mathbb{N}$,*

$$\int_{U^k} dy\,\prod_{j=1}^p \Phi_k^j(y) \leq 2^p pk(2C_0)^k C_M^{\eta k} + 2^p(1+\varepsilon_\eta)^k\sum_{j=1}^p\sum_{m=\lceil k(1-p\eta)\rceil}^k \int_{U^m} dy\,\left(\Phi_{m,M}^j(y)\right)^p, \tag{38}$$

where $\lim_{M\uparrow\infty} C_M=\lim_{\eta\downarrow 0}\varepsilon_\eta=0$.

PROOF. Indeed, we show (38) with C_0 given by

$$C_0=\sup_{x\in U}\int_U G^p(x,y)\,dy\vee 1\,.$$

To define C_M we choose a minimal $\delta_M>0$ such that $G(x,y)\leq M$ whenever $x,y\in U$ with $|x-y|\geq\delta_M$. Clearly, we have that $\lim_{M\uparrow\infty}\delta_M=0$. Now define

$$C_M=\sup_{x\in U}\sup_{a\in U}\int_{B(a,\delta_M)\cap U} G^p(x,y)\,dy.$$

From the local integrability of $G^p(x,\cdot)$ around x, one sees that C_0 and C_M are well-defined and $\lim_{M\uparrow\infty}C_M=0$. We assume without loss of generality that $C_M\leq 1$.

For any fixed $j\in\{1,\dots,p\}$ and any vector $y=(y_1,\dots,y_k)\in U^k$ we use the convention $y_0=x^j$. Put $\sigma(0)=0$ for $\sigma\in\mathfrak{S}_k$. Introduce

$$T^j(y)=\{n\in\{1,\dots,k\}\,:\,\text{ there is } 0\leq i<n \text{ such that } |y_i-y_n|\leq\delta_M\} \tag{39}$$

and

$$A^j(y) = \{\sigma \in \mathfrak{S}_k \,:\, \#T^j(y_\sigma) > \eta k\}.$$

In (31) we interchange the summation over σ with the integration over $y_1, \dots, y_k$ and split the sum over all $\sigma \in \mathfrak{S}_k$ into the sum over σ in $A^j(y)$ and in $\mathfrak{S}_k \setminus A^j(y)$.

On the first set of summation, we take advantage of the fact that at least ηk of the k integrals are taken just over a finite union of no more than k balls of radius δ_M, and the other $(1-\eta)k$ integrals yield some bounded exponential rate. On the second set of summation, we use that in at least $(1-\eta)k$ integrals we may replace G by its cut-off version G_M, and the exponential rate of the contribution coming from the remaining integrals is small if η is small. Let us turn to the details.

Using the estimate $\prod_{j=1}^p (a_j + b_j) \le 2^p \sum_{j=1}^p (a_j^p + b_j^p)$ for $a_j, b_j \ge 0$, we obtain that

$$\prod_{j=1}^p \Phi_k^j(y) \le 2^p \sum_{j=1}^p \big(\mathrm{I}_k^j(y) + \mathrm{II}_k^j(y)\big), \tag{40}$$

where

$$\mathrm{I}_k^j(y) \;=\; \Big(\frac{1}{k!} \sum_{\sigma \in A^j(y)} G\big(x^j, y_{\sigma(1)}\big) \prod_{i=2}^k G\big(y_{\sigma(i-1)}, y_{\sigma(i)}\big)\Big)^p, \tag{41}$$

$$\mathrm{II}_k^j(y) \;=\; \Big(\frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k \setminus A^j(y)} G\big(x^j, y_{\sigma(1)}\big) \prod_{i=2}^k G\big(y_{\sigma(i-1)}, y_{\sigma(i)}\big)\Big)^p. \tag{42}$$

We first fix j and turn to an estimate for $\int_{U^k} dy\, \mathrm{I}_k^j(y)$. Use Jensen's inequality for the uniform distribution on $\mathfrak{S}_k$ to see that

$$\mathrm{I}_k^j(y) \le \frac{1}{k!} \sum_{\sigma \in A^j(y)} \prod_{i=1}^k G^p\big(y_{\sigma(i-1)}, y_{\sigma(i)}\big).$$

Now integrate over $y \in U^k$ and note that $\sigma \in A^j(y)$ if and only if $\mathrm{id} \in A^j(y_\sigma)$, where $\mathrm{id} \in \mathfrak{S}_k$ denotes the identical permutation. Use the permutation invariance of the integrand to obtain,

$$\int_{U^k} dy\, \mathrm{I}_k^j(y) \le \int_{U^k} dy\, \mathbb{1}_{\{\mathrm{id} \in A^j(y)\}} \prod_{i=1}^k G^p(y_{i-1}, y_i).$$

Now substitute

$$\mathbb{1}_{\{\mathrm{id} \in A^j(y)\}} = \mathbb{1}_{\{\#T^j(y) > \eta k\}} = \sum_{\substack{B \subset \{1,\dots,k\} \\ \#B > \eta k}} \mathbb{1}_{\{T^j(y) = B\}},$$

and carry out successively the integration over $y_k, y_{k-1}, \dots, y_1$. Note that at least $\lfloor \eta k \rfloor$ of the corresponding integration regions for y_n (more precisely: the ones for $n \in B$), are contained in $U \cap B(y_i, \delta_M)$ for some $i < n$, the others are over the full set U.

With C_M resp. C_0 defined at the beginning of the proof, we arrive at

$$2^p \sum_{j=1}^p \int_{U^k} dy\, \mathrm{I}_k^j(y) \le 2^p p \sum_{\substack{B \subset \{1,\dots,k\} \\ \#B > \eta k}} k\, C_M^{\eta k} C_0^{(1-\eta)k} \le 2^p p k\, (2C_0)^k C_M^{\eta k},$$

recalling that $C_M \le 1$ and $C_0 \ge 1$. This yields the first term on the right hand side of (38).

We now turn to an estimate of the second term. Again fix $j \in \{1, \dots, p\}$ and recall the conventions $y_0 = x^j$ and $\sigma(0) = 0$ for all $\sigma \in \mathfrak{S}_k$. We turn the p-th power in the definition of $\mathrm{II}_k^j(y)$ into a sum over p indices $\sigma_1, \dots, \sigma_p$, i.e.,

$$\mathrm{II}_k^j(y) = \frac{1}{(k!)^p} \sum_{\sigma_1,\dots,\sigma_p \in \mathfrak{S}_k} \sum_{\substack{B_1,\dots,B_p \subset \{1,\dots,k\} \\ \forall l:\ \#B_l \le \eta k}} \prod_{l=1}^{p} \left[1_{\{T^j(y_{\sigma_l}) = B_l\}} \prod_{i_l=1}^{k} G(y_{\sigma_l(i_l-1)}, y_{\sigma_l(i_l)}) \right]. \tag{43}$$

Given $\sigma_1, \dots, \sigma_p$ and $B_1, \dots, B_p$, we define $B = \bigcup_{l=1}^{p} \sigma_l(B_l)$. Let $m = k - \#B$ and note that $m \ge k - p\eta k$. We write the integration vector in (43) as $y = (y_B, y_{B^c})$, where $B^c = \{1, \dots, k\} \setminus B$ and $y_B = (y_i)_{i \in B}$ and $y_{B^c} = (y_i)_{i \in B^c}$.

Given $\sigma_l \in \mathfrak{S}_k$ and the set $B \subset \{1, \dots, k\}$, we define a bijective mapping $\tau_l \colon \{1, \dots, m\} \to B^c$ as the total ordering that is induced by σ_l on B^c. In other words, if B^c is equal to the set $\{\sigma_l(i_1), \dots, \sigma_l(i_m)\}$ with $i_1 < \dots < i_m$, then $\tau_l(n) = \sigma_l(i_n)$ for any n. Note that, on $\bigcap_{l=1}^{p} \{T^j(y_{\sigma_l}) = B_l\}$, we have $|y_{\tau_l(i_l-1)} - y_{\tau_l(i_l)}| \ge \delta_M$ for all $l = 1, \dots, p$ and all $i_l = 1, \dots, m$.

Integrating (43), we get

$$\begin{aligned} \int_{U^k} dy\, \mathrm{II}_k^j(y) \le \frac{1}{(k!)^p} \sum_{\sigma_1,\dots,\sigma_p} \sum_{B_1,\dots,B_p} \\ \int_{U^m} dy_{B^c} \prod_{l=1}^{p} \prod_{i_l=1}^{m} 1_{\{|y_{\tau_l(i_l-1)} - y_{\tau_l(i_l)}| \ge \delta_M\}} \int_{U^{k-m}} dy_B \prod_{l=1}^{p} \prod_{i_l=1}^{k} G\big(y_{\sigma_l(i_l-1)}, y_{\sigma_l(i_l)}\big). \end{aligned} \tag{44}$$

Now we apply Lemma 3.2 to each of the $k - m$ single integrals over y_j with $j \in B$ in the second line. Here we conceive all y_j with $j \in B^c$ as parameters which play the rôle of $a_1, \dots, a_{2p}$ of Lemma 3.2. The $(k-m)$-fold application of this lemma yields

$$\int_{U^{k-m}} dy_B \prod_{l=1}^{p} \prod_{i_l=1}^{k} G\big(y_{\sigma_l(i_l-1)}, y_{\sigma_l(i_l)}\big) \le K_1^{k-m} \prod_{l=1}^{p} \prod_{i_l=1}^{m} G\big(y_{\tau_l(i_l-1)}, y_{\tau_l(i_l)}\big). \tag{45}$$

We substitute (45) in (44) and replace every G by G_M on the right hand side of (45). Furthermore, in order to streamline the integration set, we write $y_{B^c} \in U^m$ as $y \in U^m$. This necessitates that we replace every bijection $\tau_l \colon \{1, \dots, m\} \to B^c$ by a permutation $\tilde\tau_l \in \mathfrak{S}_m$ in such a way that $\tau_l(n)$ is the $\tilde\tau_l(n)$-th largest element in B^c for every $n \in \{1, \dots, m\}$.

Altogether, we obtain

$$\int_{U^k} dy\, \mathrm{II}_k^j(y) \le K_1^{k-m} \frac{1}{(k!)^p} \sum_{\sigma_1,\dots,\sigma_p} \sum_{B_1,\dots,B_p} \int_{U^m} dy \prod_{l=1}^{p} \prod_{i_l=1}^{m} G_M(y_{\tilde\tau_l(i_l-1)}, y_{\tilde\tau_l(i_l)}). \tag{46}$$

The integral on the right hand side of (46) depends on the set tuple $(B_1, \dots, B_p)$ only via the cardinality $k - m$ of B. We now replace the sum over the set tuples by the sum $m = \lceil k(1 - p\eta) \rceil, \dots, k$. We estimate the number of set tuples from above by

$$\left[\sum_{n=1}^{\lceil \eta k \rceil} \binom{k}{n} \right]^p \le k^p \binom{k}{\lceil \eta k \rceil}^p.$$

Furthermore, note that, when passing from the sum over $\sigma_1, \dots, \sigma_p \in \mathfrak{S}_k$ to the sum over $\tilde\tau_1, \dots, \tilde\tau_p \in \mathfrak{S}_m$, we obtain a counting factor of $(k!/m!)^p$. Then we rewrite

$$\frac{1}{m!^p} \sum_{\tilde\tau_1,\dots,\tilde\tau_p \in \mathfrak{S}_m} \prod_{l=1}^{p} \prod_{i_l=1}^{m} G_M(y_{\tilde\tau_l(i_l-1)}, y_{\tilde\tau_l(i_l)}) = \big(\Phi^j_{m,M}(y)\big)^p.$$

This leads to the estimate

$$\int_{U^k} dy\, \mathrm{II}^j_k(y) \le K_1^{p\eta k} k^p \binom{k}{\lceil \eta k \rceil}^p \sum_{m=\lceil k(1-p\eta)\rceil}^{k} \int_{U^m} dy\, \big(\Phi^j_{M,m}(y)\big)^p.$$

Now use Stirling's formula to see that the factors on the right hand side in front of the integral are bounded from above by $(1+\varepsilon_\eta)^k$ with some $\varepsilon_\eta \downarrow 0$ as $\eta \downarrow 0$. This yields the second term on the right hand side of (38), which ends the proof. $\square$

3.3. Discretization. Our second main technical tool is a reduction to a discrete counting argument. For this purpose, we introduce a finite partition of U, which is carefully chosen in order to represent many details of the continuous picture.

To introduce appropriate notation, let $\Sigma_n = \{1, \dots, n\}$ and denote the partition sets by $U_1, \dots, U_n$. We assume that every U_l is measurable and has positive Lebesgue measure $\ell(U_l)$. We call π the canonical projection $\pi\colon U \to \Sigma_n$, that is, $x \in U_{\pi x}$ for any $x \in U$. We write $\pi y = (\pi y_1, \dots, \pi y_k)$ for any $k \in \mathbb{N}$ and $y = (y_1, \dots, y_k) \in U^k$. Furthermore, if μ is a probability measure on U, then $\pi\mu \in \mathcal{M}_1(\Sigma_n)$ is its projection on Σ_n. Similarly for $\nu \in \mathcal{M}_1(U^2)$ we denote the projection on Σ_n^2 by $\pi\nu \in \mathcal{M}_1(\Sigma_n^2)$. If v is in the set $\mathcal{M}_1^*(\Sigma_n)$ of probability measures on Σ_n^2 with equal marginals, we denote by $\overline{v} \in \mathcal{M}_1(\Sigma_n)$ its left or right marginal measure. Note that $\pi\overline{\nu} = \overline{\pi\nu}$ for any $\nu \in \mathcal{M}_1^*(U)$, where we write $\overline{\nu}$ for the marginal measure of ν.

For measures $u \in \mathcal{M}_1(\Sigma_n)$ and $v \in \mathcal{M}_1(\Sigma_n^2)$ we define discrete analogues of the relative entropy functionals I and I^2_μ by

$$(47) \qquad \widetilde{I}(u) = \sum_{l \in \Sigma_n} u_l \log\Big(\frac{u_l}{\ell(U_l)}\Big) \qquad \text{and} \qquad \widetilde{I}^2_u(v) = \sum_{l,m \in \Sigma_n} v_{l,m} \log\Big(\frac{v_{l,m}}{\overline{v}_l u_m}\Big),$$

using the usual convention $0 \log 0 = 0$. Recall that $G_M = G \wedge M$ and define the approximate Green functions $G_M^+, G^-\colon \Sigma_n^2 \to \mathbb{R}$ by

$$(48) \qquad G_M^+(l,m) = \sup_{\substack{x \in U_l\\ y \in U_m}} G_M(x,y) \quad \text{and} \quad G^-(l,m) = \inf_{\substack{x \in U_l\\ y \in U_m}} G(x,y).$$

Functions $\mathcal{G}_M^+$ and $\mathcal{G}^-$ on $\mathcal{M}_1(\Sigma_n)$ analogous to $\mathcal{G}$ in (27) are defined by

$$(49) \quad \mathcal{G}_M^+(u) = \inf_{\substack{v \in \mathcal{M}_1^*(\Sigma_n)\\ \overline{v}=u}} \Big\{\widetilde{I}^2_u(v) - \big\langle v, \log G_M^+\big\rangle\Big\} \text{ and } \mathcal{G}^-(u) = \inf_{\substack{v \in \mathcal{M}_1^*(\Sigma_n)\\ \overline{v}=u}} \Big\{\widetilde{I}^2_u(v) - \big\langle v, \log G^-\big\rangle\Big\},$$

where we used the notation $\langle v, F\rangle = \sum_{l,m \in \Sigma_n} v_{l,m} F(l,m)$. The functions $\mathcal{G}_M^+$ and $\mathcal{G}^-$ are continuous. Indeed, for fixed u, if the set $\widetilde{V} \subset \mathcal{M}_1^*(\Sigma_n)$ is a neighbourhood of the set $\{v \in \mathcal{M}_1^*(\Sigma_n) : \overline{v} = u\}$, there exists a neighbourhood $\widetilde{U}$ of u with $\{v \in \mathcal{M}_1^*(\Sigma_n) : \overline{v} = \tilde u\} \subset \widetilde{V}$ for all $\tilde u \in \widetilde{U}$. Together with the obvious continuity of $\widetilde{I}^2_u(v)$ in both arguments u and v and of $v \mapsto \langle v, F\rangle$ this implies continuity of $\mathcal{G}_M^+$ and $\mathcal{G}^-$.

Our aim is to determine a partition such that the coarsened variational formula is a good approximation of the variational formula on the right hand side of (33).

LEMMA 3.4. *Given $\delta > 0$ and $M > 0$, there is a measurable partition $U_1, \dots, U_n$ of U, such that,*

$$\begin{aligned}(50)\qquad \inf_{\mu\in\mathcal{M}_1(U)} \big\{\widetilde{I}(\pi\mu) + p\mathcal{G}^-(\pi\mu)\big\} - \delta &\le \inf_{\mu\in\mathcal{M}_1(U)} \big\{I(\mu) + p\mathcal{G}(\mu)\big\} \\ &\le \inf_{\mu\in\mathcal{M}_1(U)} \big\{\widetilde{I}(\pi\mu) + p\mathcal{G}_M^+(\pi\mu)\big\} + \delta\,.\end{aligned}$$

PROOF. We start with the upper estimate. Choose the partition such that

$$\big|\log G_M^+(\pi x, \pi y) - \log G_M(x,y)\big| < \delta/p, \text{ for all } x, y \in U\,.$$

This is possible since $\log G_M$ is continuous on the closure of U^2. To every $\mu \in \mathcal{M}_1(U)$ we associate $\widetilde{\mu} \in \mathcal{M}_1(U)$ with constant density on the partition sets and $\pi\mu = \pi\widetilde{\mu}$. Analogously we associate to each $\nu \in \mathcal{M}_1^*(U)$ a $\widetilde{\nu} \in \mathcal{M}_1^*(U)$ with constant density on the products of the partition sets and $\pi\nu = \pi\widetilde{\nu}$. Then $I(\widetilde{\mu}) = \widetilde{I}(\pi\mu)$ and $\widetilde{I}^2_{\pi\mu}(\pi\nu) = I^2_{\widetilde{\mu}}(\widetilde{\nu})$, and we infer that

$$\begin{aligned}\mathcal{G}_M^+(\pi\mu) &= \inf_{\substack{\nu\in\mathcal{M}_1^*(U)\\ \pi\overline{\nu}=\pi\mu}} \Big\{\widetilde{I}^2_{\pi\mu}(\pi\nu) - \big\langle\pi\nu, \log G_M^+\big\rangle\Big\} \ge \inf_{\substack{\nu\in\mathcal{M}_1^*(U)\\ \overline{\widetilde{\nu}}=\widetilde{\mu}}} \Big\{I^2_{\widetilde{\mu}}(\widetilde{\nu}) - \langle\widetilde{\nu}, \log G_M\rangle\Big\} - \delta/p \\ &\ge \mathcal{G}(\widetilde{\mu}) - \delta/p\,,\end{aligned}$$

and hence

$$\inf_{\mu\in\mathcal{M}_1(U)} \big\{I(\mu) + p\mathcal{G}(\mu)\big\} \le \inf_{\mu\in\mathcal{M}_1(U)} \big\{I(\widetilde{\mu}) + p\mathcal{G}(\widetilde{\mu})\big\} \le \inf_{\mu\in\mathcal{M}_1(U)} \big\{\widetilde{I}(\pi\mu) + p\mathcal{G}_M^+(\pi\mu)\big\} + \delta\,,$$

which shows the upper bound in (50).

Let us turn to the lower estimate. Given $\delta > 0$, we select an approximate minimizer $\nu \in \mathcal{M}_1^*(U)$ with marginal $\mu = \overline{\nu}$ such that

$$(51)\qquad \inf_{\mu\in\mathcal{M}_1(U)} \big\{I(\mu) + p\mathcal{G}(\mu)\big\} \ge I(\mu) + p\big(I^2_{\overline{\nu}}(\nu) - \langle\nu, \log G\rangle\big) - \delta/2\,.$$

We use now Jensen's inequality to show that, for any partition, we have

$$(52)\qquad I(\mu) \ge \widetilde{I}(\pi\mu) \qquad \text{and} \qquad I^2_{\overline{\nu}}(\nu) \ge \widetilde{I}^2_{\pi\overline{\nu}}(\pi\nu).$$

To prove this, abbreviate $\phi(x) = x\log x$ and note that

$$I(\mu) = \Big\langle \ell, \phi\circ\frac{d\mu}{dx}\Big\rangle = \sum_{l\in\Sigma_n} \ell(U_l) \int_{U_l} d\overline{\ell}_l \;\phi\circ\frac{d\mu}{dx},$$

where $\overline{\ell}_l$ denotes the normalized restriction of the Lebesgue measure to U_l. Now use Jensen's inequality for the convex function ϕ and summarize to arrive at the first inequality in (52). The proof of the second one goes along the same lines, recalling that $I^2_{\overline{\nu}}(\nu) = \langle\overline{\nu}\otimes\overline{\nu}, \phi\circ(d\nu/d(\overline{\nu}\otimes\overline{\nu}))\rangle$.

Furthermore, for the fixed ν, every partition can be refined such that $\langle\nu, \log G\rangle \le \langle\pi\nu, \log G^-\rangle + \delta/(2p)$. This can be seen by choosing N so large that $\langle\nu, \log G\rangle - \langle\nu, \log G_N\rangle < \delta/(4p)$ and using uniform continuity of $\log G_N$ on the closure of U^2 to split the domain of integration into partition sets on which the variation of $\log G_N$ is less that $\delta/(4p)$. Substituting this and (52) into (51), we arrive at the lower bound in (50). ☐

3.4. The combinatorial argument. Having removed technical obstacles, like unboundedness of the integrand, and discretized the integrals, we are ready to turn to the combinatorial core argument in the proof of Proposition 2.2. The main observation is that the k-fold product in (32) only depends on the numbers of transitions the sequence $y_{\sigma(0)}, y_{\sigma(1)}, \dots, y_{\sigma(k)}$ makes from one set of the partition to the other. In other words, one can summarize the permutations that lead to the same empirical pair measure of the sequence $\pi y_{\sigma(0)}, \pi y_{\sigma(1)}, \dots, \pi y_{\sigma(k)}$ and sum over all the empirical pair measures instead. Since the combinatorics turn out to be manageable, this observation leads to a great simplification and enables us to use large deviation theory for empirical pair measures. Let us turn to the details.

We fix a large $M > 0$ and small $\delta > 0$ and a partition $U_1, \dots, U_n$ chosen according to Lemma 3.4. Recall that $\mathcal{M}_1^*(\Sigma_n)$ denotes the set of probability measures on Σ_n^2 having equal marginals. In the next lemma, we give lower and upper bounds for $\Phi_k^j(y)$ resp. $\Phi_{k,M}^j(y)$ in terms of a variational problem on $\mathcal{M}_1^*(\Sigma_n)$.

Recall our notation $\pi y = (\pi y_1, \dots, \pi y_k)$ for $y \in U^k$, and denote by $L^1_{\pi y,k} \in \mathcal{M}_1(\Sigma_n)$ the *empirical measure*,

$$L^1_{\pi y,k}(A) = \frac{1}{k}\sum_{i=1}^k \delta_{y_i}(A), \text{ for all } A \subset \Sigma_n .$$

LEMMA 3.5. *For all $j \in \{1, \dots, p\}$, as $k \uparrow \infty$, uniformly in $y = (y_1, \dots, y_k) \in U^k$,*

$$\Phi_{k,M}^j(y) \le e^{o(k)} \exp\left(-k\mathcal{G}_M^+\left(L^1_{\pi y,k}\right)\right), \tag{53}$$

and

$$\Phi_k^j(y) \ge e^{o(k)} \exp\left(-k\mathcal{G}^-\left(L^1_{\pi y,k}\right)\right). \tag{54}$$

PROOF. Fix j and recall the convention $y_0 = x^j$ and $\sigma(0) = 0$ for all $\sigma \in \mathfrak{S}_k$. Assume without loss of generality that $y_0 \in U_1$. Note that, for any $y = (y_1, \dots, y_k) \in U^k$, we have $G(y_{\sigma(i-1)}, y_{\sigma(i)}) \ge G^-(\pi y_{\sigma(i-1)}, \pi y_{\sigma(i)})$ and $G_M(y_{\sigma(i-1)}, y_{\sigma(i)}) \le G_M^+(\pi y_{\sigma(i-1)}, \pi y_{\sigma(i)})$. We introduce the *empirical pair measure*

$$L^2_{\psi,k} = \frac{1}{k}\sum_{i=1}^k \delta_{\{(\psi_{i-1},\psi_i)\}} \in \mathcal{M}_1(\Sigma_n^2), \qquad \text{for } \psi = (\psi_1, \dots, \psi_k) \in \Sigma_n^k,$$

where $\psi_0 = 1$ is the index of the set U_1. This implies that, with $\pi y_\sigma = (\pi y_{\sigma(1)}, \dots, \pi y_{\sigma(k)})$,

$$\Phi_k^j(y) \ge \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \exp\left(k\langle L^2_{\pi y_\sigma,k}, \log G^-\rangle\right) \text{ and } \Phi_{k,M}^j(y) \le \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \exp\left(k\langle L^2_{\pi y_\sigma,k}, \log G_M^+\rangle\right). \tag{55}$$

Denote by $\mathcal{M}_1^{(k)}(\Sigma_n)$ the set of probability measures u on Σ_n such that the numbers $ku(1), \dots, ku(n)$ are integers and sum up to k; analogously we define $\mathcal{M}_1^{(k)}(\Sigma_n^2)$. Abbreviate $u = L^1_{\pi y,k} \in \mathcal{M}_1^{(k)}(\Sigma_n)$ for the course of the proof. Note that, for any $\sigma \in \mathfrak{S}_k$, the total variation distance between u and any of the two marginal measures of $L^2_{\pi y_\sigma,k}$ is not bigger than $2/k$, which is denoted below as $\overline{L}^2_{\pi y_\sigma,k} \approx u$.

Now we reorganize the sum over $\sigma \in \mathfrak{S}_k$ by summing over all vectors $\psi \in \Sigma_n^k$ and simply counting the permutations σ such that $\psi = \pi y_\sigma$. Hence, for $F \in \{G^-, G_M^+\}$, we have

$$\begin{aligned}(56)\qquad &\sum_{\sigma\in\mathfrak{S}_k} \exp\big(k\langle L^2_{\pi y_\sigma,k}, \log F\rangle\big)\\ &= \sum_{\psi\in\Sigma_n^k}\sum_{\sigma\in\mathfrak{S}_k} 1_{\{\pi y_\sigma=\psi\}} 1_{\{\overline{L}^2_{\psi,k}\approx u\}} \exp\big(k\langle L^2_{\psi,k}, \log F\rangle\big)\\ &= \sum_{\substack{v\in\mathcal{M}_1^{(k)}(\Sigma_n^2)\\ \overline{v}\approx u}} \exp\big(k\langle v, \log F\rangle\big) \sum_{\psi\in\Sigma_n^k} 1_{\{L^2_{\psi,k}=v\}} \#\{\sigma\in\mathfrak{S}_k : \pi y_\sigma=\psi\}.\end{aligned}$$

Note that, for any $\psi \in \Sigma_n^k$ satisfying $L^2_{\psi,k} = v$ for some v satisfying $\overline{v} \approx u$, we have

$$(57)\qquad \#\{\sigma\in\mathfrak{S}_k : \pi y_\sigma = \psi\} = \prod_{l=1}^n (ku_l)!.$$

In other words, if the empirical measures of πy and ψ coincide, then there are $\prod_{l=1}^n (ku_l)!$ reorderings of $(\pi y_1, \dots, \pi y_k)$ that are equal to $(\psi_1, \dots, \psi_k)$.

Furthermore, well-known combinatorial considerations (see e.g. [**dHo00**, II.2]) yield that

$$(58)\qquad \#\{\psi\in\Sigma_n^k : L^2_{\psi,k} = v\} = e^{o(k)} \frac{\prod_{l\in\Sigma_n}(ku_l)!}{\prod_{l,m\in\Sigma_n}(kv_{l,m})!},$$

where $e^{o(k)}$ is uniform in $v \in \mathcal{M}_1^{(k)}(\Sigma_n^2)$. Substituting (57) and (58) in (56), we obtain

$$(59)\qquad \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \exp\big(k\langle L^2_{\pi y_\sigma,k}, \log F\rangle\big) = e^{o(k)} \sum_{\substack{v\in\mathcal{M}_1^{(k)}(\Sigma_n^2)\\ \overline{v}\approx u}} \exp\big(k\langle v, \log F\rangle\big) \frac{\prod_{l\in\Sigma_n}(ku_l)!^2}{k!\prod_{l,m\in\Sigma_n}(kv_{l,m})!}.$$

Stirling's formula yields that, uniformly in u and v, the last ratio is $e^{o(k)}\exp(-k\widetilde{I}_u^2(v))$. Hence, uniformly in $y \in U^k$,

$$(60)\qquad \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} \exp\big(k\langle L^2_{\pi y_\sigma,k}, \log F\rangle\big) = e^{o(k)} \sum_{\substack{v\in\mathcal{M}_1^{(k)}(\Sigma_n^2)\\ \overline{v}\approx u}} \exp\Big(-k\big(\widetilde{I}_u^2(v) - \langle v, \log F\rangle\big)\Big).$$

Now recall that $\widetilde{I}_u^2(\cdot) - \langle\,\cdot\,, \log F\rangle$ is continuous, that the total variation distance between $\{v \in \mathcal{M}_1(\Sigma_n^2) \colon \overline{v} \approx u\}$ and $\mathcal{M}_1^*(\Sigma_n)$ is not bigger than $2/k$ and that $\#\mathcal{M}_1^{(k)}(\Sigma_n^2)$ is bounded above by a polynomial in k. From (55) we obtain our two assertions, for $F = G_M^+$ resp. $F = G^-$. $\square$

In the next lemma, we integrate the upper resp. lower bound stated in Lemma 3.5 over $y \in U^k$ and obtain an upper resp. lower bound for the logarithmic asymptotics in (33) in terms of a variational formula, which is the coarsened form of the right hand side of (33).

LEMMA 3.6. *For each $j \in \{1, \dots, p\}$ and each $M > 0$,*

$$(61)\qquad \limsup_{k\uparrow\infty} \frac{1}{k}\log\int_{U^k} dy\, \Phi^j_{k,M}(y)^p \le -\inf_{\mu\in\mathcal{M}_1(U)} \big\{\widetilde{I}(\pi\mu) + p\mathcal{G}_M^+(\pi\mu)\big\}.$$

Moreover,

$$(62)\qquad \liminf_{k\uparrow\infty} \frac{1}{k}\log\int_{U^k} dy \prod_{j=1}^p \Phi^j_k(y) \ge -\inf_{\mu\in\mathcal{M}_1(U)} \big\{\widetilde{I}(\pi\mu) + p\mathcal{G}^-(\pi\mu)\big\}.$$

PROOF. In order to derive (61), we raise the upper bound (53) to the p-th power and integrate over $y \in U^k$. Note that the right hand side of (53) does not depend on y, but only on $\psi = \pi y = (\pi y_1, \dots, \pi y_k)$. Hence, it makes sense to subdivide the domain of integration according to the tuples that arise. In this way we get the bound

$$\begin{aligned}\int_{U^k} dy\, \Phi^j_{k,M}(y)^p &\le e^{o(k)} \sum_{\psi \in \Sigma_n^k} \int_{U_{\psi_1}} \cdots \int_{U_{\psi_k}} dy_k \dots dy_1 \exp\big(-kp\mathcal{G}^+_M(L^1_{\psi,k})\big) \\ &= e^{o(k)} \sum_{\psi \in \Sigma_n^k} \exp\big(-kp\mathcal{G}^+_M(L^1_{\psi,k})\big) \prod_{l \in \Sigma_n} \ell(U_l)^{kL^1_{\psi,k}(l)} \\ &\le e^{o(k)} \sum_{u \in \mathcal{M}_1^{(k)}(\Sigma_n)} \#\{\psi \in \Sigma_n^k : L^1_{\psi,k} = u\} e^{-kp\mathcal{G}^+_M(u)} \exp\Big(k \sum_{l \in \Sigma_n} u_l \log \ell(U_l)\Big).\end{aligned}$$

Now use Stirling's formula to see that the counting factor in the last line is $e^{o(k)} \exp(-k \sum_{l \in \Sigma_n} u_l \log u_l)$, uniformly in $u \in \mathcal{M}_1^{(k)}(\Sigma_n)$. We recall (47) and infer that

$$\int_{U^k} dy\, \Phi^j_{k,M}(y)^p \le e^{o(k)} \sum_{u \in \mathcal{M}_1^{(k)}(\Sigma_n)} \exp\Big(-k\big(p\mathcal{G}^+_M(u) - \widetilde{I}(u)\big)\Big).$$

From the two facts that $\widetilde{I} - p\mathcal{G}^+_M$ is continuous, and that $\#\mathcal{M}_1^{(k)}(\Sigma_n)$ is bounded above by a polynomial in k, we infer that (61) holds.

The proof of the lower bound (62) is analogous, using (54). $\square$

3.5. Conclusion. In this section we finish the proof of Proposition 2.2. Recall that only (33) is left to be shown.

The lower bound is immediate from (62) and (50). For the upper bound take logarithms in (38), divide by k and let $k \uparrow \infty$. According to Lemma 3.3, after letting $M \uparrow \infty$, one obtains

$$\begin{aligned}&\limsup_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \prod_{j=1}^p \Phi^j_k(y) \\ &\quad\le \log(1+\varepsilon_\eta) + \lim_{M\uparrow\infty} \max_{j=1}^p \limsup_{k\uparrow\infty} \frac{1}{k} \log \sum_{m=\lceil k(1-p\eta)\rceil}^k \int_{U^m} dy\, \Phi^j_{m,M}(y)^p.\end{aligned} \tag{63}$$

Estimate the last integral on the right hand side from above against the maximum on $m \in \{\lceil k(1-p\eta)\rceil, \dots, k\}$. Now we pick $\delta > 0$, choose a partition according to Lemma 3.4, and use Lemma 3.6 to infer

$$\limsup_{m\uparrow\infty} \frac{1}{m} \log \int_{U^m} dy\, \Phi^j_{m,M}(y)^p \le -\inf_{\mu \in \mathcal{M}_1(U)} \Big\{\widetilde{I}(\pi\mu) + p\mathcal{G}^+_M(\pi\mu)\Big\}.$$

Now use (50) and let first $M \uparrow \infty$, then $\eta \downarrow 0$, to obtain

$$\begin{aligned}&\limsup_{k\uparrow\infty} \frac{1}{k} \log \int_{U^k} dy \prod_{j=1}^p \Phi^j_k(y) \\ &\le \lim_{\eta\downarrow 0} \log(1+\varepsilon_\eta) - \lim_{\eta\downarrow 0}(1-p\eta) \inf_{\mu \in \mathcal{M}_1(U)} \big\{I(\mu) + p\mathcal{G}(\mu)\big\} + \lim_{\eta\downarrow 0} \delta(1-p\eta) \\ &= \inf_{\mu \in \mathcal{M}_1(U)} \big\{I(\mu) + p\mathcal{G}(\mu)\big\} + \delta.\end{aligned}$$

Now let $\delta \downarrow 0$. This finishes the proof of the upper bound in (33).

4. Analysis of the variational formulae

In this section we prove our results related to the characterization of the limit in (8). These are Proposition 2.1, Theorem 1.3, and finally Theorem 1.4, which we discuss in Subsections 4.1–4.3, respectively.

4.1. Proof of Proposition 2.1.

It turns out that the variational formula (9) is much easier to analyse than (28). Indeed, in contrast to (28), for (9) compactness and continuity arguments are available, such that the existence of maximizers, their positivity and the Euler-Lagrange equations can be derived in a fairly standard way. Furthermore, (9) majorizes (28) in a certain strong sense, such that the maximizers of (9) turn out to stand in an elementary one-to-one correspondence to the minimizers of (28).

Before going into the details of the proof, we discuss general continuity properties of the operator $\mathfrak{A}$, which are of use in the forthcoming proofs.

LEMMA 4.1.

(i) *If $d = 2$ and $q > 1$, then $\mathfrak{A}$ is a bounded linear map from $L^q(U)$ into $L^\infty(U)$.*

(ii) *If $d \geq 3$ and $1 < q < d/2$, then $\mathfrak{A}$ is a bounded linear map from $L^q(U)$ into $L^{\frac{dq}{d-2q}}(U)$.*

PROOF. The first statement follows easily from Hölder's inequality using that in $d = 2$ we have $\sup_x \int G(x,y)^q\, dy < \infty$ for all $q > 1$.

If $d \geq 3$ we recall, e.g. from [**LL97**, 4.3], the Hardy-Littlewood-Sobolev inequality. For all $s, r > 1$, $0 < \lambda < d$ with $1/r + \lambda/d + 1/s = 2$ there is a constant $C > 0$ with

$$\Big| \int_U\int_U f(x)\, |x-y|^{-\lambda} h(y)\, dx\, dy\Big| \leq C\, \|f\|_s \|h\|_r\,. \tag{64}$$

Recall that $G(x,y) \leq c_d|x-y|^{2-d}$ and use the Hardy-Littlewood-Sobolev inequality with $\lambda = d-2$ and $s = q$, $r = dq/(dq + 2q - d)$, which yields $\langle h, \mathfrak{A} f\rangle \leq C\|f\|_q\, \|h\|_r$ for any $f \in L^q(U)$ and $h \in L^r(U)$. Hence $\mathfrak{A}$ maps f continuously into the dual of $L^r(U)$, which is $L^{\frac{dq}{d-2q}}(U)$. This proves (ii). ☐

For our purposes, it is convenient to rewrite (9) as

$$\varrho^* = \sup\big\{\langle f, \mathfrak{A} f\rangle \;:\; f \in L^{2p/(2p-1)}(U) \text{ and } \|f\|_{2p/(2p-1)} \leq 1\big\}. \tag{65}$$

It is clear that the supremum in (65) may be restricted to positive normalized functions $f \in L^{2p/(2p-1)}(U)$. We start by showing that the operator $\mathfrak{A}\colon L^{2p/(2p-1)}(U) \to L^{2p}(U)$ is continuous, and establish (29).

LEMMA 4.2. *Suppose p is a positive integer with $p < d/(d-2)$.*

(i) *$\mathfrak{A}$ is a bounded linear map from $L^{2p/(2p-1)}(U)$ into $L^{2p}(U)$. In particular, $\varrho^* \leq \|\mathfrak{A}\|$.*

(ii) *For all $\mu \in \mathcal{M}_1(U)$ with $g^{2p}(x)\, dx = \mu(dx)$ we have*

$$\exp\Big(-\frac{1}{p}\big(I(\mu) + p\mathcal{G}(\mu)\big)\Big) \leq \langle g^{2p-1}, \mathfrak{A} g^{2p-1}\rangle\,. \tag{66}$$

(iii) *Equality in (66) holds if and only if there is $\varrho > 0$ with the property*

$$\mathfrak{A} g^{2p-1}(x) = \varrho g(x) \quad \text{for } \mu\text{-almost every } x \in U. \tag{67}$$

Moreover, in this case $I(\mu) + p\mathcal{G}(\mu) = -p\log\varrho$ and $\mathcal{G}(\mu) = -\log\varrho - 2\langle\mu, \log g\rangle$.

PROOF. Assertion (i) is a direct consequence of Lemma 4.1 and the fact that the spaces $L^q(U)$ are continuously embedded in each other for decreasing q. Indeed, if $d = 2$, one can choose $q = 2p/(2p-1)$ and if $d \geq 3$ and $p < d/(d-2)$ it is always possible to choose the parameter q in such a way that the three conditions

$$\frac{dq}{d-2q} \geq 2p, \qquad q \leq \frac{2p}{2p-1}, \qquad 1 < q < \frac{d}{2},$$

are satisfied, from which (i) readily follows.

We now show (ii). Assume without loss of generality that $I(\mu) + p\mathcal{G}(\mu) < \infty$. Let $\delta > 0$ and choose $\nu \in \mathcal{M}_1^*(U)$ such that the marginals are both equal to μ and $\mathcal{G}(\mu) > I_\mu^2(\nu) - \langle \log G\,, \nu\rangle - \delta$. As a first step, we infer from this that $\langle \log G, \nu\rangle$ is finite. Indeed, abbreviate $f(x,y) = \nu(dxdy)/dx\,dy$ and choose $\varepsilon > 0$ so small that

$$C = \log \sup_{x\in U} \int_U G^{p+\varepsilon}(x,y)\,dy < \infty.$$

Then, estimating $pI_\mu^2(\nu) \geq I_\mu^2(\nu)$, we obtain

(68)

$$\infty > I(\mu) + I_\mu^2(\nu) + \varepsilon\langle \log G, \nu\rangle - \langle \log G^{p+\varepsilon}, \nu\rangle = \varepsilon\langle \log G, \nu\rangle - \Big\langle f, \log \frac{(g^{2p}\otimes g^{2p})^{1/2} G^{p+\varepsilon}}{f}\Big\rangle.$$

For the second term we use Jensen's inequality to estimate

$$\begin{aligned}(69)\qquad \Big\langle f, \log \frac{(g^{2p}\otimes g^{2p})^{1/2} G^{p+\varepsilon}}{f}\Big\rangle &= \int_U dx\, g^{2p}(x) \int_U dy\, \frac{f(x,y)}{g^{2p}(x)} \log \frac{g^{2p}(x) G^{p+\varepsilon}(x,y)}{f(x,y)} \\ &\leq \int_U dx\, g^{2p}(x) \log\Big(\int_U G^{p+\varepsilon}(x,y)\,dy\Big) \leq C.\end{aligned}$$

From this we infer that $\langle \log G, \nu\rangle$ is finite.

Now we use the lower bound in a variational principle for I_μ^2, which we recall from [**DeZ93**, 6.5.10]. For all measurable $u : U^2 \to (0,\infty)$ that are bounded from 0 and infinity,

$$I_\mu^2(\nu) \geq \int_{U^2} \nu(dx\,dy) \log\Big(\frac{u(x,y)}{\int u(y,z)\,d\mu(z)}\Big). \qquad (70)$$

For arbitrarily large $M > 0$ and small $\varepsilon > 0$ the function

$$u(x,y) = \frac{G(x,y)}{(g\wedge M)(y) \vee \varepsilon G(x,y)},$$

is admissible in (70), which yields

(71)

$$I_\mu^2(\nu) \geq \int \nu(dxdy) \log\Big(\frac{G(x,y)}{(g\wedge M)(y)\vee \varepsilon G(x,y)}\Big) - \int \mu(dy) \log\Big(\int \frac{G(y,z)\,\mu(dz)}{(g\wedge M)(z)\vee\varepsilon G(y,z)}\Big).$$

Now recall that $\langle \log G, \nu\rangle < \infty$ and $\int \nu(dxdy) \log g(y) = \langle \mu, \log g\rangle < \infty$. Hence, using dominated convergence, we get

$$\int_{U^2} \nu(dxdy) \log\Big((g\wedge M)(y) \vee \varepsilon G(x,y)\Big) \leq \int_{U^2} \nu(dxdy) \log\big(g(y)\vee \varepsilon G(x,y)\big) \xrightarrow{\varepsilon\downarrow 0} \langle \mu, \log g\rangle.$$

Using bounded convergence we also get

$$\begin{aligned}\int \mu(dy) \log\Big(\int \frac{G(y,z)\,\mu(dz)}{(g\wedge M)(z)\vee \varepsilon G(y,z)}\Big) &\overset{M\uparrow\infty}{\longrightarrow} \int \mu(dy)\log\Big(\int \frac{G(y,z)\,\mu(dz)}{g(z)\vee\varepsilon G(y,z)}\Big)\\ \le \int \mu(dy)\log\Big(\int \frac{G(y,z)}{g(z)}\mu(dz)\Big) &= \langle \mu, \log(\mathfrak{A} g^{2p-1})\rangle.\end{aligned} \tag{72}$$

Hence, taking limits in (71), we get

$$I^2_\mu(\nu) \ge \langle \nu, \log G\rangle - \big\langle \mu, \log(g\cdot \mathfrak{A} g^{2p-1})\big\rangle.$$

Recalling the choice of ν and letting $\delta\downarrow 0$, we obtain

$$\mathcal{G}(\mu) \ge -\big\langle \mu, \log(g\cdot\mathfrak{A} g^{2p-1})\big\rangle. \tag{73}$$

We apply this and Jensen's inequality, to obtain

$$\begin{aligned}\exp\Big(-\frac{1}{p}\Big(I(\mu)+p\mathcal{G}(\mu)\Big)\Big) &\le \exp\Big(-\frac{1}{p}\langle \mu, \log g^{2p}\rangle + \big\langle \mu, \log(g\cdot \mathfrak{A} g^{2p-1})\big\rangle\Big)\\ &= \exp\Big(\Big\langle \mu, \log\frac{\mathfrak{A} g^{2p-1}}{g}\Big\rangle\Big) \le \Big\langle \mu, \frac{\mathfrak{A} g^{2p-1}}{g}\Big\rangle\\ &= \langle g^{2p-1}, \mathfrak{A} g^{2p-1}\rangle.\end{aligned} \tag{74}$$

This proves (ii).

Finally, to prove (iii), assume that we have equality everywhere in (74). By strict convexity of the logarithm, equality in the second line implies that, for some constant $\varrho > 0$, we have $\mathfrak{A} g^{2p-1} = \varrho g$, for μ-almost every $x\in U$. Together with equality in the first line of (74), which is equality in (73), this yields that

$$\mathcal{G}(\mu) = -\big\langle \mu, \log(g\cdot\mathfrak{A} g^{2p-1})\big\rangle = -\log\varrho - 2\langle \mu, \log g\rangle.$$

Conversely, if (67) holds, we have equality in the second line of (74). To check equality in the first line, we define a probability measure $\nu\in\mathcal{M}^*(U)$ by

$$\nu(dxdy) = h(x,y)\,dx\,dy = \frac{1}{\varrho} g^{2p-1}(x) g^{2p-1}(y) G(x,y)\,dx\,dy.$$

The measure ν is well-defined by (67) and plugging this into (27) yields

$$\mathcal{G}(\mu) \le -\langle \mu, \log(\varrho g^2)\rangle.$$

This means that equality holds in (73) and hence also in the first line of (74), completing the proof of (iii). $\square$

We now prove the existence of maximizers in (65).

LEMMA 4.3. *Every maximizing sequence for the variational problem in* (65) *has a subsequence that converges weakly in* $L^1(U)$ *towards some maximizer of this problem.*

PROOF. Write short $\|\cdot\|$ for $\|\cdot\|_{2p/(2p-1)}$ and put $K = \{f\in L^1(U) : f\ge 0, \|f\|\le 1\}$. It is easy to see that it is sufficient to establish the following two statements:

$$K \text{ is weakly compact in } L^1(U), \tag{75}$$

$$\begin{gathered}\text{the mapping } f\mapsto \langle f, \mathfrak{A} f\rangle \text{ is upper semicontinuous on } K\\ \text{in the weak topology on } L^1(U).\end{gathered} \tag{76}$$

To establish (75) we first note that K is weakly relatively compact in $L^1(U)$, because the family K is uniformly integrable, see [**DeZ93**, C7]. It remains to show that K is weakly closed. For this purpose it suffices to show weak closedness in $\{f \in L^1(U) \,:\, f \geq 0\}$.

Let $f \geq 0$ satisfy $f \notin K$. Then $\|f\| > 1$. Let

$$\varphi_n = \Big(\frac{f \wedge n}{\|f \wedge n\|}\Big)^{\frac{1}{2p-1}} \in L^\infty(U).$$

Then we have,

$$\liminf_{n\uparrow\infty} \int \varphi_n f \, dx \geq \liminf_{n\uparrow\infty} \|f \wedge n\|^{\frac{1}{1-2p}} \int (f \wedge n)^{\frac{2p}{2p-1}} \, dx = \liminf_{n\uparrow\infty} \|f \wedge n\| = \|f\| > 1\,,$$

and, for all $g \in K$, using Hölder's inequality,

$$\int \varphi_n g \, dx \leq \|\varphi_n\|_{2p} \|g\| \leq \|f \wedge n\|^{\frac{1}{1-2p}} \Big(\int (f \wedge n)^{\frac{2p}{2p-1}} \, dx\Big)^{\frac{1}{2p}} = 1.$$

Hence, for n sufficiently large, the function $\varphi = \varphi_n \in L^\infty(U)$ satisfies, for all sequences $(g_k \,:\, k \in \mathbb{N})$ in K,

$$\limsup_{k\uparrow\infty} \langle \varphi, g_k \rangle \leq 1 < \langle \varphi, f \rangle\,,$$

which means that f is not in the weak L^1-closure of K. We infer that K must be weakly closed in $L^1(U)$, which proves (75).

To show (76) we assume that $(f_n \colon n \in \mathbb{N})$ and f are in K such that $f_n \longrightarrow f$ weakly in $L^1(U)$. Our aim is to show that $\limsup_{n\to\infty} \langle f_n, \mathfrak{A} f_n \rangle \leq \langle f, \mathfrak{A} f \rangle$. For some large $M > 0$, we split $\mathfrak{A} = \mathfrak{A}_M + (\mathfrak{A} - \mathfrak{A}_M)$, where the operator $\mathfrak{A}_M$ is defined like $\mathfrak{A}$, but with $G 1_{G \leq M}$ instead of G. We show that $\lim_{M\to\infty} \sup_{n\in\mathbb{N}} \langle f_n, (\mathfrak{A} - \mathfrak{A}_M) f_n \rangle = 0$. In $d = 2$ this is straightforward using Hölder's inequality. If $d \geq 3$, we recall that $G(x, y) \leq c_d |x - y|^{2-d}$ to estimate $G(x, y) \leq C' |x - y|^{3/2-d} G(x, y)^{1/(4-2d)}$ for some $C' > 0$. Apply the Hardy-Littlewood-Sobolev inequality (64) with $r = s = 4d/(2d + 3)$ and $\lambda = d - 3/2$ to obtain, for some $C > 0$,

$$\langle f_n, (\mathfrak{A} - \mathfrak{A}_M) f_n \rangle \leq C' M^{1/(4-2d)} \iint f_n(x) |x - y|^{-\lambda} f_n(y) \, dx \, dy \leq C' C M^{1/(4-2d)} \|f_n\|_r^2. \tag{77}$$

Since $r \leq 2p/(2p - 1)$, this vanishes as $M \to \infty$, uniformly in $n \in \mathbb{N}$.

Hence, we only have to show that, for fixed $M > 0$, we have $\lim_{n\to\infty} \langle f_n, \mathfrak{A}_M f_n \rangle = \langle f, \mathfrak{A}_M f \rangle$ and taking the limit $M \to \infty$ yields the desired conclusion. The proof of this follows from a standard monotone class argument. Let $\mathcal{H}$ be the class of all $\phi \in L^\infty(U^2)$ with the property that

$$\lim_{n\uparrow\infty} \iint f_n(x) f_n(y) \phi(x, y) \, dx \, dy = \iint f(x) f(y) \phi(x, y) \, dx \, dy\,.$$

This class is a vector space containing all functions of the form $\phi_1(x)\phi_2(y)$ for $\phi_1, \phi_2 \in L^\infty(U)$. Finally, suppose that $0 \leq \phi_k \uparrow \phi \in L^\infty(U^2)$ and $\phi_k \in \mathcal{H}$. We can estimate, using the triangle inequality and Hölder's inequality,

$$
\begin{aligned}
&\Big|\iint f(x)f(y)\phi(x,y)\,dx\,dy - \iint f_n(x)f_n(y)\phi(x,y)\,dx\,dy\Big| \\
&\leq \Big|\iint f(x)f(y)(\phi-\phi_k)(x,y)\,dx\,dy\Big| + \Big|\iint \big(f(x)f(y)-f_n(x)f_n(y)\big)\phi_k(x,y)\,dx\,dy\Big| \\
&\quad + \Big|\iint f_n(x)f_n(y)(\phi-\phi_k)(x,y)\,dx\,dy\Big| \\
&\leq \|f\|^2\,\|\phi-\phi_k\|_{2p} + \Big|\iint (f(x)f(y)-f_n(x)f_n(y))\phi_k(x,y)\,dx\,dy\Big| + \|f_n\|^2\|\phi-\phi_k\|_{2p},
\end{aligned}
\tag{78}
$$

where we recall that $\|\cdot\| = \|\cdot\|_{2p/(2p-1)}$ and extend the notation $\|\cdot\|_{2p}$ to $L^{2p}(U^2)$. Now it is easy to see that the last line of (78) can be made arbitrarily small by choosing a large k and letting $n\to\infty$. This shows that $\phi\in\mathcal{H}$ and, by the monotone class theorem, we have shown that $\mathcal{H}=L^\infty(U^2)$. Hence $G1_{\{G\leq M\}}\in\mathcal{H}$ and this finishes the proof of (76). ☐

Now we derive the Euler-Lagrange equation for the minimizers in (65).

Lemma 4.4. *Any nonnegative maximizer of the variational problem in* (65) *is essentially bounded away from* 0. *Writing this maximizer as g^{2p-1}, the function $g\in L^{2p}(U)$ satisfies (67) with some $\varrho>0$.*

Proof. Let $f\in L^{2p/(2p-1)}(U)$ be a nonnegative and normalized minimizer in (65). We first show that f is essentially bounded from 0. Assume the contrary, which means that $\ell\{f\leq\varepsilon\}>0$ for all $\varepsilon>0$.

We fix some $c>0$ such that $\ell\{f\geq c\}>0$. We define a function $\widetilde{f}\colon U\to[0,\infty)$ as follows

$$
\widetilde{f}(x)=\begin{cases} f(x)+a, & \text{if } f(x)\leq\varepsilon,\\ f(x)-b, & \text{if } f(x)\geq c,\\ f(x), & \text{otherwise.}\end{cases}
$$

Our plan is to choose $a,b>0$ and $\varepsilon>0$ so small that $\|\widetilde{f}\|_{2p/(2p-1)}=1$ but $\langle\widetilde{f},\mathfrak{A}\widetilde{f}\rangle>\langle f,\mathfrak{A}f\rangle$, which contradicts the maximality of f.

For every sufficiently small a and $\varepsilon>0$ we may find $b\leq c/2$, such that $\|\widetilde{f}\|_{1+\eta}=1$, where we abbreviate $\eta=1/(2p-1)$. This condition implies that

$$
0=\|\widetilde{f}\|_{1+\eta}^{1+\eta}-\|f\|_{1+\eta}^{1+\eta}=\int_{\{f\leq\varepsilon\}}\big[(f(x)+a)^{1+\eta}-f(x)^{1+\eta}\big]\,dx+\int_{\{f\geq c\}}\big[(f(x)-b)^{1+\eta}-f(x)^{1+\eta}\big]\,dx,
$$

and one one can see that this implies, for some constant $C>0$, that does not depend on a or ε,

$$
b\leq Ca(a+\varepsilon)^\eta\ell\{f\leq\varepsilon\}. \tag{79}
$$

On the other hand, one can use this to check in the same way that, again for constants $C_1,C_2>0$, which do not depend on a or ε,

$$
\langle f,\mathfrak{A}f\rangle-\langle\widetilde{f},\mathfrak{A}\widetilde{f}\rangle\leq a\ell\{f\leq\varepsilon\}\big(C_1(a+\varepsilon)^\eta-C_2\big).
$$

We may now choose a,ε suitably, such that the right hand side is negative. This yields the desired contradiction.

In order to prove the second assertion, let $\varphi\colon U\to\mathbb{R}$ be bounded with $\int\varphi(x)\,dx=0$. For sufficiently small $\varepsilon>0$, we have $\|f+\varepsilon\varphi\|_1=1$ and $f+\varepsilon\varphi\geq0$ on U. Using the maximality of

f, we infer

$$0 = \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} \big\langle (f+\varepsilon\varphi)^{\frac{2p-1}{2p}}, \mathfrak{A}(f+\varepsilon\varphi)^{\frac{2p-1}{2p}} \big\rangle = \tfrac{2p-1}{p} \big\langle \varphi, f^{\frac{-1}{2p}} \mathfrak{A} f^{\frac{2p-1}{2p}} \big\rangle. \tag{80}$$

We infer that, in $L^2(U)$, the function $f^{\frac{-1}{2p}}\mathfrak{A}\big(f^{\frac{2p-1}{2p}}\big)$ is orthogonal to the orthogonal complement of the span of the constants. Hence there is a constant ϱ such that

$$\varrho f^{\frac{1}{2p}}(x) = \mathfrak{A} f^{\frac{2p-1}{2p}}(x) \text{ for } \ell\text{-almost every } x \in U.$$

As f is essentially positive, we have $\varrho > 0$. Writing now g^{2p-1} for f, we have verified (67). □

Putting together the previous three lemmas we obtain the existence of minimizers for the variational problem in (28) and the convergence of minimizing sequences.

LEMMA 4.5. *Every minimizing sequence for the variational problem in* (28) *has a subsequence that converges weakly towards some minimizer of this problem. If* g^{2p} *denotes its density, then* $g \in L^{2p}(U)$ *satisfies (67).*

PROOF. Suppose that $(\mu_n : n \in \mathbb{N}) \subset \mathcal{M}_1(U)$ is any minimizing sequence for the variational problem in (28). At least for sufficiently large n, the measures μ_n have Lebesgue densities, which we denote by $g_n^{2p} \in L^{2p}(U)$. By Lemma 4.2, $(g_n^{2p-1} : n \in \mathbb{N})$ is a maximizing sequence of the variational problem in (65). By Lemma 4.3, we can extract a subsequence converging weakly in $L^1(U)$ to some $g^{2p-1} \in L^{2p/(2p-1)}(U)$. Defining μ by $\mu(dx) = g^{2p}(x)\,dx$ yields a maximizer of the problem (28) we started with. It is obvious that μ is the weak limit of the corresponding subsequence of $(\mu_n : n \in \mathbb{N})$. □

We have now established the existence of a minimizer $\mu(dx) = g^{2p}(x)\,dx$ for (28) satisfying (67) with $\varrho = \varrho^*$ defined by (9) resp. by (65). Also, g is a maximizer of (9) if and only if the measure $\mu(dx) = g^{2p}(x)\,dx$ is a minimizer of (28). Hence, Proposition 2.1 and the existence of the maximizers have been proved.

4.2. Proof of Theorem 1.3. Next, we establish smoothness properties for every g satisfying (67).

LEMMA 4.6. *Every g satisfying (67) has a version that is twice continuously differentiable on U. By*

$$g(x) = \frac{1}{\varrho^*}\int_U G(x,y) g^{2p-1}(y)\,dy\,, \text{ for } x \in B(0,R)\,, \tag{81}$$

one can define an extension of g to $B(0,R)$, which is continuous on the closure of $B(0,R)$ and continuously differentiable in the interior of $B(0,R)$.

PROOF. We start by showing that (67) implies that $g \in L^\infty(U)$. Indeed, if $d = 2$ we know that $g \in L^{2p}(U)$ and thus Lemma 4.1 shows that $g = (1/\varrho)\mathfrak{A}g^{2p-1} \in L^\infty(U)$. In the case $d = 3$ and $p = 2$ we get from Lemma 4.1 that $g = (1/\varrho)\mathfrak{A}g^3 \in L^{12}(U)$. By Hölder's inequality we infer that

$$\mathfrak{A}g^3(x) \le \Big(\int_U G(x,y)^{4/3}\,dy\Big)^{3/4}\Big(\int_U g^{12}(y)\,dy\Big)^{1/4}.$$

Because $G(x,y) \le c_d|x-y|^{-1}$ the right hand side is bounded, and we use (67) again to infer that $g \in L^\infty(U)$. Finally, in the case $d \ge 3$ and $p = 1$ we iterate the use of Lemma 4.1 and (67) to obtain $g \in L^s(U)$ for some $s > d/2$. Let $q = s/(s-1)$ and note that $1 < q < d/(d-2)$.

By Hölder's inequality again we infer that $\mathfrak{A}g(x) \leq \|G(x,\cdot)\|_q \, \|g\|_{q/(q-1)}$. The right hand side is bounded, and we infer from (67) that $g \in L^\infty(U)$.

Now extend g by (81) to the whole ball $B(0,R)$. Using that $g\mathbb{1}_U \in L^\infty(U)$, we know from [**PoS78**, 4.6.6.] that g is continuously differentiable on $B(0,R)$ and continuous on the closure of $B(0,R)$. To see that it is even twice continuously differentiable on U let

$$\mathfrak{B}g(y) = \int_U G_U(x,y)g(x)\,dx, \text{ for } y \in B(0,R),$$

where G_U denotes the Green's function with zero boundary conditions on U rather than on $B(0,R)$. Denote by h_U the harmonic measure on ∂U. For almost every $x \in U$ we have

$$g(x) = \frac{1}{\varrho}\mathfrak{A}g^{2p-1}(x) = \frac{1}{\varrho}\mathfrak{B}g^{2p-1}(x) + \frac{1}{\varrho}\int_{\partial U}\mathfrak{A}g^{2p-1}(z)h_U(x,dz)\,. \tag{82}$$

As g^{2p-1} is Lipschitz continuous on U, the first summand on the right hand side is twice continuously differentiable by [**PoS78**, 4.6.6.] and the second summand is harmonic in U and hence infinitely often differentiable. ☐

We have by now shown that every maximizer g in (9) has a version satisfying Theorem 1.3 (a). We now prove the equivalence of the two characterizations of the minimizer given in Theorem 1.3.

LEMMA 4.7. *The equation in Theorem 1.3 (a) and the partial differential equation in Theorem 1.3 (b) are equivalent.*

PROOF. Suppose (g,ϱ) is a solution of (10). By the Meyer-Tanaka formula, for a Brownian motion $B = (B_s : s \geq 0)$, the process M defined by

$$M_t = g(B_t) + \frac{1}{\varrho}\int_0^t g^{2p-1}(B_s)\mathbb{1}_U(B_s)\,ds$$

is a local martingale on $[0,T)$, where T is the first exit time from $B(0,R)$. Hence, by the optional stopping theorem, for all $x \in U$,

$$g(x) = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}\Big[\frac{1}{\varrho}\int_0^T g^{2p-1}(B_s)\mathbb{1}_U(B_s)\,ds\Big] = \frac{1}{\varrho}\mathfrak{A}g^{2p-1}(x)\,.$$

Now assume conversely that the operator equation of Theorem 1.3 (a) holds in U. The continuation of g to the closure of $B(0,R)$ defined in (81) has the necessary differentiability and continuity properties. Recall equation (82). Applying $-\Delta/2$ to both sides of the equation and recalling that $-\Delta\mathfrak{B}f(x) = 2f(x)$ for all bounded $f \in C^2(U)$ (see [**PoS78**, Theorem 4.6.6.]) gives (10) for $x \in U$. To get (10) for $x \notin U$ it suffices to observe that (81) is harmonic on the interior of $B(0,R) \setminus U$. ☐

Finally, we prove uniqueness of the maximizer in (9) in the case $p = 1$. Note that we do not know whether uniqueness holds if $p > 1$, except in the special case when U is a ball and $R = \infty$, see Subsection 4.3 below.

LEMMA 4.8. *In the case $p = 1$ the solution g of (67) is uniquely determined.*

PROOF. Recall the Krein-Rutman theorem (see e.g. [**Ama76**, Theorem 3.2]). $L^\infty(U)$ is an ordered Banach space whose positive cone has nonempty interior and $\mathfrak{A}$ is a strongly positive compact linear operator $\mathfrak{A} : L^\infty(U) \to L^\infty(U)$. Hence $\mathfrak{A}$ has exactly one normalized strictly positive eigenvector and this eigenvector is the unique minimizer g. ☐

4.3. Proof of Theorem 1.4. Suppose now that $U = B(0,1)$ is the open unit ball centred at the origin and consider unstopped Brownian motions in $\mathbb{R}^d$, $d \geq 3$. By (6) in this situation $G(x,y)$ is a rotationally symmetric function of $x-y$, and the maximizers for the problem in (65) are necessarily rotationally symmetric. This is a direct consequence of Riesz's strict rearrangement inequality; see [**LL97**, 3.7, 3.9].

Let $f \in L^{2p/(2p-1)}(U)$ be a minimizer of the problem in (65), put $g^{2p-1} = f$ on $B(0,1)$ and extend g to a C^1-function on the whole of $\mathbb{R}^d$. Recall that this is possible by Theorem 1.3 (b). Define $z : [0,\infty) \to (0,\infty)$ by $g(x) = cz(a|x|)$, choosing the positive parameters a, c such that

$$z(0) = 1 \qquad \text{and} \qquad \varrho^* = 2c^{2p-2}/a^2.$$

Writing the Laplacian in polar coordinates gives for $|x| < 1$,

$$-ca^2 z(a|x|)^{2p-1} = -\frac{2}{\varrho^*} g^{2p-1}(x) = \Delta g(x) = \frac{d-1}{|x|} caz'(a|x|) + ca^2 z''(a|x|). \tag{83}$$

In other words, z satisfies (11) on $[0,a]$. Differentiability of g implies $z'(0) = 0$, and we can infer that z satisfies the requirements of Theorem 1.4 (i) on the interval $[0,a]$.

We now show uniqueness of the solutions of (11) with $z(0) = 1$ and $z'(0) = 0$. In the case $p = 1$ the transformation (12) can be inverted and this reduces the problem to uniqueness of solutions of the Bessel differential equation, see Remark 1.5. To show uniqueness in the remaining case $p = 2$, $d = 3$ we argue as follows. Since the map $x \mapsto xz(x)$ is strictly increasing on $[0,a]$, we may define $f\colon [0, az(a)] \to [0,\infty)$ by putting $f(xz(x)) = xz(x) + x^2 z'(x)$. Note that f has a simple zero at the origin, and that f satisfies the differential equation $f'(t) = 1 - t^3/f(t)$. Indeed, using (11) in the penultimate step, we get

(84)

$$f'(xz(x)) = \frac{\frac{d}{dx} f(xz(x))}{xz'(x) + z(x)} = 1 + \frac{2xz'(x) + x^2 z''(x)}{xz'(x) + z(x)} = 1 - \frac{x^2 z(x)^3}{xz'(x) + z(x)} = 1 - \frac{(xz(x))^3}{f(xz(x))},$$

and the result follows by substitution of $t = xz(x)$.

The function f is uniquely determined by the differential equation $f'(t) = 1 - t^3/f(t)$, at least close to the origin. To show this, let $\|\cdot\|_T$ be the supremum norm on $[0,T]$. Then, if f_1 and f_2 satisfy the equation and $f_1(0) = f_2(0) = 0$, we obtain, for suitable constants $C', C > 0$,

$$\begin{aligned} \|f_1 - f_2\|_T &\leq \int_0^T |f_1'(s) - f_2'(s)|\, ds &= \int_0^T \frac{s^3}{f_1(s) f_2(s)} |f_1(s) - f_2(s)|\, ds \\ &\leq C' \int_0^T s \|f_1 - f_2\|_T\, ds &\leq CT^2 \|f_1 - f_2\|_T. \end{aligned}$$

For sufficiently small $T > 0$, this shows that f is unique on $[0,T]$. A standard Picard-Lindelöf argument shows the uniqueness on the whole interval on which f_1 and f_2 are defined and positive.

The uniqueness of z now follows by noting that z can be uniquely recomputed from f by the formula

$$\int_{az(a)}^{xz(x)} \frac{ds}{f(s)} = C + \log x, \text{ for } x \in [0,a],$$

where $C = \lim_{\varepsilon \downarrow 0} \Big(\int_{az(a)}^{\varepsilon} \frac{ds}{f(s)} - \log \varepsilon \Big)$.

We now check that a is indeed the smallest value such that $z(a) = az'(a)/(2-d)$. Because g is harmonic on $\mathbb{R}^d \setminus \overline{B(0,1)}$ and vanishing at infinity, there exists a constant $C > 0$ such that $g(x) = C|x|^{2-d}$ for all $|x| \geq 1$. Hence, $z(t) = Kt^{2-d}$ for some $K > 0$ and any $t \geq a$, and therefore $z'(a) = (2-d)z(a)/a$, as claimed. Now suppose that there is a smaller value $0 < \widetilde{a} < a$ satisfying $z(\widetilde{a}) = \widetilde{a}z'(\widetilde{a})/(2-d)$. Define $h(x) = \widetilde{c}z(\widetilde{a}|x|)$ where $\widetilde{c}$ is chosen such that $\|h\|_{2p} = 1$. This implies, writing σ_{d-1} for the area of the $(d-1)$-sphere,

$$\begin{aligned}\frac{\widetilde{a}^d}{\widetilde{c}^{2p}} &= \widetilde{a}^d \int_U z^{2p}\big(\widetilde{a}|x|\big)\,dx = \widetilde{a}^d \sigma_{d-1} \int_0^1 r^{d-1} z^{2p}(\widetilde{a}r)\,dr = \sigma_{d-1}\int_0^{\widetilde{a}} s^{d-1}z^{2p}(s)\,ds \\ &< \sigma_{d-1}\int_0^a s^{d-1}z^{2p}(s)\,ds = \frac{a^d}{c^{2p}}.\end{aligned} \tag{85}$$

One can check easily, by reverting the arguments above, that h is a solution of

$$\frac{1}{2}\Delta h(x) = -\frac{\widetilde{a}^2}{2\widetilde{c}^{2p-2}} h^{2p-1}(x) 1_U(x).$$

In particular, recalling the equivalence of this differential equation and the operator equation in Theorem 1.3 (a) we get

$$\langle h^{2p-1}, \mathfrak{A}h^{2p-1}\rangle = \frac{2\widetilde{c}^{2p-2}}{\widetilde{a}^2} > \frac{2c^{2p-2}}{a^2} = \varrho^*\,,$$

where we have used the relationship (85). This contradicts the fact that g is a maximizer in (9).

We have thus characterized the minimizer uniquely, the formulae relating the value of ϱ^* with the differential equation follow easily from the relation $\varrho^* = 2c^{2p-2}/a^2$ by evaluating the normalization constant using polar coordinates as in (85). Hence, the proof of Theorem 1.4 is finished.

5. The dimension spectrum of thick points

In this section we give the proof of Theorem 1.6. The upper bound for both (17) and (18) follows from standard methods, for the lower bound we rely on the method of *hitting with a percolation limit set* and adapt ideas of [**LyR90**] and [**KPX00**] for our purpose. A main problem in the investigation of the Hausdorff dimension of subsets of S comes from the fact that there is no natural parametrization of S by a non-random set. This makes the investigation more involved than, say, in the case of a single Brownian path, where such a parametrization is available.

We assume that under $\mathbb{P}$ the processes W_1 and W_2 are two unstopped independent Brownian motions in $\mathbb{R}^3$, started in the origin and λ their intersection local time. For the limit in (8) we introduce the abbreviation $\theta = 2/\varrho^*$ with ϱ^* given by (9) for $d = 3$, $p = 2$, $U = B(0,1)$ and $R = \infty$, or equivalently by Theorem 1.4 for $p = 2$.

5.1. The upper bounds. In this subsection we prove all upper estimates needed in the proof of Theorem 1.6. Fix a unit cube $\mathtt{C} = b + [0,1]^3$ at positive distance from the origin. Fix $a > 0$ and denote the set of *a-thick points* in $\mathtt{C}$ by

$$F(a) = \Big\{x \in \mathtt{C} \,:\, \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} \geq a\Big\}. \tag{86}$$

The remainder of this subsection is devoted to the proof of the following proposition.

PROPOSITION 5.1. *We have, almost surely,*

(i) *if* $a > 1/\theta^2$, *then* $F(a) = \emptyset$;
(ii) *if* $a \leq 1/\theta^2$, *then* $\dim F(a) \leq 1 - \theta\sqrt{a}$.

Moreover, almost surely,

(iii) $\limsup_{r\downarrow 0} \sup_{x\in\mathbb{R}^3} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} \le \frac{1}{\theta^2}$.

We write $\mathbb{P}_{(x,y)}$ for probabilities with respect to two independent Brownian motions W_1, W_2 started in the points x and y. In particular, $\mathbb{P} = \mathbb{P}_{(0,0)}$. The following lemma draws the conclusion from Theorem 1.1 needed for the proof of Proposition 5.1.

LEMMA 5.2. *For every $\delta > 0$ there is an $\varepsilon > 0$ such that, for all $0 < r < \varepsilon$, and all $y \in \partial B(x,r)$,*

$$\mathbb{P}_{(x,y)}\Big\{\lambda(B(x,r)) \ge ar[\log(1/r)]^2\Big\} \le r^{\sqrt{a}\theta-\delta}.$$

PROOF. A simple coupling argument shows that the nonnegative random variable $\lambda(B(x,r))$ is stochastically maximal if the starting points x and y agree. Hence, using also Brownian scaling, we infer that

$$\begin{aligned}\mathbb{P}_{(x,y)}\Big\{\lambda(B(x,r)) \ge ar[\log(1/r)]^2\Big\} &\le \mathbb{P}_{(x,x)}\Big\{\lambda(B(x,r)) \ge ar[\log(1/r)]^2\Big\}\\ &= \mathbb{P}_{(0,0)}\Big\{\lambda(B(0,1)) \ge a[\log(1/r)]^2\Big\}.\end{aligned}$$

From Theorem 1.1 in the case $U = B(0,1)$ and $R = \infty$ we get the estimate for the right hand side with the value $\theta = 2/\varrho^*$ identified in Theorem 1.4 (ii). □

In the following, we define a random sequence of collections $(\mathcal{J}_n)_n$ of smaller and smaller cubes such that the set $F(a)$ is covered by $\bigcup_{n\ge j}\mathcal{J}_n$ for all j.

Fix $\varepsilon > 0$ smaller than a. Pick $c > 0$ so small that, for sufficiently large N,

$$h_n = e^{-cn} \text{ and } k_n = h_n\big(1+\tfrac{1}{n}\big) \quad \text{satisfy} \quad \frac{h_{n+1}}{k_n} \ge \frac{a-\varepsilon}{a-\varepsilon/2}, \text{ for all } n \ge N. \tag{87}$$

Abbreviate $s_n = h_n/(\sqrt{3}n)$ and observe that $\lim_{n\to\infty} s_n = 0$. Introduce the following collection of cubes of sidelength s_n,

$$\mathcal{I}_n = \Big\{b + s_n\big((k,l,m) + [0,1]^3\big) \,:\, 0 \le k,l,m < s_n^{-1} \text{ integer}\Big\}.$$

Note that $\mathcal{I}_n$ is a covering of the cube $\mathtt{C}$.

For every $I \in \mathcal{I}_n$ denote by $T_n(I)$ the first hitting time of the cube I by W_1 and, if $T_n(I) < \infty$, let $z(I) = W_1(T_n(I)) \in \partial I$ be the entry point. We pick now those minicubes from $\mathcal{I}_n$ that are hit by W_1 and such that the intersection local time of the two motions in a neighbourhood of the entry point exceeds a certain threshold,

$$\mathcal{J}_n = \Big\{I \in \mathcal{I}_n \,:\, T_n(I) < \infty \text{ and } \lambda(B(z(I),k_n)) \ge (a-\varepsilon)k_n[\log(1/k_n)]^2\Big\}.$$

We first prove that $F(a)$ is covered by the cubes in $\bigcup_{n\ge j}\mathcal{J}_n$ for all j.

LEMMA 5.3.

(i) *For all sufficiently large $n \in \mathbb{N}$ and $r \in [h_{n+1}, h_n]$,*

$$V(r) = \Big\{x \in \mathtt{C} \cap S \,:\, \lambda\big(B(x,r)\big) \ge \big(a - \tfrac{\varepsilon}{2}\big)r[\log(1/r)]^2\Big\} \subset \bigcup_{I\in\mathcal{J}_n} I.$$

(ii) $F(a) \subset \bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{I\in\mathcal{J}_n} I.$

Proof. Fix a large n and $r \in [h_{n+1}, h_n]$ and let $x \in V(r)$. As $\mathcal{I}_n$ is a covering of $\mathtt{C}$, certainly x lies in some $I \in \mathcal{I}_n$. We have to show that this small cube I lies in $\mathcal{J}_n$. Note that $T_n(I) < \infty$ since $x \in S$. Put $z = z(I)$. As $|x - z| \le \sqrt{3} s_n$ we have, using also (87), that

$$\begin{aligned}
\lambda(B(z,k_n)) &= \lambda(B(z, h_n + \sqrt{3}s_n)) \ge \lambda(B(x,r)) \\
&\ge \left(a - \tfrac{\varepsilon}{2}\right) r [\log(1/r)]^2 \ge \left(a - \tfrac{\varepsilon}{2}\right) h_{n+1} [\log(1/h_{n+1})]^2 \\
&\ge (a-\varepsilon) k_n [\log(1/h_{n+1})]^2 \ge (a - \varepsilon) k_n [\log(1/k_n)]^2.
\end{aligned}$$

Hence, $I \in \mathcal{J}_n$ and this proves (i).

Now let x be in $F(a)$. Then, for infinitely many $n \in \mathbb{N}$, there is an $r \in [h_{n+1}, h_n]$ such that $x \in V(r)$. Then x lies in $\bigcup_{I \in \mathcal{J}_n} I$ for infinitely many n, and this implies (ii). □

Lemma 5.4. *There is a sequence $\varepsilon_n \downarrow 0$ such that, for every $I \in \mathcal{I}_n$, we have*

$$\mathbb{P}\Big\{ I \in \mathcal{J}_n \Big\} \le h_n^{2 + \sqrt{a-\varepsilon}\theta - \varepsilon_n}.$$

Proof. The probability that a 3-dimensional Brownian motion started in the origin hits a ball $B(y,r)$, whose closure does not contain the origin, is equal to $r/|y|$. In particular, as the cube $\mathtt{C}$ has positive distance to the origin, there is a constant $K > 0$ such that, for sufficiently large n,

$$\mathbb{P}\big\{ W_1 \text{ hits } I \big\} \le K s_n \text{ for all } I \in \mathcal{I}_n.$$

Moreover, if we apply the same argument to the second Brownian motion, we can find another constant $\tilde{K} > 0$ such that, for sufficiently large n,

$$\mathbb{P}\big\{ W_1 \text{ hits } I \text{ and } W_2 \text{ hits } B\big(z(I), k_n\big) \big\} \le \tilde{K} s_n k_n \text{ for all } I \in \mathcal{I}_n.$$

Look at a pair of Brownian motions started in z_1 resp. z_2 with $|z_1 - z_2| = k_n$. From Lemma 5.2 we get a sequence $\delta_n \downarrow 0$, which does not depend on the choice of z_1, z_2, such that, for sufficiently large n,

$$\mathbb{P}_{(z_1,z_2)}\big\{ \lambda(B(z_1,k_n)) \ge (a-\varepsilon) k_n [\log(1/k_n)]^2 \big\} \le k_n^{\sqrt{a-\varepsilon}\theta - \delta_n}.$$

We apply this to the points $z_1 = z(I)$ and z_2 defined as the point where W_2 enters $B(z_1, k_n)$. By the strong Markov property, if n is sufficiently large,

$$\begin{aligned}
\mathbb{P}\Big\{ I \in \mathcal{J}_n \Big\} &= \mathbb{P}\Big\{ W_1 \text{ hits } I \text{ and } \lambda\big(B(z(I),k_n)\big) \ge (a-\varepsilon) k_n [\log(1/k_n)]^2 \Big\} \\
&\le \tilde{K} s_n k_n^{1 + \sqrt{a-\varepsilon}\theta - \delta_n} \le h_n^{2 + \sqrt{a-\varepsilon}\theta - \varepsilon_n},
\end{aligned}$$

recalling $s_n = h_n/(\sqrt{3}n)$, $k_n = h_n(1 + \frac{1}{n})$ and defining $\varepsilon_n \downarrow 0$ accordingly. □

Proof of Proposition 5.1. Lemma 5.4 gives, for any $\gamma \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\Big[\sum_{n=1}^{\infty} \# \mathcal{J}_n s_n^{\gamma} \Big] &= \sum_{n=1}^{\infty} s_n^{\gamma} \sum_{I \in \mathcal{I}_n} \mathbb{P}\Big\{ I \in \mathcal{J}_n \Big\} \le \sum_{n=1}^{\infty} s_n^{\gamma} h_n^{2 + \sqrt{a-\varepsilon}\theta - \varepsilon_n} \# \mathcal{I}_n \\
&\le \sum_{n=1}^{\infty} h_n^{\gamma - 1 + \sqrt{a-\varepsilon}\theta - \varepsilon_n} (\sqrt{3}n)^{-\gamma},
\end{aligned} \tag{88}$$

where we recall that $s_n = h_n/(\sqrt{3}n)$. Because $h_n = e^{-cn}$ the right hand side of (88) is finite for all $\gamma > 1 - \theta\sqrt{a - \varepsilon}$, and we can infer that

$$\sum_{n=1}^{\infty} \# \mathcal{J}_n s_n^{\gamma} < \infty \text{ almost surely.} \tag{89}$$

By Lemma 5.3 (ii), the set $F(a)$ can be covered, for each j, by the family $\bigcup_{n\geq j} \mathcal{J}_n$ consisting, for each $n \geq j$, of $\#\mathcal{J}_n$ cubes of sidelength s_n. Hence (89) implies that $\dim F(a) \leq \gamma$. As this holds for all $\gamma > 1-\theta\sqrt{a-\varepsilon}$ and $\varepsilon > 0$ we can infer that

$$\dim F(a) \leq 1 - \theta\sqrt{a}.$$

If the right hand side is negative, $F(a)$ must be empty almost surely, so that Propostion 5.1 (i) and (ii) are proved.

For the proof of (iii) we use the above estimates for arbitrary $a > 1/\theta^2$, some small $\varepsilon > 0$ and $\gamma = 0$. By Chebyshev's inequality we infer that

$$\sum_{n=1}^{\infty} \mathbb{P}\{\#\mathcal{J}_n \geq 1\} \leq \sum_{n=1}^{\infty} \mathbb{E}[\#\mathcal{J}_n] < \infty.$$

By the Borel-Cantelli lemma, $\mathcal{J}_n$ must be empty for all sufficiently large n, almost surely. Lemma 5.3 (i) thus implies

$$\limsup_{r\downarrow 0} \sup_{x\in S} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} \leq a.$$

Letting a tend to $1/\theta^2$ finishes the proof of Proposition 5.1 (iii). □

5.2. Hitting with a percolation limit set. To obtain lower bounds we use the method of intersection with independent random sets, see for example [**KPX00**] for an extensive account of this.

Suppose that $\mathtt{C} \subset \mathbb{R}^3$ is a fixed compact unit cube not containing the origin. We denote by $\mathfrak{D}_n$ the collection of compact dyadic subcubes (relative to $\mathtt{C}$) of sidelength 2^{-n}. We also let $\mathfrak{D} = \bigcup_{n=0}^{\infty} \mathfrak{D}_n$.

Given $\gamma \in [0,3]$ we construct a random compact set $\Gamma[\gamma] \subset \mathtt{C}$ inductively as follows: We keep each of the eight compact cubes in $\mathfrak{D}_1$ independently with probability $p = 2^{-\gamma}$. Let $\mathcal{S}_1$ be the collection of cubes kept in this procedure and pass from $\mathcal{S}_n$ to $\mathcal{S}_{n+1}$ by keeping each cube of $\mathfrak{D}_{n+1}$, which is not contained in a previously rejected cube, independently with probability p. Then the random set

$$\Gamma[\gamma] := \bigcap_{n=1}^{\infty} \bigcup_{C\in\mathcal{S}_n} C$$

is called *percolation limit set.* The usefulness of percolation limit sets in fractal geometry comes from the following lemma, see [**Per96**] or [**Haw81**] for a proof.

LEMMA 5.5. *For every $\gamma \in [0,3]$ and every Borel set $A \subset \mathtt{C}$ the following properties hold*

(i) *if* $\dim A < \gamma$, *then almost surely,* $A \cap \Gamma[\gamma] = \emptyset$,
(ii) *if* $\dim A > \gamma$, *then* $A \cap \Gamma[\gamma] \neq \emptyset$ *with positive probability,*
(iii) *if* $\dim A > \gamma$, *then almost surely* $\dim\big(A \cap \Gamma[\gamma]\big) \leq \dim A - \gamma$ *and, for all* $\varepsilon > 0$, *with positive probability* $\dim\big(A\cap\Gamma[\gamma]\big) \geq \dim A - \gamma - \varepsilon$.

Observe that the first part of the lemma gives a *lower* bound γ for the Hausdorff dimension of a set A, if we can show that $A \cap \Gamma[\gamma] \neq \emptyset$ with positive probability. To make use of this observation, recall that $\theta = 2/\varrho^*$, fix $a \in [0, 1/\theta^2]$, and put $\gamma = 1-\theta\sqrt{a}$. We suppose that the random set $\Gamma[\gamma]$ and two unstopped Brownian motions W_1 and W_2, started at the origin, are realized independently on the same probability space, and we write $\mathbb{P}$ for the joint distribution of the motions and $\Gamma[\gamma]$.

Observe that Lemma 5.5 (i), applied to the set $F(a)$ defined in (86), yields the lower bound $\dim\big(F(a)\big) \geq \gamma = 1 - \theta\sqrt{a}$, *if* we show that the set $F(a) \cap \Gamma[\gamma]$ is non-empty with positive probability. This is shown in Proposition 5.6 below. A lower bound for the dimension of the set of *strictly* a-thick points follows with a little more effort, see Subsection 5.3 below.

PROPOSITION 5.6 (Hitting by thick points).

$$\mathbb{P}\big\{F(a) \cap \Gamma[\gamma] \neq \emptyset\big\} > 0\,. \tag{90}$$

The remainder of this subsection is devoted to the proof of Proposition 5.6.

The idea is to construct a compact random subset $S^* \subset S \cap \Gamma[\gamma]$ with the regularity properties,

(a) $S^* \neq \emptyset$ with positive probability,
(b) almost surely, for every open set $U \subset \mathtt{C}$ with $U \cap S^* \neq \emptyset$ we have $\dim\big(U \cap S^*\big) \geq 1 - \gamma$.

Property (b) is instrumental in the proof that $F(a) \cap S^*$ is dense in S^* almost surely, which together with (a) implies that $F(a) \cap \Gamma[\gamma] \neq \emptyset$ with positive probability.

To construct such a set S^* we fix a countable base $\mathfrak{B}$ of open subsets of $\mathtt{C}$ and define, for $0 < \delta \leq 1 - \gamma$, compact random sets

$$S^*[\delta] = \big(S \cap \Gamma[\gamma]\big) \setminus \bigcup \big\{B \in \mathfrak{B} \,:\, \dim(B \cap S \cap \Gamma[\gamma]) < \delta\big\}.$$

Obviously the sets $S^*[\delta]$ have the property that, almost surely for every open set $U \subset \mathtt{C}$,

$$U \cap S^*[\delta] \neq \emptyset \qquad \text{implies} \qquad \dim\big(U \cap S \cap \Gamma[\gamma]\big) \geq \delta\,. \tag{91}$$

In the case $\delta = 1 - \gamma$ this property is closest to property (b), but it still weaker than (b) and there is also no direct argument to see that $S^*[1-\gamma]$ is nonempty with positive probability.

On the other hand, let $0 < \delta < 1 - \gamma$ and recall from Subsection 2.1 that $\dim(S \cap \mathtt{C}) = 1$ if $\lambda(\mathtt{C}) > 0$, and the latter event has positive probability. Hence, by Lemma 5.5 (ii), we have $\dim(S \cap \Gamma[\gamma]) > \delta$ with positive probability. Removing countably many sets of dimension strictly smaller than δ does not decrease the dimension of the set and hence we have $\dim S^*[\delta] > \delta$ and, in particular, $S^*[\delta] \neq \emptyset$, with positive probability.

Property (b) is not immediate for $S^*[\delta]$, so we have to make do with (91). The proof of Lemma 5.9 below, however, implicitly show that (b) holds, and the sets $S^*[\delta]$ are the same for all $0 < \delta \leq 1 - \gamma$. In order to avoid unnecessary repetition of arguments, though, we do not make this statement explicit, but fix some $0 < \delta < 1 - \gamma$ once and for all, put $S^* := S^*[\delta]$ and use (91) instead of (b).

We now provide the two main technical lemmas in the proof of Proposition 5.6. The first of them is an extension of the lower bound of our upper tail asymptotics of Theorem 1.1 to the following situation. For given $y \in \mathbb{R}^3$ and integer $n \geq 1$, we consider three balls centred at y:

- At two points in $\partial B(y, 2^{-n}/n)$, we start the two motions,
- we measure their intersection local time in $B(y, 2^{-n}n)$,
- we consider the intersections of the paths only until the motions leave $B(y, 2^{-n}n^2)$, and we condition on their respective leaving positions.

Readers unfamiliar with Brownian motion conditioned on leaving a domain at a fixed exit point are recommended to consult [**Bas95**, III Prop.(2.7)].

To formulate precisely what we need, we assume for simplicity of notation that $y = 0$. By λ_R we denote the intersection local time for the two independent Brownian motions W_1 and W_2 stopped at the time σ_1^R resp. σ_2^R of their first exit from $B(0, R)$. We write $\mathbb{P}_{(x_1,x_2)}$ for the distribution of the two motions, started at x_1 and x_2, respectively.

LEMMA 5.7 (Localization). *For any $\varepsilon > 0$, there is an $N > 0$ such that, for any $n > N$ and for any $x_1, x_2, u_1, u_2 \in \partial B(0,1)$,*

(92)
$$\mathbb{P}_{(x_1,x_2)2^{-n}/n}\Big\{\lambda_{2^{-n}n^2}\big(B(0,2^{-n}n)\big) \geq a2^{-n}n[\log(2^n/n)]^2 \,\Big|\, W_i(\sigma_i^{2^{-n}n^2}) = u_i 2^{-n}n^2 \text{ for } i \in \{1,2\}\Big\} \geq 2^{-n(\sqrt{a}\theta+\varepsilon)}.$$

PROOF. It is clear from the Brownian scaling property that the left hand side of (92) is equal to
$$\mathbb{P}_{(x_1,x_2)/n^2}\Big\{\lambda_n(B(0,1)) \geq a[\log(2^n/n)]^2 \,\Big|\, W_i(\sigma_i^n) = u_i n \text{ for } i \in \{1,2\}\Big\}.$$
In the following we write P^n for the joint distribution of $\big(\frac{1}{n}W_1(\sigma_1^n), \frac{1}{n}W_2(\sigma_2^n)\big)$ under $\mathbb{P}_{(x_1,x_2)/n^2}$. Let A_1, A_2 be arbitrary Borel subsets of $\partial B(0,1)$. Fix some $R > 1$ and assume that $n > 2R$. Since $\lambda_n \geq \lambda_R$, we obtain a lower bound by replacing λ_n by λ_R. Use the strong Markov property at time σ_1^R and σ_2^R to obtain
$$\begin{aligned}\frac{1}{\mathsf{P}^n(A_1\times A_2)} \iint\limits_{A_1\times A_2} \mathbb{P}_{(x_1,x_2)/n^2}\Big\{\lambda_R(B(0,1)) \geq a[\log(2^n/n)]^2 \,\Big|& \\ W_i(\sigma_i^n) = nu_i \text{ for } i \in \{1,2\}\Big\}\, &\mathsf{P}^n(du_1\, du_2) \\ = \mathbb{P}_{(x_1,x_2)/n^2}\Big\{\lambda_R(B(0,1)) \geq a[\log(2^n/n)]^2 \,\Big|\,& W_i(\sigma_i^n) \in nA_i \text{ for } i \in \{1,2\}\Big\} \\ \geq \mathbb{P}_{(x_1,x_2)/n^2}\Big\{\lambda_R(B(0,1)) \geq a[\log(2^n/n)]^2\Big\} &\prod_{i=1}^2 \frac{\min_{|u|=R}\mathbb{P}_u\big\{W_i(\sigma_i^n) \in nA_i\big\}}{\max_{|u|=R}\mathbb{P}_u\big\{W_i(\sigma_i^n) \in nA_i\big\}}.\end{aligned}$$
From the explicit formula for the harmonic measure, see e.g. [**Bas95**, II Theorem (1.17)], one can see that the product on the right converges uniformly in A_1 and A_2 to 1.

Obviously,
$$\mathbb{P}_{(x_1,x_2)/n^2}\Big\{\lambda_R(B(0,1)) \geq a[\log(2^n/n)]^2\Big\} \geq \mathbb{P}_{(x_1,x_2)/n^2}\Big\{\lambda_R\big(B(x_1/n^2, 1-\tfrac{1}{n^2})\big) \geq a[\log(2^n/n)]^2\Big\}.$$
By a simple coupling argument, one can see that the random variable $\lambda_R(B(x_1/n^2, 1-\frac{1}{n^2}))$ is stochastically minimal if the distance of the starting points x_1/n^2 and x_2/n^2 is maximal. Therefore, fixing some $y \in \partial B(0,1)$,
$$\mathbb{P}_{(x_1,x_2)/n^2}\Big\{\lambda_R\big(B(x_1/n^2, 1-\tfrac{1}{n^2})\big) \geq a[\log(2^n/n)]^2\Big\} \geq \mathbb{P}_{(0,y)}\Big\{\lambda_R\big(B(0, 1-\tfrac{1}{n^2})\big) \geq a[\log(2^n/n)]^2\Big\}.$$
Altogether, there exists an integer N such that, for all $u_1, u_2, x_1, x_2 \in \partial B(0,1)$ and $n \geq N$,

(93)
$$\begin{aligned}\mathbb{P}_{(x_1,x_2)/n^2}\Big\{\lambda_n(B(0,1)) \geq a[\log(2^n/n)]^2 \,\Big|\,& W_i(\sigma_i^n) = nu_i \text{ for } i \in \{1,2\}\Big\} \\ \geq \tfrac{1}{2}\mathbb{P}_{(0,y)}\Big\{\lambda_R\big(B(0,1-\tfrac{1}{n^2})\big) \geq a[\log(2^n/n)]^2\Big\}&.\end{aligned}$$

Recall that the left hand side is equal to the left hand side of (92). On the other hand, the right hand side satisfies, as $n \to \infty$,
$$\tfrac{1}{2}\mathbb{P}_{(0,y)}\Big\{\lambda_R\big(B(0,1-\tfrac{1}{n^2})\big) \geq a[\log(2^n/n)]^2\Big\} \geq e^{-\sqrt{a}\log(2^n/n)(2/\varrho^*(R)+o(1))} \geq 2^{-n\sqrt{a}(2/\varrho^*(R)+o(1))},$$
where we used the lower bound in (8) and denoted $\varrho^*(R)$ the quantity in (9) for motions stopped at the first exit time from $B(0,R)$. Recall from Remark 1.2 that the limit as $R \to \infty$ of $\varrho^*(R)$ is equal to the value of ϱ^* for the unstopped motions. Since $2/\varrho^* = \theta$, we arrive at the assertion. $\square$

The second main technical lemma is a lower bound on the probability that the path of two conditioned Brownian paths intersect $\Gamma[\gamma]$ in a set of dimension close to $1-\gamma$.

We look at the following situation. Let $n \geq 1$ and $V \in \mathcal{S}_n$ be a cube, which is kept in the percolation procedure, and $\xi \in V$ its centre.

- At two points in ∂V, we start the two motions,
- we consider the dimension of the intersection of the paths with $\Gamma[\gamma]$ only until the motions leave the annulus $B(\xi, 2^{-n}n^2) \setminus B(\xi, 2^{-n}/n)$, and we condition on their respective first exit positions.

For the precise statement, we again simplify notation by assuming that the cube is centred in the origin. We consider two independent Brownian motions W_1 and W_2 and write again $\mathbb{P}_{(x_1,x_2)}$ for the distribution of the two motions, started at x_1 and x_2, respectively. For $0<r<s$ we let $\omega_1^{r,s}$ resp. $\omega_2^{r,s}$ be the time of first exit of W_1 resp. W_2 from the annulus $B(0,s) \setminus B(0,r)$. Write $\mathbb{U}$ for the centred compact cube of sidelength one.

Lemma 5.8. *For every $\varepsilon > 0$ there is an $N = N(\varepsilon) > 0$ and $p = p(\varepsilon) > 0$ such that, for any $n > N$ and for any $x_1, x_2 \in \partial\mathbb{U}$ and $u_1, u_2 \in \partial B(0,1/n) \cup \partial B(0,n^2)$,*

$$\begin{aligned}\mathbb{P}_{(x_1,x_2)2^{-n}}\Big\{ \dim\big(W_1[0,\omega_1^{2^{-n}/n,2^{-n}n^2}] \cap W_2[0,\omega_2^{2^{-n}/n,2^{-n}n^2}] \cap 2^{-n}\mathbb{U} \cap \Gamma[\gamma]\big) > 1-\gamma-\varepsilon \,\Big|\, \\ W_i(\omega_i^{2^{-n}/n,2^{-n}n^2}) = 2^{-n}u_i \text{ for } i \in \{1,2\},\ 2^{-n}\mathbb{U} \in \mathcal{S}_n \Big\} > p(\varepsilon)\,.\end{aligned} \tag{94}$$

Proof. By the scaling invariance of Brownian motion and the canonical self-similarity of percolation fractals, the left hand side of (94) is equal to

$$\mathbb{P}_{(x_1,x_2)}\Big\{ \dim\big(W_1[0,\omega_1^{1/n,n^2}] \cap W_2[0,\omega_2^{1/n,n^2}] \cap \Gamma[\gamma]\big) > 1-\gamma-\varepsilon \,\Big|\, W_i(\omega_i^{1/n,n^2}) = u_i \text{ for } i \in \{1,2\}\Big\},$$

where $\Gamma[\gamma]$ is a percolation fractal in the unit cube $\mathbb{U}$.

Using the same line of argument as in the beginning of Lemma 5.7 one can show that there exists an integer N such that, for all $n \geq N$, $x_1, x_2 \in \partial\mathbb{U}$ and $u_1, u_2 \in \partial B(0,1/n) \cup \partial B(0,n^2)$,

$$\begin{aligned}&\mathbb{P}_{(x_1,x_2)}\Big\{ \dim\big(W_1[0,\omega_1^{1/n,n^2}] \cap W_2[0,\omega_2^{1/n,n^2}] \cap \Gamma[\gamma]\big) > 1-\gamma-\varepsilon \,\Big| \\ &\qquad\qquad W_i(\omega_i^{1/n,n^2}) = u_i \text{ for } i \in \{1,2\}\Big\} \\ &\geq \tfrac{1}{2}\,\mathbb{P}_{(x_1,x_2)}\Big\{ \dim\big(W_1[0,\omega_1^{1/4,2}] \cap W_2[0,\omega_2^{1/4,2}] \cap \Gamma[\gamma]\big) > 1-\gamma-\varepsilon\Big\},\end{aligned}$$

thus getting rid of the conditioning on the exiting points. Using Lemma 5.5 (iii) it is easy to see that the latter probability is bounded from below by a positive constant depending only on ε. This completes the proof. □

The following lemma constitutes the main step in the proof of Proposition 5.6. Define, for $m \geq 1$,

$$U(m) = \Big\{ x \in S^* \,:\, \text{ there is } 0 < h < 2^{-m+1} \text{ with } \frac{\lambda(B(x,h))}{h[\log(1/h)]^2} > a - \frac{1}{m}\Big\}.$$

Lemma 5.9. *Almost surely, $U(m)$ is dense in S^* for all $m \in \mathbb{N}$.*

Proof. Fix $m \in \mathbb{N}$ and an open set $O \subset \mathtt{C}$. It suffices to show that, almost surely, the event $O \cap S^* \neq \emptyset$ implies that $U(m) \cap O \neq \emptyset$. By (91) the event $O \cap S^* \neq \emptyset$ implies that $\dim(O \cap S \cap \Gamma[\gamma]) \geq \delta$. Hence there exists a dyadic cube $V \in \mathfrak{D}$ with $V \subset O$ such that

$\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2$. It thus suffices to show that, almost surely for any dyadic cube $V \in \mathfrak{D}$ the event $\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2$ implies $U(m) \cap V \neq \emptyset$.

Fix $n \in \mathbb{N}$ and a dyadic cube $V \in \mathfrak{D}_n$. We introduce some terminology for the proof. For every integer $k \geq n$ we let $\mathfrak{C}^k$ be the collection of 2^{k-n} cubes in $\mathfrak{D}_k$ that are contained in V. We subdivide $\mathfrak{C}^k$ into $m(k) = (k^3+1)^3$ disjoint subcollections

$$\mathfrak{C}_1^k, \dots, \mathfrak{C}_{m(k)}^k$$

such that, for all $1 \leq j \leq m(k)$, any two distinct cubes in $\mathfrak{C}_j^k$ have distance at least $k^3 2^{-k}$.

Recall that $\sqrt{a}\,\theta = 1-\gamma$ and $\delta < 1-\gamma$. Fix a small number $\varepsilon > 0$ and some η such that

$$\delta < \eta \qquad \text{and} \qquad \sqrt{a - \tfrac{1}{2m}}\,\theta + \varepsilon < \eta < 1-\gamma\,. \tag{95}$$

For every cube $U \in \mathfrak{C}_j^k$ with centre ξ we let ρ_1^U be the first entry time of W_1 into U, τ_1^U the first entry time of W_1 into $B(\xi, 2^{-k}/k)$ and σ_1^U the first exit time after ρ_1^U of W_1 from $B(\xi, 2^{-k}k^2)$. Let $\omega_1^U = \tau_1^U \wedge \sigma_1^U$ be the first exit time after ρ_1^U from the annulus $B(\xi, 2^{-k}k^2) \setminus B(\xi, 2^{-k}/k)$.

Analogously for the second Brownian motion W_2 define the stopping times $\rho_2^U, \tau_2^U, \sigma_2^U$ and ω_2^U. Let λ^U be the intersection local time of the two Brownian motions on the (possibly degenerate) time intervals $[\omega_1^U, \sigma_1^U]$ resp. $[\omega_2^U, \sigma_2^U]$.

We call a cube $U \in \mathfrak{C}_j^k$ *admissible* if $U \in \mathcal{S}_k$ and both Brownian motions hit the cube, i.e. if $\rho_1^U, \rho_2^U < \infty$. Denote by M_j^k the number of admissible cubes $U \in \mathfrak{C}_j^k$.

An admissible cube $U \in \mathfrak{C}_j^k$ with centre ξ is called *successful* if

(A) both motions hit $B(\xi, 2^{-k}/k)$ before leaving $B(\xi, 2^{-k}k^2)$, in other words $\tau_1^U < \sigma_1^U$ and $\tau_2^U < \sigma_2^U$,
(B) the dimension of the intersection of $\Gamma[\gamma] \cap U$ with the paths before they hit $B(\xi, 2^{-k}/k)$ is bigger than η, formally $\dim\big(W_1[\rho_1^U, \omega_1^U] \cap W_2[\rho_2^U, \omega_2^U] \cap \Gamma[\gamma] \cap U\big) > \eta$,
(C) the intersection local time in $B(\xi, 2^{-k}k)$ of the paths started on first hitting $B(\xi, 2^{-k}/k)$ and stopped upon leaving $B(\xi, 2^{-k}k^2)$ is exceptionally large, more precisely

$$\frac{\lambda^U\big(B(\xi, 2^{-k}k)\big)}{2^{-k}k[\log(2^k/k)]^2} > a - \tfrac{1}{2m}.$$

We say that the collection $\mathfrak{C}_j^k$ is *bad* if there exists *no* successful cube in $\mathfrak{C}_j^k$.

Let us next explain that it is sufficient for the proof of this lemma to show that, almost surely given $\dim\big(V \cap S \cap \Gamma[\gamma]\big) > \delta/2$, for arbitrary large k there exists a successful cube $U \in \mathfrak{C}^k$.

Indeed, if $U \in \mathfrak{C}^k$ is successful, then, by condition (B), $\dim\big(S \cap \Gamma[\gamma] \cap U\big) > \eta > \delta$. Hence there does not exist a covering of $S \cap \Gamma[\gamma] \cap U$ by sets $B \in \mathfrak{B}$ with $\dim(S \cap \Gamma[\gamma] \cap B) < \delta$. This implies that $S^* \cap U \neq \emptyset$ by construction of S^*. Pick some $y \in U \cap S^*$ and infer from condition (C), for $h = 2^{-k}(k + \sqrt{3})$,

$$\lambda\big(B(y,h)\big) \geq \lambda^U\big(B(\xi, 2^{-k}k)\big) > \big(a - \tfrac{1}{2m}\big)2^{-k}k[\log(2^k/k)]^2.$$

Hence, if k is so large that $h < 2^{-m}$, and

$$\big(a - \tfrac{1}{2m}\big)2^{-k}k[\log(2^k/k)]^2 \geq \big(a - \tfrac{1}{m}\big)h[\log(1/h)]^2\,,$$

we have $y \in V \cap U(m)$ and the proof is finished.

Thus, writing $A_k = \{\mathfrak{E}_j^k \text{ is bad for all } 1 \le j \le m(k)\}$ it is sufficient to show that

$$\lim_{k\to\infty} \mathbb{P}\big\{A_k \,\big|\, \dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\big\} = 0.$$

We estimate the probability that all collections $\mathfrak{E}_j^k$ are bad as follows. We use the estimate

$$\begin{aligned} &\mathbb{P}\{A_k \,\big|\, \dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\} \\ &\le \mathbb{P}\Big\{M_j^k \le 2^{\eta n} \text{ for all } 1 \le j \le m(k) \,\Big|\, \dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\Big\} \\ &\quad + \sum_{j=1}^{m(k)} \frac{\mathbb{P}\Big\{\mathfrak{E}_j^k \text{ is bad } \,\Big|\, M_j^k > 2^{\eta n}\Big\}}{\mathbb{P}\{\dim(V \cap S \cap \Gamma[\gamma]) > \delta/2\}}. \end{aligned} \tag{96}$$

It suffices to show that the two terms on the last two lines of (96) vanish as $k \to \infty$. We begin with the term on the *second* line.

Recall the definition of an admissible cube, and the number M_j^k of admissible cubes $U \in \mathfrak{E}_j^k$. Further recall that $m(k)$ is a polynomial in k and assume that $\dim\big(V \cap S \cap \Gamma[\gamma]\big) > \delta/2$. Together with basic properties of the Hausdorff dimension this implies that

$$\liminf_{k\to\infty} \frac{1}{\log(2^k)} \log \max_{j=1}^{m(k)} M_j^k \ge \dim\big(V \cap S \cap \Gamma[\gamma]\big) > \delta/2\,,$$

Hence, for all sufficiently large k, there exists j with $M_j^k > 2^{k\delta/2}$.

The next step in the proof is to use self-similarity in order to improve this lower bound and get $M_j^k > 2^{k\eta}$ for some j, for all sufficiently large k. For this purpose fix k and $1 \le j \le m(k)$. We find a maximal finite sequence of stopping times

$$0 < \rho_1(1) < \sigma_1(1) < \rho_1(2) < \sigma_1(2) < \ldots < \rho_1(\mathfrak{m}) < \sigma_1(\mathfrak{m}) < \infty$$

successively as follows: Let $\rho_1(1)$ be the time of first entry of W_1 into some cube in $\mathfrak{E}_j^k$, denote its centre by $\xi_1(1)$ and let $\sigma_1(1)$ be the first exit time from $B(\xi_1(1), 2^{-k}k^2)$. Having constructed $\sigma_1(l-1)$, we let $\rho_1(l)$ be the first time of entry of W_1 into a cube in $\mathfrak{E}_j^k$, which is different from all previous cubes, denote its centre by $\xi_1(l)$ and let $\sigma_1(l)$ be the first exit time from $B(\xi_1(l), 2^{-k}k^2)$. We proceed until $\rho_1(\mathfrak{m}+1) = \infty$.

Denote by $\mathcal{F}_1(j,k)$ the σ-field generated by W_1 restricted to the time domain

$$\big[0, \rho_1(1)\big] \cup \big[\sigma_1(1), \rho_1(2)\big] \cup \ldots \cup \big[\sigma_1(\mathfrak{m}), \infty\big)\,.$$

The analogous stopping times for W_2 are denoted by $\rho_2(l), \sigma_2(l)$ and the σ-field for W_2 is $\mathcal{F}_2(j,k)$. The σ-field generated by $\mathcal{S}_k$ is called $\mathcal{G}(k)$ and we let $\mathcal{F}(j,k) = \mathcal{F}_1(j,k) \vee \mathcal{F}_2(j,k) \vee \mathcal{G}(k)$. It is important to note at this place that whether or not a cube $U \in \mathfrak{E}_j^k$ is admissible is an event in $\mathcal{F}(j,k)$ and hence M_j^k is $\mathcal{F}(j,k)$-measurable.

Conditional on $\mathcal{F}(j,k)$, there is a fixed maximal collection $\{U_1, \ldots, U_M\} \subset \mathfrak{E}_j^k$ of admissible cubes (where $M = M_j^k$) whose centres we denote by $\xi_1, \ldots, \xi_M$. For each $1 \le l \le M$ there are unique values $\zeta_i(l) \in \mathbb{N}$ such that $\rho_i(\zeta_i(l))$ is the first entry time of W_i in U_l and $\sigma_i(\zeta_i(l))$ the exit time from the ball $B(\xi_l, 2^{-k}k^2)$. Moreover, each Brownian motion W_i in the time interval $[\rho_i(\zeta_i(l)), \sigma_i(\zeta_i(l))]$ is conditioned to start from a fixed point on ∂U_l and is stopped upon exiting the ball $B(\xi_i(l), 2^{-k}k^2)$ at a fixed point of the boundary. Consider the events

$$E_l = \Big\{\dim\big(W_1[\rho_1^{U_l}, \omega_1^{U_l}] \cap W_2[\rho_2^{U_l}, \omega_2^{U_l}] \cap U \cap \Gamma[\gamma]\big) > \eta\Big\}.$$

Conditional on $\mathcal{F}(j,k)$, the events $E_1,\dots,E_M$ are independent, and, by Lemma 5.8, the conditional probability of each E_l is

$$\mathbb{P}\big\{E_l \,\big|\, \mathcal{F}(j,k)\big\} \geq p(1-\gamma-\eta)\,.$$

Hence the conditional probability that all events $E_1,\dots,E_M$ fail is at most

$$\Big(1-p(1-\gamma-\eta)\Big)^{M_j^k}\,.$$

We infer that the conditional probability, given $\dim\big(V\cap S\cap\Gamma[\gamma]\big)>\delta/2$, that *no* dyadic subcube $U\subset V$ has $\dim(U\cap S\cap\Gamma[\gamma])>\eta$ is bounded from above by

$$\limsup_{k\to\infty}\, m(k)\Big(1-p(1-\gamma-\eta)\Big)^{2^{k\delta/2}}=0\,.$$

We have thus shown that

$$\mathbb{P}\big\{\dim\big(V\cap S\cap\Gamma[\gamma]\big)>\eta\,\big|\,\dim\big(V\cap S\cap\Gamma[\gamma]\big)>\delta/2\big\}=1\,. \tag{97}$$

As above, this implies that, for all suficiently large k, we have $M_j^k>2^{k\eta}$ for some $1\leq j\leq m(k)$, almost surely on $\dim\big(V\cap S\cap\Gamma[\gamma]\big)>\delta/2$. In particular, the term on the second line of (96) converges to 0 as $k\to\infty$.

To deal with the term on the *third* line of (96), we again fix $k\geq 2$ and $j\leq m(k)$. The problem is to control the probability, conditional on $\mathcal{F}(j,k)$, that the collection of admissible cubes does not contain a successful cube. More precisely, we show that, on $M_j^k>2^{\eta n}$, the conditional probability that $\mathfrak{C}_j^k$ is bad is exponentially decreasing.

Conditional on $\mathcal{F}(j,k)$, we again look at the maximal collection $\{U_1,\dots,U_M\}\subset\mathfrak{C}_j^k$ of admissible cubes and recall the notation, ξ_l for the centres of the cubes, and $\zeta_i(l)\in\mathbb{N}$ such that $\rho_i(\zeta_i(l))$ is the first entry time of W_i in U_l and $\sigma_i(\zeta_i(l))$ the exit time from $B(\xi_l,2^{-k}k^2)$. Additionally, let $\tau_i(\zeta_i(l))$ the first entry time into $B(\xi_l,2^{-k}/k)$ and $\omega_i(\zeta_i(l))=\sigma_i(\zeta_i(l))\wedge\tau_i(\zeta_i(l))$.

Conditional on $\mathcal{F}(j,k)$ each Brownian motion W_i in the time interval $[\rho_i(\zeta_i(l)),\sigma_i(\zeta_i(l))]$ is conditioned to start on a fixed point on the boundary ∂U_l and exit the ball around ξ_l with radius $2^{-k}k^2$ at a fixed point $W_i(\sigma_i(\zeta_i(l)))$.

For $l\in\{1,\dots,M\}$ recall that λ^{U_l} denotes the intersection local time of the two motions in the intervals $[\tau_1(\zeta_1(l)),\sigma_1(\zeta_1(l))]$ resp. $[\tau_2(\zeta_2(l)),\sigma_2(\zeta_2(l))]$. Consider the event E_l that $\tau_1(\zeta_1(l))<\sigma_1(\zeta_1(l))$, $\tau_2(\zeta_2(l))<\sigma_2(\zeta_2(l))$, and

$$\dim\Big(W_1[\rho_1(\zeta_1(l)),\omega_1(\zeta_1(l)]\cap W_2[\rho_2(\zeta_2(l)),\omega_2(\zeta_2(l)]\cap U\cap\Gamma[\gamma]\Big)>\eta,$$

and also

$$\frac{\lambda^{U_l}\big(B(\xi,2^{-k}k)\big)}{2^{-k}k[\log(2^k/k)]^2}>a-\tfrac{1}{2m}.$$

Clearly the event E_l implies that U_l is a successful cube. Moreover, conditional on $\mathcal{F}(j,k)$, the events $E_1,\dots,E_M$ are independent. To estimate the probability of E_l from below, first note that, for a Brownian motion started in $x_i\in\partial U_l$,

$$\mathbb{P}_{x_i}\Big\{\tau_i^{U_l}<\sigma_i^{U_l}\Big\}\geq\frac{\sqrt{2}-k^{-2}}{k-k^{-2}}\geq\frac{c}{k}\,,$$

for an absolute constant $c > 0$. From Lemma 5.8 we know that, given $u_i \in \partial B(\xi_l, 2^{-k}/k)$, that

$$\mathbb{P}\Big\{ \dim\big(W_1[\rho_1(\zeta_1(l)), \omega_1(\zeta_1(l))] \cap W_2[\rho_2(\zeta_2(l)), \omega_2(\zeta_2(l))] \cap U \cap \Gamma[\gamma]\big) > \eta \,\Big|\, W_i(\omega_i^{U_l}) = u_i \text{ for } i \in \{1,2\}\Big\} > p(1-\gamma-\eta)\,,$$

and finally, by Lemma 5.7,

$$\mathbb{P}_{(u_1,u_2)}\Big\{ \frac{\lambda^{U_l}\big(B(\xi, 2^{-k}k)\big)}{2^{-k}k[\log(2^k/k)]^2} > a - \tfrac{1}{2m} \Big\} \geq 2^{-k\left(\sqrt{a-\frac{1}{2m}}\theta+\varepsilon\right)}.$$

Altogether, the conditional probability of each E_l is bounded from below by

$$\mathbb{P}\big\{E_l \,\big|\, \mathcal{F}(j,k)\big\} \geq \frac{C}{k^2}\, 2^{-k\left(\sqrt{a-\frac{1}{2m}}\theta+\varepsilon\right)}\,,$$

for $C = c^2 p(1-\gamma-\eta)$. Hence the conditional probability that the collection of admissible cubes does *not* contain a successful cube is at most

$$\Big(1 - \frac{C}{k^2} 2^{-k\left(\sqrt{a-\frac{1}{2m}}\theta+\varepsilon\right)}\Big)^M.$$

We infer that, for all $1 \leq j \leq m(k)$,

$$\begin{aligned}\mathbb{P}\Big\{\mathfrak{C}_j^k \text{ is bad} \,\Big|\, M_j^k > 2^{\eta k}\Big\} &\leq \mathbb{E}\Big[\Big(1 - \frac{C}{k^2} 2^{-k\left(\sqrt{a-\frac{1}{2m}}\theta+\varepsilon\right)}\Big)^{M_j^k} \,\Big|\, M_j^k > 2^{\eta k}\Big] \\ &\leq \Big(1 - \frac{C}{k^2} 2^{-k\left(\sqrt{a-\frac{1}{2m}}\theta+\varepsilon\right)}\Big)^{2^{\eta k}} \leq \exp\Big(-\frac{C}{k^2} 2^{k\left(\eta-\sqrt{a-\frac{1}{2m}}\theta-\varepsilon\right)}\Big).\end{aligned} \tag{98}$$

Recalling (95) we see that the right hand side decreases exponentially fast. As the number of summands in the third line of (96) is just polynomial, we infer that this term converges to 0 as $k \to \infty$. ☐

LEMMA 5.10. *$F(a) \cap S^*$ is almost surely dense in S^*. In particular, we have*

$$\mathbb{P}\big\{F(a) \cap S^* \neq \emptyset \,\big|\, S^* \neq \emptyset\big\} = 1\,.$$

PROOF. Note that $U(m)$ is relatively open in S^* and, by Lemma 5.9, also dense in S^* for any m. As S^* is compact, hence complete, one can infer from Baire's Theorem that

$$F(a) \cap S^* = \bigcap_{m=1}^{\infty} U(m)$$

is dense in S^* almost surely. Hence $\mathbb{P}\big\{F(a) \cap S^* \neq \emptyset \,\big|\, S^* \neq \emptyset\big\} = 1$. ☐

PROOF OF PROPOSITION 5.6. Recall that

$$\begin{aligned}\mathbb{P}\big\{F(a) \cap \Gamma[\gamma] \neq \emptyset\big\} &\geq \mathbb{P}\big\{F(a) \cap S^* \neq \emptyset\big\} = \mathbb{P}\big\{F(a) \cap S^* \neq \emptyset \,\big|\, S^* \neq \emptyset\big\}\mathbb{P}\big\{S^* \neq \emptyset\big\} \\ &= \mathbb{P}\big\{S^* \neq \emptyset\big\} > 0\,.\end{aligned}$$

This proves Proposition 5.6. ☐

5.3. Completion of the proof of Theorem 1.6. Fix $0 \le a \le 1/\theta^2$ and denote the set of *strictly a-thick points* by

$$H(a) = \Big\{ x \in \mathbb{R}^3 \,:\, \limsup_{r\downarrow 0} \frac{\lambda(B(x,r))}{r[\log(1/r)]^2} = a \Big\}.$$

For the upper bound in (18) first fix a unit cube $\mathtt{C}$ at positive distance from the origin. Then, by Proposition 5.1, we have

$$\dim(H(a)\cap \mathtt{C}) \le \dim F(a) \le 1-\theta\sqrt{a}.$$

Taking a countable family of such cubes covering $\mathbb{R}^3\setminus\{0\}$ we get $\dim H(a) \le 1-\theta\sqrt{a}$.

Turning to the proof of the lower bound in (18), we look at a large open ball $B\subset\mathbb{R}^3$ centred at the origin and fix a compact unit cube $\mathtt{C}\subset B$ at positive distance from the origin. Let $\gamma = 1-\theta\sqrt{a}$. By Proposition 5.1 (ii),

$$\dim\big(F(a+\tfrac1n)\big) \le 1-\theta\sqrt{a+\tfrac1n} < \gamma,$$

and, by Lemma 5.5 (ii), we have that $F(a+1/n)\cap\Gamma[\gamma]=\emptyset$ almost surely for all n. Hence, almost surely,

$$H(a)\cap\Gamma[\gamma] = F(a)\cap\Gamma[\gamma]\cap\bigcap_{n=1}^{\infty} F(a+1/n)^{\mathrm{c}} = F(a)\cap\Gamma[\gamma]. \tag{99}$$

Recall from Proposition 5.6 that the set on the right hand side of (99) is nonempty with positive probability. Hence we have shown that $\mathbb{P}\big\{H(a)\cap\Gamma[\gamma]\ne\emptyset\big\}>0$. Together with Lemma 5.5 (i) this implies that $\dim\big(H(a)\cap B\big)\ge\gamma$ with positive probability.

By the Brownian scaling property this probability does not depend on the radius of the ball B, so that it holds for arbitrarily small balls. In particular,

$$\mathbb{P}\Big\{\dim\big(H(a)\cap B(0,r)\big)\ge\gamma \text{ for all } r>0\Big\} = \lim_{r\downarrow0}\mathbb{P}\Big\{\dim\big(H(a)\cap B(0,r)\big)\ge\gamma\Big\}>0.$$

We now use Blumenthal's zero-one law to see that this probability is actually equal to one. Indeed, recall that *three* Brownian paths almost surely do not have a point of intersection, hence after the first exit times T_1 resp. T_2 from any centred ball B both Brownian motions do not hit $W[0,T_1]\cap W[0,T_2]$ again and, by compactness, even keep a positive distance from $W[0,T_1]\cap W[0,T_2]$. We infer that almost surely every point, which is strictly a-thick for the intersection local time of the Brownian motions stopped at T_1 resp. T_2, is also strictly a-thick for the unstopped motions. Moreover, by transience, there exists a small (random) centred ball, which is not visited by any Brownian motion after T_1 resp. T_2. Thus the event $\{\dim\big(H(a)\cap B(0,r)\big)\ge\gamma \text{ for all } r>0\}$ is in the completion of the σ-field generated jointly by the Brownian motion W_1 stopped in T_1 and the Brownian motion W_2 stopped in T_2. This holds for every ball B and hence, by Blumenthal's zero-one law, we infer that

$$\mathbb{P}\Big\{\dim H(a)\ge\gamma\Big\}\ge\mathbb{P}\Big\{\dim\big(H(a)\cap B(0,r)\big)\ge\gamma \text{ for all } r>0\Big\}=1.$$

This finishes the proof of the lower bound in (18).

In order to prove (17) we observe that

$$\frac{1}{\theta^2}\le\sup_{x\in S}\limsup_{r\downarrow0}\frac{\lambda(B(x,r))}{r[\log(1/r)]^2}\le\limsup_{r\downarrow0}\sup_{x\in S}\frac{\lambda(B(x,r))}{r[\log(1/r)]^2}\le\frac{1}{\theta^2}.$$

Indeed, the first equality follows from the fact that, for every $a < 1/\theta^2$, the set of a-thick points has positive dimension and hence is nonempty. The second inequality is obvious, and the third inequality is Lemma 5.1 (iii). This proves (17) and thus finishes the proof of Theorem 1.6.

6. Outlook on future work

The characterization of the limit in (8) in terms of (9) naturally raises the question how the maximizers g of (9) can be interpreted.

For a first answer define, on $\{\lambda(U) > 0\}$, the random probability measure L on U as the normalized restriction of λ to U, more precisely let

$$L(A) = \lambda(A)/\lambda(U), \text{ for } A \subset U \text{ Borel.}$$

We ask how the measure L distributes the unit mass over the set U if we condition the Brownian paths to have a large amount of occupation measure $\lambda(U)$. An answer to this question is given by the following result. Let $\mathtt{d} : U \times U \to [0, \infty)$ be any metric that induces the weak topology on $\mathcal{M}_1(U)$.

THEOREM 6.1 (Law of large masses). *Let $\mathfrak{M} \subset \mathcal{M}_1(U)$ be the set of probability measures $\mu(dx) = g^{2p}(x)dx$ on U with g a maximizer in the variational problem (9). Then, for all $\varepsilon > 0$,*

$$\lim_{a\uparrow\infty} \mathbb{P}\Big\{\mathtt{d}\left(L, \mathfrak{M}\right) > \varepsilon \,\Big|\, \lambda(U) > a\Big\} = 0. \tag{100}$$

The convergence in (100) is exponentially fast with rate $a^{1/p}$. Theorem 6.1 itself is a consequence of the identification of the exact rate of decay of $\mathbb{P}\{L \in A \,|\, \lambda(U) > a\}$ for sets $A \subset \mathcal{M}_1(U)$ in terms of a *large deviation principle.* This problem goes beyond the scope of the present paper and will be treated in a forthcoming paper.

CHAPTER 4

How fast are the particles of super-Brownian motion?

TO APPEAR IN PROBABILITY THEORY AND RELATED FIELDS.

Abstract. In this paper we investigate fast particles in the range and support of super-Brownian motion in the historical setting. In this setting each particle of super-Brownian motion alive at time t is represented by a path $w : [0,t] \to \mathbb{R}^d$ and the state of historical super-Brownian motion is a measure on the set of paths. Typical particles have Brownian paths, however in the uncountable collection of particles in the range of a super-Brownian motion there are some which at exceptional times move faster than Brownian motion. We determine the maximal speed of all particles during a given time period E, which turns out to be a function of the packing dimension of E. A path w in the support of historical super-Brownian motion at time t is called *a-fast* if $\limsup_{h\downarrow 0} |w(t) - w(t-h)|/\sqrt{h \log(1/h)} \geq a$. We calculate the Hausdorff dimension of the set of a-fast paths in the support and the range of historical super-Brownian motion. A valuable tool in the proofs is a uniform dimension formula for the Brownian snake, which reduces dimension problems in the space of stopped paths to dimension problems on the line.

Mathematics Subject Classification (2000): *60G17, 60J65, 60J80, 28A78.*
Key words: *super-Brownian motion, Brownian snake, measure-valued diffusion, historical process, limsup random fractal, Hausdorff dimension, dimension spectrum, packing dimension, fast point, modulus of continuity, uniform dimension formula.*

1. Introduction and statement of the main results

Super-Brownian motion is the high density limit of a system of branching Brownian motions with a critical, finite variance branching mechanism. It has been an object of intensive research in the last fifteen years, leading to interesting results about its fractal geometry and its potential theoretic properties. Recent deep results emphasize its importance as the natural scaling limit of many interacting particle systems, see for example [**CDP99**], [**Sla99**] and [**LeG99**] for surveys.

In the setting of historical processes, introduced by Dawson and Perkins in [**DaP91**], super-Brownian motion is a continuous Markov process $\{\mathfrak{X}_t\}$ whose value at time t is a finite measure on the space

$$\mathfrak{P} := \Big\{(w,\zeta) \,:\, w : [0,\infty) \to \mathbb{R}^d \text{ continuous with } w(s) = w(\zeta) \text{ for all } s \geq \zeta \geq 0\Big\}$$

of *stopped paths w with total lifetime ζ*. Sometimes we write short w for (w,ζ). The state $\mathfrak{X}_t$ of super-Brownian motion is a random measure on $\mathfrak{P}$ supported by a set of paths with lifetime $\zeta = t$. Intuitively, each path in $\operatorname{supp} \mathfrak{X}_t$ represents the trail of a particle alive at time t, so that the model describes the individual movements of all the particles. The historical setting also contains the complete description of the genealogical structure of super-Brownian motion: Any two particles $w_1, w_2 \in \operatorname{supp} \mathfrak{X}_t$ are obtained from the branching of the particle $w : [0,\infty) \to \mathbb{R}^d$ at time $\zeta = \sup\{r : w_1 = w_2 \text{ on } [0,r]\}$ with $w(s) = w_1(s) = w_2(s)$ for all $s \le \zeta$ and $w(s) = w(\zeta)$ for all $s \ge \zeta$.

In a super-Brownian motion, typical particles move according to Brownian motion, which means that the posterior distribution of a path picked according to a (normalised) sample measure $\mathfrak{X}_t$ is the Wiener measure. As is well known, the maximal speed of a Brownian particle is given by Paul Lévy's uniform modulus of continuity, which in our setting states that for every $t > 0$, almost surely, for $\mathfrak{X}_t$-almost every path w,

$$\sup_{s>0} \limsup_{h\downarrow 0} \frac{|w(s) - w(s-h)|}{\sqrt{h \log(1/h)}} = \sqrt{2}\,. \tag{1}$$

Recall that the *range* of a (historical) super-Brownian motion

$$\operatorname{range} \mathfrak{X} = \bigcup_{\varepsilon>0} \overline{\bigcup_{t>\varepsilon} \operatorname{supp} \mathfrak{X}_t}$$

is an uncountable collection of paths, so that one might wonder whether for exceptional paths in the range of a super-Brownian motion a higher speed is possible. This question is answered affirmatively by a remarkable result of Dawson and Perkins [**DaP91**, 8.7], who show that special particles can move faster than a Brownian motion. More precisely, almost surely,

$$\sup_{s>0} \sup_{w\in \operatorname{range}\mathfrak{X}} \limsup_{h\downarrow 0} \frac{|w(s) - w(s-h)|}{\sqrt{h \log(1/h)}} = 2\,. \tag{2}$$

The question of the maximal speed of all particles at a particular instant in time can be treated similarly, Verzani [**Ver95**] points out that, at any fixed time $t > 0$, almost surely on $\{\mathfrak{X}_t \neq 0\}$,

$$\sup_{w\in \operatorname{range}\mathfrak{X}} \limsup_{h\downarrow 0} \frac{|w(t) - w(t-h)|}{\sqrt{h \log(1/h)}} = \sqrt{2}. \tag{3}$$

For a deeper understanding of the values appearing at the right hand sides in formulas (1), (2) and (3) we consider a Borel set $E \subset (0,\infty)$ of times and ask for the maximum speed of all particles at all times in E. It turns out that the maximum speed of all particles at all times in E is related to the packing dimension, denoted $\dim_P(E)$, of the set $E \subset \mathbb{R}$.

The most natural definition of *packing dimension* is based on the family of s-dimensional packing measures. A packing of the set E is a collection of disjoint balls centred in E and s-dimensional packing measure can be defined in essentially the same way as the s-dimensional Hausdorff measure replacing minimal covers by maximal packings of E. The packing dimension of E is the critical parameter s where the packing measure switches from infinity to 0. Readers unfamiliar with the concept are recommended to consult [**Mat95**, 5.10] for more precise statements.

THEOREM 1.1 (Speed in a given time period). *Suppose that $\{\mathfrak{X}_t\}$ is a super-Brownian motion in dimension $d \ge 1$ and $E \subset (0,t)$ a Borel set. Then, almost surely on $\{\mathfrak{X}_t \neq 0\}$,*

$$\sup_{s\in E} \sup_{w\in \operatorname{range}\mathfrak{X}} \limsup_{h\downarrow 0} \frac{|w(s) - w(s-h)|}{\sqrt{h \log(1/h)}} = \sqrt{2 + 2\dim_P(E)}\,.$$

REMARK 1.2. a) Observe that the theorem is independent of the choice of a finite starting mass for the super-Brownian motion.

b) Theorem 1.1 contains (3), by choosing E as a singleton, and (2), by exhausting $(0,\infty)$.

Theorem 1.1 should be compared to recent related results of Khoshnevisan, Peres and Xiao [**KPX00**] on the modulus of continuity of classical Brownian paths in a given time set. Continuing work of Khoshnevisan and Shi [**KhS00**], they showed that for each analytic set $E \subset (0,1)$, almost surely,

$$\sup_{t\in E} \limsup_{h\downarrow 0} \frac{|B(t)-B(t-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2\dim_P(E)}\,. \tag{4}$$

Our proof adapts their technique to the superprocess setting, see Section 2 for an outline of the proof and a refined statement (Proposition 2.5).

We now assume that super-Brownian motion is started with a unit mass at the origin so that $w(0)=0$ for all $w \in \operatorname{range} \mathfrak{X}$. The second result of this paper determines a *Hausdorff-dimension spectrum* for fast paths in the range and support of historical super-Brownian motion, which is analogous to the result

$$\dim\Big\{t\in[0,1] \,:\, \limsup_{h\downarrow 0} \frac{|B(t)-B(t-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} = 1-\frac{a^2}{2}, \text{ for } a\in[0,\sqrt{2}], \tag{5}$$

of Orey and Taylor [**OrT74**] about the fast times of classical Brownian motion. Here and throughout the paper dim stands for the Hausdorff dimension of a metric space. For the formulation of our results we equip the space $\mathfrak{P}$ of stopped paths with the metric

$$d\Big((w_1,\zeta_1),(w_2,\zeta_2)\Big) = \|w_1-w_2\|_\infty + \sqrt{|\zeta_1-\zeta_2|}\,. \tag{6}$$

Note that, comparing with the choice of the metric in [**LeG99**] and most other papers on the subject, I have introduced a square root in the second summand, in order to sharpen the statements. See Remark 1.4(b) for other metrics.

Of course, the space $\mathfrak{P}$ of stopped paths is of infinite Hausdorff dimension, but the range of historical super-Brownian motion has dimension no more than 4. This is intuitively plausible because for historical super-Brownian motion all particles alive at time t are obtained by branching from just a finite number of paths at time $t-\varepsilon$.

We consider a closed set $E\subset(0,\infty)$ of times and denote the *range of E* by

$$\mathfrak{R}(E) = \overline{\bigcup_{t\in E} \operatorname{supp} \mathfrak{X}_t}\,.$$

Intuitively this is the collection of particles with total lifetime in the set E. The result of Theorem 1.1 also gives that

$$\sup_{w\in\mathfrak{R}(E)} \limsup_{h\downarrow 0} \frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} = \sqrt{2+2\dim_P(E)}\,.$$

We can give bounds for the dimension of the set of a-fast paths in the range of E, which are sharp for all closed sets E, for which Hausdorff- and packing dimension agree.

THEOREM 1.3 (Dimension spectrum). *Suppose that $E\subset(0,t)$ is a fixed closed set and $\{\mathfrak{X}_t\}$ is a historical super-Brownian motion in dimension $d\geq 2$. Then, almost surely on $\{\mathfrak{X}_t\neq 0\}$, for all $a\in[0,\sqrt{2+2\dim_P(E)}]$,*

$$2+2\dim(E)-a^2 \leq \dim\Big\{(w,\zeta)\in\mathfrak{R}(E) \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} \leq 2+2\dim_P(E)-a^2\,.$$

Moreover, if $\dim_P(E) < a^2/2 - 1$ *the set of a-fast paths in* $\mathfrak{R}(E)$ *is empty. In particular, almost surely for all* $a \in [0, 2]$,

$$\dim\Big\{(w,\zeta) \in \operatorname{range}\mathfrak{X} \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta) - w(\zeta - h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} = 4 - a^2\,, \tag{7}$$

and, for all $t > 0$, *almost surely on* $\{\mathfrak{X}_t \neq 0\}$, *for all* $a \in [0, \sqrt{2}]$,

$$\dim\Big\{(w,\zeta) \in \operatorname{supp}\mathfrak{X}_t \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta) - w(\zeta - h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} = 2 - a^2\,. \tag{8}$$

REMARK 1.4. (a) In all the results of Theorem 1.3 one can get the same dimension spectra replacing the set of a-fast paths by

$$\Big\{(w,\zeta) \in \mathfrak{R}(E) \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta) - w(\zeta - h)|}{\sqrt{h\log(1/h)}} = a\Big\}$$

of *strictly a-fast paths.*

(b) In $d \geq 4$ the results of Theorem 1.3 remain true if in the definition (6) of the metric d on $\mathfrak{P}$ either the square root is removed from the second term or the term is completely removed. The results can also be stated in a non-historical setting by looking at *fast points* $w(\zeta) \in \mathbb{R}^d$, which are the endpoints of the fast paths (w, ζ). In $d \geq 4$ we obtain the same spectrum for the Hausdorff dimension of the fast points. The analogue of (8) even holds for the fast points in all dimensions $d \geq 2$. These variants of Theorem 1.3 follow by using the variants of Theorem 2.3, which are outlined in Remark 2.4 and can be proved with the same method.

In the following section we briefly review the construction of the historical super-Brownian motion by means of the Brownian snake and present a technique of reducing dimension problems in the space $\mathfrak{P}$ to dimension problems on the line. We then review a recent technique for obtaining hitting estimates and dimension bounds for a class of random fractals on the line. Details of the proofs are carried out in Sections 3, 4, 5 and 6 and we finish the paper with some final remarks and a brief discussion of related work.

2. Outline and method of proof

2.1. Historical super-Brownian motion and a uniform dimension formula. In this subsection we briefly review the construction of historical super-Brownian motion by means of the Brownian snake. This construction is due to Le Gall, see for example [**LeG91**], [**LeG93**], [**LeG99**]. The Brownian snake allows us to reduce dimension problems in path space to dimension problems on the line by means of a uniform dimension formula, see Theorem 2.3.

We restrict attention to super-Brownian motion started with a unit mass concentrated at the origin. With only a little more formal effort one can generalise the construction for super-Brownian motion with any finite starting mass. The Brownian snake is a continuous strong Markov process $\{W_s = (w_s, \zeta_s) : s \geq 0\}$ with values in the set $\mathfrak{P}$ of stopped paths. Denote by $\Omega = \mathcal{C}([0,\infty), \mathfrak{P})$ the space of continuous mappings from $[0, \infty)$ to $\mathfrak{P}$ equipped with the Borel-σ-field $\mathcal{A}$ coming from the compact-open topology and by $W = \{W_s\}_{s\geq 0}$ the coordinate process on Ω. By $\mathbb{P}$ denote the law on $(\Omega, \mathcal{A})$ of the path-valued process associated with d-dimensional Brownian motion, or *Brownian snake*, starting in the constant path at the origin of lifetime 0. This law was introduced in [**LeG91**], [**LeG93**] and can be described as follows. First, the *lifetime process* $\{\zeta_s\}$ is a reflected Brownian motion. We denote the σ-field generated on Ω by the process $\{\zeta_s\}$ by $\mathcal{F}$. Second, the distribution of $\{w_s\}$ given $\mathcal{F}$ is that of an inhomogeneous Markov process whose transition kernel from $s > 0$ to $r > s$ is described by

(a) $w_r(u) = w_s(u)$ for all $u \le m := \inf_{s \le v \le r} \zeta_v$.
(b) conditionally given m the process $\{w_r(m+t) - w_s(m) \,:\, 0 \le t \le \zeta_r - m\}$ is independent of w_s and distributed as a Brownian path started in the origin and stopped at time $\zeta_r - m$.

The intuitive picture is that $\{w_s\}$ grows like a Brownian motion in $\mathbb{R}^d$ when $\{\zeta_s\}$ is increasing and is erased, when $\{\zeta_s\}$ is decreasing (this is merely heuristical, as $\{\zeta_s\}$, of course, has neither points of increase, nor of decrease). We denote by $\{L_s^t \,:\, s \ge 0\}$ the continuous local time of $\{\zeta_s\}$ at level t, normalised to be a density of the occupation measure of $\{\zeta_s\}$, and by $\tau = \tau(\zeta) = \inf\{s \ge 0 \,:\, L_s^0 = 1\}$ the inverse local time at 1. Let $\widetilde{W}_s = w_s(\zeta_s)$ be the endpoint of W_s. Then

$$X_t(A) = \int_0^\tau 1_A(\widetilde{W}_s)\, L_\cdot^t(ds) \quad \text{, for } A \subseteq \mathbb{R}^d \text{ Borel,} \tag{9}$$

defines a continuous Markov process $\{X_t\}$ with values in the space $\mathcal{M}_F(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$ with the weak topology. This process is a *super-Brownian motion started with a unit mass in the origin.* The corresponding *historical super-Brownian motion* can be defined by

$$\mathfrak{X}_t(A) = \int_0^\tau 1_A(W_s)\, L_\cdot^t(ds) \quad \text{, for } A \subseteq \mathfrak{P} \text{ Borel.} \tag{10}$$

Observe that projection of paths onto their endpoints maps the process $\{\mathfrak{X}_t\}$ onto $\{X_t\}$. Conversely, in the historical setting we attach to each particle in $\operatorname{supp} X_t$ its path up to time $t > 0$. This richer model allows the formulation of more precise results than super-Brownian motion alone. Almost surely, for each $t > 0$, the support of $\mathfrak{X}_t$ agrees up to at most one path with $W\{s \in [0,\tau] \,:\, \zeta_s = t\}$. The exceptional paths arise, when $\zeta_s = t$ is a local maximum of ζ. As the isolated points of a fixed closed set $E \subset (0,\infty)$ are almost surely not local maxima of the reflected Brownian motion ζ, we obtain, almost surely,

$$\mathfrak{R}(E) = W(\zeta^{-1}(E)) \text{ and, in particular, } \operatorname{range}\mathfrak{X} = W[0,\tau]\,.$$

Observe at this point that

$$\sup_{t\in E}\ \sup_{w\in\operatorname{range}\mathfrak{X}}\ \limsup_{h\downarrow 0} \frac{|w(t)-w(t-h)|}{\sqrt{h\log(1/h)}} \quad = \quad \sup_{\substack{s\in[0,\tau]\cap\\ \zeta^{-1}(E)}}\ \limsup_{h\downarrow 0} \frac{|w_s(\zeta_s)-w_s(\zeta_s-h)|}{\sqrt{h\log(1/h)}}\,.$$

Define, for any $a \ge 0$, the set of *a-fast times* or simply *fast times* as

$$F(a) = \Big\{ s \in [0,\tau] \,:\, \limsup_{h\downarrow 0} \frac{|w_s(\zeta_s)-w_s(\zeta_s-h)|}{\sqrt{h\log(1/h)}} \ge a \Big\}\,,$$

and note that the fast times are not times for the super-Brownian motion, but for the Brownian snake. The statement of Theorem 1.1 is now reduced to the problem to identify those parameters a for which the set of a-fast times hits the preimage $\zeta^{-1}(E)$. To solve this, we prove a general result about hitting probabilities of the random set of a-fast times. Recall that $\mathcal{F}$ is the σ-field on Ω generated by the lifetime process $\{\zeta_s\}$.

Proposition 2.1 (Hitting Probabilities). *For the Brownian snake $W = \{(w_s,\zeta_s) \,:\, s \ge 0\}$ and any $a \ge 0$ let $F(a)$ be the set of a-fast times. Then, for every $\mathcal{F}$-measurable random Borel set $D \subset [0,\tau]$, almost surely,*

$$\mathbb{P}\Big\{ F(a) \cap D \ne \emptyset \Big| \mathcal{F} \Big\} = \begin{cases} 1 & \text{if } \dim_P D > a^2/4, \\ 0 & \text{if } \dim_P D < a^2/4. \end{cases}$$

Observe that the set D is allowed to depend on the lifetime process ζ, so that it is feasible to apply it to $D = \zeta^{-1}(E)$. In order to carry this out we finally need a formula relating the packing dimension of $\zeta^{-1}(E)$ to the packing dimension of E. The necessary result is analogous to the result for Hausdorff dimension, which was proved by Serlet [**Ser95**], see the appendix there.

LEMMA 2.2. *Let $\{\zeta_s : 0 \le s \le \tau\}$ be a reflected Brownian motion and $t > 0$. Then, almost surely on $\sup\{\zeta_s : 0 \le s \le \tau\} > t$, we have for every nonempty Borel subset $E \subset (0,t)$,*

$$\dim_P \zeta^{-1}(E) = \frac{1}{2} + \frac{\dim_P E}{2}.$$

Now the proof of Theorem 1.1 follows easily from Proposition 2.1 and Lemma 2.2. Just observe that, almost surely on $\{\mathfrak{X}_t \neq \emptyset\}$,

$$\begin{aligned}\sup_{\substack{w \in \text{range}\, \mathfrak{X} \\ t \in E}} \limsup_{h \downarrow 0} \frac{|w(t) - w(t-h)|}{\sqrt{h \log(1/h)}} &= \sup_{s \in \zeta^{-1}(E)} \limsup_{h \downarrow 0} \frac{|w_s(\zeta_s) - w_s(\zeta_s - h)|}{\sqrt{h \log(1/h)}} \\ &= \sup\{a : F(a) \cap \zeta^{-1}(E) \neq \emptyset\} = 2\sqrt{\dim_P \zeta^{-1}(E)} = \sqrt{2 + 2\dim_P(E)}.\end{aligned}$$

Proposition 2.1 also covers the classical Lévy modulus (1) and allows to shed some light on the equality of the right hand sides in (1) and (3). Fix $s > 0$. Then, conditional on $s < \tau$, let D be the associated set of record points

$$D = \{r \in [s,\tau] : \zeta_r = \min_{[s,r]} \zeta\}.$$

Then $W(D)$ consists of all restrictions of the path $w_s : [0,\zeta_s] \to \mathbb{R}^d$ on the intervals $[0,r]$ for times $r \in [0,\zeta_s]$ and the proposition yields that

$$\sup_{r \in [0,\zeta_s]} \limsup_{h \downarrow 0} \frac{|w_s(r) - w_s(r-h)|}{\sqrt{h \log(1/h)}} = \sup_{t \in D} \limsup_{h \downarrow 0} \frac{|w_t(\zeta_t) - w_t(\zeta_t - h)|}{\sqrt{h \log(1/h)}} = 2\sqrt{\dim_P(D)} = \sqrt{2}.$$

The equality of the right hand sides in (1) and (3) is hence a consequence of Lévy's famous distributional equivalence of the zero set and the record set of Brownian motion.

We now formulate the crucial tool in the proof of the Hausdorff dimension results. In order to reduce the study of sets in the range or support of $\mathfrak{X}_t$ to the study of subsets of the real line, we have to relate the dimension of a set $A \subset [0,\tau]$ and its image $W(A)$ under the Brownian snake.

THEOREM 2.3 (Uniform dimension formula). *Suppose that W is a Brownian snake in dimension $d \ge 2$. Then, $\mathbb{P}$-almost surely, for every set $A \subset [0,\tau]$ such that $0 \notin \zeta(A)$, we have*

$$\dim W(A) = 4 \dim(A).$$

REMARK 2.4. A closely related result was found by Serlet [**Ser95**], who showed that, in the case $d \ge 4$, $\mathbb{P}$-almost surely, for every set $A \subset [0,\tau]$ such that $0 \notin \zeta(A)$, we have $\dim \widetilde{W}(A) = 4\dim(A)$. By the methods of Section 4, this result can be extended to dimension $d = 2,3$ if the set A is a subset of an arbitrary levelset of ζ, see [**Mör99**]. Note that, for every t, almost surely, the subsets of levelsets $\zeta^{-1}(t)$ are mapped by the Brownian snake onto the subsets of the support $\operatorname{supp} \mathfrak{X}_t$, of super-Brownian motion at time t.

Because the exceptional nullset of paths in the previous theorem does not depend on A we may use it for sets A which depend on W in an arbitrary manner. It is due to this fact that the theorems can be used to reduce the study of random subsets of the range or support to the study of random subsets A of the real line.

Using Theorem 2.3 we can observe that, for closed sets $E \subset (0,t)$,

$$\begin{aligned}
&\dim\Big\{(w,\zeta) \in \mathfrak{R}(E) \,:\, \limsup_{h\downarrow 0} \frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\} \\
&= 4\dim\big\{s\in[0,\tau] \,:\, \zeta_s\in E \text{ and } s\in F(a)\big\} \\
&= 4\dim(F(a)\cap\zeta^{-1}(E))\,.
\end{aligned}$$

Now recall that, almost surely on $\sup\{\zeta_s : 0 \leq s \leq \tau\} > t$, for every nonempty Borel subset $E \subset (0,t)$,

$$\dim \zeta^{-1}(E) = \frac{1}{2} + \frac{\dim E}{2} \text{ and } \dim_P \zeta^{-1}(E) = \frac{1}{2} + \frac{\dim_P E}{2}\,.$$

Hence, we have reduced the proof of Theorem 1.3 to the statement of the following proposition.

PROPOSITION 2.5 (Dimension estimate). *Suppose that $D \subset [0,\tau]$ is an $\mathcal{F}$-measurable random closed set. Then, almost surely, for all $a \in [0, 2\sqrt{\dim_P E}]$,*

$$\dim(D) - \frac{a^2}{4} \leq \dim(F(a)\cap D) \leq \dim_P(D) - \frac{a^2}{4}\,. \tag{11}$$

If $\dim_P(D) < \frac{a^2}{4}$ we have $F(a)\cap D = \emptyset$.

Note that with the appropriate choice of D this also contains the result (5) on the fast points of Brownian motion. In the next subsection we review a technique to obtain results like Proposition 2.1 and Proposition 2.5.

2.2. Limsup random fractals and hitting probabilities. In their paper [**KPX00**] Khoshnevisan, Peres and Xiao give a general framework for the discussion of so-called discrete limsup random fractals. In Section 3 we shall see that the set

$$W^{-1}\Big\{(w,\zeta)\in\mathfrak{P} \,:\, \limsup_{h\downarrow 0}\frac{|w(\zeta)-w(\zeta-h)|}{\sqrt{h\log(1/h)}} \geq a\Big\}$$

contains a suitably defined discrete limsup random fractal. In this subsection we briefly review the framework and results of [**KPX00**].

For every $n\in\mathbb{N}$ denote by $\mathcal{D}_n$ the collection of all dyadic intervals of the form $[k/2^n,(k+1)/2^n]$ for $k = 0,\ldots,2^n-1$. Suppose further that we have a collection $\{Z_n(I) : I\in\mathcal{D}_n\}$ of random variables taking values in $\{0,1\}$. The associated *discrete limsup random fractal* is defined as $A := \limsup_{n\to\infty} A_n$, where

$$A_n = \bigcup_{I\in\mathcal{D}_n, Z_n(I)=1} \operatorname{int}(I)\,,$$

and $\operatorname{int}(I)$ denotes the interior of I. The framework of Khoshnevisan, Peres and Xiao is based on the following two assumptions:

Index assumption: Suppose that for each $n \geq 1$ the distribution of $Z_n(I)$ is the same for each $I\in\mathcal{D}_n$ and that

$$\lim_{n\to\infty}\frac{\log_2 \mathbb{E}\{Z_n(I)\}}{n} = -\gamma$$

for some $\gamma > 0$, where $\log_2$ is the logarithm of base 2,

Correlation assumption: For each $\varepsilon > 0$ define

$$f(n,\varepsilon) = \max_{I\in\mathcal{D}_n} \#\Big\{ J \in \mathcal{D}_n \,:\, \mathrm{Cov}\{Z_n(I), Z_n(J)\} \ge \varepsilon\, \mathbb{E}\{Z_n(I)\}\, \mathbb{E}\{Z_n(J)\} \Big\}.$$

Suppose that, for all $\varepsilon > 0$,

$$\limsup_{n\to\infty} \frac{\log_2 f(n,\varepsilon)}{n} = 0\,.$$

The result of [**KPX00**] subsumes and refines earlier methods and approaches. For example the idea of limsup random fractals is present in the work of Orey and Taylor [**OrT74**], Deheuvels and Mason [**DeM98**] or Dembo, Peres, Rosen and Zeitouni [**DPR00a**] and examples for the idea of finding dimension bounds by means of intersections with independent random sets can be found in the work of Taylor [**Tay66**], Hawkes [**Haw81**], Perkins [**Per83**], Peres [**Per96**] or Khoshnevisan and Shi [**KhS00**]. The **proof** of the following lemma can be found in [**KPX00**, Theorem 3.1 and Corollary 3.3].

LEMMA 2.6. *Suppose that A is a discrete limsup random fractal that fulfils the two assumptions above. Then, for every analytic set $D \subset [0,1]$,*

$$\mathbb{P}\Big\{ A \cap D \neq \emptyset \Big\} = \begin{cases} 1 & \text{if } \dim_P(D) > \gamma, \\ 0 & \text{if } \dim_P(D) < \gamma. \end{cases}$$

Moreover, almost surely,

$$\dim(D) - \gamma \le \dim(A\cap D) \le \dim_P(D) - \gamma\,.$$

We remark without going into details that [**KPX00**] also offers a refined version of this result, which permits to deal with the problem of strictly fast points, see Remark 1.4(a).

In Section 3 we directly prove the upper bounds and use the method of limsup random fractals to prove the lower bounds of the hitting probabilities (Proposition 2.1) and the dimension estimate (Proposition 2.5). Section 4 contains the proof of Theorem 2.3. Section 5 provides two auxiliary lemmas and, finally, a brief proof of Lemma 2.2 is given in Section 6.

3. Proof of the hitting probabilities and dimension results

3.1. Preliminaries. Throughout the proofs we will use the following positive random quantities,

$$\Delta_1 := \sup\left\{ \delta < \frac{1}{4} \,:\, |\zeta_s - \zeta_t| \le 2\sqrt{|s-t|\log(1/|s-t|)} \text{ for all } |s-t| \le \delta \right\},$$

and

$$\Delta_2 := \sup\Big\{ \delta < \frac{1}{4} \,:\, |w_s(t) - w_s(t-h)| \le 4\sqrt{h\log(1/h)}\ \forall s \in [0,\tau], t \ge h \text{ and } 0 \le h \le \delta \Big\}.$$

These quantities are well-defined and positive by the results of Lévy resp. Dawson and Perkins (see [**DaP91**, 8.7]) on the modulus of continuity of Brownian motion resp. super-Brownian motion. For later use we formulate the basic estimate as a lemma, for formal reasons put $w_s(-t) = w_s(0)$ for all $t \ge 0$.

LEMMA 3.1 (Basic estimate). *Suppose that $|s-t| < \Delta_1$ and $2\sqrt{|s-t|\log(1/|s-t|)} < \Delta_2$, then*

$$|w_t(\zeta_t) - w_t(\zeta_t - h)| \ge |w_s(\zeta_s) - w_s(\zeta_s - h)| - 16|s-t|^{1/4}\log(1/|s-t|)\,.$$

PROOF. We may assume that $s < t$. The triangle inequality gives

$$|w_t(\zeta_t) - w_t(\zeta_t - h)| \geq |w_s(\zeta_s) - w_s(\zeta_s - h)| - |w_s(\zeta_s) - w_t(\zeta_t)| - |w_s(\zeta_s - h) - w_t(\zeta_t - h)|.$$

By the construction of the Brownian snake one can estimate, choosing $m \in [s,t]$ such that $\zeta_m = \inf_{[s,t]} \zeta$,

$$\begin{aligned} |w_s(\zeta_s) - w_t(\zeta_t)| &\leq |w_s(\zeta_s) - w_s(\zeta_m)| + |w_t(\zeta_t) - w_t(\zeta_m)| \\ &\leq 4\sqrt{(\zeta_s - \zeta_m)\log(1/(\zeta_s - \zeta_m))} + 4\sqrt{(\zeta_t - \zeta_m)\log(1/(\zeta_t - \zeta_m))} \\ &\leq 8\sqrt{2\sqrt{(t-s)\log(1/(t-s))}\log(1/(2\sqrt{(t-s)\log(1/(t-s))}))} \\ &\leq 8(t-s)^{1/4}\log(1/(t-s)). \end{aligned}$$

The same procedure works for the last term, and this yields the basic estimate. □

3.2. Upper bounds. In this subsection we show that, conditional on $\mathcal{F}$, $\dim_P(D) < a^2/4$ implies that $\mathbb{P}\{F(a) \cap D \neq \emptyset \,|\, \mathcal{F}\} = 0$ and prove the upper bound in Proposition 2.5. These parts of the proof follow standard routes and may be proved directly without reference to Lemma 2.6.

Suppose $a \leq 4$ and $\varepsilon > 0$ is held fixed. We choose $\gamma > 0$ so small that $e^{\gamma} - 1 \leq (\varepsilon/16)^2$ and define $h_n = e^{-\gamma n}$ and $s_n \downarrow 0$ such that, for all sufficiently large n,

$$h_n = \sqrt{s_n}\Big(\log\frac{1}{s_n}\Big)^2.$$

We shall use intervals of length s_n to construct sets similar to limsup fractals, which contain $F(a)$. The choice of s_n guarantees that, for sufficiently large n,

$$16\sqrt[4]{s_n}\log(1/s_n) = 16\sqrt{h_n} \leq (\varepsilon/8)\sqrt{h_n\log(1/h_n)}, \tag{12}$$

and the choice of h_n gives, for all $h \in [h_n, h_{n-1}]$,

$$4\sqrt{(h-h_n)\log(1/(h-h_n))} \leq (\varepsilon/8)\sqrt{h_n\log(1/h_n)}, \tag{13}$$

together with

$$\sqrt{h\log(1/h)} \leq (1+\varepsilon/32)\sqrt{h_n\log(1/h_n)}. \tag{14}$$

Let $\mathcal{I}_n$ be the collection of intervals $[ks_n,(k+1)s_n]$ for $0 \leq k < \tau/s_n$ integer and $\mathcal{J}_n$ be the subclass of intervals, such that for the left endpoint $t = ks_n$,

$$|w_t(\zeta_t) - w_t(\zeta_t - h_n)| \geq (a-\varepsilon)\sqrt{h_n\log(1/h_n)}. \tag{15}$$

We show that, almost surely,

$$F(a) \subset \bigcap_{j=1}^{\infty}\bigcup_{n=j}^{\infty}\bigcup_{I\in\mathcal{J}_n} I. \tag{16}$$

For this purpose, choose N so large that $s_n \leq \Delta_1$, $2\sqrt{s_n\log(1/s_n)} \leq \Delta_2$ and (12), (13), (14) hold for all $n \geq N$. Suppose that $s \in F(a)$. Then, there exists an arbitrarily large $n > N$ and an $h > 0$ with $h_n \leq h \leq h_{n-1}$ such that

$$|w_s(\zeta_s) - w_s(\zeta_s - h)| \geq (a - \varepsilon/8)\sqrt{h\log(1/h)}.$$

For the interval $I = [ks_n,(k+1)s_n] \in \mathcal{I}_n$ with $s \in I$ and left endpoint $t := ks_n$ we have to show that $I \in \mathcal{J}_n$. Using the basic estimate, we infer

$$|w_t(\zeta_t) - w_t(\zeta_t - h_n)| \geq |w_s(\zeta_s) - w_s(\zeta_s - h_n)| - 16\sqrt[4]{s_n}\log(1/s_n).$$

For the first term we obtain

$$\begin{aligned}|w_s(\zeta_s) - w_s(\zeta_s - h_n)| &\geq |w_s(\zeta_s) - w_s(\zeta_s - h)| - |w_s(\zeta_s - h) - w_s(\zeta_s - h_n)| \\ &\geq (a - \varepsilon/8)\sqrt{h \log(1/h)} - 4\sqrt{(h - h_n)\log(1/(h - h_n))}\,.\end{aligned}$$

Putting this together with (12), (13) and (14) and recalling $a \leq 4$ yields (15) and hence (16) is proved.

Let us now look at the hitting probabilities. We recall the definition of the upper Minkowski dimension in the form most convenient for our purpose. Supposing that D is a nonempty subset of $[0, \tau]$, we let $N(D, n)$ be the number of intervals in $\mathcal{I}_n$, which hit the set D. The *upper Minkowski dimension of D* is defined as

$$\overline{\dim}_M(D) = \inf\Big\{ s \geq 0 \,:\, \limsup_{n\uparrow\infty} N(D, n) s_n^s = 0 \Big\}.$$

Alternative definitions and properties of the upper Minkowski dimension can be found in [**Mat95**, 5.3]. The crucial fact for our purpose is that *packing* dimension can be characterised as a regularisation of upper Minkowski dimension, more precisely

$$\dim_P(A) = \inf\Big\{ \sup_n \overline{\dim}_M(D_n) \,:\, A \subset \bigcup_{n=1}^{\infty} D_n, D_n \text{ a bounded set} \Big\}\,, \tag{17}$$

see [**Mat95**, 5.9]. Hence it suffices to show that, whenever $\overline{\dim}_M(D) < a^2/4$, then $F(a) \cap D = \emptyset$. Suppose that $\vartheta > 0$ is such that $\overline{\dim}_M(D) < \vartheta < (a - \varepsilon)^2/4$. By the definition of the upper Minkowski dimension, we have, for all sufficiently large n,

$$\#\Big\{ I \in \mathcal{I}_n \,:\, I \cap D \neq \emptyset \Big\} = N(D, n) \leq s_n^{-\vartheta}\,. \tag{18}$$

Almost surely, for all left endpoints $t = ks_n$, w_t is a Brownian path on $[0, \zeta_t]$. Hence, we know that, for every $I \in \mathcal{I}_n$,

$$\mathbb{P}\Big\{ I \in \mathcal{J}_n \,\Big|\, \mathcal{F} \Big\} = \mathbb{P}\Big\{ |w_t(\zeta_t) - w_t(\zeta_t - h_n)| \geq (a - \varepsilon)\sqrt{h_n \log(1/h_n)} \Big| \mathcal{F} \Big\} = h_n^{(a-\varepsilon)^2/2 + \mathsf{o}(1)}\,. \tag{19}$$

Denote by S_n the number of $I \in \mathcal{J}_n$ with $I \cap D \neq \emptyset$. From (18) and (19) we infer that

$$\begin{aligned}\mathbb{P}\Big\{ S_n > 0 \,\Big|\, \mathcal{F} \Big\} &\leq s_n^{-\vartheta} \max_{I \in \mathcal{I}_n} \mathbb{P}\Big\{ I \in \mathcal{J}_n \,\Big|\, \mathcal{F} \Big\} \\ &\leq s_n^{-\vartheta} \Big(\sqrt{s_n} \Big(\log \frac{1}{s_n} \Big)^2 \Big)^{(a-\varepsilon)^2/2 + \mathsf{o}(1)}\,.\end{aligned} \tag{20}$$

As $(a - \varepsilon)^2/4 - \vartheta > 0$ the latter quantity is summable and we obtain, using (16) and the Borel-Cantelli lemma,

$$\mathbb{P}\Big\{ D \cap F(a) \neq \emptyset \,\Big|\, \mathcal{F} \Big\} \leq \mathbb{P}\Big\{ \forall_{j=1}^{\infty} \exists_{n=j}^{\infty} \exists I \in \mathcal{J}_n \text{ with } I \cap D \neq \emptyset \Big| \mathcal{F} \Big\} = 0\,.$$

This proves the first part of the hitting estimates. To obtain the upper bound on the Hausdorff dimension in (11), we first observe that, using again regularisation (17), it suffices to prove

$$\dim(F(a) \cap D) \leq \overline{\dim}_M(D) - a^2/4 \text{ almost surely.}$$

Choose any $\vartheta > \overline{\dim}_M(D)$ and $\eta < (a - \varepsilon)^2/4$ and recall that S_n denotes the number of intervals $I \in \mathcal{J}_n$ with $D \cap I \neq \emptyset$. Then, using (19) again,

$$\mathbb{E}\Big\{ \sum_{n=1}^{\infty} S_n s_n^{\vartheta - \eta} \Big| \mathcal{F} \Big\} \leq \sum_{n=1}^{\infty} s_n^{-\eta} \Big(\sqrt{s_n} \Big(\log \frac{1}{s_n} \Big)^2 \Big)^{(a-\varepsilon)^2/2 + \mathsf{o}(1)} < \infty\,.$$

Hence, almost surely, $\sum_{n=1}^{\infty} S_n s_n^{\vartheta-\eta}$ is finite. As, for arbitrarily large N, the set $D \cap F(a)$ can be covered by the union over $n \geq N$ of the collection of S_n intervals of length s_n, we infer that $\dim(D \cap F(a)) \leq \vartheta - \eta$ and as this holds for every small $\varepsilon > 0$, $\eta < (a-\varepsilon)^2/4$ and $\vartheta > \overline{\dim}_M(D)$, this yields the desired result.

3.3. Lower bounds. In this section we prove the lower bounds in Proposition 2.1 and 2.5 with the help of Lemma 2.6. We start by showing that the set

$$F(a) = \Big\{ s \in [0,\tau] \,:\, \limsup_{h \downarrow 0} \frac{|w_s(\zeta_s) - w_s(\zeta_s - h)|}{\sqrt{h \log(1/h)}} \geq a \Big\},$$

contains a suitably defined discrete limsup random fractal.

LEMMA 3.2. *Let $\varepsilon > 0$ be given and define $h_n = 2^{-n/2} n^4$. For $I = [k2^{-n}, (k+1)2^{-n}] \in \mathcal{D}_n$ with left endpoint $t = k2^{-n}$ define the random variables $\{Z_n(I)\}$ to be the indicator function of the event*

$$\frac{|w_t(\zeta_t) - w_t(\zeta_t - h_n)|}{\sqrt{h_n \log(1/h_n)}} \geq a + \varepsilon \,.$$

Then the discrete limsup random fractal A defined through the random variables $\{Z_n(I)\}$ is contained in $F(a)$.

PROOF. Suppose that $s \in I$ and $Z_n(I) = 1$. Then, by the basic estimate,

$$|w_s(\zeta_s) - w_s(\zeta_s - h_n)| \geq (a+\varepsilon)\sqrt{h_n \log(1/h_n)} - 16(\log 2) 2^{-n/4} n \geq a \sqrt{h_n \log(1/h_n)}\,,$$

provided n is sufficiently large. □

Note that the lower bounds in Proposition 2.1 and Proposition 2.5 follow if we show that the limsup random fractal in the lemma fulfils the index assumption with index $\gamma = (a+\varepsilon)^2/4$ and the correlation assumption. The former is easy to see.

LEMMA 3.3. *The conditional distribution of $Z_n(I)$ given $\mathcal{F}$ is the same for each $I \in \mathcal{D}_n$ and*

$$\lim_{n \to \infty} \frac{\log_2 \mathbb{E}\{Z_n(I) \,|\, \mathcal{F}\}}{n} = -(a+\varepsilon)^2/4\,.$$

PROOF. We have

$$\begin{aligned} \mathbb{E}\{Z_n(I) \,|\, \mathcal{F}\} &= \mathbb{P}\Big\{ |w_t(\zeta_t) - w_t(\zeta_t - h_n)| \geq (a+\varepsilon)\sqrt{h_n \log(1/h_n)} \,\Big|\, \mathcal{F} \Big\} \\ &= h_n^{(a+\varepsilon)^2/2 + \mathsf{o}(1)} = 2^{-n(a+\varepsilon)^2/4 + \mathsf{o}(1)}\,. \end{aligned}$$

This implies the statement. □

It remains to check that our limsup random fractals fulfils the *correlation assumption.* Here is a significant difference to the example cases discussed in [**KPX00**], in which the correlation assumption was verified by means of tail correlation bounds that guaranteed a small correlation of intervals, which are far apart relative to their (common) size. Here suppose that s_1, s_2 are the beginning and end of an excursion of the reflected Brownian motion $\{\zeta_s\}$ above level t. For large n let $I, J \in \mathcal{D}_n$ be the two intervals containing s_1 resp. s_2. Then the correlation of $Z_n(I)$ and $Z_n(J)$ is large although the distance of I and J relative to their length is large. The next lemma shows that this phenomenon of long range dependence is sufficiently rare, so that the correlation assumption still holds true.

LEMMA 3.4. *Define*

$$f(n)=\max_{I\in\mathcal{D}_n}\#\Big\{J\in\mathcal{D}_n\,:\,\mathrm{Cov}\{Z_n(I),Z_n(J)\,|\,\mathcal{F}\}>0\Big\}.$$

Then, almost surely,

$$\limsup_{n\to\infty}\frac{\log_2 f(n)}{n}=0\,.$$

PROOF. Look at intervals $I=[k/2^n,(k+1)/2^n]$ and $J=[l/2^n,(l+1)/2^n]$ with $k<l$. Recalling the construction of the Brownian snake we observe that, conditional on $\mathcal{F}$, $Z_n(I)$ and $Z_n(J)$ are independent if either

$$\zeta_{l/2^n}-\zeta_{k/2^n}\geq h_n$$

or

$$\zeta_{k/2^n}-\min\Big\{\zeta_s\,:\,s\in[k/2^n,l/2^n]\Big\}\geq h_n\,.$$

We therefore have to show that, for all $\delta>0$, almost surely, there exists $N\in\mathbb{N}$ such that, for all $n\geq N$ and $k=1,\dots,2^n$, $A(n,k)\leq 2^{n\delta}$, where

$$A(n,k):=\#\Big\{l\in\{k,\dots,2^n\}\,:\,\zeta_{l/2^n}-\zeta_{k/2^n}\leq h_n\text{ and }\max_{s\in[k/2^n,l/2^n]}\Big(\zeta_{k/2^n}-\zeta_s\Big)\leq h_n\Big\}.$$

Now fix n sufficiently large to satisfy $2^{-n}<\Delta_1$ and fix $1\leq k\leq 2^n$. We may assume that $\zeta(k/2^n)\geq h_n$, the opposite case is easier. Look at the stopped Brownian motion $B(t)=\zeta(k/2^n+t)-\zeta(k/2^n)$ stopped at time

$$\tau:=\inf\Big\{l\in\{k,\dots,2^n\}\,:\,\min\{B(t)\,:\,0\leq t\leq(l-k)/2^n\}\leq -h_n\Big\}\,.$$

Then,

$$A(n,k)=\#\Big\{l\in\{k,\dots,\tau\}\,:\,B((l-k)/2^n)\leq h_n\Big\}\,.$$

Now suppose that $A(n,k)>2^{n\delta}$. If $B((l-k)/2^n)\leq h_n$, then, recalling that $2^{-n}<\Delta_1$, we have $B(s)\leq 2h_n$ for all $s\in[(l-k)/2^n,(l-k+1)/2^n]$. We let

$$T_n:=\inf\Big\{t\geq 0\,:\,B(t)\leq -2h_n\Big\}\,,$$

and note that $T_n\geq\tau$. Our assumption implies

$$\int_0^{T_n}1_{\{|B(s)|\leq 2h_n\}}\,ds\geq 2^{n(\delta-1)}\,.$$

We thus get, for all N,

$$\begin{aligned}&\mathbb{P}\Big\{\text{ there is }n\geq N\text{ with }2^{-n}<\Delta_1\text{ and }A(n,k)\geq 2^{n\delta}\text{ for some }k\Big\}\\&\quad\leq\sum_{n=N}^{\infty}2^n\mathbb{P}\Big\{\int_0^{T_n}1_{\{|B_s|\leq 2h_n\}}\,ds\geq 2^{n(\delta-1)}\Big\}\\&\quad=\sum_{n=N}^{\infty}2^n\mathbb{P}\Big\{(2h_n)^2\int_0^{T}1_{\{|B_s|\leq 1\}}\,ds\geq 2^{n(\delta-1)}\Big\}\,.\end{aligned}\tag{21}$$

In the last step we use Brownian scaling and define the stopping time T as the first hitting time of level -1 by $\{B_s\}$. We now use the following auxiliary lemma, which will be proved in Section 5 together with a similar lemma. Observe that the stopping time T does not even have finite first moment, so that the statement is not trivial.

LEMMA 3.5. *Let $\{B_s\}$ be a linear Brownian motion and T the first hitting time of level -1 by $\{B_s\}$. Then, for all $0 \le \lambda < 1/4e$, we have*

$$\mathbb{E}\Big\{ \exp\Big(\lambda \int_0^T 1_{\{|B_s|\le 1\}}\, ds\Big)\Big\} \le \frac{e^{4e\lambda} - e^{4e\lambda-1}}{1-e^{4e\lambda-1}} .$$

Using Chebyshev's inequality we now obtain,

$$\mathbb{P}\Big\{(2h_n)^2 \int_0^T 1_{\{|B_s|\le 1\}}\, ds \ge 2^{n(\delta-1)}\Big\} = \mathbb{P}\Big\{4n^8 \int_0^T 1_{\{|B_s|\le 1\}}\, ds \ge 2^{n\delta}\Big\}$$
$$\le \exp\Big(-\lambda 2^{n\delta}/4n^8\Big)\mathbb{E}\Big\{ \exp\Big(\lambda \int_0^T 1_{\{|B_s|\le 1\}}\, ds\Big)\Big\} .$$

From Lemma 3.5 we infer that, for a suitably small $\lambda > 0$, the series (21) converges and hence, almost surely, there is a random N such that $A(n,k) \le 2^{n\delta}$ for all $k = 1, \dots, 2^n$ and $n \ge N$. This finishes the proof. □

4. A uniform dimension formula for the Brownian snake

The proof of the upper bound is easy. We recall from [**LeG99**, IV.4, Prop. 5] that, for every $\eta > 0$, there is an $\mathcal{F}$-measurable random variable $C_\eta(\zeta) > 0$ such that, for all integers $p > 4$ and $s, t \in [0, \tau]$,

$$\mathbb{E}\Big\{ d(W_s, W_t)^p \Big| \mathcal{F}\Big\} \le C_\eta(\zeta)|s-t|^{\frac{p}{2}(\frac{1}{2}-\eta)} ,$$

where we note that the different definition of the metric on $\mathfrak{P}$ does not affect the calculation. Hence, by Kolmogorov's Theorem, for every $\delta > 0$, the mapping $s \mapsto W_s$ is locally Hölder continuous with exponent $(4+\delta)^{-1}$ almost surely and we get the upper bound $\dim W(A) \le 4 \dim A$ without any effort.

It remains to prove the lower bound. For this purpose we have to show that, on a set of full $\mathbb{P}$-measure, for every $H \subset \mathfrak{P}$ and every $t > 0$, we have

$$\dim\{s \in (0,\tau) \,:\, W_s \in H\} \le \frac{\dim H}{4} . \tag{22}$$

The lower bound then follows by choosing $H = \{W_s \,:\, s \in A\}$. Without loss of generality we may assume that there is a unit cube $C = a + [0,1)^d \subset \mathbb{R}^d$ at positive distance from the origin and a unit interval $I = b + [0,1]$ with $b > 0$ such that, for all stopped paths $(w, \zeta) \in H$, we have $\zeta \in I$ and $w(\zeta) \in C$.

We first define a set of full measure such that the dimension property (22) holds for all W in the set. The basic setup will be as in Serlet's proof of the uniform dimension result for $\widetilde{W}$, but we have to add one refined property to the set.

For this purpose let $\{x_m \,:\, m \ge 1\}$ be the set of dyadic points in the cube C indexed such that, for every n,

$$\Big\{x_m \,:\, 1 \le m \le 2^{nd}\Big\} = \Big\{a + \Big(\frac{k_1}{2^n}, \dots, \frac{k_d}{2^n}\Big) \,:\, 0 \le k_i \le 2^n - 1\Big\} .$$

Next, let $\{a_m \,:\, m \ge 0\}$ be the set of dyadic points in the interval I indexed such that, for every n, $\{a_m \,:\, 0 \le m \le 2^n\} = \{b + \frac{k}{2^n} \,:\, 0 \le k \le 2^n\}$. Define $I_m(\varepsilon) = [a_m - \varepsilon, a_m + \varepsilon]$. The *occupation measure* Z for the Brownian snake is defined by

$$Z(J, A) = \int_0^\tau 1_J(\zeta_s) 1_A(\widetilde{W}_s)\, ds , \quad \text{for all } A \subset \mathbb{R}^d \text{ and } J \subset (0,\infty) \text{ Borel.}$$

Also, we have to define the successive entrance times in the ball $B(x,\varepsilon)$ and exit times from the ball $B(x,2\varepsilon)$ for the process $\{\widetilde{W}_s : s \geq 0\}$ by

$$T_1(x,\varepsilon) = \inf\{s \geq 0 : \widetilde{W}_s \in B(x,\varepsilon)\},$$

and, for $k \geq 1$,

$$\begin{aligned} S_k(x,\varepsilon) &= \inf\{s > T_k(x,\varepsilon) : \widetilde{W}_s \notin B(x,2\varepsilon)\}, \\ T_{k+1}(x,\varepsilon) &= \inf\{s > S_k(x,\varepsilon) : \widetilde{W}_s \in B(x,\varepsilon)\}. \end{aligned}$$

Finally, throughout the paper, denote $L(x) = 1 + |\log(x)|$.

LEMMA 4.1. *For every $\delta > 0$ there is a set $\Omega(\delta) \subset \Omega$ of full $\mathbb{P}$-measure, a constant $c > 1$ and a random variable Q, which is finite on $\Omega(\delta)$, such that, for every path $W = \{(w_s, \zeta_s) : s \geq 0\} \in \Omega(\delta)$, every $\varepsilon \in \{2^{-q} : q \geq Q\}$ and all integers $m, n \geq 1$,*

(i) $Z\Big(I_n(\varepsilon^2), B(x_m, 2\varepsilon)\Big) \leq cL(n)^2 L(m)^2 L(\varepsilon)^6 \varepsilon^4$,
(ii) $S_k(x_m,\varepsilon) - T_k(x_m,\varepsilon) \leq cL(m)^3 L(k)^3 L(\varepsilon)^3 \varepsilon^4$, *for every k such that $T_k(x_m,\varepsilon) < \tau$,*
(iii) $|\widetilde{W}_s - \widetilde{W}_t| \leq |s-t|^{1/(4+\delta)}$, *for all $0 \leq s,t \leq \tau$ such that $|s-t| \leq 2^{-Q}$,*
(iv) $|\zeta_s - \zeta_t| \leq |s-t|^{1/(2+\delta)}$, *for all $0 \leq s,t \leq \tau$ such that $|s-t| \leq 2^{-Q}$.*

PROOF. It is clear from the fact that $s \mapsto \widetilde{W}_s$ is locally $1/(4+\delta)$-Hölder continuous and that $s \mapsto \zeta_s$ is locally $1/(2+\delta)$-Hölder continuous that properties (iii) and (iv) can be realized on a set of full measure. From Serlet [**Ser95**, Lemma 9] we know that property (ii) can be realized on a set of full measure. The novel part is the proof of property (i). We formulate two lemmas to prepare the proof.

We first need the following statement about reflected Brownian motion stopped when its local time at level 0 first hits 1. Recall that the local time of $\{|B_s|\}$ at level 0 is twice the local time of $\{B_s\}$ at level 0.

LEMMA 4.2. *Let $\{B_s\}$ be a linear Brownian motion, L_s^0 its local time at level 0 and $\tau = \tau(B) = \inf\{s : L_s^0 = 1/2\}$. Then, for all $0 \leq \lambda < 1/3e$, we have*

$$\mathbb{E}\Big\{ \exp\Big(\lambda \int_0^\tau 1_{\{|B_s| \leq 1\}}\, ds\Big)\Big\} \leq \frac{e^{3e\lambda} - e^{3e\lambda-1}}{1 - e^{3e\lambda-1}}.$$

A similar lemma was used in the proof of the hitting estimates and they are proved together in Section 5. We also need the existence of exponential moments for the random measure of balls. The following lemma is well known, see [**LGP95**, Lemma 3.1] for an easy proof. Denote by $p(x,s) = (1/\sqrt{2\pi s})^d \exp(-\|x\|^2/2s)$ the standard heat kernel.

LEMMA 4.3. *Denote $G(r,t) = \int_0^t [\int_{\|y\| \leq r} p(y,s)\, dy]\, ds$. Then, for all $t > 0, r > 0$ and $x \in C$,*

$$\mathbb{P}\Big\{ \exp\Big(\lambda X_t(B(x,r))\Big)\Big\} \leq \exp\Big(2\lambda \int_{\|y\| \leq r} p(y,t)\, dy\Big),$$

for all $0 \leq \lambda \leq (4G(r,t))^{-1}$.

We can now formulate the main lemma behind the dimension formula. Recall the definition of Δ_2 from the beginning of Section 3. The idea of the lemma is the following: For $0 < \varepsilon < e^{-36}$, if a particle at time $t + \varepsilon^2$ is outside $B(x, \varepsilon L(\varepsilon))$ at time t and $\Delta_2(W) > \varepsilon^2$, then at time t the particle has a distance to x, which is at least

$$\varepsilon L(\varepsilon) - 4\sqrt{\varepsilon^2 \log(1/\varepsilon^2)} \geq \varepsilon + \varepsilon \log(1/\varepsilon) - 6\varepsilon\sqrt{\log(1/\varepsilon)} \geq \varepsilon.$$

In other words, the offspring of a particle outside $B(x,\varepsilon L(\varepsilon))$ at time t cannot move into $B(x,\varepsilon)$ in less than ε^2 time units. Heuristically this means that, if $\int_t^{t+\varepsilon^2} X_s(B(x,\varepsilon))\,ds$ is large and $\Delta_2(W) > \varepsilon^2$, then either $X_t(B(x,\varepsilon L(\varepsilon)))$ is large or the particles which are in $B(x,\varepsilon L(\varepsilon))$ at time t are unusually fertile. Both these events are shown to be unlikely.

As the next proof uses the Markov property of super-Brownian motion, we suppose that $\{X_t\}$ under $\mathbb{P}_\mu$ is super-Brownian motion started at time $t=0$ with mass μ and abbreviate $\mathbb{P}_x := \mathbb{P}_{\delta_x}$.

LEMMA 4.4. *For every bounded interval $I \subset (0,\infty)$ there exist positive constants c_1 and c_2 such that, for all $t \in I$, $x \in C$, all $\lambda > 0$ and all $\varepsilon = 2^{-q}$, $q \geq 72$,*

$$\mathbb{P}_0\Big\{\Delta_2(W) > \varepsilon^2\,,\ \int_t^{t+\varepsilon^2} X_s(B(x,\varepsilon))\,ds \geq \lambda\varepsilon^4 L(\varepsilon)^4\Big\} \leq c_1 \exp(-c_2\sqrt{\lambda})\,.$$

PROOF. Denote by

$$M = \Big\{\mu \in \mathcal{M}_F(\mathbb{R}^d)\ :\ \mu(\mathbb{R}^d) \leq \sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4\Big\}\,.$$

We have, by the Markov property of super-Brownian motion, and our initial consideration,

$$\begin{aligned}&\mathbb{P}_0\Big\{\Delta_2(W) > \varepsilon^2, \int_t^{t+\varepsilon^2} X_s(B(x,\varepsilon))\,ds \geq \lambda\varepsilon^4 L(\varepsilon)^4\Big\}\\ &\leq\ \mathbb{P}_0\Big\{X_t(B(x,\varepsilon L(\varepsilon))) \geq \sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4\Big\} + \sup_{\mu\in M}\mathbb{P}_\mu\Big\{\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt \geq \lambda\varepsilon^4 L(\varepsilon)^4\Big\}\,.\end{aligned}$$

We give separate estimates of these two probabilities. To investigate the first probability we use Lemma 4.3. We have

$$G(r,t) = \int_{\|y\|\leq r}\Big[\int_0^t p(y,s)\,ds\Big]\,dy = r^2\int_{\|y\|\leq 1}\Big[\int_0^{t/r^2} p(y,s)\,ds\Big]\,dy\,,$$

which, for all $t \in I$ and $0 < r < 1/2$, is bounded from above by a constant multiple of $r^2\log(1/r)$ in dimension $d=2$ and by a constant multiple of r^2 in dimension $d \geq 3$. In all cases we find a positive constant k_1 such that

$$t\int_{B(0,\varepsilon L(\varepsilon))} p(y,t)\,dy \leq G(\varepsilon L(\varepsilon),t) \leq k_1\varepsilon^2 L(\varepsilon)^4\,.$$

Choose $k_2 = 1/(4k_1)$ and k_3 such that $\exp(1/(2t)) < k_3$ for all $t \in I$. Then, using Chebyshev and Lemma 4.3,

$$\begin{aligned}&\mathbb{P}_0\Big\{X_t(B(x,\varepsilon L(\varepsilon))) \geq \sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4\Big\}\\ &\leq\ \exp\Big(-\frac{\sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4}{4G(\varepsilon L(\varepsilon),t)}\Big)\,\mathbb{P}_0\Big\{\exp\Big(\frac{X_t(B(x,\varepsilon L(\varepsilon)))}{4G(\varepsilon L(\varepsilon),t)}\Big)\Big\}\\ &\leq\ \exp\Big(-\sqrt{\lambda}/(4k_1)\Big)\cdot\exp\Big(2\frac{\int_{\|y\|\leq\varepsilon L(\varepsilon)} p(y,t)\,dy}{4G(\varepsilon L(\varepsilon),t)}\Big)\\ &\leq\ k_3\exp(-k_2\sqrt{\lambda})\,.\end{aligned}\tag{23}$$

This is the required estimate for the first term. To investigate the second probability we use Lemma 4.2. We choose a positive constant $k_4 < 1/3e$. Then, by Chebyshev,

$$\mathbb{P}_\mu\Big\{\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt \geq \lambda\varepsilon^4 L(\varepsilon)^4\Big\} \leq e^{-k_4\lambda}\,\mathbb{P}_\mu\Big\{\exp\Big(k_4\frac{\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt}{\varepsilon^4 L(\varepsilon)^4}\Big)\Big\}\,.\tag{24}$$

Now let $a = k_4/L(\varepsilon)^4$ so that $0 < a < 1/3e$. We have, by the branching property of super-Brownian motion, for all $\mu \in M$,

$$\mathbb{P}_\mu\Big\{\exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt\Big)\Big\} = \exp\Big(\int \log \mathbb{P}_x\Big\{\exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt\Big)\Big\}\,d\mu(x)\Big). \tag{25}$$

Now, recalling either the snake representation or the Ray-Knight Theorem,

$$\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt = \int_0^\tau 1_{\{\zeta_s \le \varepsilon^2\}}\,ds\,,$$

and we infer that

$$\begin{aligned}
&\int \log \mathbb{P}_x\Big\{\exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt\Big)\Big\}\,d\mu(x)\\
&= \int \log P_0\Big\{\exp\Big(\frac{a}{\varepsilon^4}\int_0^\tau 1_{\{|B_s|\le\varepsilon^2\}}\,ds\Big)\Big\}\,d\mu(x)\\
&\le \sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4\cdot \log P_0\Big\{\exp\Big(\frac{a}{\varepsilon^4}\int_0^\tau 1_{\{|B_s|\le\varepsilon^2\}}\,ds\Big)\Big\}\,. \qquad (26)
\end{aligned}$$

We now use a scaling argument. Define $\tau_0 = 0$ and $\{B_s^0\} = \{B_s\}$. Proceed inductively by putting, for $j \ge 0$, $\tau_{j+1} = \inf\{s : L_s^0(B_s^j) = 1/2\}$ and $\{B_s^{j+1}\} = \{B_{\tau_{j+1}+s}\}$. Finally, let $\tilde\tau = \inf\{s : L_s^0 = 1/(2\varepsilon^2)\}$. Then, using Lemma 4.2 in the final step,

$$\begin{aligned}
\log P_0\Big\{\exp\Big(\frac{a}{\varepsilon^4}\int_0^\tau 1_{\{|B_s|\le\varepsilon^2\}}\,ds\Big)\Big\} &= \log P_0\Big\{\exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^4\tilde\tau} 1_{\{|B_{s/\varepsilon^4}|\le 1\}}\,ds\Big)\Big\}\\
= \log P_0\Big\{\exp\Big(a\sum_{j=1}^{1/\varepsilon^2}\int_0^{\tau_j} 1_{\{|B_s^{j-1}|\le 1\}}ds\Big)\Big\} &= \sum_{j=1}^{1/\varepsilon^2}\log P_0\Big\{\exp\Big(a\int_0^{\tau_j} 1_{\{|B_s^{j-1}|\le 1\}}ds\Big)\Big\}\\
\le \frac{1}{\varepsilon^2}\cdot\log\Big(\frac{e^{3ea}-e^{3ea-1}}{1-e^{3ea-1}}\Big)\,. \qquad (27)
\end{aligned}$$

Putting together (24), (25), (26), (27) and using that, for some constant $k_5 > 0$, $\log((e^{3ea} - e^{3ea-1})/(1-e^{3ea-1})) \le k_5 a$, we infer

$$\begin{aligned}
\mathbb{P}_\mu\Big\{\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt \ge \sqrt{\lambda}\varepsilon^4 L(\varepsilon)^4\Big\} &\le e^{-k_4\lambda}\cdot\exp\Big(\sqrt{\lambda}L(\varepsilon)^4\log\Big(\frac{e^{3ea}-e^{3ea-1}}{1-e^{3ea-1}}\Big)\Big)\\
&\le \exp\Big(-\lambda k_4(1-k_5/\sqrt{\lambda})\Big)\,. \qquad (28)
\end{aligned}$$

The required result follows from (23) and (28). ◻

We can now add the probabilities and obtain, for some $c > 0$, the convergence of the series

$$\sum_{q=1}^\infty \mathbb{P}_0\Big\{\Delta_2(W) > 2^{-2q}, \int_{I_n(2^{-2q})} X_t(B(x_m,2^{-q}))\,dt \ge c2^{-4q}L(n)^2L(m)^2L(2^q)^6 \text{ for some } n,m\Big\}.$$

Hence Borel-Cantelli yields a random integer Q_1 such that, whenever $\Delta_2(W) > 2^{-2Q_1}$, we have

$$\int_{I_n(2^{-2q})} X_t(B(x_m,2^{-q}))\,dt \le c2^{-4q}L(n)^2L(m)^2L(2^q)^6\,,$$

for all integers n,m and $q \ge Q_1$. By the modulus of continuity of Dawson and Perkins (i) can be realized on a set of full measure and hence we obtain the statement of Lemma 4.1. ◻

We finish the proof of Theorem 2.3 by showing that the dimension property holds for all W which are in the intersection of the sets $\Omega(\delta)$. Pick a set $H \subset \mathfrak{P}$ with $\zeta \in I$ and $w(\zeta) \in C$ for all $(w,\zeta) \in H$. We may assume that $\dim H < 4$, and let $4 > \gamma > \dim H$ and $0 < \delta < \gamma - \dim H$. We further fix an integer K with $2^K \geq \sqrt{d} + 1$. We choose a covering of H by a sequence $\{B(w_j, r_j) \,:\, j \geq 1\}$ of balls whose radii satisfy $r_j \leq 2^{-(Q+K+1)}$, such that

$$\sum_{j=1}^{\infty} r_j^{\gamma}\, L(2^K r_j)^{28} \leq 2^{-(K+1)\gamma}\,, \tag{29}$$

and $\zeta_j = \zeta(w_j) \in I$ and $y_j = w_j(\zeta_j) \in C$. Then

$$\Big\{ B(y_j, r_j) \times [\zeta_j - r_j^2, \zeta_j + r_j^2] \,:\, j \geq 1 \Big\}$$

is a covering of the set $\{(w(\zeta), \zeta) \,:\, w \in H\}$. This covering can be replaced by a covering consisting of sets $B(x_{m_j}, \varepsilon_j) \times I_{n_j}(\varepsilon_j^2)$ such that $\varepsilon_j \in \{2^{-q} \,:\, q \geq Q\}$ with $2^{-(K+1)}\varepsilon_j \leq r_j \leq 2^{-K}\varepsilon_j$ and

$$|y_j - x_{m_j}| \leq \sqrt{d}\frac{\varepsilon_j}{2^K} \text{ for some } m_j \leq \Big(\frac{2^K}{\varepsilon_j}\Big)^d,$$

$$|\zeta_j - a_{n_j}| \leq \frac{\varepsilon_j^2}{2^K} \text{ for some } n_j \leq \frac{2^K}{\varepsilon_j^2}\,.$$

We write T_k^j, S_k^j for $T_k(x_{m_j}, \varepsilon_j)$, $S_k(x_{m_j}, \varepsilon_j)$, and denote by

$$\mathcal{N}_j = \Big\{ k \geq 1 \,:\, T_k^j < \tau \text{ and } \zeta[T_k^j, S_k^j] \cap I_{n_j}(\varepsilon_j^2) \neq \emptyset \Big\}\,,$$

and

$$\mathcal{M}_j = \Big\{ k \geq 1 \,:\, T_k^j < \tau \text{ and } \zeta[T_k^j, S_k^j] \subset \bigcup_{m=0}^{2^K \varepsilon_j^{-2}} I_m(\varepsilon_j^2) \Big\}\,.$$

Let $N_j = \#\mathcal{N}_j$ and $M_j = \#\mathcal{M}_j$. Because

$$B(y_j, r_j) \times [\zeta_j - r_j^2, \zeta_j + r_j^2] \subset B(x_{m_j}, \varepsilon_j) \times I_{n_j}(\varepsilon_j^2),$$

we have that

$$\Big\{ s \in [0,\tau] \,:\, W_s \in H \Big\} \subset \bigcup_{j=1}^{\infty} \bigcup_{k \in \mathcal{N}_j} [T_k^j, S_k^j]\,.$$

This is the covering which will give a good upper bound of the Hausdorff measure. Property (ii) ensures that we have control over the length of the individual covering intervals $[T_k^j, S_k^j]$. We now argue that also the cardinality of the sets $\mathcal{N}_j$ is not too large. We begin by observing that $|\widetilde{W}_{S_k^j} - \widetilde{W}_{T_k^j}| \geq \varepsilon_j$. As $\varepsilon_j \leq 2^{-Q}$ we infer from property (iii) that

$$S_k^j - T_k^j \geq \varepsilon_j^{4+\delta}\,. \tag{30}$$

From (i) applied to the family of intervals $I_m(\varepsilon_j^2)$, $m = 0, \ldots, 2^K \varepsilon_j^{-2}$, we infer that

$$\begin{aligned} M_j \varepsilon_j^{4+\delta} &\leq \sum_{k \in \mathcal{M}_j} S_k^j - T_k^j \leq Z\Big(\bigcup_{m=0}^{2^K \varepsilon_j^{-2}} I_m(\varepsilon_j^2), B(x_{m_j}, 2\varepsilon_j) \Big) \\ &\leq cL(2^K \varepsilon_j^{-2})^2 L(m_j)^2 L(\varepsilon_j)^6 \varepsilon_j^4 (2^K \varepsilon_j^{-2} + 1), \end{aligned}$$

and hence, for some positive constant c_3,

$$M_j \le c_3 L(\varepsilon_j)^{10} \varepsilon_j^{-2-\delta}.$$

Properties (iv) and (ii) imply that, for some positive constant c_4, for every k,

$$\text{(31)} \quad \operatorname{diam} \zeta[T_k^j, S_k^j] \le (S_k^j - T_k^j)^{1/(2+\delta)} \le cL(m_j)^3 L(M_j)^3 L(\varepsilon_j)^3 \varepsilon_j^{4/(2+\delta)} \le c_4 L(\varepsilon_j)^9 \varepsilon_j^{4/(2+\delta)}.$$

We infer from this and the numbering of the points that there is $m \le 2^n$ such that for all k,

$$I_{n_j}(\varepsilon_j^2) \cap \zeta[T_k^j, S_k^j] \ne \emptyset \text{ implies } \zeta[T_k^j, S_k^j] \subset [a_m - 2^{-n}, a_m + 2^{-n}],$$

where n satisfies

$$2^{-(n+1)} \le c_4 L(\varepsilon_j)^9 \varepsilon_j^{4/(2+\delta)} < 2^{-n}.$$

On the one hand we then have

$$Z\Big([a_m - 2^{-n}, a_m + 2^{-n}], B(x_{m_j}, 2\varepsilon_j)\Big) \ge \sum_{k \in \mathcal{N}_j} (S_k^j - T_k^j) \ge N_j \cdot \varepsilon_j^{4+\delta},$$

and on the other hand, $[a_m - 2^{-n}, a_m + 2^{-n}]$ can be covered by at most $2c_4 L(\varepsilon_j)^9 \varepsilon_j^{-\delta}$ intervals from the collection $\{I_k(\varepsilon_j^2) : 0 \le k \le 1/\varepsilon_j^2\}$. From (i), we hence know that

$$Z\Big([a_m - 2^n, a_m + 2^n], B(x_{m_j}, 2\varepsilon_j)\Big) \le 2cc_4 L(m)^2 L(m_j)^2 L(\varepsilon_j)^{15} \varepsilon_j^{4-\delta},$$

which together gives, for some positive constant c_5,

$$\text{(32)} \qquad N_j \le c_5 L(\varepsilon_j)^{19} \varepsilon_j^{-2\delta}.$$

Using again (ii) and (32), we get, for every $\alpha \in [0, 1]$, and a suitable positive constant c_6,

$$\sum_{j=1}^{\infty} \sum_{k \in \mathcal{N}_j} (S_k^j - T_k^j)^\alpha \le c \sum_{j=1}^{\infty} N_j \Big(L(m_j)^3 L(M_j)^3 L(\varepsilon_j)^3 \varepsilon_j^4\Big)^\alpha \le c_6 \sum_{j=1}^{\infty} \varepsilon_j^{4\alpha - 2\delta} L(\varepsilon_j)^{28}.$$

We choose $\alpha = (\gamma + 2\delta)/4$, and get, upon recalling (29) and $2^{-K}\varepsilon_j \ge r_j \ge 2^{-(K+1)}\varepsilon_j$,

$$\sum_{j=1}^{\infty} \sum_{k \in N_j} (S_k^j - T_k^j)^{(\gamma+2\delta)/4} \le c_6 \sum_{j=1}^{\infty} \varepsilon_j^\gamma L(\varepsilon_j)^{28} \le c_6 \sum_{j=1}^{\infty} (2^{K+1} r_j)^\gamma L(2^K r_j)^{28} \le c_6 < \infty.$$

This implies that $\dim\{s : W_s \in H\} \le (\gamma + 2\delta)/4$ and, as this holds for all $\delta > 0$, the required result follows and Theorem 2.3 is proved. □

5. Proof of two auxiliary lemmas

In this section we prove Lemmas 3.5 and 4.2, which were used in the previous proofs. The proofs follow the same approach, we do the proof of Lemma 4.2 in full detail and indicate the necessary changes to obtain Lemma 3.5.

Let $\{B_s\}$ be a linear Brownian motion, L_s^0 its local time at level 0 and $\tau = \tau(B) = \inf\{s : L_s^0 = 1/2\}$. We intend to show that, for all $0 \le \lambda < 1/3e$, we have

$$\mathbb{E}\Big\{ \exp\Big(\lambda \int_0^\tau 1_{\{|B_s| \le 1\}}\, ds\Big)\Big\} \le \frac{e^{3e\lambda+1} - e^{3e\lambda}}{e - e^{3e\lambda}}.$$

We first check that, for the process $\{X_t\}$ defined as

$$X_t = \int_0^{t \wedge \tau} 1_{\{|B_s| \le 1\}}\, ds,$$

we have, for all stopping times S and associated σ-algebras $\mathcal{F}_S$, that $\mathbb{E}\{|X_\infty - X_S|\,|\mathcal{F}_S\} \le 1$ almost surely. Indeed,

$$\mathbb{E}\{X_\infty - X_S|\mathcal{F}_S\} \le \sup_{|a|\le 1} P_a\Big\{\int_0^{\tau(B)} 1_{\{|B_s|\le 1\}}\,ds\Big\} = \sup_{|a|\le 1}\int_0^\infty P_a\{|B_s|\le 1, L_s^0 < 1/2\}\,ds,$$

where P_a refers to a Brownian motion B starting in $B_0 = a$. Recall the explicit density of the joint distribution of (B_s, L_s^0), see for example [**BoS96**, (1.3.8)], which yields

$$\begin{aligned}
&\sup_{|a|\le 1}\int_0^\infty P_a\{|B_s|\le 1, L_s^0 < 1/2\}\,ds\\
&= \sup_{a\in[0,1]}\int_0^\infty \frac{ds}{\sqrt{2\pi s^3}}\int_{-1}^1 dz\int_0^{1/2} dy\Big\{(|z|+y+a)\exp(-(|z|+y+a)^2/2s)\Big\}\\
&\quad + \int_0^\infty \frac{ds}{\sqrt{2\pi s}}\int_{-1}^1 dz\Big\{\exp(-(z-a)^2/2s) - \exp(-(-|z|+a)^2/2s)\Big\}\\
&= \sup_{a\in[0,1]}\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-t/2}}{\sqrt{t}}\,dt + \int_0^1 (|x+a| - |x-a|)\,dx = 3\,.
\end{aligned}$$

With this property of $\{X_t\}$ at hand, we can see that, for every $a > 0$ and $S = \inf\{t : X_t = a\}$,

$$\begin{aligned}
P_0\{X_\infty > a + 3e\} &= \mathbb{E}\Big\{P_0\{X_\infty - X_S > 3e\,|\,\mathcal{F}_S\}\,\Big|\,S<\infty\Big\}\cdot P\{S<\infty\}\\
&\le \frac{1}{3e}\mathbb{E}\Big\{\mathbb{E}\{X_\infty - X_S\,|\,\mathcal{F}_S\}\,\Big|\,S<\infty\Big\}\cdot P\{S<\infty\}\\
&\le P_0\{X_\infty \ge a\}/e\,.
\end{aligned}$$

From this it is easy to see that $\mathbb{E}\{\exp(\lambda X_\infty)\} < \infty$ for all $0 \le \lambda < 1/3e$ and, similarly, we see that

$$\begin{aligned}
\mathbb{E}\{\exp(\lambda X_\infty)\} &= \int_1^\infty P_0\Big\{X_\infty > \log a/\lambda\Big\}\,da + 1\\
&= e^{3e\lambda}\cdot\int_{e^{-3e\lambda}}^\infty P_0\{X_\infty > 3e + \log a/\lambda\}\,da + 1\\
&= e^{3e\lambda}\cdot\int_1^\infty P_0\{X_\infty > 3e + \log a/\lambda\}\,da + e^{3e\lambda}\\
&\le e^{3e\lambda-1}\cdot\int_1^\infty P_0\{X_\infty \ge \log a/\lambda\}\,da + e^{3e\lambda}\\
&= e^{3e\lambda-1}\,\mathbb{E}\{\exp(\lambda X_\infty)\} + e^{3e\lambda} - e^{3e\lambda-1}\,,
\end{aligned}$$

from which we infer that $\mathbb{E}\{\exp(\lambda X_\infty)\} \le (e^{3e\lambda} - e^{3e\lambda-1})/(1 - e^{3e\lambda-1})$, as required to finish the proof of Lemma 4.2.

We proceed similarly to obtain Lemma 3.5. Replacing τ by T in the definition of the process $\{X_t\}$ we get, for every stopping time S,

$$\mathbb{E}\{X_\infty - X_S|\mathcal{F}_S\} \le \sup_{|a|\le 1}\int_0^\infty P_a\{|B_s|\le 1, \min_{0\le t\le s} B_t > -1\}\,ds\,.$$

From the joint distribution of Brownian motion and its minimum up to date, see e.g. [**BoS96**, 1.2.8], we get that the value of the integral on the right is $3 - a^2 + 2a$, so that the supremum is

4. Hence,

$$P_0\{X_\infty > a + 4e\} \le \frac{1}{e} P_0\{X_\infty \ge a\},$$

and we can proceed as before to get the result of Lemma 3.5. □

6. A packing dimension result for linear Brownian motion

In this section we prove Lemma 2.2. For the lower bound recall from Joyce and Preiss [**JoP95**] that there is a Frostman type lemma for packing dimension: $\dim_P(E) > a$ is equivalent to the existence of a positive, finite measure μ on E such that,

$$\limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} > a \text{ for } \mu\text{-almost all } x \in E.$$

Let ν be such a measure on E. For each $t \in E$ let $L^t(s)$ be the local time at level t and $\mu_t(ds) = L^t(ds)$ be the associated local time measure on the level set $\zeta^{-1}(\{t\})$. Recall that, given $\varepsilon > 0$, there is a random constant $c > 0$ such that

$$|L^t(u) - L^t(v)| \le c|u - v|^{1/2-\varepsilon}, \text{ for all } t > 0 \text{ and } u, v \in [0, \tau]. \tag{33}$$

Define a positive measure on $\zeta^{-1}(E)$ by

$$\mu = \int_E \mu_t \, \nu(dt).$$

Fix $x \in E$ such that there is a sequence $r_n \downarrow 0$ with $\nu(B(x, r_n)) \le r_n^{\dim_P E - \varepsilon}$ for all n. Pick a sequence $s_n \downarrow 0$ such that $r_n = 3s_n\sqrt{\log(1/s_n)}$ and fix $y \in \zeta^{-1}(E)$ with $\zeta(y) = x$. By the property (33) of local time, there is, for every $\varepsilon > 0$, an N such that

$$\mu_t(B(y, s_n^2)) \le 2cs_n^{1-2\varepsilon} \text{ for all } t > 0 \text{ and } n \ge N.$$

Hence we infer that, for sufficiently large n,

$$\mu(B(y, s_n^2)) = \int_{B(\zeta(y), r_n)} \mu_t(B(y, s_n^2)) \, \nu(dt) \le 2cr_n^{1+\dim_P(E)-3\varepsilon}.$$

Therefore the packing dimension of $\zeta^{-1}(E)$ is bounded below by $1/2 + \dim_P(E)/2$. The corresponding upper bound can be obtained in a similar manner as for the Hausdorff case in [**Ser95**]. Recall for this purpose that, for a Brownian motion $\{B_t\}$ in dimension $d \ge 2$, almost surely for all $A \subset [0, \infty)$,

$$\dim_P(B(A)) = 2 \dim_P(A).$$

This result was proved by Kaufman [**Kau69**] for Hausdorff dimension, but the result extends without problems to the case of packing dimension. Now, for a planar Brownian motion $B_t = (\zeta_t, \xi_t)$, we have

$$\dim_P\{t : \zeta_t \in E\} = \dim_P\{t : B_t \in E \times \mathbb{R}\} \le \frac{1}{2} \dim_P\{E \times \mathbb{R}\} \le \frac{1}{2} + \frac{\dim_P(E)}{2},$$

using $\dim_P(A \times B) \le \dim_P(A) + \dim_P(B)$. This completes the proof. □

7. Final remarks

- Obvious changes to the final part of the proof show that the uniform dimension formula for the Brownian snake also holds for packing dimension instead of Hausdorff dimension. However, one cannot expect an interesting packing dimension spectrum for fast paths, because, in general, the packing dimension of nonempty limsup-random fractals is full. For example, one can check that for all dimensions $d \geq 2$, for all $t > 0$, almost surely on $\{\mathfrak{X}_t \neq 0\}$, for every $0 \leq a \leq \sqrt{2}$ the set of a-fast paths in supp $\mathfrak{X}_t$ has packing dimension 2. This interesting feature of limsup-random fractals is discussed for example in [**DPR00a**] and [**KPX00**].
- Combining our approach with the arguments of [**KhS00**, Theorem 1.1] and [**KPX00**, Remark 1.5] one can obtain the following variant of (3). Almost surely on $\{\mathfrak{X}_t \neq 0\}$,

$$\lim_{h \downarrow 0} \sup_{w \in \operatorname{supp} \mathfrak{X}_t} \frac{|w(t) - w(t-h)|}{\sqrt{h \log(1/h)}} = \sqrt{2}\,.$$

  This result on the *past* movements of all particles at time t can be compared with an analogous result of Tribe [**Tri89**] on the *future* movements of their offspring. Almost surely on $\{\mathfrak{X}_t \neq 0\}$,

$$\lim_{h \downarrow 0} \sup_{w \in \operatorname{range} \mathfrak{X}} \frac{|w(t+h) - w(t)|}{\sqrt{h \log(1/h)}} = \sqrt{2}\,.$$

  The methods of [**Tri89**] and later refinements of Dhersin and Le Gall [**DLG98**], [**Dhe97**] are quite different from our methods and, generally, questions of past and future movements of particles seem to be of different nature.

Acknowledgements: Parts of this research were carried out while I was a post-doctoral research fellow at the DFG-Graduiertenkolleg “Stochastische Prozesse und probabilistische Analysis” in Berlin, another part of the research was supported by the DFG project “Dimensionsspektren für die Super-Brownsche Bewegung” at Universität Kaiserslautern, the support is gratefully acknowledged. I would also like to thank Klaus Fleischmann in Berlin and Jochen Blath in Kaiserslautern for helpful discussions.

CHAPTER 5

Strong clumping of super-Brownian motion in a stable catalytic medium

WITH DONALD A. DAWSON, CARLETON UNIVERSITY
AND KLAUS FLEISCHMANN, WEIERSTRASSS INSTITUT BERLIN

WIAS PREPRINT NO. 636. SUBMITTED FOR PUBLICATION.

Abstract. A typical feature of the long time behaviour of continuous super-Brownian motion in a stable catalytic medium is the development of large mass clumps or clusters at spatially rare sites. We describe this phenomenon by means of a functional limit law under renormalisation. The limiting process is a Poisson point field of mass clumps with no spatial motion component and with infinite variance. The mass of each cluster evolves independently according to a continuous process trapped at mass zero, which we describe explicitly by means of a Brownian snake construction in a random medium. We also determine the survival probability and asymptotic size of the clumps.

AMS Subject Classification (2000): *Primary 60K37, 60K35, 60J80; Secondary 60G57, 60F05.*
Key words: *catalytic super-Brownian motion, stable catalysts, critical branching, measure-valued branching, random medium, clumping, functional limit law, historical superprocess, Brownian snake in a random medium, subordination, exit measures, good and bad paths, stopped measures, collision local time, heavy tails, Feynman-Kac formula, annealed and quenched random medium approach.*

1. Introduction

1.1. Motivation. Models of particle movement and branching in random media have been widely studied in the last twenty years. A class which received particular interest are models of measure-valued processes where, heuristically speaking, the individual branching rates of the moving particles depend on the amount of contact between the particle, called the *reactant*, and a singular random medium, called the *catalyst.* In dimension one even very thin catalysts, for example point catalysts, can be considered. A particularly natural choice of a catalytic medium are *stable* random measures Γ on $\mathbb{R}$ of index $0 < \gamma < 1$, which are the prototypes of a singular catalyst with infinite asymptotic density, see formula (3) below, and [**DaF91**, Subsections 1.3–1.4] for further motivation for this choice of catalytic medium. A rather general one-dimensional model combining super-stable motions of the reactant particles with possibly moving random catalysts, covering the case of the stable medium Γ was developed in [**DaF91, DaF92**]. For an up-to-date introduction to catalytic super-Brownian motion, we refer to [**DaF00**].

Recent research on super-Brownian motion with a stable catalytic medium lead to several interesting results; we restrict our attention to the case of a Brownian moving reactant, which branches with finite variance in the presence of a non-moving stable catalyst in $\mathbb{R}$. In this case, starting from a finite initial mass, the compact support property was shown in [**DLM95**], and finite time extinction in [**DFM00**], see [**FlM00**] for a quick route. Already in [**DaF91**], in the case of an infinite initial measure, the long-term clumping behaviour of the reactant was exposed in a mass-time-space rescaling limit theorem. It states that at a fixed macroscopic time t the suitably mass-space-rescaled clumps form a random measure with *independent increments*, see [**DaF91**, Theorem 1.9.4]. But it could not be settled, see [**DaF91**, p.251], whether or not the clumps are macroscopically spatially isolated, that is whether the limiting measure is carried by a Poisson point field on $\mathbb{R}$ as known in the constant medium case [**DaF88**].

The main motivation for the present paper was to attack this problem. We show that in fact the clumps *are* isolated, that is, the limiting measure is a homogeneous Poisson point field of mass clumps, Theorem 1.1 (ii) below. This is achieved by a refinement of a method of good and bad historical reactant paths, which was developed in [**DFM00**] and goes back to [**DaP91, FlM97, MuP92**].

Beyond this problem, we describe the mass of the rescaled clumps as a process in macroscopic time. For this purpose we provide a *functional* limit approach, Theorem 1.1 (i) below, which shows convergence of the rescaled processes on a path space of *continuous* measure-valued processes. The time evolution of these masses is described in terms of exit measures of a Brownian snake in a random medium with a motion process featuring the inverse of the collision local times of the reactant paths with the medium, see Theorem 1.5 (ii) below. Whereas the clumps of the original process have finite variance given the medium, this property is lost in the limit, a remarkable property conjectured in [**DaF91**, p.253]. In fact, the clump sizes of the limit have probability tails of index $1+\gamma<2$, see Theorem 1.10 (iii) below. This is in contrast to the constant medium case studied in [**DaF88**] and due to the fact that the stable catalyst does not have locally finite expectations, so that the atomic catalyst sizes are highly fluctuating. We also determine the (macroscopic) survival probability of clumps, see Theorem 1.10 (ii).

A main tool for the functional limit theorem is the representation of both the catalytic super-Brownian motion and the limit process in terms of exit measures of a *Brownian snake in the stable medium* Γ. The use of exit measures and subordination for the historical particles to describe general branching mechanisms goes back to [**BGJ97**], though the present paper seems to be the first instance where this approach is used to deal with the case of catalytic, space-dependent, branching.

Revealing the even macroscopically isolated nature of the clumps embeds the present investigation in the realm of the concept of *intermittency.* Roughly speaking, intermittency means in our context that in the long time the catalytic superprocess exhibits a spatially extremely irregular structure consisting of islands of high mass peaks, which are located at great distance from each other. See for instance [**GdH99, GäM98**] or [**Mol94**] for other work in this direction.

1.2. Statement of the main results.

1.2.1. *Super-Brownian motion in a stable catalytic medium: preliminaries.* Let $\mathcal{M}(\mathbb{R})$ denote the space of all locally finite measures on $\mathbb{R}$, equipped with the vague topology generated by the mappings $\varphi\mapsto\langle\mu,\varphi\rangle$, for all $\varphi:\mathbb{R}\to[0,\infty)$ continuous with compact support. Here and throughout the paper we use both notations $\langle\mu,\varphi\rangle$ and $\int_{\mathbb{R}}\varphi\,d\mu$ to denote integrals. There is a sequence $\{\varphi_n\colon n\in\mathbb{N}\}$ of such functions such that

$$d(\mu,\nu):=\sum_{n=1}^{\infty}2^{-n}\Big(\big|\langle\mu,\varphi_n\rangle-\langle\nu,\varphi_n\rangle\big|\wedge 1\Big),\ \text{for}\ \ \mu,\nu\in\mathcal{M}(\mathbb{R}), \tag{1}$$

defines a metric, which makes $\mathcal{M}(\mathbb{R})$ Polish.

Define Φ to be the set of all continuous functions $\varphi : \mathbb{R} \to [0,\infty)$ such that there are constants $a, b > 0$ with $\varphi(x) \le a \exp(-bx^2)$ for all $x \in \mathbb{R}$. For all measure-valued processes in this paper we choose the state space to be the space of tempered measures

$$\mathcal{M}_{\text{tem}} := \mathcal{M}_{\text{tem}}(\mathbb{R}) := \Big\{\mu \in \mathcal{M}(\mathbb{R}) : \langle \mu, \varphi\rangle < \infty \text{ for all } \varphi \in \Phi\Big\}. \tag{2}$$

Note that in particular the Lebesgue measure ℓ belongs to $\mathcal{M}_{\text{tem}}$. We let $\mathcal{M}_{\text{tem}} \subseteq \mathcal{M}(\mathbb{R})$ inherit the vague topology of $\mathcal{M}(\mathbb{R})$.

Suppose that Γ is a *stable random measure* on $\mathbb{R}$ of index $0 < \gamma < 1$, i.e. for every measurable $\varphi : \mathbb{R} \to [0,\infty)$ we have

$$\mathsf{E}\big\{\exp\langle \Gamma, -\varphi\rangle\big\} = \exp\Big(- \int_{\mathbb{R}} \varphi(x)^\gamma \, dx\Big). \tag{3}$$

Almost surely, Γ belongs to $\mathcal{M}_{\text{tem}}$. This follows from the fact that the integral on the right hand side of (3) is always finite for $\varphi \in \Phi$. Moreover, Γ is almost surely a purely atomic measure with atoms densely located in $\mathbb{R}$.

The measure-valued processes under consideration may be considered as random variables with values in the space $C((0,\infty), \mathcal{M}_{\text{tem}})$ of continuous functions $\nu : (0,\infty) \to \mathcal{M}_{\text{tem}}$, where for topological reasons it is sometimes convenient to exclude the time $t = 0$. We endow this space with the topology of *uniform convergence on compact intervals*, which is induced by the metric

$$\mathbf{d}(\mu,\nu) := \sum_{n=1}^{\infty} 2^{-n} \sup_{1/n \le t \le n} d\big(\mu(t), \nu(t)\big), \ \text{ for } \mu,\nu \in C\big((0,\infty), \mathcal{M}_{\text{tem}}\big), \tag{4}$$

and is easily seen to be Polish.

Let $X := X[\Gamma] := \{X_t : t \ge 0\}$ denote the continuous super-Brownian motion in $\mathbb{R}$ in the catalytic random medium Γ. Throughout the paper we refer to probabilities and expectations with respect to the random medium Γ with letters P and E and to the probabilities and expectations of the process with given medium Γ by $\mathbb{P}^\Gamma$ and $\mathbb{E}^\Gamma$, sometimes with a subscript indicating the starting measure. With this convention, for *given* Γ, the process $X = X[\Gamma]$ is the continuous, time-homogeneous Markov process with Laplace transition functionals

$$\mathbb{E}^\Gamma\big\{\exp\langle X_t, -\varphi\rangle \,\big|\, X_s = \mu\big\} = \exp\langle \mu, -V^\Gamma_{t-s}\varphi\rangle, \ \text{ for } t > s \ge 0, \tag{5}$$

where $\mu \in \mathcal{M}_{\text{tem}}$, $\varphi \in \Phi$, and $V^\Gamma\varphi = \big\{V^\Gamma_t\varphi\,(x) : t \ge 0,\, x \in \mathbb{R}\big\}$ is the unique nonnegative solution of the equation

$$V^\Gamma_t\varphi\,(y) = S_t\varphi\,(y) - 2\int_0^t ds \int_{\mathbb{R}} p_s(x-y)\,\big[V^\Gamma_{t-s}\varphi\,(x)\big]^2\,\Gamma(dx), \ \text{ for } t \ge 0,\ y \in \mathbb{R}. \tag{6}$$

Here p denotes the standard heat kernel in $\mathbb{R}$, and $S = \{S_t : t \ge 0\}$ the heat flow semigroup defined by $S_t\varphi\,(y) = \int_{\mathbb{R}} p_t(x-y)\varphi(x)\,dx$. The nonlinear semigroup $V^\Gamma = \{V^\Gamma_t : t \ge 0\}$ operates in Φ. The interpretation of the process X as a process whose reactant particles branch at site $x \in \mathbb{R}$ with rate $2\,\Gamma(dx)$ corresponds to the fact that, loosely speaking, given Γ, the function $v = V^\Gamma\varphi$ solves the symbolic partial differential equation

$$\frac{\partial}{\partial t} v = \frac{1}{2}\,\frac{\partial^2}{\partial x^2} v - 2\,\Gamma v^2, \quad \text{with initial condition } v\big|_{t=0} = \varphi. \tag{7}$$

Existence and uniqueness of nonnegative solutions V^Γ of (6) were established in [**DaF92**], X was constructed as a Markov process in [**DaF91**, Section 2], and its continuity follows from [**DaF97**, Corollary 2 (p.257), Proposition 12 (p.230), and Theorem 1(b) (p.235)], even in a stronger topology. Note also that in the case of a finite starting measure the total mass process $\|X\| := \{\|X_t\| : t \geq 0\}$ is a continuous martingale [**DaF97**, Proposition 3, p.236].

We stress the fact that we always use a *quenched* approach in dealing with the model $X = X[\Gamma]$ in the random medium Γ: First the catalyst Γ is sampled, and then the reactant process $X[\Gamma]$ is run, given the catalyst Γ. In particular, the law $\mathbb{P}^\Gamma$ of the reactant is random, and the randomness is inherited from the law P of Γ.

1.2.2. *Strong clumping of catalytic super-Brownian motion.* Define the *scaling index*

$$\eta := (\gamma + 1)/(2\gamma) \tag{8}$$

and observe that this number is larger than 1. For every $k > 0$ we introduce the renormalised measure-valued process $X^k = X^k[\Gamma] = \{X_t^k : t \geq 0\}$ by

$$X_t^k(B) := k^{-\eta} X_{kt}(k^\eta B), \ \text{for } B \subseteq \mathbb{R} \text{ Borel}, \ t \geq 0. \tag{9}$$

The next theorem summarises the main results obtained before the present paper.

THEOREM 1.0 (Results of [**DaF91**, Theorem 1.9.4]).

(i) Convergence. *Starting $X = X[\Gamma]$ in the Lebesgue measure $X_0 = \ell$, for every* fixed t *there is a random variable X_t^∞ defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P}_\ell)$, which is independent of the medium Γ, such that, in P–probability, the following weak convergence of probability measures on $\mathcal{M}_{\text{tem}}$ holds:*

$$\lim_{k\uparrow\infty} \mathbb{P}_\ell^\Gamma\{X_t^k \in \bullet\} = \mathbf{P}_\ell\{X_t^\infty \in \bullet\}. \tag{10}$$

(ii) Characterization of the limit. *For every bounded, measurable function $\varphi : \mathbb{R} \to [0, \infty)$ let*

$$U^\Gamma \varphi = \{U_r^\Gamma \varphi\,(x) : r \geq 0, \ x \in \mathbb{R}\} \tag{11}$$

be the nonnegative solution of the equation

$$U_r^\Gamma \varphi\,(y) = S_r \varphi\,(y) - 2 \int_0^r ds \int_{\mathbb{R}} p_s(x - y)\, \big[U_{r-s}^\Gamma \varphi\,(x)\big]^2\, \Gamma(dx), \ \text{for } r \geq 0, \ y \in \mathbb{R}, \tag{12}$$

which is constructed in [**DaF92**, Theorem 2.14]. *Then the Laplace functional of X_t^∞ satisfies*

$$\mathbf{E}_\ell\big\{\exp\big(-\theta X_t^\infty(A)\big)\big\} = \exp\big(-\ell(A)\, \mathsf{E} U_t^\Gamma \theta\,(0)\big), \ \text{for } A \subseteq \mathbb{R} \text{ Borel}, \ \theta \geq 0. \tag{13}$$

(iii) Properties of the limit. *$\mathbf{P}_\ell$–almost surely, X_t^∞ is non-degenerate and has independent, stationary increments. The scaling procedure is* persistent *in the sense that $\mathbf{E}_\ell X_t^\infty = \ell$.*

The important feature of this result is that the non-degenerate limit is obtained by a different, stronger scaling as in the classical case of a constant medium [**DaF88**], hence the qualitative nature of the clumping behaviour is different.

Crucial questions about the spatial structure of the limit measures X_t^∞ were left open in [**DaF91**]. A question of particular interest in this realm was posed in [**DaF91**, p.251]: the problem is whether or not the X_t^∞ are compound Poisson point fields on $\mathbb{R}$, i.e. whether on the macroscopic level the clumps are spatially separated. Our first main result answers this question in the affirmative. Our second aim in this paper is to give a full description of the spatial and temporal evolution of the field of clumps at a macroscopic level. This requires, as a first step, a functional limit theorem. This question was not investigated in [**DaF91**] and is particularly interesting, as the limit process turns out to be non-Markovian and continuous. Here is the precise statement:

THEOREM 1.1 (Main result). *Let X be the continuous super-Brownian motion in the stable random medium Γ started with $X_0 = \ell$, and $X^k = X^k[\Gamma] = \{X_t^k : t > 0\}$, for $k > 0$, the renormalised processes defined in* (9).

(i) Functional limit theorem. *In* P*–probability, the random laws of the renormalised processes $X^k[\Gamma]$ converge weakly on the function space $C((0,\infty), \mathcal{M}_{\text{tem}})$ as $k \uparrow \infty$ to the law of a limit process $X^\infty = \{X_t^\infty : t > 0\}$ defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P}_\ell)$, which is independent of the sampled medium Γ, and is started in $X_0^\infty = \ell$.*

(ii) Compound Poisson structure. *Let X^∞ be the limit process of part* (i). *Then, for each time $t > 0$, the state X_t^∞ of the limit process is a compound Poisson point field, i.e. a random discrete measure on $\mathbb{R}$ with atoms located in the points of a Poisson point field and with independent identically distributed atomic weights. The temporal development of X^∞ is as follows: Almost surely, the atoms do not move in space, no new atoms are born, but each atom dies in finite time.*

REMARK 1.2 (The rôle of $t = 0$). If $Z = \{Z_t : t > 0\}$ is a random variable in $C\big((0,\infty), \mathcal{M}_{\text{tem}}\big)$, and Z_0 a random variable in $\mathcal{M}_{\text{tem}}$, we say that Z is *started in* Z_0, provided that Z_t converges to Z_0 in law, as $t \downarrow 0$. In the functional limit theorem we get convergence of the processes X^k started in $X_0^k = \ell$ (to X^∞ started in $X_0^\infty = \ell$). This requirement can be relaxed slightly but not completely omitted (see Remark 2.3 below). Nevertheless it is possible to construct the process X^∞ canonically for *any* starting measure $X_0^\infty = \mu \in \mathcal{M}_{\text{tem}}$, see Theorem 1.5 or Corollary 1.9 below, leaving *open* the question of sample path continuity of X^∞ at time $t = 0$. Moreover, part (ii) of the theorem holds for the process X^∞ started in any measure $X_0^\infty = \mu \in \mathcal{M}_{\text{tem}}$. $\diamond$

REMARK 1.3 (Continuous limit process). Note that convergence in our functional limit theorem holds in the strong sense of convergence of laws on the space $C\big((0,\infty), \mathcal{M}_{\text{tem}}\big)$ of continuous $\mathcal{M}_{\text{tem}}(\mathbb{R})$–valued paths. In particular, the limit process X^∞ is *continuous* on $(0,\infty)$ as well. $\diamond$

1.2.3. *The crossing property.* An interesting path property of the unscaled process X, which enters in the proof of the functional limit theorem and may be of independent interest, is the following *crossing property*, which is reminiscent of the compact support property investigated in [**DLM95**]. For the precise formulation, denote by $\ell_{(a,b)}$ for $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ the restriction of Lebesgue measure ℓ on $\mathbb{R}$ to the open interval (a, b). We show that for the catalytic super-Brownian motion X started with $\ell_{(0,\infty)}$ the amount of total mass at a time which has travelled across the origin to the nonpositive halfline is bounded in time.

THEOREM 1.4 (Crossing property). *Suppose that $\{X_t : t \geq 0\}$ is the catalytic super-Brownian motion in the stable medium Γ and $X_0 = \ell_{(0,\infty)}$. Then, for* P*–almost every Γ,*

$$\sup_{t \geq 0} X_t \big((-\infty, 0]\big) < \infty, \ \ \mathbb{P}^\Gamma\text{–almost surely.} \tag{14}$$

1.2.4. *Snake representations of X and X^∞.* As a further major tool in the proof of Theorem 1.1 we construct representations of both the original process X and the limiting process X^∞ in terms of exit measures of a Brownian snake in the stable medium Γ. This makes the limit process explicit and permits a comparison of the two processes X and X^∞. As this is also of independent interest, we present the results here.

The idea of using the path-valued process or Brownian snake to represent classical super-Brownian motion is due to Le Gall and has since been generalised to various other types of superprocesses. Bertoin, Le Gall and Le Jan have extended this technique to represent superprocesses with more general, but not space-dependent, branching mechanisms. Roughly speaking, they use individual time-changes for each particle, which allow to pass from one branching mechanism to a different one by subordination on the particle level. References are [**LeG91, LeG93**] for the first explicit snake construction and [**BGJ97**] for the extension.

In the present paper we extend this idea to our particular case of a *space-dependent* branching mechanism — recall that in rough terms the branching rate at site x is given by $2\,\Gamma(dx)$. To formulate the result we briefly introduce the basic notation of the Brownian snake $w = w[\Gamma]$ in our random medium case, and its excursion measures $\mathbb{N}_x^{\Gamma}$, both in the quenched situation of a fixed sample Γ of the random medium. More details can be found in Subsection 2.1 below.

To describe our approach, let us first look at a generic reactant particle, which moves along a Brownian path $W = \bigl\{W(t) : t \geq 0\bigr\}$ in $\mathbb{R}$ until its death. Of course, the motion process could as well be described by the two-dimensional Markov process $t \mapsto \bigl(t, W(t)\bigr)$ with phase space $D := [0,\infty) \times \mathbb{R}$. The branching of the reactant particle, however, is governed by its *collision local time* $L_{[\Gamma,W]}$ with the medium Γ, defined by

$$L_{[\Gamma,W]}(r) = \int_{\mathbb{R}} \Gamma(dy)\mathsf{L}^y(r), \ \text{ for } \ r \geq 0, \tag{15}$$

where $r \mapsto \mathsf{L}^y(r)$ is the continuous local time of W at level $y \in \mathbb{R}$. $L_{[\Gamma,W]}$ is an nondecreasing continuous additive functional of Brownian motion W. As the positions of the atoms of Γ are dense in $\mathbb{R}$, it is easy to see that $L_{[\Gamma,W]}$ is (strictly) increasing.

We use the continuous inverse function $L^{-1}_{[\Gamma,W]}$ of $L_{[\Gamma,W]}$ to introduce a new time scale for the reactant particle on which its collision local time grows linearly. More precisely, instead of $t \mapsto \bigl(t, W(t)\bigr)$, we define a time-homogeneous continuous Markov process $\xi := \{\xi_r : r \geq 0\}$ with values in $D = [0,\infty) \times \mathbb{R}$ by

$$\xi_r := \bigl(L^{-1}_{[\Gamma,W]}(r),\, W \circ L^{-1}_{[\Gamma,W]}(r)\bigr), \ \text{ for } \ r \geq 0, \tag{16}$$

where W is a Brownian motion started in x. The first component of this process can be interpreted as the new individual clock of the Brownian reactant particle, travelling in the medium Γ, and the second component as its position along the new time scale. For all $t > 0$, define the *first exit time*

$$\tau_t := \tau_t(\xi) := \inf\bigl\{r > 0 : \xi_r \notin [0,t) \times \mathbb{R}\bigr\} \tag{17}$$

of the path ξ from the domain $D^t := [0,t) \times \mathbb{R}$. At time τ_t the process ξ is in the state $\bigl(t, W(t)\bigr)$, and the reactant particle has accumulated the collision local time

$$L_{[\Gamma,W]}(t) = \tau_t \tag{18}$$

and is placed in $W(t)$. Less formally, a single generic reactant particle of X_t may be represented by a path ξ stopped at the random time τ_t.

The Brownian snake can be interpreted as a natural parametrisation of the collection of all reactant particles in the range of X, where each particle is represented by a stopped path. For this purpose, define the *set of stopped paths* by

$$\mathfrak{P} := \Bigl\{f \in C\bigl([0,\infty), D\bigr) : \ \text{there exists } \zeta \geq 0 \text{ with } f(r) = f(\zeta) \text{ for all } r \geq \zeta\Bigr\}. \tag{19}$$

With every $f \in \mathfrak{P}$ we can associate the *lifetime* $\zeta = \zeta(f)$, which is the minimal $\zeta \geq 0$ such that the path f is constant on $[\zeta,\infty)$. We equip $\mathfrak{P}$ with the following metric: for $f, f' \in \mathfrak{P}$ let ζ, ζ'

the associated lifetimes and

$$\mathfrak{d}(f,f') := \big|f(0)-f'(0)\big| + |\zeta-\zeta'| + \int_0^{\zeta\wedge\zeta'} \Big(\sup_{x\in[0,u]} \big|f(x)-f'(x)\big| \wedge 1\Big)\,du. \tag{20}$$

The *Brownian snake* w with start in $(0,x)\in D$ and motion process ξ is a certain continuous strong Markov process $w:[0,\infty)\to\mathfrak{P}$ whose state space is the set of all stopped paths $f\in\mathfrak{P}$ with $f(0)=(0,x)$. Brownian snakes with general Markov processes as motion process were constructed in [**BGJ97**].

With every $\mathfrak{P}$-valued Markov process we can associate the *lifetime process* $\zeta:[0,\infty)\to[0,\infty)$ defined by $\zeta_s := \zeta(w_s)$. For the Brownian snake w, the lifetime process ζ is by definition a reflected Brownian motion. Moreover, given ζ, two paths w_{s_1} and w_{s_2}, $s_1<s_2$, agree up to time $m:=\min_{[s_1,s_2]}\zeta$, and the two continuations $\{w_{s_1}(m+r): 0\le r\le \zeta_{s_1}-m\}$ and $\{w_{s_2}(m+r): 0\le r\le\zeta_{s_2}-m\}$ with fixed starting point $w_{s_1}(m)=w_{s_2}(m)$ are independent (see also Figure 1 below). Heuristically, if $m=0$ the particles represented by w_{s_1} and w_{s_2} belong to different families, whereas if $m>0$ and $s\in[s_1,s_2]$ satisfies $\zeta_s=m$, the path w_s represents the last common ancestor of w_{s_1} and w_{s_2}.

The constant path $f\in\mathfrak{P}$ given by $f(r)=(0,x)$ for all $r\ge 0$ is a regular recurrent point for the Markov process w. Indeed, this follows immediately from the fact that the lifetime process is a reflected Brownian motion. Hence we can define $\mathbb{N}_x^\Gamma$ to be the suitably normalised *excursion measure* of the Brownian snake w from the constant path $f=(0,x)$; see e.g. [**Blu92**] for the excursion theory of Markov processes. Every sample of such an *excursion from* $(0,x)$ is a continuous path-valued function $w:[0,\sigma]\to\mathfrak{P}$ for some finite $\sigma=\sigma(w)>0$, the *length* of the excursion, such that $w_0=w_\sigma$ is the constant path remaining in $(0,x)$, and w_s is not constant for all $s\in(0,\sigma)$. Then $\mathbb{N}_x^\Gamma$ is a σ-finite measure on the set

$$\mathfrak{W} := \Big\{ w\in C\big([0,\sigma],\mathfrak{P}\big) : \text{ for some } \sigma>0\Big\} \tag{21}$$

of path-valued functions. Although it is stretching the usual terminology a bit, we use the words sample and process also in the case of the underlying non-probability measure $\mathbb{N}_x^\Gamma$.

With each excursion $w:[0,\sigma]\to\mathfrak{P}$ we can again associate the *lifetime process* $\zeta:[0,\sigma]\to[0,\infty)$ by letting $\zeta_s:=\zeta(w_s)$, which under $\mathbb{N}_x^\Gamma$ is a Brownian excursion from 0. Heuristically, an excursion w represents the whole family tree of an reactant particle, which at time 0 is located at x.

Following [**BGJ97**] or [**Dyn91**], we can define, for every $t>0$, the *exit local time at level* t of an excursion $w\in\mathfrak{W}$ as the process $L^t[w]:=L^t:=\big\{L_s^t: s\in[0,\sigma]\big\}$ such that

$$L_s^t := L_s^t[w] := \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^s 1_{\{\tau_t(w_u)<\zeta_u<\tau_t(w_u)+\varepsilon\}}\,du, \tag{22}$$

for every s, $\mathbb{N}_x^\Gamma$–almost surely, where τ_t is defined as in (17). The *total exit local time at level* $t>0$ *of an excursion* w is $L_\sigma^t[w]:=L_{\sigma[w]}^t[w]$. Note that the measure associated with the monotone function $s\mapsto L_s^t[w]$ is supported by those s where $\tau_t(w_s)=\zeta_s$ and recall that exactly those paths w_s represent particles of X_t. The *exit measure at level* $t>0$ is the measure $Z^t:=Z^t[w]$ on $\partial D^t:=\{t\}\times\mathbb{R}$ defined by

$$\big\langle Z^t[w],\varphi\big\rangle := \int_0^\sigma \varphi\big(w_s(\zeta_s)\big)\,L_\bullet^t(ds), \ \text{ for } \varphi:\partial D^t\to[0,\infty) \text{ measurable}, \tag{23}$$

where the integral is a Stieltjes integral with respect to the nondecreasing function $s\mapsto L_s^t[w]$. Slightly abusing notation, for fixed $t>0$, we can identify ∂D^t with $\mathbb{R}$, that is, we can consider

$Z^t[w]$ and φ as a measure respectively function on $\mathbb{R}$. Such identifications will often be used in the following. The measure $Z^t[w]$ can be interpreted as the spatial distribution of the descendants at time t of a reactant particle, which at time 0 is located as x. The quantity $L^t_\sigma[w]$ describes the mass of the totally produced reactant progeny of this particle at time t.

We now have the means to describe both X and X^∞ in terms of the excursion measures $\mathbb{N}^\Gamma_x$. In the case of a measure $\mu \in \mathcal{M}_{\text{tem}}$ different from Lebesgue measure, (27) below should be understood as the natural definition of X^∞ with $X^\infty_0 = \mu$.

THEOREM 1.5 (Snake representations). *Let $\mu \in \mathcal{M}_{\text{tem}}$ be an arbitrary starting measure.*

(i) Representation of X. *Given Γ, let $\Pi = \Pi[\Gamma]$ be a Poisson point field on the space $\mathfrak{W} \times \mathbb{R}$ with intensity measure $\pi = \pi[\Gamma]$ defined by*

$$\pi(dw\,dx) := \int_{\mathbb{R}} \mathbb{N}^\Gamma_y(dw) \otimes \delta_y(dx)\,\mu(dy). \tag{24}$$

Then, for P–almost all Γ, the superprocess $X = X[\Gamma]$ with $X_0 = \mu$ can be represented as

$$\langle X_t, \varphi\rangle = \int_{\mathfrak{W}\times\mathbb{R}} \big\langle Z^t[w], \varphi\big\rangle\, \Pi(dw\,dx), \tag{25}$$

for all $t > 0$ and $\varphi : \mathbb{R} \to [0,\infty)$ measurable.

(ii) Representation of X^∞. *Let Π^∞ be a Poisson point field on $\mathfrak{W} \times \mathbb{R}$ with intensity measure*

$$\pi^\infty(dw\,dx) := \Big(\int_{\mathcal{M}_{\text{tem}}} \mathbb{N}^\Upsilon_0(dw)\,\mathsf{P}(d\Upsilon)\Big) \otimes \mu(dx). \tag{26}$$

Then the limit process X^∞ with $X^\infty_0 = \mu$ has the representation

$$\langle X^\infty_t, \varphi\rangle = \int_{\mathfrak{W}\times\mathbb{R}} L^t_\sigma[w]\varphi(x)\,\Pi^\infty(dw\,dx), \tag{27}$$

for all $t > 0$ and $\varphi : \mathbb{R} \to [0,\infty)$ measurable.

REMARK 1.6 (Only dependence on the marginal measure). Note that only the marginal measures $\Pi(dw \times \mathbb{R})$ enter in the representation (25). We have included the space coordinate into the definition of the Poisson point field Π in order to simplify a comparison of the intensity measures (24) and (26). $\diamond$

REMARK 1.7 (Continuous versions of $Z[w]$ and $L_\sigma[w]$). As the process X has a continuous version, there is a continuous version of the process $Z[w] = \big\{Z^t[w] : t > 0\big\}$ of exit measures as well. We may henceforth assume that $Z[w]$ under $\mathbb{N}^\Gamma_x$ is this continuous version. Similarly, from the continuity of the total mass process $\|X\|$ in the case of a finite starting measure, we can see that also the process $L_\sigma[w] = \big\{L^t_\sigma[w] : t > 0\big\}$ has a continuous version, which we henceforth use. $\diamond$

REMARK 1.8 (A finiteness property). From the representation (27) it can be seen easily, that X^∞ has the compound Poisson structure stated in Theorem 1.1 (ii) if and only if the intensity measure π^∞ in (26) satisfies

$$\int_{\mathcal{M}_{\text{tem}}} \mathbb{N}^\Upsilon_0\,\{w : L^t_\sigma[w] > 0\}\;\mathsf{P}(d\Upsilon) < \infty, \quad \text{for } t > 0. \tag{28}$$

This *finiteness property* of π^∞ implies that after an arbitrarily small, positive amount t of macroscopic time, locally only finitely many macroscopic clumps exist. Moreover, together with the continuity of $L_\sigma[w]$ this also implies the continuity of X^∞ in the representation (27). $\diamond$

The snake representations enable us to make a comparison of X and X^∞, and, moreover, draw a revealing heuristic picture of the limit process X^∞. For both processes, mass is initially spread on $\mathbb{R}$ according to μ. In the case of the original process X, starting from each infinitesimal small mass point $\mu(dx)$ a potential family of reactant particle is evolving, whereas in the case of the limit process X^∞ at each infinitesimal small mass point $\mu(dx)$ a potential macroscopic clump can be created. After an arbitrarily small, positive amount of time locally only finitely many families of X survive and, similarly, after an arbitrarily small, positive amount of *macroscopic* time locally only finitely macroscopic clumps survive in X^∞. The further development of the *total mass* of the offspring progeny of any reactant particle in X or of any macroscopic clump in X^∞ is in both cases governed by the laws of $t \mapsto L^t_\sigma[w]$ under the excursion measure $\mathbb{N}^\Gamma_\bullet$ in the random medium Γ.

There are however a number of *significant differences*:

- In X each particle family uses the excursion measures $\mathbb{N}^\Gamma_x$ for the same sample Γ (though around different places x). The clumps of X^∞ however are based on the samples w of the measure $\mathsf{E}\mathbb{N}^\Upsilon_0(dw)$ which is independent of the position x of the clump and of the medium sample Γ. For each individual clump the sample w is in fact the result of a two stage experiment: First Υ is sampled with the law P of the stable medium, and then w is sampled according to the law $\mathbb{N}^\Upsilon_0(dw)$.
- Whereas the reactant particle families of X have a spatial spread and their motion component is visible, this is not the case with the clumps of X^∞. Macroscopic clumps are mass points, which remain at their original spatial position, only their mass is variable. Indeed, whereas the full measure $Z^t[w]$ enters into (25), only $L^t_\sigma[w]$ enters into formula (27) and the spatial structure of $Z^t[w]$ is suppressed. This in particular leads to the loss of the Markov property in the limit process X^∞. Heuristically speaking, the clumps have a hidden *micro-life*, governing the branching behaviour, but invisible from the outside, since the excursion measure $\mathbb{N}^\Upsilon_0$ in the random medium Υ is used in the annealed sense $\mathsf{E}\mathbb{N}^\Upsilon_0$ only.

From the representation (27) of the limit process X^∞, we easily get the Laplace functionals of its finite dimensional marginals — note that the result is consistent with the representation of the one-dimensional marginals mentioned already in (13).

Corollary 1.9 (Finite dimensional distributions). *The finite dimensional marginals of the limit process X^∞ with starting measure $X^\infty_0 = \mu$, defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P}_\mu)$, are determined by*

$$\mathbf{E}_\mu\Big\{\exp\Big(-\sum_{i=1}^n \langle X^\infty_{t_i}, \varphi_i\rangle\Big)\Big\} = \exp\Big(-\int_{\mathbb{R}} \mathsf{E}U^\Gamma_{t_1,\dots,t_n}\left[\varphi_1(x),\dots,\varphi_n(x)\right](0)\,\mu(dx)\Big), \tag{29}$$

for $0 \le t_1 \le \dots \le t_n$ and measurable $\varphi_1,\dots,\varphi_n : \mathbb{R} \to [0,\infty)$ for $n \ge 1$. Here $U^\Gamma_{t_1}[a_1] := U^\Gamma_{t_1} a_1$ is taken from (12) with constant function $\varphi = a_1 \ge 0$, and $U^\Gamma_{t_1,\dots,t_n}[a_1,\dots,a_n]$ is defined inductively: for $n \ge 2$,

$$U^\Gamma_{t_1,\dots,t_n}[a_1,\dots,a_n] = U^\Gamma_{t_1}\Big[a_1 + U^\Gamma_{t_2-t_1,\dots,t_n-t_1}[a_2,\dots,a_n]\Big], \tag{30}$$

for all $0 \le t_1 \le \dots \le t_n$ and $a_1,\dots,a_n \ge 0$.

1.2.5. *Further properties of the limit process X^∞.* To round out the picture of the limit model we describe the major indices related to the survival probability and the tail behaviour of the mass clumps on the macroscopic level. Recall the index $\gamma \in (0,1)$ of our stable medium Γ, and the scaling index η introduced in (8).

THEOREM 1.10 (Properties of the limit process X^∞). *Let X^∞ be the limit process started in ℓ.*

(i) Self-similarity. *X^∞ satisfies, for every $k>0$,*

$$X_t^\infty(B) = k^{-\eta} X_{kt}^\infty(k^\eta B) \text{ in distribution, for } t \geq 0 \text{ and } B \subseteq \mathbb{R} \text{ Borel.}$$

(ii) Survival probability. *The ratio of the intensities $\lambda(t)$ and $\lambda(s)$ of the Poisson point fields carrying the clumps at various macroscopic times $t>s>0$, respectively, satisfies*

$$\lambda(s)/\lambda(t) = \big(t/s\big)^\eta.$$

Hence, denoting by $\daleth_s(t)$ the mass at the macroscopic time t of a clump at time s, the survival probability of $\daleth_s(t)$ is given by

$$\mathbf{P}_\ell\big\{\daleth_s(t) > 0\big\} = (s/t)^\eta, \text{ for all } t>s>0.$$

Moreover, we have, for all $t>s>0$,

$$\daleth_t(t) = \Big(\frac{t}{s}\Big)^\eta \daleth_s(s) \text{ in distribution.}$$

(iii) Clump size tails. *The tail behaviour of the clump size $\daleth_t(t)$ is governed by*

$$\mathbf{P}_\ell\big\{\daleth_t(t) > a\big\} \asymp t^{\eta(\gamma+1)}\, a^{-\gamma-1} \text{ as } a \uparrow \infty.$$

Here $\asymp$ means that the ratio of the quantities involved is bounded away from zero and infinity as $a \uparrow \infty$ by constants independent of $t>0$.

REMARK 1.11 (Infinite variance). It is quite remarkable, that the clump size is heavy-tailed, in particular it has *infinite variance.* The latter fact was conjectured in [**DaF91**, Subsection 1.14]. $\diamond$

REMARK 1.12 (Open problem). Note that the intensity $\lambda(t)$ of the carrying Poisson point field at time $t>0$ occurring in (ii) is positive, but it is *open* to determine its exact value. $\diamond$

1.3. Outline of the paper. Here we indicate the further structure of the paper, give a guideline to where various parts of the proofs can be found, and briefly review the main methods of proof.

Section 2 is devoted to those aspects of the paper related to the Brownian snake construction in a random medium. In Subsection 2.1 we establish the Brownian snake representation of super-Brownian motion X in the catalytic medium Γ [Theorem 1.5 (i)]. Subsection 2.2 contains the proofs of the Laplace functionals in Corollary 1.9 and the description of X^∞ in Theorem 1.1 (ii), which both rely on the definition of X^∞ in terms of its snake representation [Theorem 1.5 (ii)]. Both snake representations are used in Subsection 2.3, together with Birkhoff's individual ergodic theorem, to prove the functional limit theorem. The proof also relies on two further steps of independent interest, whose proofs are deferred to Section 3: the finiteness property (28) of the intensity measure π^∞, and the crossing property, Theorem 1.4.

Section 3 concerns the aspects of proof related to the method of good and bad paths. In Subsections 3.1 and 3.2 we formulate a quantitative extension of this method. The key step is to give an upper bound on the survival probability of the catalytic super-Brownian motion with a finite starting mass in terms of a quantitative characteristic of the random medium Γ. This is then applied in Subsection 3.3 to prove the crossing property and in 3.4 to verify the finiteness statement (28) and thus derive the compound Poisson structure of the limit process

X^{∞}. We also like to point out that our approach to the method of good and bad paths (other than the approach of [**DFM00**]) does not rely on the compact support property of catalytic super-Brownian motion established in [**DLM95**], and conversely seems to be a good starting point for an independent, new probabilistic proof of the compact support property.

Section 4 deals with the more analytical proof techniques. We first investigate the time evolution of the mass of the clumps in our limit model. The calculations of the Poisson intensities and survival probabilities stated in Theorem 1.10 exploit the natural scaling invariance of the limit process together with the Poisson carrier structure, see Subsection 4.1. The calculation of the tail behaviour in Theorem 1.10 is based on a Feynman-Kac representation of the solutions of the log-Laplace equation (6), provided in Subsection 4.2 and a simple version of the Tauberian Theorem of Bingham and Doney [**BGT87**, Theorem 8.16].

2. The Brownian snake approach in the case of a catalytic medium

In this section we prove the snake representations, Theorem 1.5, and the functional limit theorem, Theorem 1.1 (i).

2.1. The Brownian snake representation of catalytic super-Brownian motion. We now formalize the construction of the Brownian snake and verify the snake representation of X, Theorem 1.5 (i).

As announced, we first take a fixed sample of the catalytic medium Γ. Recall that $L^{-1}_{[\Gamma,W]}$ denotes the inverse function of the collision local time $L_{[\Gamma,W]}$ of a Brownian path W with Γ, which was introduced in (15).

The continuous time-homogeneous Markov process $\xi = \{\xi_r : r \geq 0\}$ on $D = [0,\infty) \times \mathbb{R}$ with start in $(a,x) \in D$ is defined by

$$\xi_r = \big(a + L^{-1}_{[\Gamma,W]}(r), W \circ L^{-1}_{[\Gamma,W]}(r)\big), \quad \text{for } r \geq 0, \tag{31}$$

where W is a Brownian motion started in $x \in \mathbb{R}$. Let $P_{(a,x)}$ denote the law of ξ started at time $t=0$ in (a,x) and, for $b \geq 0$, denote by $P^b_{(a,x)}$ the law of the related stopped paths $\{\xi_{r\wedge b} : r \geq 0\}$.

We now briefly describe the definition of the Brownian snake with motion process ξ, following the construction of the Brownian snake for an arbitrary continuous Markovian motion process in [**LeG93**]. Consider a stopped path $f \in \mathfrak{P}$ with lifetime $\zeta(f) > 0$ and such that $f(0) = (0,x)$ as introduced around (19).

If $0 \leq a \leq \zeta(f)$ and $b \geq a$ we define $\mathbf{Q}_{a,b}(f, d\tilde{f})$ to be the unique probability measure on $\mathfrak{P}$ such that

- $\mathbf{Q}_{a,b}(f, d\tilde{f})$–almost surely $\tilde{f}(r) = f(r)$, for all $r \in [0,a]$,
- the law under $\mathbf{Q}_{a,b}(f, d\tilde{f})$ of $\{\tilde{f}(a+r) : r \geq 0\}$ is the law of $\{\xi_r : r \geq 0\}$ under $P^{b-a}_{f(a)}$.

This transition can be thought of as follows. From its endpoint $\zeta(f)$, the path f is erased backwards in the original time until the absolute time a, and then renewed according to the random motion process ξ, but stopped at the absolute time b. In particular, $\mathbf{Q}_{0,b}(f, d\tilde{f}) = P^b_{(0,x)}(d\tilde{f})$. By convention we also let $\mathbf{Q}_{0,b}(x, d\tilde{f}) := P^b_{(0,x)}(d\tilde{f})$.

The parameters a, b entering into the transition laws $\mathbf{Q}_{a,b}$ are used to control erasing and renewal of the paths. In snake constructions, these parameters are determined continuously by a stochastic process, for the Brownian snake this rôle is played by a reflected Brownian motion. To be more precise, for $r, s \geq 0$, denote by $\vartheta^r_s(da\,db)$ the joint distribution of the pair

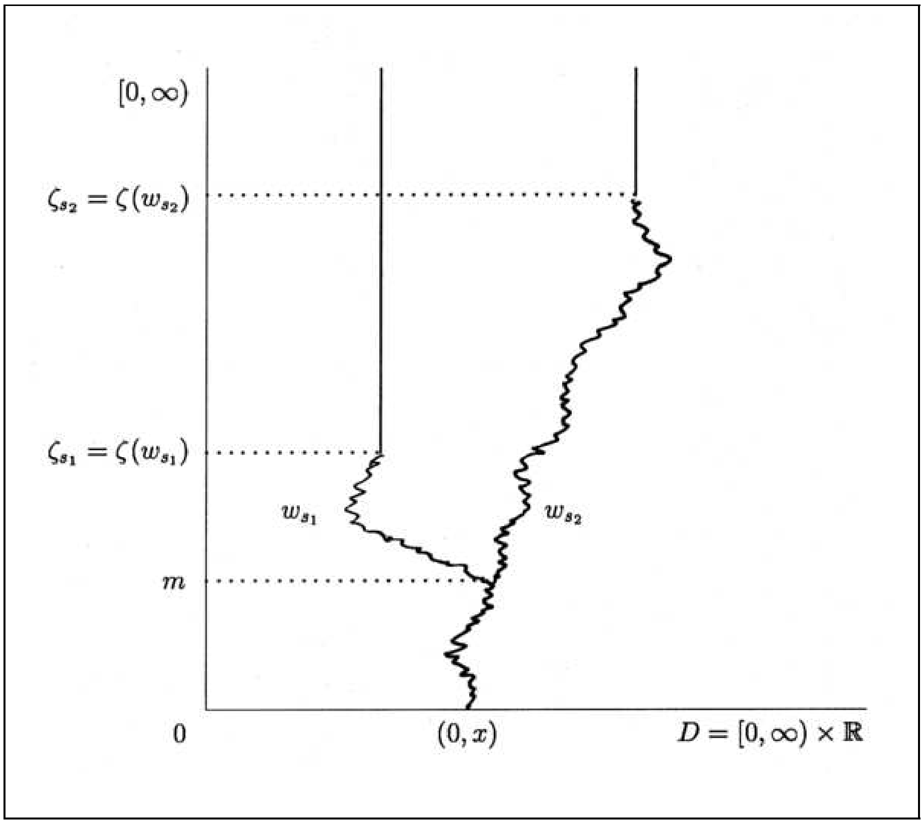

FIGURE 1. Erasing the path w_{s_1} to renew it to w_{s_2}

$\big(\min_{\tilde{s}\in[0,s]} |B_{\tilde{s}}|, |B_s|\big)$, where $B = \{B_t : t \geq 0\}$ is a Brownian motion on $\mathbb{R}$ with $B_0 = r$. Note that $\vartheta_s^r\big\{(a,b) \in [0,r] \times \mathbb{R} : a \leq b\big\} \equiv 1$.

The *Brownian snake with motion process* ξ and start in $(0,x)$ is defined to be the time-homogeneous continuous strong Markov process $w = \{w_s : s \geq 0\}$ whose transition kernels are given by

$$\mathbf{Q}_s(f, d\tilde{f}) = \int_0^\infty \int_0^\infty \vartheta_s^{\zeta(f)}(da\, db)\, \mathbf{Q}_{a,b}(f, d\tilde{f}), \quad \text{for } s \geq 0 \text{ and } f \in \mathfrak{P} \text{ with } f(0) = (0,x), \tag{32}$$

see [**BGJ97**, Proposition 5]. Recall that the *lifetime process* $\zeta = \{\zeta_s : s \geq 0\}$ is defined by $\zeta_s = \zeta(w_s)$. Under the law of w determined by the transition kernels $\mathbf{Q}_s$, $s \geq 0$, the lifetime process ζ is a reflected Brownian motion, just by construction.

To interpret the dynamics of the snake w, observe that if $s_1 < s_2$ the path w_{s_2} is obtained from w_{s_1} by erasing from its endpoint ζ_{s_1} down to the absolute time $m := \min_{[s_1,s_2]} \zeta$ and adding an independent tip of length $\zeta_{s_2} - m$ at the end. Figure 1 tries to visualise this. The paths w_{s_1} and w_{s_2}, which are stopped versions of ξ, have to be chosen to be identical on the time interval $[0,m]$ (indicated by the thick lines), but to be independent of the intervals $[m, \zeta_{s_1}]$ and $[m, \zeta_{s_2}]$, respectively, except the common starting point $w_{s_1}(m) = w_{s_2}(m)$. In particular, if $m = 0$, a new path is created, starting again in $(0,x)$. In this case the paths do not have a common part, which means that the reactant particles they represent do not have a common ancestor. This can also interpreted in the sense that the *excursions* from $(0,x)$ of the Markov process w correspond to different families of particles.

To be more precise, note that the constant path $(0,x)$ is a regular recurrent point of the Markov process w. Denote by $\mathbb{N}_x^\Gamma$ the *excursion measure* $\mathbb{N}_x^\Gamma$ of w from this path, which is a σ-finite measure on the space $\mathfrak{W}$ defined in (21). Again we can associate with every excursion $w\colon [0,\sigma] \to \mathfrak{P}$ a lifetime process $\zeta(w) := \zeta := \{\zeta_s : s \in [0,\sigma]\}$. Observe that under the measure $\mathbb{N}_x^\Gamma$ the process ζ is a Brownian excursion. Under $\mathbb{N}_x^\Gamma$, every excursion $w : [0,\sigma] \to \mathfrak{P}$ has a finite life length $\sigma(w) = \sigma > 0$, and $\zeta_s > 0$ on $(0,\sigma)$. As usual, $\mathbb{N}_x^\Gamma$ is normalised such that

$$\mathbb{N}_x^\Gamma\Big\{\sup_{s\in[0,\sigma]} \zeta_s > \varepsilon\Big\} = \frac{1}{2\varepsilon}. \tag{33}$$

At this point it is worth looking back at the definition of the intensity measures $\pi = \pi[\Gamma]$ and π^∞ in (24) respectively (26) and noting that (33) implies that both are in fact σ-finite measures, as needed for the definition of the Poisson point fields.

Proof of Theorem 1.5 (i). Recall that a sample Γ is fixed. For each measure $\mu \in \mathcal{M}_{\text{tem}}$ we consider a Poisson point field Π with intensity measure π as in (24), defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P}_\mu^\Gamma)$. We have to verify that the process X *defined* on this space by (25) is indeed a catalytic super-Brownian motion in the medium Γ, started in μ.

For $\varphi \in \Phi$, $t > 0$, and $x \in \mathbb{R}$, using $\mu = \delta_x$, let

$$U_t\varphi\,(x) := \mathbb{E}_{\delta_x}^\Gamma\left\{\exp\langle X_t, -\varphi\rangle\right\}. \tag{34}$$

It suffices to verify the following two points:

(a) $U\varphi := \big\{U_t\varphi\,(x) : \ t > 0,\ x \in \mathbb{R}\big\}$ defines a nonnegative solution of equation (6).
(b) For all $0 \le h < t$ and $\varphi \in \Phi$,

$$\mathbb{E}_\mu^\Gamma\Big\{\exp\langle X_t\,, -\varphi\rangle \,\Big|\, X_u\,,\, u \le h\Big\} = \exp\langle U_{t-h}\varphi, -X_h\rangle.$$

Fix $\varphi \in \Phi$ and $t > 0$ for the remaining proof. In order to give the *proof of* (a) we need the following facts concerning the exit measures $Z^t[w]$ of (23) under the excursion measures, see [**BGJ97**, Proposition 6]. For $x \in \mathbb{R}$ and $0 \le s < t$, define

$$u_t(s,x) := \int_{\mathfrak{W}} \mathbb{N}_x^\Gamma(dw)\Big(1 - \exp\big\langle Z^{t-s}[w], -\varphi\big\rangle\Big), \tag{35}$$

then u_t satisfies the equation

$$u_t(0,x) = E_{(0,x)}\big\{\varphi(\xi_{\tau_t})1_{\{\tau_t<\infty\}}\big\} - 2E_{(0,x)}\Big\{\int_0^{\tau_t} \big[u_t(\xi_s)\big]^2\,ds\Big\}, \quad \text{for } x \in \mathbb{R}, \tag{36}$$

with $\tau_t = \tau_t(\xi)$ from (17), using $\xi_{\tau_t} \in \partial D^t = \{t\} \times \mathbb{R}$ and again the identification of ∂D^t and $\mathbb{R}$. On the other hand, by the Laplace functional formula for Poisson point fields, from (34) and (25) we have

$$U_t\varphi\,(x) = \int_{\mathfrak{W}} \mathbb{N}_x^\Gamma(dw)\Big(1 - \exp\big\langle Z^t[w], -\varphi\big\rangle\Big). \tag{37}$$

Consequently, $u_t(s,x) = U_{t-s}\varphi\,(x)$. Then (35) and (36) show that

$$U_t\varphi\,(x) = E_{(0,x)}\big\{\varphi(\xi_{\tau_t})1_{\{\tau_t<\infty\}}\big\} - 2E_{(0,x)}\Big\{\int_0^{\tau_t} \big[u_t(\xi_s)\big]^2\,ds\Big\}. \tag{38}$$

Recalling that ξ with law $P_{(0,x)}$ can by definition be expressed by a Brownian motion W starting at time 0 from x, whose law we denote by $\mathcal{P}_{0,x}$, and that at time $\tau_t = L_{[\Gamma,W]}(t)$ the process ξ is in the state $\big(t, W(t)\big)$, which is identified with $W(t)$, formula line (38) can be rewritten as

$$U_t\varphi\,(x) = \mathcal{E}_{0,x}\left\{\varphi\big(W(t)\big)\right\} - 2\,\mathcal{E}_{0,x}\bigg\{\int_0^{L_{[\Gamma,W]}(t)} \Big[U_{t-L^{-1}_{[\Gamma,W]}(s)}\varphi\left(W\circ L^{-1}_{[\Gamma,W]}(s)\right)\Big]^2\, ds\bigg\}.$$

We now substitute s for $L^{-1}_{[\Gamma,W]}(s)$ in the second summand. Thus

$$U_t\varphi\,(x) = \mathcal{E}_{0,x}\left\{\varphi\big(W(t)\big)\right\} - 2\,\mathcal{E}_{0,x}\bigg\{\int_0^{t} \left[U_{t-s}\varphi\left(W(s)\right)\right]^2\, dL_{[\Gamma,W]}(s)\bigg\}. \tag{39}$$

But this is a probabilistic representation of (6), that is, we have proved (a).

Now we give the *proof of* (b). Recall that $t > 0$ and $\varphi \in \Phi$ are fixed. By the definition (25) of X, it obeys the branching property, hence it clearly suffices to consider finite starting measures μ. The main ingredient is the *special Markov property* of the exit measures $Z^t[w]$ of the Brownian snake w with respect to its recursion measures $\mathbb{N}^\Gamma_x$. Instead of giving the most general statement of this property, we just quote the special case we need, which follows directly from the formulation in [**BGJ97**, Proposition 7]. In our case, the special Markov property states that, for all $0 < h < t$ and $x \in \mathbb{R}$,

$$\begin{aligned} &\mathbb{N}^\Gamma_x\Big\{\exp\big(-\langle Z^t,\varphi\rangle\big)\,\Big|\, Z^u, u\le h\Big\} \\ &\quad = \exp\bigg[-\int_{\mathbb{R}} Z^h(dz)\int_{\mathfrak{W}} \mathbb{N}^\Gamma_z(dw)\Big(1-\exp\big\langle Z^{t-h}[w],-\varphi\big\rangle\Big)\bigg]. \end{aligned} \tag{40}$$

By (37) and (34), the right hand side of (40) equals $\exp\big\langle Z^h, -U^\Gamma_{t-h}\varphi\big\rangle$, and we infer that with respect to $\mathbb{N}^\Gamma_x$ the conditional distribution of $\langle Z^t,\varphi\rangle$ given $\{Z^u : u \le h\}$ is equal to the distribution of $\langle X_{t-h},\varphi\rangle$ for the starting measure Z^h. Now $\langle X_t,\varphi\rangle$ can be written as the sum of a Poissonian number of random variables $\langle Z^t_i,\varphi\rangle$, where Z_i are independent with distribution $\int_{\mathbb{R}} \mathbb{N}^\Gamma_x\big\{Z[w]\in\bullet \,\big|\, Z^h[w]\ne 0\big\}\,\mu(dx)$. Hence, the conditional distribution of $\langle X_t,\varphi\rangle$ given $\{X_u,\, u\le h\}$ is the sum of independent samples of $\langle X_{t-h},\varphi\rangle$ with starting measures adding up to X_h. Hence (b) follows from the branching property, and this finishes the proof of Theorem 1.5 (i). ☐

2.2. The Brownian snake representation of the limit process. We now assume that an arbitrary starting measure $\mu \in \mathcal{M}_{\text{tem}}$ is fixed and a Poisson point field Π^∞ with intensity measure π^∞ as in (26) is defined on a probability space $(\Omega, \mathcal{A}, \mathbf{P}_\mu)$. Recall from (33) that π^∞ is σ-finite and hence the Poisson point field is well-defined. We *define* the process X^∞ on this space by (27). In this subsection we show, using results of Section 3 below, that this process has the properties claimed in Theorem 1.1 (ii) and Corollary 1.9. The proof of the convergence $X^k \to X^\infty$ is deferred to Subsection 2.3 below, which then completes the proof of Theorem 1.5 (ii).

Proof of Corollary 1.9. Consider μ, $t_1,\dots,t_n$ and $\varphi_1,\dots,\varphi_n$ as in the corollary. By the definition (27) of X^∞, recalling the formula for the Laplace functional of a Poisson point field,

$$\mathbf{E}_\mu\Big\{\exp\Big(-\sum_{i=1}^n\langle X_{t_i}^\infty,\varphi_i\rangle\Big)\Big\} = \mathbf{E}_\mu\Big\{\exp\Big(-\int_{\mathfrak{W}\times\mathbb{R}}\sum_{i=1}^n L_\sigma^{t_i}[w]\,\varphi_i(x)\,\Pi^\infty(dw\,dx)\Big)\Big\}$$

$$= \exp\Big(\int_{\mathbb{R}}\int_{\mathcal{M}_{\text{tem}}}\int_{\mathfrak{W}}\Big(\exp\big(-\sum_{i=1}^n L_\sigma^{t_i}[w]\,\varphi_i(x)\big)-1\Big)\,\mathbb{N}_0^\Upsilon(dw)\,\mathsf{P}(d\Upsilon)\,\mu(dx)\Big). \tag{41}$$

The total mass process of $\{X_t : t\ge 0\}$ started in a finite measure $X_0=\nu$ has, as is easily seen by induction using (5), the Laplace transform

$$\mathbb{E}^\Gamma\Big\{\exp\big(-\sum_{i=1}^n c_i\|X_{t_i}\|\big)\Big\} = \exp\Big(-\int_{\mathbb{R}} U^\Gamma_{t_1,\dots,t_n}[c_1,\dots,c_n]\,(z)\,\nu(dz)\Big) \tag{42}$$

with $U^\Gamma_{t_1,\dots,t_n}[c_1,\dots,c_n]$ from (30). On the other hand, by the snake representation (25) of X, this Laplace transform can also be written as

$$\mathbb{E}^\Gamma\Big\{\exp\Big(-\sum_{i=1}^n c_i\|X_{t_i}\|\Big)\Big\} = \exp\Big(\int_{\mathbb{R}}\int_{\mathfrak{W}}\Big(\exp\big(-\sum_{i=1}^n c_iL_\sigma^{t_i}[w]\big)-1\Big)\,\mathbb{N}_x^\Gamma(dw)\,\nu(dx)\Big). \tag{43}$$

Comparing (42) and (43) as well as taking expectations with respect to the medium Γ,

$$\mathsf{E}\int_{\mathbb{R}}\int_{\mathfrak{W}}\Big(\exp\big(-\sum_{i=1}^n c_iL_\sigma^{t_i}[w]\big)-1\Big)\,\mathbb{N}_x^\Gamma(dw)\,\nu(dx) = -\int_{\mathbb{R}}\mathsf{E}U^\Gamma_{t_1,\dots,t_n}[c_1,\dots,c_n]\,(z)\,\nu(dz). \tag{44}$$

Specializing to $\nu=\delta_0$ gives

$$\int_{\mathcal{M}_{\text{tem}}}\int_{\mathfrak{W}}\Big(\exp\big(-\sum_{i=1}^n c_iL_\sigma^{t_i}[w]\big)-1\Big)\,\mathbb{N}_0^\Upsilon(dw)\,\mathsf{P}(d\Upsilon) = -\mathsf{E}U^\Gamma_{t_1,\dots,t_n}[c_1,\dots,c_n]\,(0). \tag{45}$$

Plugging this into (41) yields the formula stated in Corollary 1.9. $\square$

Proof of Theorem 1.1 (ii) [subject to the proof of (28), which will be given in Section 3 below.] We still allow an arbitrary starting measure μ, cf. Remark 1.2. From the definition (27) of X^∞ in terms of the Poisson point field Π^∞ it is clear that, for every $t>0$, the measure X_t^∞ is supported by the points of a Poisson point field on $\mathbb{R}$ with intensity measure

$$\Big(\int_{\mathcal{M}_{\text{tem}}}\mathbb{N}_0^\Upsilon\{w:\,L_\sigma^t[w]>0\}\,\mathsf{P}(d\Upsilon)\Big)\,\mu(dx). \tag{46}$$

By the finiteness property (28) the factor in front of the measure μ is *finite*, say $c>0$. Moreover, the masses of the atoms at these locations are independent with common distribution

$$\frac{1}{c}\int_{\mathcal{M}_{\text{tem}}}\mathbb{N}_0^\Upsilon\{L_\sigma^t[w]\in\bullet\}\,\mathsf{P}(d\Upsilon). \tag{47}$$

This establishes the compound Poisson property.

Suppose that $I\subseteq\mathbb{R}$ is a bounded interval and $t>0$. Again by (28), $\mathbf{P}_\mu$–almost surely, the point field Π^∞ restricted to the set $\big\{(w,x)\in\mathfrak{W}\times I:\,L_\sigma^t[w]>0\big\}$ is supported by finitely many points in $\mathfrak{W}\times I$, say

$$(w_1,x_1),\dots,(w_n,x_n),\qquad\text{with } x_1\le\dots\le x_n\,. \tag{48}$$

For every $s\ge t$, the measure X_s^∞ is supported by the set $\big\{x_i:\,1\le i\le n,\,L_\sigma^s[w_i]>0\big\}$. Hence atoms cannot move in space. To show that no new atoms can be born it would suffice to show that zero is an absorbing state for the process $s \mapsto L^s_\sigma[w_i]$. However, it is easier to argue via the Laplace transform of Corollary 1.9. Indeed, for all $s,t>0$,

$$\begin{aligned}(49)\qquad \mathbf{P}_\mu\big\{X^\infty_t(I)=0\big\} &= \lim_{\theta\uparrow\infty}\exp\big(-\mu(I)\,\mathsf{E}U^\Gamma_t\theta\,(0)\big) = \lim_{\theta\uparrow\infty}\exp\big(-\mu(I)\,\mathsf{E}U^\Gamma_t[\theta+U^\Gamma_s\theta]\,(0)\big)\\ &= \mathbf{P}_\mu\big\{X^\infty_t(I)=0 \text{ and } X^\infty_{t+s}(I)=0\big\}\end{aligned}$$

since $0\le U^\Gamma_s\theta\le\theta$ and by monotonicity. This shows that zero is an absorbing state for $t\mapsto X^\infty_t(I)$ and hence also for $t\mapsto L^t_\sigma[w]$.

Finally, for the proof that macroscopic clumps have almost surely finite lifetime, it suffices to show that, for every bounded interval I,

$$(50)\qquad \lim_{t\to\infty}\mathbf{P}_\mu\big\{X^\infty_t(I)=0\big\} = \lim_{t\to\infty}\lim_{\theta\uparrow\infty}\exp\big(-\mu(I)\,\mathsf{E}U^\Gamma_t\theta\,(0)\big)=1\,.$$

This does not depend on the starting measure μ, so that we can assume $\mu=\ell$. In this case the result follows directly from the stronger statement $\mathbf{P}_\ell\big\{\beth_s(t)>0\big\}=(s/t)^\eta$, which we prove in Subsection 4.1 below. ☐

2.3. The functional limit theorem. In this section we prove the weak convergence in P–probability of the random distributions of $X^k[\Gamma]$, as $k\uparrow\infty$, which was claimed in Theorem 1.1 (i). For this purpose we also rescale the catalytic medium, but with a *different* spatial scaling, namely

$$(51)\qquad \Gamma^k(\,\bullet\,) := k^{1/(2\gamma)}\Gamma\big(\,\bullet\,/\sqrt{k}\big)\ \text{ for } k>0.$$

But note that by self-similarity the rescaled stable medium Γ^k has the same distribution as Γ. Our strategy is to look at the distributions of the renormalised process $X^k[\Gamma^k]=\big\{X^k_t[\Gamma^k]: t\ge 0\big\}$ with changing medium Γ^k (instead of Γ), and show, using the representation of Theorem 1.5 (i), P–almost surely(!) the weak convergence of the random distributions of $X^k[\Gamma^k]$. This clearly implies weak convergence in P–probability of the random distributions of the rescaled processes $X^k[\Gamma]$ in the unscaled medium.

We start by looking at the case of the constant test function $\varphi\equiv 1$. i.e. at the total mass process $t\mapsto\|X^k_t\|$, and start X with the restricted Lebesgue measure, $X^k_0=\ell_{(a,b)}$ for $a<b$ real. The following proposition is the core of our proof of the functional limit theorem. We equip the space $C((0,\infty),\mathbb{R})$ with the Polish topology of uniform convergence on compact intervals, which matches the earlier definition of the topology on $C((0,\infty),\mathcal{M}_{\text{tem}})$.

PROPOSITION 2.1 (Total mass process). *Fix real numbers $a<b$.*

(i) Convergence. P–*almost surely, the random laws of the renormalised total mass processes $\big\|X^k[\Gamma^k]\big\|=\big\{\|X^k_t[\Gamma^k]\|: t>0\big\}$ with $X^k_0[\Gamma^k]=\ell_{(a,b)}$ converge weakly on the path space $C\big((0,\infty),\mathbb{R}\big)$ as $k\uparrow\infty$ to the law of a limit process $\{X^\infty_t(a,b): t>0\}$.*

(ii) Identification of the limit. *Let $\Pi^\infty_{a,b}$ be a Poisson point field on $\mathfrak{W}$ with intensity measure*

$$\pi^\infty_{a,b}(dw) := (b-a)\int_{\mathcal{M}_{\text{tem}}}\mathbb{N}^\Upsilon_0(dw)\,\mathsf{P}(d\Upsilon).$$

Then the limit process satisfies

$$(52)\qquad X^\infty_t(a,b)=\int_{\mathfrak{W}}L^t_\sigma[w]\,\Pi^\infty_{a,b}(dw),\quad \textit{for } t>0,$$

in particular, it is independent of the medium Γ.

Proof. Fix $a < b$. To begin with, we infer from (33) that $\pi^{\infty}_{a,b}$ is σ-finite and hence the Poisson point field $\Pi^{\infty}_{a,b}$ is well-defined. We can thus assume that the process $X^{\infty}(a,b) := \{X^{\infty}_t(a,b) : t > 0\}$ is *defined* by (52), and our aim is to show that P–almost surely the processes $\|X^k[\Gamma^k]\|$ with $X^k_0[\Gamma^k] = \ell_{(a,b)}$ converge in law on $C\big((0,\infty),\mathbb{R}\big)$ to $X^{\infty}(a,b)$ as $k \uparrow \infty$.

The first step is to derive a representation of $\|X^k[\Gamma^k]\|$ as a k-independent functional of a Poisson point field, with k-dependent intensity measure. To do this fix $k > 0$ and the medium Γ throughout the first step. From the Brownian snake representation of Theorem 1.5 (i) we infer that

$$\|X^k_t[\Gamma^k]\| = k^{-\eta} \int_{\mathfrak{W}} L^{kt}_{\sigma}[w]\,\Pi(dw), \quad \text{for } t > 0, \tag{53}$$

where $\Pi = \Pi[\Gamma^k]$ is a Poisson point field on $\mathfrak{W}$ with intensity measure $\int_{k^{\eta}a}^{k^{\eta}b} \mathbb{N}^{\Gamma^k}_x\,dx$. As the total exit local time $L^{kt}[w]$ of a snake excursion w does not depend on the second component of the motion process ξ, we can equivalently use the intensity measure

$$\int_{k^{\eta}a}^{k^{\eta}b} \mathbb{N}^{T^x\Gamma^k}_0\,dx. \tag{54}$$

We now show that

$$\begin{aligned} &\text{the distributions of } \big\{k^{-\eta}L^{kt}_{\sigma}[w] : t > 0\big\} \text{ under } \mathbb{N}^{T^x\Gamma^k}_0 \text{ and} \\ &\text{of } \big\{L^{t}_{\sigma}[w] : t > 0\big\} \text{ under } k^{-\eta}\mathbb{N}^{T^{x/\sqrt{k}}\Gamma}_0 \text{ coincide.} \end{aligned} \tag{55}$$

Indeed, a Brownian scaling of time and space yields for the collision local times

$$L_{[\Gamma^k, x+W]}(kt) = k^{\eta} L_{\left[\Gamma, x/\sqrt{k}+W^k\right]}(t), \quad \text{for } t > 0, \tag{56}$$

where W^k is defined by $W^k_t = (1/\sqrt{k})W_{tk}$, for $t \geq 0$. We now define a scaling $\mathfrak{W} \to \mathfrak{W}$ mapping w to w^k in such a way that

- the lifetime process ζ^k of w^k is given by $r \mapsto \zeta^k_r = k^{-\eta}\zeta_{k^{2\eta}r}$,
- the motion process of w^k is ξ^k given as

$$r \mapsto \xi^k_r = \Big(L^{-1}_{\left[\Gamma, x/\sqrt{k}+W^k\right]}(r),\, W^k \circ L^{-1}_{\left[\Gamma, x/\sqrt{k}+W^k\right]}(r)\Big). \tag{57}$$

Hence, if w has the distribution $\mathbb{N}^{T^x\Gamma^k}_0$, then w^k has the distribution $k^{-\eta}\mathbb{N}^{T^{x/\sqrt{k}}\Gamma}_0$. Note that $\sigma(w^k) = k^{-2\eta}\sigma(w)$ is the length of the excursion w^k. For the stopping times τ_t we obtain from the formula lines (56), (57) and (18) the relation

$$\tau_{kt}(w_u) = k^{\eta}\tau_t\big(w^k_{k^{-2\eta}u}\big), \quad \text{for all } u \in [0,\sigma],\ t > 0,\ w \in \mathfrak{W}. \tag{58}$$

Looking at the total exit local times $L^{kt}_{\sigma(w)}[w]$ and using (22) and (58) and substitutions $v = k^{-2\eta}u$ and $\delta = k^{-\eta}\varepsilon$ gives, for all $t > 0$,

$$\begin{aligned} L^{kt}_{\sigma(w)}[w] &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\sigma} 1_{\{\tau_{kt}(w_u)<\zeta_u<\tau_{kt}(w_u)+\varepsilon\}}\,du \\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{\sigma} 1_{\{k^\eta\tau_t(w^k_{k^{-2\eta}u})<\zeta_u<k^\eta\tau_t(w^k_{k^{-2\eta}u})+\varepsilon\}}\,du \\ &= k^{2\eta}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^{k^{-2\eta}\sigma} 1_{\{\tau_t(w^k_v)<k^{-\eta}\zeta(k^{2\eta}v)<\tau_t(w^k_v)+k^{-\eta}\varepsilon\}}\,dv \\ &= k^{\eta}\lim_{\delta\downarrow 0}\frac{1}{\delta}\int_0^{\sigma(w^k)} 1_{\{\tau_t(w^k_v)<\zeta^k(v)<\tau_t(w^k_v)+\delta\}}\,dv = k^\eta L^t_{\sigma(w^k)}[w^k]. \end{aligned} \tag{59}$$

Hence, (55) is proved, and from (53) and (55) we get the representation

$$\big\|X^k_t[\Gamma^k]\big\| = \int_{\mathfrak{W}} L^t_\sigma[w]\,\Pi^k_{a,b}(dw), \ \text{ for } \ t>0, \tag{60}$$

where $\Pi^k_{a,b} = \Pi^k_{a,b}[\Gamma]$ is a Poisson point field on $\mathfrak{W}$ with intensity measure

$$\pi^k_{a,b} = \pi^k_{a,b}[\Gamma] := k^{-\eta}\int_{k^\eta a}^{k^\eta b} \mathbb{N}_0^{T^{x/\sqrt{k}}\Gamma}\,dx = k^{-\rho}\int_{k^\rho a}^{k^\rho b}\mathbb{N}_0^{T^x\Gamma}\,dx, \tag{61}$$

with $\rho := \eta - 1/2 > 0$. This finishes the first step in the proof.

Comparing (52) and (60) we note that $\big\|X^k[\Gamma^k]\big\|$ and the right hand side in (52) are defined by the same functional of a Poisson point field on $\mathfrak{W}$, of course with different intensity measures.

To do the second step in the proof and show that P-almost surely the processes $\big\|X^k[\Gamma^k]\big\|$ converge in law on $C\big((0,\infty),\mathbb{R}\big)$ to $X^\infty(a,b)$, one has to show, by definition of the topology on $C\big((0,\infty),\mathbb{R}\big)$, that for every compact set $I\subset(0,\infty)$, P–almost surely, the processes $\big\{\|X^k_t[\Gamma^k]\| : t\in I\big\}$ converge in law on the space $C(I,\mathbb{R})$ with the uniform topology to $\big\{X^\infty_t(a,b) : t\in I\big\}$. Clearly, it suffices to show this for compact sets of the form $I=[1/n,n]$, so fix an arbitrary positive integer n.

Abbreviate $C_n := C([1/n,n],\mathbb{R})$ and, for $w\in\mathfrak{W}$, let $L_n[w]\in C_n$ denote the function defined by $L_n[w](s) := L^s_\sigma[w]$ for all $s\in[1/n,n]$. By *Birkhoff's Individual Ergodic Theorem* applied to the group of spatial shifts acting ergodically on the stable random measure Γ we obtain, for each measurable $F : C_n\to[0,\infty)$, P–almost surely,

$$\lim_{k\uparrow\infty} k^{-\rho}\int_{k^\rho a}^{k^\rho b}\int_{\mathfrak{W}} F\big(L_n[w]\big)\,\mathbb{N}_0^{T^x\Gamma}(dw)\,dx = (b-a)\int_{\mathcal{M}_{\mathrm{tem}}}\mathsf{P}(d\Upsilon)\int_{\mathfrak{W}}F\big(L_n[w]\big)\,\mathbb{N}_0^{\Upsilon}(dw). \tag{62}$$

Define random measures μ_k on C_n by

$$\mu_k(B) := k^{-\rho}\int_{k^\rho a}^{k^\rho b}\int_{\mathfrak{W}} 1_{B\setminus\{0\}}\big(L_n[w]\big)\,\mathbb{N}_0^{T^x\Gamma}(dw)\,dx, \quad \text{for } B\subseteq C_n \text{ Borel.} \tag{63}$$

Define, similarly, a measure μ on C_n by

$$\mu(B) := (b-a)\int_{\mathcal{M}_{\mathrm{tem}}}\mathsf{P}(d\Upsilon)\int_{\mathfrak{W}} 1_{B\setminus\{0\}}\big(L_n[w]\big)\,\mathbb{N}_0^{\Upsilon}(dw), \quad \text{for } B\subseteq C_n \text{ Borel.} \tag{64}$$

Note that, by (28), μ_k and μ are finite measures since we did not allow them to have mass at the zero function in C_n.

As the space C_n is Polish, there is a countable family $\{F_k : C_n \to [0,\infty) : k \geq 1\}$ of continuous and bounded functions, which are convergence determining for the weak convergence of finite measures on C_n. This fact together with (62) implies that P–almost surely the measures μ_k converge weakly to the measure μ on C_n.

Formula lines (60) and (61) together state that $\|X^k[\Gamma^k]\| = \{\|X^k_t[\Gamma^k]\| : t \in [1/n, n]\}$ is equal in law to the sum of all functions in C_n in the support of a Poisson point field with intensity measure μ_k. By the finiteness of μ and elementary properties of Poisson point fields we infer that P–almost surely this sum converges in distribution to the sum of all functions in the support of a Poisson point field in C_n with intensity measure μ. In other words, P–almost surely, in distribution on the space C_n with the topology of uniform convergence,

$$\lim_{k\uparrow\infty} \|X^k[\Gamma_k]\| = \lim_{k\uparrow\infty} \int_{\mathfrak{W}} L_n[w]\, \Pi^k_{a,b}(dw) = \int_{\mathfrak{W}} L_n[w]\, \Pi^\infty_{a,b}(dw). \tag{65}$$

This finishes the proof of the second step and thus proves the statements (i) and (ii) in the proposition. □

In order to be able to deal with the real-valued processes $t \mapsto X^k_t(a,b)$, started in $X_0 = \ell$, we use the crossing property, Theorem 1.4 (which is proved in Section 3 below) to derive the following corollary.

Corollary 2.2 (No mass transport on macroscopic scales). *Let (a,b) be a bounded interval and consider the rescaled processes $\{X^k_t : t > 0\}$ with $X^k_0 = \ell_{(-\infty,a)}$ or $X^k_0 = \ell_{(b,\infty)}$. Then, in P–probability, the processes $\{X^k_t(a,b) : t > 0\}$ converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the zero function as $k \uparrow \infty$.*

Proof. By translation and (if needed) reflection, we see that it is equivalent to show that, in P–probability, the processes

$$\{X^k_t(a-b,0) : t > 0\}, \text{ for } X_0 = \ell_{(0,\infty)}, \tag{66}$$

converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the zero function. Now observe that

$$\sup_{t\geq 0} X^k_t(a-b,0) = \sup_{t\geq 0} k^{-\eta} X_{kt}\big(k^\eta(a-b),0\big) \leq k^{-\eta} \sup_{t\geq 0} X_t(-\infty,0] \xrightarrow[k\uparrow\infty]{} 0, \tag{67}$$

$\mathbb{P}^\Gamma$–almost surely, for P–almost all Γ, by Theorem 1.4. This finishes the proof. □

We now have the means to complete the *proof of Theorem* 1.1 subject to the proof of the crossing property, Theorem 1.4, and the finite mass property (28). We start the processes X in $X_0 = \ell$ and show the convergence of the rescaled processes X^k to the process X^∞ defined by (27) with starting mass $X^\infty_0 = \ell$.

Let again $a < b$. Given Γ, by the branching property, $\{X^k_t(a,b) : t \geq 0\}$ is the sum of independent processes started in $X^k_0 = \ell_{(a,b)}$, $\ell_{(b,\infty)}$ and $\ell_{(-\infty,a)}$. Combining the total mass convergence, Proposition 2.1 (i), and Corollary 2.2 we see that, in P–probability, the processes $\{X^k_t(a,b) : t \geq 0\}$ converge in distribution on $C\big((0,\infty),\mathbb{R}\big)$ to the limit process $\{X^\infty_t(a,b) : t \geq 0\}$, which is described in Proposition 2.1 (ii) and coincides, of course, with the limit process applied to the interval (a,b).

It remains to lift the result from the indicator functions $1_{(a,b)}$ to any continuous function $\varphi\colon \mathbb{R} \to [0,\infty)$ with bounded support, say the support is contained in (a,b). Let $\delta > 0$. We choose step functions (i.e. linear combinations of indicator functions on bounded, open intervals) $g, h\colon (a,b) \to [0,\infty)$ with $h \le \varphi \le g$ and $\|g-h\|_\infty < \delta$. Then, for all positive integers n,

$$\sup_{1/n \le t \le n} \langle X_t^k, h\rangle \le \sup_{1/n \le t \le n} \langle X_t^k, \varphi\rangle \le \sup_{1/n \le t \le n} \langle X_t^k, g\rangle. \tag{68}$$

In P–probability, the left and right hand side as well as $\sup_{1/n \le t \le n} \langle X_t^k, g-h\rangle$ converge in distribution as $k \uparrow \infty$ to

$$\sup_{1/n \le t \le n} \langle X_t^\infty, h\rangle \le \sup_{1/n \le t \le n} \langle X_t^\infty, g\rangle \quad \text{and} \quad \sup_{1/n \le t \le n} \langle X_t^\infty, g-h\rangle, \text{ respectively.} \tag{69}$$

Moreover, the latter term is bounded by $\delta \sup_{1/n \le t \le n} X_t^\infty(a,b)$. As, by Proposition 2.1 (i), the process $X^\infty(a,b)$ has almost surely continuous paths, this can be made arbitrarily small by choice of δ. Recalling the definition of the metric from (1), we see that this implies convergence of the processes X^k on $C\big((0,\infty), \mathcal{M}(\mathbb{R})\big)$. But the states of the limit process are again in $\mathcal{M}_{\text{tem}}$, because

$$\mathbf{E}_\ell \langle X_t^\infty, \varphi\rangle = \langle \ell, \varphi\rangle < \infty, \text{ for all } \varphi \in \Phi \text{ and } t > 0, \tag{70}$$

recall Theorem 1.0 (iii), and path continuity. This finishes the proof. $\square$

Remark 2.3 (Other starting measures). It is possible to start the process X with k–dependent initial measures $X_0 = \mu^{(k)}$ on $\mathbb{R}$ such that $X_0^k \equiv \mu$, where μ is a sufficiently *diffuse* measure in $\mathcal{M}_{\text{tem}}$. For example it is sufficient to require that μ has a continuous density g with the property that, for some constants $a, b \ge 0$,

$$\lim_{x\uparrow\infty} \int_x^\infty \big|g(y) - a\big|\, dy = \lim_{x\downarrow-\infty} \int_{-\infty}^x \big|g(y) - b\big|\, dy = 0. \tag{71}$$

To show this, observe that one can extend the convergence easily from the case $X_0^k = \ell$ to $X_0^k = \ell_{(a,b)}$. By uniform approximation from above and below one can then get convergence for all starting measures satisfying (71). However, the functional limit law does *not* hold without any condition for the scaled starting measure μ. Starting, for example, with the counting measure $X_0^k \equiv \sum_{z\in\mathbb{Z}} \delta_z$ does *not* lead to a limit process that is independent of Γ. $\diamond$

3. The method of good and bad paths

In Subsection 3.1 and 3.2 we formulate a quantitative approach to the method of good and bad paths extending a recent result of [**DFM00**]. The main result of this part, Theorem 3.4, enables us to prove the crossing property of Theorem 1.4, in Subsection 3.3, and the compound Poisson property of Theorem 1.1 (ii), in Subsection 3.4.

3.1. Regularity of the catalytic medium. In this subsection we introduce a characteristic quantity $N(\Gamma)$, which measures the regularity of the particular sample Γ of the catalyst. This is used, in Subsection 3.2 below, to formulate an upper bound on the survival property of the superprocess $X = X[\Gamma]$ either at a fixed time or of an associated stopped measure at a Brownian stopping time.

Recall from (3) the definition of the stable random measure Γ of index $0 < \gamma < 1$. For every $n \ge 1$ we denote by $\pi_n = \pi_n[\Gamma] \subseteq \mathbb{R}$ the set of spatial positions of the atoms of Γ whose weights are in $[2^{-n}, 2^{-n+1})$. Then π_n is the support of a homogeneous Poisson point field on $\mathbb{R}$ with intensity

$$I_n := 2^{\gamma n} c_\gamma\,, \quad \text{where } c_\gamma := \frac{1-2^{-\gamma}}{\gamma \displaystyle\int_0^\infty \frac{1-\mathrm{e}^{-r}}{r^{1+\gamma}}\, dr}\,, \tag{72}$$

see e.g. [**DaF92**].

DEFINITION 3.1 (p–perfect Γ samples). Fix a value $\beta \in (0, \gamma \log 2)$ once and forever. Given a positive integer m and a positive real k, we denote by $A(m,k)$ the event that the maximal connected component of $[-k,k] \setminus \pi_m$ is shorter than $\Delta_m := \mathrm{e}^{-\beta m}$. We call a Γ sample *p–perfect* if it satisfies $A(m,k)$ for all k and $m \geq p$. $\diamond$

The following lemma is adapted from results of [**DFM00**].

LEMMA 3.2 (Large gaps in $\pi_n(\Gamma)$). *Fix $0 < \varrho < \gamma \log 2 - \beta$ and suppose that $k(n)$ is an increasing sequence of positive reals with $\log\log k(n) = o(n)$ as $n \uparrow \infty$. Then there are constants $c, d > 0$ such that, for every $N \geq 1$,*

$$\mathsf{P}\Big\{\textit{there is an } n \geq N \textit{ such that } A\big(n, k(n)\big) \textit{ fails}\Big\} \leq c \exp\big(-d\,\mathrm{e}^{\varrho N}\big). \tag{73}$$

Consequently, neighbouring atoms of Γ in $\big[-k(n),\, k(n)\big]$ of weight about 2^{-n} are more than Δ_n away only with an exponentially small probability.

PROOF. We write $J(n)$ for the number of points in $[-1/2, 1/2] \cap \pi_n$, then $J(n)$ has a Poisson distribution with parameter $a(n) := c_\gamma 2^{\gamma n}$. Denote x_0 the largest point of π_n left of $[-1/2, 1/2]$, moreover $x_1 \leq \ldots \leq x_{J(n)}$ the points in $[-1/2, 1/2] \cap \pi_n$, and finally $x_{J(n)+1}$ the smallest point to the right of the interval. Define the distances $y_k := x_{k+1} - x_k$ for $k = 0, \ldots, J(n)$. The y_k are independent exponentially distributed random variables with parameter $a(n)$. Following the arguments in [**DFM00**, Subsection 5.2] we find that there are constants $c_1, c_2, c_3 > 0$ such that

$$\mathsf{P}\Big\{J(n) + 1 > 2\,a(n)\Big\} \leq \exp(-c_1 2^{\gamma n}), \tag{74}$$

and, writing $\sigma := \gamma \log 2 - \beta > 0$,

$$\mathsf{P}\Big\{\max_{0 \leq i \leq 2\,a(n)} y_i > \mathrm{e}^{-\beta n}\Big\} \leq c_2\, \mathrm{e}^{n\gamma \log 2} \exp(-c_3\, \mathrm{e}^{\sigma n}). \tag{75}$$

From this we infer that, for some $c_4, c_5 > 0$,

$$\mathsf{P}\Big\{A(n, 1/2) \text{ fails}\Big\} \leq c_4 \exp(-c_5\, \mathrm{e}^{n\sigma}). \tag{76}$$

This is invariant under shifts of the interval $(-1/2, 1/2)$ and thus, by adding up the events, we obtain for some c_6, $c_7 > 0$, and all $n \geq 1$,

$$\mathsf{P}\Big\{A\big(n, k(n)\big) \text{ fails }\Big\} \leq c_4 k(n) \exp\big(-c_5\, \mathrm{e}^{\sigma n}\big) \leq c_6 \exp\big(-c_7\, \mathrm{e}^{\varrho n}\big). \tag{77}$$

Finally, we can add up the events $A\big(n, k(n)\big)$ over all $n \geq N$ and find a suitable c and $d := c_7$, such that the statement (73) holds. $\square$

DEFINITION 3.3 (Characteristic $N(\Gamma)$). Fix $k(n) := \exp\exp\sqrt{n}$ for $n \geq 1$ and define the *characteristic* $N(\Gamma)$ of the sampled medium Γ by

$$N(\Gamma) := \min\big\{N \geq 1 : A\big(n, k(n)\big) \text{ holds for all } n \geq N\big\}\,. \tag{78}$$

Using the Borel-Cantelli Lemma and Lemma 3.2, one can see that $N(\Gamma)$ is a well-defined integer, P–almost surely. $\diamond$

3.2. An upper bound on the survival probability. In this subsection we determine an upper bound on the survival probability of a super-Brownian motion in a fixed catalytic medium sample Γ with characteristic $N(\Gamma)$. The approach taken here is similar to the key technique of [**DFM00**], in particular we also work in a historical setting and use a decomposition into good and bad reactant paths, but we have to make a more quantitative approach. Moreover, our argument does not rely on the compact support property of $X[\Gamma]$. The most crucial point is that we work out explicitly how the upper bound of the survival probability depends on $N(\Gamma)$.

For the further development, we presuppose the reader is familiar with basic ideas and the formalism of historical catalytic super-Brownian motion. Here we closely follow the presentation of [**DFM00**]. Denote by $Y = \{Y_t : t \geq 0\}$ the *historical super-Brownian motion* in the catalytic medium Γ with starting measure $\mu \in \mathcal{M}_{\text{tem}}$ defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P}^{\Gamma}_{\mu})$; if there is no danger of confusion we omit Γ or μ from the notation, that is, we write simply $\mathbb{P}$ and $\mathbb{E}$. At the same time we use, for any finite Brownian stopping time T, Dynkin's stopped measure Y_T and the pre-T σ-field $\mathcal{G}(T)$ for the historical superprocess as introduced in [**Dyn91**] and reviewed in [**DFM00**, Subsection 3.2]. We suppose that $W = \{W_s : s \geq t\}$ under the distribution $\mathcal{P}_{t,x}$ is a Brownian path started at $W_t = x$.

The central result of this subsection is the following theorem. For its formulation we extend the definition (15) of the collision local time $L_{[W,\Gamma]}(s)$ for $s \geq t$ to Brownian paths W distributed according to $\mathcal{P}_{t,x}$ meaning the collision local time of W and Γ on the time interval $[t, s]$. We introduce two formal hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ on an increasing sequence $T_n \uparrow T < \infty$ of Brownian stopping times with $T_0 = 0$. Suppose positive integers d and p, a sequence of positive thresholds l_n and small reals $\varepsilon_1, \varepsilon_2 > 0$ are given.

$\mathbf{H_1}$: The sequence of stopping times satisfies the *hypothesis* $\mathbf{H_1}(d, \varepsilon_1)$ *in* $x \in \mathbb{R}$ if

$$\sum_{n=0}^{\infty} 2^{n+1} \mathcal{P}_{0,x} \Big\{ \sup_{T_n \leq s \leq T_{n+1}} |W_s| > k(n+d) \Big\} \leq \varepsilon_1 . \tag{79}$$

$\mathbf{H_2}$: For every nonnegative integer n, define the set $B(n)$ of *bad paths* for the catalytic medium Γ on the random time interval $[T_n, T_{n+1}]$ as the set of paths W satisfying

$$\int_{T_n}^{T_{n+1}} L_{[W,\Gamma]}(ds) < l_n . \tag{80}$$

The sequence of stopping times satisfies the *hypothesis* $\mathbf{H_2}(p, l_n, \varepsilon_2)$ *in* $x \in \mathbb{R}$ if, in every p–perfect medium Γ, we have for $x_0 = 0$ and any x_n in the range of $W(T_n)$,

$$\sum_{n=0}^{\infty} \mathcal{P}_{T_n, x_n} \{ W \in B(n) \} \leq \varepsilon_2 . \tag{81}$$

Here is the announced result on the survival probability.

THEOREM 3.4 (Upper bound on survival probability). *Suppose $T_0 = 0$ and $T_n \uparrow T < \infty$ is a sequence of Brownian stopping times, μ a unit starting mass for Y, supported by a compact interval I and $\varepsilon > 0$ is fixed. Suppose further there are nonnegative integers m and d and a sequence of thresholds $l_n > 0$ such that*

$$\sum_{n=0}^{\infty} \frac{2^{-m-n}}{l_n} \leq \frac{\varepsilon}{3} , \tag{82}$$

and the hypotheses $\mathbf{H_1}\big(d,(\varepsilon/3)2^m\big)$ *and* $\mathbf{H_2}(m+d,l_n\,,\varepsilon/6)$ *are satisfied in any* $x\in I$. *Then, for* P*–almost every* Γ *with* $N(\Gamma)\le m$ *and for the stopped measure* Y_T, *we have*

$$\mathbb{P}^{\Gamma}_{2^{-m}\mu}\{Y_T\neq 0\}\le\varepsilon. \tag{83}$$

The remainder of this subsection is devoted to the proof of this theorem. We work with a fixed catalyst Γ with characteristic $N=N(\Gamma)$ and use the notation from the theorem, additionally abbreviating $M_n:=2^{-m-n}$ for the fixed m. We start with a simple lemma taken from [**DFM00**, Subsection 3.4].

LEMMA 3.5 (Extinction by partitioning). *Define events* $A_n:=\big\{\|Y_{T_n}\|\le M_n\big\}$ *and* $A:=\bigcap_{n=1}^{\infty}A_n$. *Then, for every* $\nu\in\mathcal{M}_{\text{tem}}(\mathbb{R})$, *we have* $\mathbb{P}^{\Gamma}_{\nu}$*–almost surely on* A *that* $Y_T=0$.

PROOF. By Markov's inequality, for each $n\ge1$ and arbitrary $\zeta>0$,

$$\mathbb{P}^{\Gamma}_{\nu}\Big(\big\{\|Y_T\|>\zeta\big\}\cap A\Big)\le\zeta^{-1}\,\mathbb{E}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_T\|\big\} \tag{84}$$

and, by the special Markov property,

$$\mathbb{E}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_T\|\big\}=\mathbb{E}^{\Gamma}_{\nu}\big\{1_{A_n}\mathbb{E}^{\Gamma}_{Y_{T_n}}\{\|Y_T\|\}\big\}=\mathbb{E}^{\Gamma}_{\nu}\big\{1_{A_n}\|Y_{T_n}\|\big\}\le M_n\,. \tag{85}$$

As n was arbitrary, we infer that

$$\mathbb{P}^{\Gamma}_{\nu}\Big(\big\{\|Y_T\|>\zeta\big\}\cap A\Big)=0, \tag{86}$$

and as ζ can be made arbitrarily small, we get the statement. □

The idea now is to divide the set of paths W alive at time T_{n+1} in two classes: the good paths, which have accumulated a large amount of collision local time in the time interval $[T_n\,,T_{n+1}]$, and the bad paths, which have not. Formally, define the set $E(n)$ of *good paths* on the interval $[T_n\,,T_{n+1}]$ with respect to the medium Γ as the set of paths W, which are not bad, i.e. where inequality (80) fails. For the good paths we use the comparison with the survival probability in Feller's branching diffusion from [**DFM00**, Proposition 12].

LEMMA 3.6 (Comparison with Feller's branching diffusion). *For all* $n\ge0$, *and every* ν *in* $\mathcal{M}_{\text{tem}}(\mathbb{R})$, *we have, for* P*–almost every* Γ,

$$\mathbb{P}^{\Gamma}_{\nu}\Big\{Y_{T_{n+1}}\big(E(n)\big)>0\;\Big|\;\mathcal{G}(T_n)\Big\}\le\frac{\|Y_{T_n}\|}{l_n}\,. \tag{87}$$

The lemma above takes care of the good paths, and it remains to show, that there are not too many bad paths. To show this we use $\mathbf{H_1}$ and $\mathbf{H_2}$. There are two possible reasons, why a path could be bad on $[T_n\,,T_{n+1}]$ for the medium Γ, namely the occurrence of one of the following two disjoint events.

- **Event** $B_1(n)$**:** the set of paths, that leave the interval $\big[-k(d{+}n),\,k(d{+}n)\big]$ during $[T_n\,,T_{n+1}]$ and thus enter an area where we have no control over the atoms,
- **Event** $B_2(n)$**:** the set of paths, that stay inside the interval $\big[-k(d+n),\,k(d+n)\big]$ but for which the collision local time accumulated during $[T_n\,,T_{n+1}]$ is below the threshold l_n.

Note that in case of event $B_2(n)$, if $n\ge N(\Gamma)$, the path stays in an area, where the medium coincides with an $(N+d)$–perfect medium. It is clear that we have the decompositions

$$\operatorname{supp}Y_{T_{n+1}}\subseteq E(n)\cup B(n)\;\;\text{and}\;\;B(n)\subseteq B_1(n)\cup B_2(n). \tag{88}$$

The following lemma provides estimates for the extinction probability of the bad paths.

LEMMA 3.7 (Mass of bad paths). *Consider the historical superprocess Y with the starting measure $\nu = 2^{-m}\mu$. Then, for P–almost every Γ with $N(\Gamma) \le m$,*

$$\text{(i)}\ \sum_{n=0}^{\infty} \mathbb{P}_{\nu}^{\Gamma}\Big\{Y_{T_{n+1}}\big(B_1(n)\big) \ge M_{n+1}\Big\} \le \frac{\varepsilon}{3},$$

$$\text{(ii)}\ \sum_{n=0}^{\infty} \mathbb{P}_{\nu}^{\Gamma}\Big\{Y_{T_{n+1}}\big(B_2(n)\big) \ge M_{n+1}\,\Big|\, \|Y_{T_n}\| \le M_n\Big\} \le \frac{\varepsilon}{3}.$$

PROOF. The proof is based on the expectation formula for the historical mass on a set B of stopped paths $W : [0, T_{n+1}] \to \mathbb{R}$. If B depends only on $\big\{W(s) : s \ge T_n\big\}$, we have

$$\mathbb{E}_{\nu}^{\Gamma}\left\{Y_{T_{n+1}}(B) \,\middle|\, Y_{T_n}\right\} = \int_{\mathbb{R}} \mathcal{P}_{T_n, \tilde{W}(T_n)}\{W \in B\}\, Y_{T_n}(d\tilde{W}), \tag{89}$$

see e.g. [**DFM00**, (37)]. From Markov's inequality, the expectation formula and (79), we infer,

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{P}_{\nu}^{\Gamma}\left\{Y_{T_{n+1}}\big(B_1(n)\big) \ge M_{n+1}\right\} &\le \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \mathbb{E}_{\nu}^{\Gamma}\left\{Y_{T_{n+1}}\big(B_1(n)\big)\right\} \\ &= \sum_{n=0}^{\infty} M_{n+1}^{-1} 2^{-m}\, \mathcal{P}_{0,x}\Big\{\sup_{T_n \le s \le T_{n+1}} |W_s| > k(d+n)\Big\} \le \frac{\varepsilon}{3},\end{aligned} \tag{90}$$

which is (i). Note that every path $W \notin B_1(n)$ spends the time $[T_n\,, T_{n+1}]$ inside a compact interval in which the medium Γ coincides with an $(m+d)$–perfect medium $\bar{\Gamma}$. Hence we can use the bound in (81) together with Markov's inequality, the special Markov property and the expectation formula (89) to see that,

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathbb{P}_{\nu}^{\Gamma}\Big\{Y_{T_{n+1}}\big(B_2(n)\big) \ge M_{n+1}\,\Big|\, \|Y_{T_n}\| \le M_n\Big\} \\ &\quad\le \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \mathbb{E}_{\nu}^{\bar{\Gamma}}\left\{\mathbb{E}^{\bar{\Gamma}}\Big\{Y_{T_{n+1}}\big(B_2(n)\big)\,\Big|\, Y_{T_n}\Big\}\,\Big|\, \|Y_{T_n}\| \le M_n\right\} \\ &\quad\le \sum_{n=0}^{\infty} M_{n+1}^{-1}\, \mathbb{E}_{\nu}^{\bar{\Gamma}}\Big\{\int \mathcal{P}_{T_n, \tilde{W}(T_n)}\{W \in B_2(n)\}\, dY_{T_n}(\tilde{W})\,\Big|\, \|Y_{T_n}\| \le M_n\Big\} \le \frac{\varepsilon}{3},\end{aligned}$$

which is (ii). This ends the proof of the lemma. $\square$

Now recall Lemma 3.5 and in particular the definition of the sets A_n and A. Lemmas 3.6 and 3.7 provide the ingredients we need to bound $\mathbb{P}_{\nu}^{\Gamma}(A^c)$ by ε, for the starting measure $\nu = 2^{-m}\mu$. Note that the event A_0 has probability one. Hence, for P–almost all Γ with $N(\Gamma) \le m$,

$$\mathbb{P}^\Gamma_\nu(A^c) = \sum_{n=0}^\infty \mathbb{P}^\Gamma_\nu\Big(A_0 \cap \cdots \cap A_n \cap A^c_{n+1}\Big) \tag{91a}$$

$$\le \sum_{n=0}^\infty \mathbb{P}^\Gamma_\nu\left\{Y_{T_{n+1}}\big(B_1(n)\big) \ge M_{n+1}\right\} \tag{91b}$$

$$+ \sum_{n=0}^\infty \mathbb{P}^\Gamma_\nu\Big\{Y_{T_{n+1}}\big(B_2(n)\big) \ge M_{n+1} \;\Big|\; \|Y_{T_n}\| \le M_n\Big\} \tag{91c}$$

$$+ \sum_{n=0}^\infty \mathbb{P}^\Gamma_\nu\Big\{Y_{T_{n+1}}\big(E(n)\big) \ge M_{n+1} \;\Big|\; \|Y_{T_n}\| \le M_n\Big\}. \tag{91d}$$

Now, by Lemma 3.7, the series in (91b) and (91c) are each bounded by $\varepsilon/3$. By Lemma 3.6, we obtain for (91d), using (82),

$$\sum_{n=0}^\infty \mathbb{P}^\Gamma_\nu\Big\{Y_{T_{n+1}}\big(E(n)\big) \ge M_{n+1} \;\Big|\; \|Y_{T_n}\| \le M_n\Big\} \le \sum_{n=0}^\infty \frac{M_n}{l_n} \le \frac{\varepsilon}{3}. \tag{92}$$

Hence $\mathbb{P}^\Gamma_\nu(A^c) \le \varepsilon$ and, by Lemma 3.5, this implies the statement of the theorem. □

3.3. The crossing property. In this section we prove Theorem 1.4. The following lemma constitutes the main step in the proof.

LEMMA 3.8 (Decay of crossing probability). *There is an integer m depending only on the characteristic $N(\Gamma)$ of the catalytic medium Γ, such that, for all sufficiently large x,*

$$\mathbb{P}^\Gamma_{2^{-m}\delta_x}\Big\{\int_0^\infty X_t(-\infty,0]\,dt > 0\Big\} \le \frac{1}{x^{3/2}}. \tag{93}$$

The lemma is proved by choosing the right ingredients for the use of the survival probability bound of Theorem 3.4. As a preparation for the proof, define the Brownian stopping time T to be the first hitting time of level 0, then the event we are interested in is the survival of Y_T. We denote

$$d := d(x) := \Big[\log\Big(1 + \frac{5}{2}\log x\Big)\Big]^2. \tag{94}$$

Observe that $d(x)$ is growing slower than logarithmically as $x \uparrow \infty$. We say that x is *sufficiently large* if $x \ge 1$ and

$$d(x) \le \frac{\log x}{4(\log 2 + \beta)}. \tag{95}$$

Fix x sufficiently large, let $\mu = \delta_x$ and $\varepsilon = 1/x^{3/2}$. To define the remaining quantities for Theorem 3.4, we first leave the integer parameter m open and define T_n and l_n in terms of m. Recall that $\alpha < \beta < 2\alpha$ and define

$$d_n := \frac{\mathrm{e}^{(\alpha-\beta)n}}{1-\mathrm{e}^{\alpha-\beta}} = \frac{1}{\varepsilon(m)}\exp\big((\alpha-\beta)(n+m)\big), \quad \text{where } \varepsilon(m) := \frac{\mathrm{e}^{(\alpha-\beta)m}}{1-\mathrm{e}^{\alpha-\beta}}. \tag{96}$$

Define barriers

$$x_0 = x \quad\text{and}\quad x_{n+1} = x_n - x d_n. \tag{97}$$

As $\sum_{n=1}^{\infty} d_n = 1$, we have $x_n \downarrow 0$, and we can define an increasing sequence of hitting times

$$T_n := \inf\{t > 0 : B(t) = x_n\}, \tag{98}$$

so that $T = \lim_{n\uparrow\infty} T_n$. Finally define

$$l_n := x^{3/2}\, m\, n^2\, 2^{-n-m}. \tag{99}$$

Lemma 3.8 follows from Theorem 3.4 if we verify (82) and the conditions $\mathbf{H_1}\big(d,(\varepsilon/3)2^m\big)$ and $\mathbf{H_2}(m+d, l_n, \varepsilon/6)$ in x for $\varepsilon = 1/x^{3/2}$ and a suitable m, which we choose at the end of the proof. The next lemma prepares the verification of hypothesis $\mathbf{H_1}$.

LEMMA 3.9 (Escape probability). *For all integers $n, m \geq 0$, the events*

$$D_0 := D_0(n) := \Big\{W : \sup_{t\in[T_n, T_{n+1}]} |W_t| > k(d+n+m)\Big\} \tag{100}$$

have probability

$$\mathcal{P}_{T_n, x_n}\big\{W \in D_0(n)\big\} \leq \frac{2}{x^{3/2}} \exp(-\exp\sqrt{n+m}\,). \tag{101}$$

PROOF. To begin with, note that d is chosen in such a way that, for all $n \geq 0$,

$$x^{5/2} \leq \frac{\exp\exp\sqrt{d+n}}{\exp\exp\sqrt{n}} = \frac{k(d+n)}{k(n)}. \tag{102}$$

A Brownian path starting in x_n is in D_0 if and only if it hits level $k(d+n+m)$ before x_{n+1}. The probability of this event is

$$\begin{aligned}\mathcal{P}_{T_n, x}\big\{W \in D_0(n)\big\} &\leq \frac{x d_n}{k(d+n+m) - x_{n+1}} \\ &\leq \frac{2x}{k(d+n+m)} \leq \frac{2}{x^{3/2}k(n+m)},\end{aligned} \tag{103}$$

where we have used (102), $d_n \leq 1$ and $x_{n+1} \leq k(d+n+m)/2$. □

We observe that hypothesis $\mathbf{H_1}\big(d,(\varepsilon/3)2^m\big)$ is verified in x if m is chosen such that

$$\sum_{n=0}^{\infty} 2^{n+2} \exp\big(-\exp\sqrt{n+m}\,\big) \leq \frac{2^m}{3}, \tag{104}$$

however, there will be other constraints on m coming from hypothesis $\mathbf{H_2}$ and we turn to the verification of this hypothesis now. For this purpose, let m be arbitrary and fix an $(m+d)$–perfect medium Γ. Recall $\alpha < \beta < 2\alpha$ and fix $\beta < \delta < 2\alpha$. Let

$$a_n := x^{7/4}\, \mathrm{e}^{\delta n}\, m\, \mathrm{e}^{\beta m}. \tag{105}$$

To estimate the probability from above that a path W is bad in $[T_n, T_{n+1}]$ it suffices to consider a special hitting strategy: As the medium is $(m+d)$–perfect we can select, for every $l \geq m$, atoms of mass in $[2^{-l-d}, 2^{-l-d+1})$ in such a way that all neighbouring atoms have distance in $[\Delta_{l+d}/2, 3\Delta_{l+d}/2]$, we call this set of atoms $\tilde{\pi}_l = \tilde{\pi}_l(x)$. During $[T_n, T_{n+1}]$ we only count collisions with the atoms $\tilde{\pi}_{n+m}$. If a path is bad on $[T_n, T_{n+1}]$, i.e. if (80) holds, this must be due to one of the following two events.

Event $D_1(n)$**:** the set of paths such that during the time interval $[T_n, T_{n+1}]$ the number of collisions with the catalytic atoms of $\tilde{\pi}_{n+m}$ is too small, say less than a_n,

Event $D_2(n)$: the set of paths such that it takes more than a_n visits to the chosen atoms before the collision local time exceeds the threshold l_n.

Note that $B(n) \subseteq D_1(n) \cup D_2(n)$ and hence we have to check the probability that a single Brownian path encounters one of these two events. Note, that the constants $\theta > 0$ and $C_0, \dots, C_3$ in the following two lemmas only depend on our fixed choices α, β, δ and nothing else.

LEMMA 3.10 (Probability of Event $D_1(n)$). *We define a sequence $\{\kappa_j\}$ of Brownian stopping times by $\kappa_0 := T_n$ and, for $j \geq 1$,*

$$\kappa_j := \inf\Big\{t \geq \kappa_{j-1} : W_t \text{ hits } \tilde{\pi}_{n+m} \setminus \{W_{\kappa_{j-1}}\}\Big\} \tag{106}$$

and denote

$$K_n := \max\{j : \kappa_j \leq T_{n+1}\}. \tag{107}$$

There is a $\theta > 0$ and constants $C_0, C_1 > 0$ such that, for all integers $n \geq 0$, the events

$$D_1 := D_1(n) := \{W : K_n \leq a_n\} \tag{108}$$

have probability

$$\mathcal{P}_{T_n, x_n}\big\{W \in D_1(n)\big\} \leq C_0 \exp\big(-C_1 a_n^\theta\big). \tag{109}$$

PROOF. The indicated probability is bounded above by the probability that a simple random walk S_n defined on a probability space $(\Omega, \mathcal{A}, P)$ needs less than $a_n = x^{7/4} m\, \mathrm{e}^{\delta n + \beta m}$ steps to cross the level

$$\frac{x d_n}{(1/2)\Delta_{d+n+m}} \geq \frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)}. \tag{110}$$

By the reflection principle,

$$P\Big\{\max_{1 \leq k \leq a_n} S_k > \frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)}\Big\} \leq 2P\Big\{S_{a_n} \geq \frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)}\Big\}. \tag{111}$$

We fix a $0 < \theta_0 < 1/2$ such that $\delta(1-\theta_0) \leq \alpha$ and $(7/4)(1-\theta_0) \leq 1$. By the refinement of Cramér's Theorem given in [**DeZ93**, (3.7.1)] for fixed $\theta_0 < \theta_1 < 1/2$ and $c = 2(1 - \mathrm{e}^{\alpha-\beta})$ there is a constant $C_0 > 0$ such that, for all integers k,

$$P\Big\{S_k \geq c k^{1-\theta_0}\Big\} \leq C_0 \exp\Big(-k^{1-2\theta_1}(c^2/2)\Big). \tag{112}$$

Our choice of θ_0 and c is such that

$$\frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)} \geq c a_n^{1-\theta_0}. \tag{113}$$

Hence we can use (112) and put $\theta := 1 - 2\theta_1 > 0$ to get

$$P\Big\{S_{a_n} > \frac{2x}{\varepsilon(m)}\, \mathrm{e}^{\alpha(n+m)}\Big\} \leq C_0 \exp\Big(-a_n^\theta(c^2/2)\Big), \tag{114}$$

which is the required estimate with $C_1 := 2(1 - \mathrm{e}^{\alpha-\beta})^2$. ☐

LEMMA 3.11 (Probability of Event $D_2(n)$). *For a Brownian path W started in x_n we denote by $\{y_i = W_{\kappa_i} : 1 \leq i \leq a_n\}$ the sequence of atoms in $\tilde{\pi}_{n+m}$ hit by the path. Define*

$$L_i := \int_{\kappa_i}^{\kappa_{i+1}} L^{y_i}(ds). \tag{115}$$

There are constants $C_2 > 0$ *and* $C_3 > 0$ *such that, for all integers* $n \geq 0$*, the events*

$$D_2(n) := \Big\{W : \sum_{i=1}^{a_n} 2^{-(d+n+m)} L_i < l_n\Big\} \tag{116}$$

have probability

$$\mathcal{P}_{T_n, x_n}\big\{W \in D_2(n)\big\} \leq C_2 \exp\big(-C_3 a_n\big). \tag{117}$$

Proof. By scaling, $\{L_i\}$ is bounded below by a sequence $\{\tilde{L}_i\}$ of independent, identically distributed positive random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ such that the distribution of $\tilde{L}_i/\Delta_{n+m+d}$ is independent of d, n and m. Hence,

$$\begin{aligned}\mathcal{P}_{T_n, x}\big\{W \in D_2(n)\big\} &\leq P\Big\{\sum_{j=1}^{a_n} 2^{-n-m-d}\tilde{L}_j < l_n\Big\}\\ &\leq P\Big\{\frac{1}{a_n}\sum_{j=1}^{a_n}\frac{\tilde{L}_j}{\Delta_{n+m+d}} < \frac{2^{n+m+d} l_n}{a_n \Delta_{n+m+d}}\Big\}. \end{aligned}\tag{118}$$

Note that, for all $n, m \in \mathbb{N}$ and $x > 0$,

$$\frac{2^{n+m+d} l_n}{a_n \Delta_{n+m+d}} = \frac{n^2 2^d\, \mathrm{e}^{\beta d}}{\mathrm{e}^{(\delta-\beta)n} x^{1/4}} \leq n^2 \mathrm{e}^{(\beta-\delta)n} \longrightarrow 0, \ \text{ as } n \uparrow \infty, \tag{119}$$

using that $2^d \mathrm{e}^{\beta d} \leq x^{1/4}$ by (95). Hence, by Cramér's Theorem, the right hand side in (118) is bounded above by $C_2 \exp(-C_3 a_n)$, for suitable constants $C_2, C_3 > 0$. ☐

Completion of the proof of Lemma 3.8. It is now time to choose the value of m large enough such that $m \geq N(\Gamma)$ and the following set of conditions is satisfied, for $\varepsilon = 1/x^{3/2}$:

(120a) $\displaystyle\sum_{n=0}^{\infty} 2^{n+2} \exp\big(-\exp\sqrt{n+m}\big) \leq \frac{2^m}{3}$, (120b) $\displaystyle\sum_{n=0}^{\infty} C_0 \exp\big(-C_1\, a_n^{\theta}\big) \leq \frac{\varepsilon}{12}$,

(120c) $\displaystyle\sum_{n=0}^{\infty} C_2 \exp(-C_3\, a_n) \leq \frac{\varepsilon}{12}$, (120d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{mn^2} \leq \frac{1}{3}$.

Note that a_n is a multiple of $x^{7/4}$ and hence m can be chosen independently of x. We have already seen in (104) that (120a) implies $\mathbf{H_1}\big(d, (\varepsilon/3)2^m\big)$. Moreover, (120b) and (120c) together with Lemmas 3.10 and 3.11 imply condition $\mathbf{H_2}(m+d, l_n, \varepsilon/6)$ and, finally (120d) is (82). Hence Lemma 3.8 follows from Theorem 3.4. ☐

Completion of the proof of the crossing property, Theorem 1.4. We use the branching property and the result of Lemma 3.8 to see that, for sufficiently large integers x,

$$\begin{aligned}\mathbb{P}^{\Gamma}_{2^{-m}\ell[x,x+1]}\Big\{\int_0^{\infty} X_t(-\infty,0]\,dt = 0\Big\} &= \exp\Big(\int_x^{x+1}\log \mathbb{P}^{\Gamma}_{2^{-m}\delta_y}\Big\{\int_0^{\infty} X_t(-\infty,0]\,dt = 0\Big\}\,dy\Big)\\ &\geq \exp\Big(\int_x^{x+1}\log\big(1-y^{-3/2}\big)\,dy\Big) \geq 1 - \frac{1}{x^{3/2}}. \end{aligned}\tag{121}$$

Hence $\mathbb{P}^{\Gamma}_{2^{-m}\ell[x,x+1]}\{\int_0^{\infty} X_t(-\infty,0]\,dt > 0\}$ is summable over all positive integers x. Hence, by the Borel-Cantelli Lemma, there exists some random integer $K > 0$ such that the process X

started in $2^{-m}\ell_{[K,\infty]}$ has the property $\int_0^\infty X_t(-\infty,0]\,dt=0$. Repeating this argument 2^m times independently and taking the maximal value of K one can see that it remains to argue that

$$\sup_{t\geq 0} X_t(-\infty,0]<\infty, \quad \text{for } X_0=\ell_{[0,K]}\,. \tag{122}$$

Because the process X with finite starting measure with compact support has the finite time extinction property, by[**DFM00**, Theorem 6], there is a random time $T>0$ such $X_t=0$ for all $t\geq T$. By continuity of the process, finally, $X_t(-\infty,0]$ is bounded on $[0,T]$ and we are done. ☐

3.4. The compound Poisson property. In this section we prove Theorem 1.1 (ii). The following lemma constitutes the main step in the proof.

LEMMA 3.12 (Lower bound for extinction). *For every time $t>0$ there is a constant $\theta=\theta(t)>0$, such that for P–almost all Γ,*

$$\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\big\{X_t=0\big\}\geq \theta^{(2^{N(\Gamma)})}. \tag{123}$$

To prove Lemma 3.12 by application of Theorem 3.4 we proceed similarly as in [**DFM00**]. We fix t, leave the integer parameter m open for a while and define deterministic times T_n and thresholds l_n in terms of m. We first let

$$\varepsilon(m):=\Big(\frac{2}{t}\sum_{n=m}^{\infty}\mathrm{e}^{(\alpha-\beta)n}\Big)^{1/3}. \tag{124}$$

We then define $m_n:=[\mathrm{e}^{\alpha(n+m)}/\varepsilon(m)]$ and $s_n:=\mathrm{e}^{-\beta(n+m)}/\varepsilon(m)^2$ (here $[\bullet]$ denotes the integer part). Put

$$T_0:=0 \quad\text{and}\quad T_{n+1}:=T_n+2m_ns_n\,. \tag{125}$$

Note that $t\geq T:=\lim_{n\uparrow\infty}T_n$. Finally define

$$\bar l_n:=m_n\sqrt{s_n}\,2^{-n}. \tag{126}$$

We later define l_n to be a constant multiple of $\bar l_n$. Lemma 3.12 follows from Theorem 3.4 if we verify (82) and the hypotheses $\mathbf{H_1}(d,(\varepsilon/3)2^m)$ and $\mathbf{H_2}(m+d,l_n,\varepsilon/6)$ in all $x\in[0,1]$ for $d=0$, $\varepsilon=1/2$ and a suitable integer m, which we choose at the end of the proof. Indeed, define $\theta:=(1/2)^{(2^m)}>0$. By the branching property and Theorem 3.4 we obtain, for $M=m+N(\Gamma)$,

$$\mathbb{P}^{\Gamma}_{\ell_{[0,1]}}\big\{X_t=0\big\}\geq\Big(\mathbb{P}^{\Gamma}_{\ell_{[0,1]2^{-M}}}\big\{Y_t=0\big\}\Big)^{(2^M)}\geq(1/2)^{(2^M)}=\theta^{(2^{N(\Gamma)})}, \tag{127}$$

which is the statement of Lemma 3.12.

To prepare the verification of the hypotheses we formulate three lemmas. The constants $C_0,\dots,C_3$ in these lemmas depend only on the fixed values of α and β. The first lemma is the main ingredient in the verification of hypothesis $\mathbf{H_1}$.

LEMMA 3.13 (Escape probability). *There is a constant $C_0>0$ such that, for all starting points $x\in[0,1]$ and all integers $n,m\geq 0$, the events*

$$\overline{D_0}:=\overline{D_0}(n):=\Big\{W:\sup_{s\in[T_n,T_{n+1}]}|W_s|>k(n+m)\Big\} \tag{128}$$

have probability

$$\mathcal{P}_{0,x}\big\{W\in\overline{D_0}(n)\big\}\leq C_0\exp\big(-\exp\sqrt{n+m}\,\big). \tag{129}$$

PROOF. Recall that the random variable $\sup_{0\le s\le t}|W_s|$ has finite first moment. Hence, using Markov's inequality, there is a constant $C_0>0$, depending only on t, such that

$$\mathcal{P}_{0,x}\Big\{\sup_{T_n\le s\le T_{n+1}}|W_s|>k(n+m)\Big\}\le\mathcal{P}_{0,x}\Big\{\sup_{0\le s\le t}|W_s|>k(n+m)\Big\}\le C_0\exp(-\exp\sqrt{n+m}\,)$$

for all $x\in[0,1]$ and $n,m\ge 0$. ☐

We conclude that $\mathbf{H_1}\big(d,(\varepsilon/3)2^m\big)$ holds if m is chosen such that

$$\sum_{n=0}^{\infty}C_0\,2^{n+1}\exp\big(-\exp\sqrt{n+m}\,\big)\le\frac{1}{6}\,2^m. \tag{130}$$

Again further restrictions on m follow from the verification of $\mathbf{H_2}$. For this purpose let Γ be an m–perfect medium and define the set $\bar\pi_{n+m}$ consisting of the atoms of Γ with mass in $[2^{-n-m},2^{-n-m+1})$. Note that the neighbouring pairs of atoms in $\bar\pi_{n+m}$ are no further than Δ_{n+m} apart. For our estimate we consider on the interval $[T_n\,,T_{n+1}]$ only the collisions with the atoms of $\bar\pi_{n+m}$. In fact, we can even restrict our view to a selection of collisions chosen according to a special strategy of [**DFM00**], which is based on our choice of the sequences $s_n>0$ of small times and m_n of positive integers. Heuristically, on the interval $[T_n\,,T_{n+1}]$ the strategy suggests to wait until the Brownian particle hits the first atom of $\bar\pi_{n+m}$, count the collision local time with this particular atom for s_n time units and then wait for the next collision with $\bar\pi_{n+m}$. This procedure is iterated until m_n atoms are visited. A visualisation of this procedure can be found in [**FlM00**, Figure 5]. The path W is good on $[T_n\,,T_{n+1}]$ unless one of the following two events has taken place.

Event $\overline{D_1}(n)$: the set of paths for which in our strategy the waiting times between the collisions with the catalytic atoms of $\bar\pi_{n+m}$ are too long, so that we have less than m_n visits during the time interval $[T_n\,,T_{n+1}]$,

Event $\overline{D_2}(n)$: the set of paths for which the collision local time accumulated during m_n visits of the path to the chosen atoms is below the threshold l_n.

LEMMA 3.14 (Probability of Event $\overline{D_1}(n)$). *We define a sequence $\{\kappa_n\}$ of Brownian stopping times by $\kappa_0:=T_n$ and, for $j\ge 1$,*

$$\kappa_j:=\bar\kappa_j+s_n\,,\ \text{where } \bar\kappa_j:=\inf\big\{s\ge\kappa_{j-1}:W_s \text{ hits } \bar\pi_{j+m}\big\}\,, \tag{131}$$

and denote waiting times by $H_m:=\bar\kappa_m-\kappa_{m-1}$. There is a constant $C_1>0$ such that, for all starting points x and integers $n\ge 0$, the events

$$\overline{D_1}(n):=\Big\{W:\sum_{j=1}^{m_n}H_j\ge m_ns_n\Big\} \tag{132}$$

have probability

$$\mathcal{P}_{T_n\,,x}\big\{W\in\overline{D_1}(n)\big\}\le C_1^{-1}m_n\exp\Big(-\frac{C_1s_n}{\Delta^2_{n+m}}\Big). \tag{133}$$

PROOF. This is estimate (93) in [**DFM00**]. ☐

LEMMA 3.15 (Probability of Event $\overline{D_2}(n)$). *There are constants $C_2>0$ and $C_3>0$ with the property that, for all starting points x and integers $n\ge 0$, the events*

$$\overline{D_2}(n):=\Big\{W:\int_{T_n}^{\kappa_{m_n}}L_{[W,\Gamma]}(dr)<C_2\,\bar l_n\Big\} \tag{134}$$

have probability

$$\mathcal{P}_{T_n,x}\big\{W \in \overline{D_2}(n)\big\} \le \exp\big(-2C_3 m_n\big). \tag{135}$$

PROOF. This is estimate (99) in [**DFM00**]. □

Completion of the proof of Lemma 3.12. It is now clear that with the threshold value $l_n = C_2\,\bar{l}_n$ we have the decomposition

$$B(n) \subseteq \overline{D_1}(n) \cup \overline{D_2}(n). \tag{136}$$

Having provided the estimates for the probability of a path being bad, it is now time to make precise the value of m. We choose m large enough such that the following set of conditions is satisfied

$$\sum_{n=0}^{\infty} C_0\, 2^{n+1} \exp\big(-\exp\sqrt{n+m}\,\big) \;\le\; \frac{1}{6}\,2^m, \tag{137a}$$

$$\sum_{n=m}^{\infty} \frac{1}{C_1}\,\frac{\mathrm{e}^{\alpha n}}{\varepsilon(m)}\,\exp\Big(-C_1\,\frac{\mathrm{e}^{\beta n}}{\varepsilon(m)^2}\Big) \;\le\; \frac{1}{12}, \tag{137b}$$

$$\sum_{n=m}^{\infty} \exp\Big(-2C_3\,\frac{\mathrm{e}^{\alpha n}}{\varepsilon(m)}\Big) \;\le\; \frac{1}{12}, \tag{137c}$$

$$\frac{1}{C_2}\,2^{-m}\,\varepsilon(m)^2\,\mathrm{e}^{(\beta/2-\alpha)m}\sum_{n=0}^{\infty}\mathrm{e}^{(\beta/2-\alpha)n} \;\le\; \frac{1}{6}. \tag{137d}$$

Note that it is possible to find such an m: For (137a) this is trivial, for (137c) this is due to the fact that $\varepsilon(m) \downarrow 0$ and for (137d) note that $\alpha > \beta/2$. For (137b) it suffices to check that, for $a, b \ge 1$, the function $x \mapsto (a/x)\exp(-b/x^2)$ is increasing on the interval $(0,1)$. Hence hypothesis $\mathbf{H_1}(0, 2^m/6)$ holds by (137a), see(130), and $\mathbf{H_2}(m, l_n, 1/12)$ holds by (137b), (137c) together with Lemmas 3.14 and 3.15. Finally, (137d) is (82). This finishes the proof. □

Completion of the proof of the compound Poisson property, Theorem 1.1 (ii). Fixing $t > 0$ and a starting measure $\ell_{(a,b)}$, for (a,b) an interval of unit length, we have to show that the measure

$$\pi^\infty(dw\,dx) = \int_a^b \int_{\mathcal{M}_{\mathrm{tem}}} \mathbb{N}_0^\Gamma(dw) \otimes \delta_y(dx)\,\mathsf{P}(d\Gamma)\,dy \tag{138}$$

is finite on the set

$$S = \Big\{(w,x) \in \mathfrak{W} \times (a,b) : L_\sigma^t[w] > 0\Big\}. \tag{139}$$

Then the snake representation Theorem 1.5 (ii) of the limit process describes a compound Poisson point field on (a,b) with underlying Poisson intensity $\lambda(t) := \pi^\infty(S)$. To prove finiteness of $\lambda(t)$ we have to show that the following expression is finite

$$\lambda(t) = \mathsf{E}\Big\{\mathbb{N}_0^\Gamma\,\{w : L_\sigma^t[w] > 0\}\Big\} = \mathsf{E}\Big\{\int_0^1 \mathbb{N}_x^\Gamma\,\{w : L_\sigma^t[w] > 0\}\,dx\Big\}, \tag{140}$$

where we have used the fact that the distribution of $L_\sigma^t[w]$ under $\mathbb{N}_x^\Gamma$ is independent of x. To interpret the integrand on the right hand side of (140) recall the snake representation in Theorem 1.5 (i). The process X started in $X_0 = \ell_{[0,1]}$ has become extinct at time t if and only if a Poisson point field with intensity $\int_0^1 \mathbb{N}_x^\Gamma \big\{w : L^t[w] > 0\big\}\, dx$ has the value 0, the probability of this event is

$$\exp\Big(- \int_0^1 \mathbb{N}_x^\Gamma \{w : L_\sigma^t > 0\}\, dx \Big). \tag{141}$$

This reduces our task to showing that

$$\mathsf{E}\Big\{ - \log \mathbb{P}^\Gamma_{\ell_{[0,1]}} \{X_t = 0\} \Big\} < \infty. \tag{142}$$

We now apply Fubini's Theorem to rewrite

$$\begin{aligned} \mathsf{E}\Big\{ - \log \mathbb{P}^\Gamma_{\ell_{[0,1]}} \{X_t = 0\} \Big\} &= \int_0^\infty \mathsf{P}\Big\{ - \log \mathbb{P}^\Gamma_{\ell_{[0,1]}} \{X_t = 0\} > a \Big\}\, da \\ &= \int_0^\infty \mathsf{P}\Big\{ \mathbb{P}^\Gamma_{\ell_{[0,1]}} \{X_t = 0\} < \mathrm{e}^{-a} \Big\}\, da. \end{aligned} \tag{143}$$

Hence our problem can be formulated as

$$\int_0^\infty \mathsf{P}\Big\{ \mathbb{P}^\Gamma_{\ell_{[0,1]}} \{X_t = 0\} < \mathrm{e}^{-a} \Big\}\, da < \infty. \tag{144}$$

Here comes the *key idea* of our proof: With respect to the random medium Γ the event

$$\Big\{ \mathbb{P}^\Gamma_{\ell_{[0,1]}} \{X_t = 0\} < \mathrm{e}^{-a} \Big\} \tag{145}$$

can only occur if Γ has unusually low density, or equivalently, if the points in the Poisson point fields π_n introduced before (72) are unusually far apart. This can be expressed in terms of the quantity $N(\Gamma)$. In fact, by Lemma 3.12,

$$\mathbb{P}^\Gamma_{\ell_{[0,1]}} \{X_t = 0\} \geq \exp\Big((\log\theta) 2^{N(\Gamma)} \Big), \tag{146}$$

and hence, the latter event implies

$$(\log\theta) 2^{N(\Gamma)} < -a \iff N(\Gamma) > \frac{1}{\log 2}\, \log\big(-a/\log\theta\big)\,. \tag{147}$$

We can now use the estimate (73) obtained in the large gaps lemma, Lemma 3.2, for the quantity $N(\Gamma)$,

$$\begin{aligned} &\int_0^\infty \mathsf{P}\Big\{ \mathbb{P}^\Gamma_{\ell_{[0,1]}} \{X_t = 0\} < \mathrm{e}^{-a} \Big\}\, da \\ &\leq \int_0^\infty \mathsf{P}\Big\{ N(\Gamma) > \frac{1}{\log 2}\, \log\big(-a/\log\theta\big) \Big\}\, da \\ &\leq c \int_0^\infty \exp\Big(- d \exp\Big[(\varrho/\log 2) \log\big(-a/\log\theta\big) \Big] \Big)\, da \\ &\leq c_0 + c_1 \int_0^\infty \exp\big(- c_2 a^{c_3} \big)\, da < \infty, \end{aligned} \tag{148}$$

using suitable constants $c_0, c_1, c_2, c_3 > 0$. This proves (144), and Theorem 1.1 (ii) is established. □

4. Properties of the macroscopic clumps

In this section we prove the various parts of Theorem 1.10. Part (i) and, perhaps surprisingly, part (ii) can be obtained by soft arguments, whereas part (iii) requires a new approach based on a Feynman-Kac formula for the solutions of (6).

4.1. The extinction probability of the clumps. In this subsection we prove Theorem 1.10 (i) and (ii). From the definition of the renormalised processes (9) we infer, for all $k, l > 0$ and $t \geq 0$,

$$X_t^{kl}(B) = k^{-\eta} l^{-\eta} X_{(kt)l}\big(l^\eta(k^\eta B)\big), \quad \text{for } B \subseteq \mathbb{R} \text{ Borel.} \tag{149}$$

Chosing any continuity set $B \subset \mathbb{R}$ and letting $l \uparrow \infty$ we obtain the self-similarity property

$$X_t^\infty(B) = k^{-\eta} X_{kt}^\infty(k^\eta B) \quad \text{in distribution,} \tag{150}$$

first for all continuity sets $B \subset \mathbb{R}$ and then, by approximation, for all Borel sets $B \subset \mathbb{R}$. This proves Theorem 1.10 (i) and is also the key to part (ii). By the compound Poisson structure of X_t^∞ the Laplace functionals have the form

$$\mathbf{E}_\ell\Big\{\exp\big(-\theta X_t^\infty(0,a)\big)\Big\} = \exp\Big(- \lambda(t)\, a\big(1 - \Lambda^t(\theta)\big)\Big), \tag{151}$$

where $\lambda(t)$ is the intensity of the Poisson point field underlying the compound Poisson point field and Λ^t is the Laplace functional of the weights of an atom. Using (150) one obtains

$$\begin{aligned} \exp\Big(- \lambda(t)\, a\big(1 - \Lambda^t(\theta)\big)\Big) &= \mathbf{E}_\ell\Big\{\exp\big(-\theta X_t^\infty(0,a)\big)\Big\} = \mathbf{E}_\ell\Big\{\exp\big(-\theta k^{-\eta} X_{kt}^\infty(0,k^\eta a)\big)\Big\} \\ &= \exp\Big(- \lambda(kt) k^\eta a\big(1 - \Lambda^{kt}(\theta k^{-\eta})\big)\Big). \end{aligned} \tag{152}$$

We infer that $\lambda(t) = k^\eta \lambda(kt)$ and $\Lambda^t(\theta) = \Lambda^{kt}(\theta k^{-\eta})$. The former expression gives us the decay of the intensity $\lambda(t) = t^{-\eta}\lambda(1)$ of the Poisson point field, the latter yields the equality in distribution of $\beth_t(t)$ and $(t/s)^\eta \beth_s(s)$. Using $\lambda(s)\mathbf{P}_\ell\big\{\beth_s(t) > 0\big\} = \lambda(t)$ for $t > s$, we infer that the survival probabilities of the clumps satisfy

$$\mathbf{P}_\ell\big\{\beth_s(t) > 0\big\} = \Big(\frac{s}{t}\Big)^\eta \text{ and } \mathbf{P}_\ell\big\{X_t^\infty(0,a) > 0\big\} \sim \frac{\lambda(1)\, a}{t^\eta}, \text{ as } a \uparrow \infty, \tag{153}$$

where the latter form is obtained by conditioning on the number of clumps in an interval. ☐

4.2. The tail behaviour of the clump size. This subsection is devoted to the proof of Theorem 1.10 (iii). We first note that it suffices to give the proof for a fixed value of t, because the particular dependence on t, which is claimed in Theorem 1.10 (iii), already follows from the self-similarity of the process $\big\{\beth_t(t) : t > 0\big\}$ proved in Subsection 4.1.

We use the Feynman-Kac representation of the solutions $U\theta := U^\Gamma\theta$ of

$$U_t\theta\,(y) = \theta - 2\int_0^t ds \int_{\mathbb{R}} p_s(x-y)\,\big[U_{t-s}\theta\,(x)\big]^2\,\Gamma(dx), \tag{154}$$

in order to obtain the tail asymptotics of the mass clumps. Recall that this equation can also be written probabilistically as

$$U_t\theta\,(y) = \theta - 2\,\mathcal{E}_y\Big\{\int_0^t \big[U_{t-s}\theta\,(W_s)\big]^2\, L_{[\Gamma,W]}(ds)\Big\}, \tag{155}$$

where $\mathcal{E}_{s,y}$ is used to indicate expectation with respect to a Brownian motion W started at time s in y, $\mathcal{E}_y := \mathcal{E}_{0,y}$, and $L_{[\Gamma,W]}$ is the collision local time between Γ and W, as defined in (15).

LEMMA 4.1 (Feynman-Kac-representation). *For each fixed Γ the family $U = \big\{U_t\theta\,(y) : t \geq 0,\, y \in \mathbb{R}\big\}$ is a solution of* (154) *if and only if it is a solution of*

$$U_t\theta\,(y) = \theta\,\mathcal{E}_y\Big\{\exp\Big(- 2\int_0^t U_{t-s}\theta\,(W_s)\, L_{[W,\Gamma]}(ds)\Big)\Big\}, \quad \textit{for } t \geq 0, \;\; y \in \mathbb{R}. \tag{156}$$

PROOF. Fix Γ. We first show that every solution of (156) solves (154). It suffices to consider $\theta = 1$, as the dependence on θ is simple. Let $U = \{U_t(y) : t \geq 0,\ y \in \mathbb{R}\}$ be a bounded solution of equation (156), which is continuous in t. By the fundamental theorem of calculus,

$$\begin{aligned}\exp\Big[-2\int_0^t U_{t-r}(W_r)\,L_{[W,\Gamma]}(dr)\Big] \\ = 1 - 2\int_0^t U_{t-s}(W_s)\exp\Big[-2\int_s^t U_{t-r}(W_r)\,dL_{[W,\Gamma]}(r)\Big]\,L_{[W,\Gamma]}(ds).\end{aligned} \tag{157}$$

Taking expectations,

$$U_t(y) - 1 = -2\,\mathcal{E}_y\Big\{\int_0^t U_{t-s}(W_s)\exp\Big[-2\int_s^t U_{t-r}(W_r)\,dL_{[W,\Gamma]}(r)\Big]\,L_{[W,\Gamma]}(ds)\Big\}. \tag{158}$$

The Markov property (and a glance at the definition of Stieltjes integrals) allows us to continue this with

$$\begin{aligned}&= -2\,\mathcal{E}_y\Big\{\int_0^t U_{t-s}(W_s)\,\mathcal{E}_{W_s}\Big\{\exp\Big(-2\int_0^{t-s} U_{t-s-r}(W_r)\,dL_{[W,\Gamma]}(r)\Big)\Big\}\,L_{[W,\Gamma]}(ds)\Big\} \\ &= -2\,\mathcal{E}_y\Big\{\int_0^t \{U_{t-s}(W_s)\}^2\,L_{[W,\Gamma]}(ds)\Big\} = -2\int_{\mathbb{R}}\Gamma(dx)\int_0^t p_s(y-x)\,\big[U_{t-s}(x)\big]^2\,ds,\end{aligned} \tag{159}$$

which is the formula we had to prove.

To show conversely that every solution $U_t(y)$ of (154) solves (156), we start with the formula

$$\begin{aligned}2\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big[U_{t-s}(W_s) - 1\Big]U_{t-s}(W_s)\,L_{[\Gamma,W]}(ds) \\ = -\int_0^t \big[U_{t-s}(W_s) - 1\big]\ d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big).\end{aligned} \tag{160}$$

We take the expectation, use (155), apply the Markov property as before, and finally use Fubini's Theorem to see

$$\begin{aligned}&\mathcal{E}_y\Big\{2\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\ \big[U_{t-s}(W_s)\big]^2\ L_{[\Gamma,W]}(ds)\Big\} \\ &\qquad - 2\,\mathcal{E}_y\Big\{\int_0^t \exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)U_{t-s}(W_s)\,L_{[\Gamma,W]}(ds)\Big\} \\ &= -\mathcal{E}_y\Big\{\int_0^t \mathcal{E}_{s,W_s}\Big\{-2\int_s^t \big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\Big\}d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big)\Big\} \\ &= \mathcal{E}_y\Big\{2\int_0^t\int_s^t \big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big)\Big\} \\ &= \mathcal{E}_y\Big\{2\int_0^t\int_0^v d_s\Big(\exp\Big(-2\int_0^s U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big)\ \big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\Big\} \\ &= \mathcal{E}_y\Big\{2\int_0^t \Big[\exp\Big(-2\int_0^v U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big) - 1\Big]\ \big[U_{t-v}(W_v)\big]^2 L_{[\Gamma,W]}(dv)\Big\}.\end{aligned}$$

From this and (155) we infer that

$$\mathcal{E}_y\Big\{2\int_0^t \exp\Big(-2\int_0^s U_{t-v}(W_v)\,L_{[\Gamma,W]}(dv)\Big)U_{t-s}(W_s)\,L_{[\Gamma,W]}(ds)\Big\} \\ = \mathcal{E}_y\Big\{2\int_0^t \big[U_{t-r}(W_r)\big]^2\,L_{[\Gamma,W]}(dr)\Big\} = 1 - U_t(y). \tag{161}$$

By the fundamental theorem once more, expression (161) equals

$$1 + \mathcal{E}_y\Big\{-\exp\Big(-2\int_0^t U_{t-r}(W_r)\,L_{[\Gamma,W]}(dr)\Big)\Big\}, \tag{162}$$

from which (156) follows. □

We now aim for upper and lower bounds of $\mathsf{E}U_t\theta\,(0)$, which give the tail asymptotic of the clump sizes by means of a Tauberian theorem. From (12) one immediately sees that $U_t\theta\,(y) \le \theta$ and hence $\mathsf{E}U_t\theta\,(0) \le \theta$. Now plugging the estimate into (12) leads only to a trivial lower bound for $\mathsf{E}U_t\theta\,(y)$. A better lower bound is obtained by means of the Feynman-Kac representation.

LEMMA 4.2 (Asymptotic behaviour of $\mathsf{E}U_t\theta$). *For every $t > 0$ there are positive, finite constants $C_1 = C_1(t)$ and $C_2 = C_2(t)$, such that*

$$\theta - C_1(t)\,\theta^{\gamma+1} \le \mathsf{E}U_t\theta\,(0) \le \theta - C_2(t)\,\theta^{\gamma+1}, \quad \textit{for all } \theta \in (0,1). \tag{163}$$

PROOF. We fix $t > 0$. Plugging $U_t\theta\,(y) \le \theta$ into the Feynman-Kac-representation (156) yields,

$$U_t\theta\,(y) \ge \theta\,\mathcal{E}_y\Big\{\exp\big(-\theta L_{[W,\Gamma]}(t)\big)\Big\} \quad \text{for all } y \in \mathbb{R}, \tag{164}$$

Taking expectation with respect to the medium and using the Laplace functional formula (3) for stable random measures gives

$$\begin{aligned} \mathsf{E}U_t\theta\,(0) &\ge \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta L_{[W,\Gamma]}(t)\Big)\Big\} &&= \theta\,\mathcal{E}_0\exp\Big(-2^\gamma\theta^\gamma\int_{\mathbb{R}}\mathsf{L}^x(t)^\gamma\,dx\Big) \\ &\ge \theta\Big(1 - 2^\gamma\theta^\gamma\,\mathcal{E}_0\int_{\mathbb{R}}\mathsf{L}^x(t)^\gamma\,dx\Big) &&= \theta - \theta^{\gamma+1}2^\gamma\int_{\mathbb{R}}\mathcal{E}_0\left\{\mathsf{L}^x(t)^\gamma\right\}\,dx. \end{aligned} \tag{165}$$

We now show that

$$C_1(t) := 2^\gamma\int_{\mathbb{R}}\mathcal{E}_0\{\mathsf{L}^x(t)^\gamma\}\,dx = t^{\gamma\eta}\,\frac{\gamma}{\gamma+1}\,\frac{2^{\gamma(\eta+1)}}{\sqrt{\pi}}\,G\Big(\frac{\gamma}{2}\Big) < \infty, \tag{166}$$

where G denotes the Gamma function. Indeed, for $x, y > 0$ the density function of $\mathsf{L}^x(t)$ at y is given by $\sqrt{2/\pi t}\,\exp\big(-(x+y)^2/2t\big)$. Hence

$$\int_{\mathbb{R}}\mathcal{E}_0\big\{\mathsf{L}^x(t)^\gamma\big\}\,dx = \sqrt{\frac{8}{\pi t}}\int_0^\infty y^\gamma\int_y^\infty \exp(-z^2/2t)\,dz\,dy \\ = \sqrt{\frac{8}{\pi t}}\int_0^\infty \frac{y^{\gamma+1}}{\gamma+1}\exp(-y^2/2t)\,dy, \tag{167}$$

using integration by parts. One can get the result by substituting $x = y^2/2t$ and recalling the definition of the Gamma function G.

To obtain the upper bound fix $t > 0$. We use (164) in (156) and obtain

$$U_t\theta\,(y) \le \theta\,\mathcal{E}_y\Big\{\exp\Big(-2\,\theta\int_0^t \mathcal{E}_{W_s}\Big\{\exp\Big(-2\,\theta L_{[\tilde W,\Gamma]}(t-s)\Big)\Big\}\,L_{[W,\Gamma]}(ds)\Big)\Big\}. \tag{168}$$

Next we write $\Gamma = \sum_{j=1}^{\infty} a_j\delta_{x_j}$ and use Jensen's inequality,

$$\begin{aligned} \mathsf{E}U_t\theta\,(0) &\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\int_0^t L_{[W,\Gamma]}(ds)\,\mathcal{E}_{W_s}\{\exp\big(-2\,\theta L_{[\tilde W,\Gamma]}(t-s)\big)\}\Big)\Big\} \\ &\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\sum_{i=1}^{\infty} a_i\int_0^t \mathsf{L}^{x_i}(ds)\,\mathcal{E}_{x_i}\{\exp\big(-2\,\theta\sum_{j=1}^{\infty} a_j\mathsf{L}^{x_j}(t-s)\big)\}\Big)\Big\} \\ &\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\sum_{i=1}^{\infty} a_i\int_0^t \mathsf{L}^{x_i}(ds)\,\Big[\exp\big(-2\,\theta\sum_{j=1}^{\infty} a_j\,\mathcal{E}_{x_i}\{\mathsf{L}^{x_j}(t-s)\}\big)\Big]\Big)\Big\}. \end{aligned} \tag{169}$$

Using monotonicity we can continue the estimate with

$$\begin{aligned} &\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\sum_{i=1}^{\infty} a_i\int_0^t \mathsf{L}^{x_i}(ds)\,\Big[\exp\big(-2\,\theta\sum_{j=1}^{\infty} a_j\,\mathcal{E}_{x_i}\{\mathsf{L}^{x_j}(t-s)\}\big)\Big]\Big)\Big\} \\ &\qquad\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\sum_{i=1}^{\infty} a_i\,\mathsf{L}^{x_i}(t)\Big[\exp\big(-2\,\theta\sum_{j=1}^{\infty} a_j\,\mathcal{E}_{x_i}\{\mathsf{L}^{x_j}(t)\}\big)\Big]\Big)\Big\} \\ &\qquad\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\sum_{|x_i|\le 1} a_i\,\mathsf{L}^{x_i}(t)\Big[\exp\big(-2\,\theta\sum_{j=1}^{\infty} a_j\,\mathcal{E}_{x_i}\{\mathsf{L}^{x_j}(t)\}\big)\Big]\Big)\Big\}. \end{aligned} \tag{170}$$

Now we split the sum in the innermost exponential into the sum over the atoms x_j inside, respectively outside, the unit ball. Recall that

$$\mathcal{E}_{x_i}\big\{\mathsf{L}^{x_j}(t)\big\} = f(t, x_i - x_j) \ \text{ for } \ f(t,x) := \int_0^t \frac{1}{\sqrt{2\pi s}}\exp\Big(-\frac{x^2}{2s}\Big)\,ds. \tag{171}$$

For the atoms x_j inside the unit ball we use the estimate

$$\mathcal{E}_{x_i}\big\{\mathsf{L}^{x_j}(t)\big\} \le c_0 := \sqrt{\frac{2t}{\pi}}\,, \tag{172}$$

which gives

$$\begin{aligned} &\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\sum_{|x_i|\le 1} a_i\,\mathsf{L}^{x_i}(t)\Big[\exp\big(-2\,\theta\sum_{j=1}^{\infty} a_j\,\mathcal{E}_{x_i}\{\mathsf{L}^{x_j}(t)\}\big)\Big]\Big)\Big\} \\ &\le \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{\exp\Big(-2\,\theta\sum_{|x_i|\le 1} a_i\,\mathsf{L}^{x_i}(t)\Big[\exp\Big(-2\,\theta c_0\sum_{|x_j|\,\le\,1} a_j\Big) \\ &\qquad\qquad \exp\Big(-2\,\theta\sum_{|x_j|\,>\,1} a_j f\big(t,|x_j|-1\big)\Big)\Big]\Big)\Big\}. \end{aligned} \tag{173}$$

Now denote

$$L_1 := \sum_{|x_j|\,\le\,1} a_j \qquad\text{and}\qquad L_2 := \sum_{|x_j|\,>\,1} a_j\,f(t,|x_j|-1) = \int_{\mathbb{R}} \tilde f\,d\Gamma, \tag{174}$$

for $\tilde f(x) = 1_{\{|x|>1\}} f\big(t, |x|-1\big)$. Under P the random variables L_1 and L_2 are independent, almost surely finite, and stable of index γ. We infer that, for arbitrary fixed $c>0$, the event

$$A := \Big\{ \inf_{|x|\le 1} \mathsf{L}^x(t) \ge c \Big\} \cap \Big\{ L_2 \le \frac{1}{\theta}\Big\} \cap \Big\{ \frac{1}{\theta} \le L_1 \le \frac{2}{\theta}\Big\} \tag{175}$$

has (annealed) probability

$$\begin{aligned} \mathsf{E}\,\mathcal{P}_0(A) &= \mathcal{E}_0\Big\{ \inf_{|x|\le 1} \mathsf{L}^x(t) \ge c \Big\}\, \mathsf{P}\Big\{ L_2 \le \frac{1}{\theta}\Big\}\, \mathsf{P}\Big\{ \frac{1}{\theta} \le L_1 \le \frac{2}{\theta}\Big\} \\ &\ge \mathcal{E}_0\Big\{ \inf_{|x|\le 1} \mathsf{L}^x(t) \ge c \Big\}\, \big(1 - c_1\theta^\gamma\big)\big(c_2\theta^\gamma\big) \ge c_3\theta^\gamma\,, \end{aligned} \tag{176}$$

for a suitable choice of the constants $c_1, c_2, c_3 > 0$. On A we have

$$\begin{aligned} &\exp\Big(-2\,\theta c L_1 \exp\big(-2\,\theta c_0 L_1\big) \exp\big(-2\,\theta L_2\big)\Big) \\ &\le \exp\Big(-2\,\theta \mathrm{e}^{-2} c L_1 \exp\big(-2\,\theta c_0 L_1\big)\Big) \le \exp\Big(-2\mathrm{e}^{-2} c\, \mathrm{e}^{-4c_0}\Big) < 1\,. \end{aligned} \tag{177}$$

We can thus continue the estimate (173) with

$$\begin{aligned} &\theta\,\mathsf{E}\,\mathcal{E}_0\Big\{ \exp\Big(-2\,\theta \sum_{|x_i|\le 1} a_i \mathsf{L}^{x_i}(t) \Big[\exp\Big(-2\,\theta c_0 \sum_{|x_j|\,\le 1} a_j\Big) \exp\Big(-2\,\theta \sum_{|x_j|\,>1} a_j f(t, |x_j| - 1)\Big)\Big]\Big)\Big\} \\ &\qquad \le \theta - \theta\,\mathsf{E}\,\mathcal{E}_0\Big\{ 1_A \Big(1 - \exp\Big[-2\,\theta c L_1 \exp\big(-2\,\theta c_0 L_1\big) \exp\big(-2\,\theta L_2\big)\Big]\Big)\Big\} \\ &\qquad \le \theta - \theta\,\mathsf{E}\,\mathcal{P}_0(A) \Big(1 - \exp\big(-2\mathrm{e}^{-2} c\, \mathrm{e}^{-4c_0}\big)\Big) \;=\; \theta - C_2\theta^{\gamma+1}, \end{aligned} \tag{178}$$

by (176) for a suitable choice of $C_2 = C_2(t) > 0$. This finishes the proof. ☐

The bounds for $\mathsf{E}U_t\theta$ in Lemma 4.2 translate easily into bounds for the Laplace transform

$$\Lambda^t(\theta) = \mathbf{E}_\ell\big\{\exp\big(-\theta\,\beth_t(t)\big)\big\}\,, \text{ for } \theta \ge 0, \tag{179}$$

of the mass of a clump alive at a macroscopic time t.

LEMMA 4.3 (Asymptotic behaviour of Λ^t). *For all $t>0$, as $\theta \downarrow 0$,*

$$1 - \frac{1}{\lambda(t)}\,\theta + \frac{C_2(t)}{\lambda(t)}\,\theta^{\gamma+1} \;\le\; \Lambda^t(\theta) \;=\; 1 - \frac{1}{\lambda(t)}\,\theta + \frac{C_1(t)}{\lambda(t)}\,\theta^{\gamma+1}. \tag{180}$$

PROOF. Using the Laplace transform of a general compound Poisson point field,

$$\mathsf{E}U_t\theta\,(0) = -\log \mathbf{E}_\ell\big\{ -\theta X_t^\infty[0,1]\big\} = \lambda(t)\big(1 - \Lambda^t(\theta)\big), \tag{181}$$

hence the statement follows by applying Lemma 4.2 and the scaling relation of $\lambda(t)$. ☐

Finally, to get the tail behaviour we observe that Theorem 1.10 (iii) follows directly from the previous lemma together with the following version of the Tauberian Theorem of Bingham and Doney, see [**BGT87**, Theorem 8.1.6] for the original statement. Here we can apply the relation $\asymp$ occurring in Theorem 1.10 (iii) in a t–independent situation.

LEMMA 4.4 (Tauberian Theorem). *Suppose ξ is a nonnegative random variable defined on a probability space $(\Omega, \mathcal{A}, P)$ with positive and finite mean m and Laplace transform Λ. Then*

$$\Lambda(\theta) - \big(1 - m\theta\big) \asymp \theta^{\gamma+1}, \;\; \textit{as } \theta \downarrow 0, \tag{182}$$

implies

$$P\{\xi > x\} \asymp \frac{1}{x^{\gamma+1}}\,, \quad \text{as } x \uparrow \infty. \tag{183}$$

Proof. We denote $F(x) := P\{\xi \le x\}$ and

$$h(x) := \frac{1}{m}\int_x^\infty \big(1 - F(y)\big)\; dy\,. \tag{184}$$

Then, using integration by parts twice and then plugging in the assumption,

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\theta x} h(x)\, dx \;&=\; \frac{1}{\theta} - \frac{1}{\theta m}\int_0^\infty \mathrm{e}^{-\theta x}\big(1 - F(x)\big)\; dx \\ &= \frac{1}{\theta} - \frac{1}{\theta m}\,\frac{1-\Lambda(\theta)}{\theta} \asymp \theta^{\gamma-1}.\end{aligned} \tag{185}$$

We now apply the Tauberian Theorem of de Haan and Stadtmüller [**BGT87**, Theorem 2.10.2] to infer

$$\int_0^x h(y)\, dy \asymp x^{1-\gamma}\,, \ \text{ as } \ x \uparrow \infty. \tag{186}$$

Next we use the O-version of the Monotone Density Theorem [**BGT87**, Proposition 2.10.3] twice to conclude

$$\text{first that } h(x) \asymp x^{-\gamma} \text{ and then } 1 - F(x) \asymp \frac{1}{x^{\gamma+1}}\,, \ \text{ as } \ x \uparrow \infty, \tag{187}$$

as claimed. □

Acknowledgement: Most of this work was accomplished while the second author was a postdoctoral fellow at the DFG Graduiertenkolleg "Probabilistische Analysis und stochastische Prozesse" in Berlin, during visits of the first author to Berlin and of the second and third author to the Fields Institute, supported by Don Dawson's Max Planck Award for International Cooperation. All this support is gratefully acknowledged.

Bibliography

[AbS72] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables.* Dover, New York (1972).

[ArP96] M. Arbeiter and N. Patzschke. Random self similar multifractals. *Math. Nachr.* **181**, 5–42, (1996).

[Ama76] H. Amann. Fixed-point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Review* **18**(4), 620–709, (1976).

[Ban92] C. Bandt. The tangent distribution for self-similar measures. *Lecture at the 5th Conference on Real Analysis and Measure Theory, Capri* (1992).

[Ban00] C. Bandt. Local geometry of fractals given by tangent measure distributions. Preprint Universität Greifswald (2000).

[Bas95] R.F. Bass. *Probabilistic Techniques in Analysis.* Springer, New York (1995).

[BeF92] T. Bedford and A.M. Fisher. Analogues of the Lebesgue density theorem for fractal sets of reals and integers. *Proc. London Math. Soc. (3)* **64**, 95–124, (1992).

[BeF96] T. Bedford and A.M. Fisher. On the magnification of Cantor sets and their limit models. *Mh. Math.* **121**, 11–40, (1996).

[BeF97] T. Bedford and A.M. Fisher. Ratio geometry, rigidity and the scenery process for hyperbolic Cantor sets. *Ergodic Theory Dynam. Syst.* **17**, 531–564, (1997).

[BGJ97] J. Bertoin, J.-F. Le Gall and Y. Le Jan. Spatial branching processes and subordination. *Can. J. Math.* **49(1)**, 24–54, (1997).

[BCC99] I. Berkes, E. Csàki and S. Csörgö. Almost sure limit theorems for the St.Petersburg game. *Stat. and Probab. Letters* **45**, 23–30, (1999).

[BeD93] I. Berkes and H. Dehling. Some limit theorems in log density. *Ann. Prob.* **21**, 1640–1670, (1993).

[Bes38] A.S Besicovitch. On the fundamental geometric properties of linearly measurable plane sets of points II. *Math. Ann.* **115**, 296-329, (1938).

[Bin88] N.H. Bingham. On the limit of a supercritical branching process. *Journal Appl. Prob.* **25(A)**, 215–228, (1988).

[BGT87] N.H. Bingham, C.M. Goldie and J.L. Teugels. *Regular Variation.* Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge (1987).

[Blu92] R. Blumenthal. *Excursions of Markov processes.* Birkhäuser Verlag, Boston (1992).

[Bon96] A. Bonnet. On the regularity of edges in image segmentation. *Ann. Inst. Poincaré, Analyse non lineaire* **13**, 485–528, (1996).

[BoS96] A.N. Borodin and P. Salminen. *Handbook of Brownian motion.* Birkhäuser, Basel (1996).

[Bro88] G.A. Brosamler. An almost everywhere central limit theorem. *Math. Proc. Camb. Phil. Soc.* **104**, 561–574, (1988).

[CiT62] Z. Ciesielski and S.J. Taylor. First passage times and sojourn times for Brownian motion in space and the exact Hausdorff measure of the sample path. *Trans. Amer. Math. Soc.* **103**, 434–450, (1962).

[CDP99] T. Cox, R. Durrett, and E.A. Perkins. Rescaled particle systems converging to super-Brownian motion. In: *Perplexing problems in probability*, Eds. M. Bramson, R. Durrett, pp. 269–284. Birkhäuser, Basel (1999).

[Daw93] D.A. Dawson. Measure-valued Markov processes. In: D.A. Dawson, B. Maisonneuve, J. Spencer, *Ecole d'été de probabilités XXI*, Springer LNM **1541**, pp.7–260 (1993).

[DaF88] D.A. Dawson and K. Fleischmann. Strong clumping of critical space-time branching models in subcritical dimensions. *Stoch. Proc. Appl.* **30**, 193–208, (1988).

[DaF91] D.A. Dawson and K. Fleischmann. Critical branching in a highly fluctuating random medium. *Probab. Th. Rel. Fields* **90**, 241–274, (1991).

[DaF92] D.A. Dawson and K. Fleischmann. Diffusion and reaction caused by point catalysts. *SIAM J. Appl. Math.* **52**, 163–180, (1992).

[DaF97] D.A. Dawson and K. Fleischmann. A continuous super-Brownian motion in a super-Brownian medium. *Journ. Theoret. Probab.* **10**, 213–276, (1997).

[DaF00] D.A. Dawson and K. Fleischmann. Catalytic and mutually catalytic super-Brownian motions. *WIAS Berlin, Preprint No.546*, (2000).

[DFM00] D.A. Dawson, K. Fleischmann and C. Mueller. Finite time extinction of superprocesses with catalysts. *Ann. Probab.* **28**(2), 603-642, (2000).

[DFR91] D.A. Dawson, K. Fleischmann and S. Roelly. Absolute continuity of the measure states in a branching model with catalysts. *In: Seminar on stochastic processes 1990*, 117–160, Birkhäuser, Boston (1991).

[DLM95] D.A. Dawson, Y. Li and C. Mueller. The support of measure-valued branching processes in a random environment. *Ann. Probab.* **23(4)**, 1692–1718, (1995).

[DaP91] D.A. Dawson and E.A. Perkins. Historical Processes. *Memoirs Amer. Math. Soc.* **454**, (1991).

[DeM98] P. Deheuvels and D.M. Mason. Random fractal functional laws of the iterated logarithm. *Studia Sci. Math. Hungar.* **34**, 89–106, (1998).

[DPR00a] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni. Thick points for spatial Brownian motion: multifractal analysis of occupation measure. *Ann. Probab.* **28(1)**, 1–35, (2000).

[DPR00b] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni. Thick points for planar Brownian motion and the Erdös-Taylor conjecture on random walk, *Acta Math.* (to appear).

[DPR00c] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni. Thin points for Brownian motion. *Ann. Inst. H. Poincaré, Probab. et Stat.* **36(6)**, 749–774 (2000).

[DPR01] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni. Thick points for intersections of planar sample paths. Preprint, (2001).

[DeZ93] A. Dembo and O. Zeitouni. *Large deviations techniques and applications.* Jones and Bartlett, Boston (1993).

[DLG98] J.S. Dhersin and J.F. Le Gall. Kolmogoroff's test for super-Brownian motion. *Ann. Probab.* **26**, 1041–1056 (1998).

[Dhe97] J.S. Dhersin. Lower functions for the support of super-Brownian motion. *Stoch. Proc. Appl.* **78**, 145–154 (1998).

[Dub71a] S. Dubuc. La densité de la loi limite d'un processus en cascade expansif. *Z. Wahrscheinlichkeitstheorie* **19**, 281–290, (1971).

[Dub71b] S. Dubuc. Problèmes relatifs à l'itération de fonctions suggérés par les processus en cascade. *Ann. Inst. Fourier (Grenoble)* **21**, 171–251, (1971).

[Dyn91] E.B. Dynkin. Branching particle systems and superprocesses. *Ann. Probab.* **19**, 1157–1194, (1991).

[FaS98] I. Fahrner and U. Stadtmüller. On almost sure max-limit theorems. *Stat. and Probab. Letters* **37**, 229–236, (1998).

[Fal99] K.J. Falconer. *Techniques in fractal geometry.* John Wiley, Chichester (1999).

[FaS95] K.J. Falconer and O.B. Springer. Order-two density of sets and measures with non-integral dimension, *Mathematika* **42**, 1–14, (1995).

[FaX95] K.J. Falconer and Y. Xiao. Average densities of the image and zero set of stable processes. *Stoch. Proc. Appl.* **55**, 271–283, (1995).

[Fed87] H. Federer. *Geometric Measure Theory*. Springer Verlag, Berlin (1968).

[Fis87] A.M. Fisher. Convex-invariant means and a pathwise central limit theorem. *Adv. in Math.* **63**, 213–246, (1987).

[FlM97] K. Fleischmann and C. Mueller. A super-Brownian motion with a locally infinite catalytic mass. *Probab. Theory Relat. Fields* **107**, 325–357, (1997).

[FlM00] K. Fleischmann and C. Mueller. Finite time extinction of catalytic branching processes. In Luis G. Gorostiza and B. Gail Ivanoff (eds), *Stochastic Models*, **26** CMS Conference Proceedings, pp 125–139. Amer. Math. Soc., Providence (2000).

[FlM90] K. Fleischmann and S.A. Molchanov. Exact asymptotics in a mean field model with random potential. *Probab. Theory Relat. Fields* **86**, 239–251, (1990).

[Fuk84] M. Fukushima. Basic properties of Brownian motion and a capacity on the Wiener space. *J. Math. Soc. Japan* **36(1)**, 161–175, (1984).

[Fur70] H. Furstenberg. Intersections of Cantor sets and transversality of semigroups. In: *Probl. Analysis, Sympos. in Honor of Salomon Bochner*, Princeton Univ. 1969, pp. 41-59 (1970).

[GdH99] J. Gärtner and F. den Hollander. Correlation structure of intermittency in the parabolic Anderson model. *Probab. Theory Relat. Fields* **114**, 1–54 (1999).

[GäM98] J. Gärtner and S.A. Molchanov. Parabolic problems for the Anderson model. II. Second order asymptotics and structure of high peaks. *Probab. Theory Relat. Fields* **111**, 17–55, (1998).

[GHR84] D. Geman, J. Horowitz and J. Rosen. A local time analysis of intersections of Brownian motion in the plane. *Ann. Probab.* **12**, 86–107, (1984).

[Gra95] S. Graf. On Bandt's tangential distribution for self-similar measures. *Mh. Math.* **120**, 223–246, (1995).

[Haw81] J. Hawkes. Trees generated by a simple branching process. *J. London Math. Soc.* **24**, 373–384, (1981).

[dHo00] F. den Hollander. *Large Deviations.* American Mathematical Society, Providence, Rhode Island (2000).

[Huo97] P. Huovinen. Singular integrals and rectifiability of measures in the plane. *Ann. Acad. Sci. Fennicae, Diss.* **109**, 1–63, (1997).

[JoP95] H. Joyce and D. Preiss. On the existence of subsets of finite positive packing measure. *Mathematika* **42**, 15–24, (1995).

[Kal83] O. Kallenberg. *Random Measures.* Akademie-Verlag, Berlin (1983).

[Kau69] R. Kaufman. Une propriété métrique du mouvement Brownien. *C. R. Acad. Sci. Paris* **268**, 727-728, (1969).

[Kes86] H. Kesten. Subdiffusive behaviour of random walk on a random cluster. *Ann. Inst. Henri Poincaré, Probabilités et Statistiques* **22(4)**, 425–487, (1986).

[KPX00] D. Khoshnevisan, Y. Peres and Y. Xiao. Limsup random fractals. *El. J. Probab.* **5** Paper 4, pp.1–24, (2000).

[KhS00] D. Khoshnevisan and Z. Shi. Fast sets and points for fractional Brownian motion. In: *Séminaire de Probabilités XXXIV*, Springer LNM **1729**, pp. 393–416, (2000).

[Kir88] B. Kirchheim. Uniformly distributed measures, tangent measures and analytic varieties. In: *Proc. Conf. Topology and Measure V*, Wissenschaftliche Beiträge der Ernst-Moritz-Arndt-Universität Greifswald, pp. 54–60, (1988).

[Kle01] A. Klenke. Catalytic branching and the Brownian snake. Preprint Universität Erlangen (2001).

[KrM98] D. Krieg and P. Mörters. The tangent measure distributions of hyperbolic Cantor sets. *Mh. Math.* **126**, 313–328, (1998).

[LeG86] J.-F. Le Gall. Sur la saucisse de Wiener et les points multiples du mouvement brownien. *Ann. Probab.* **14**, 1219–1244, (1986).

[LeG87] J.-F. Le Gall. The exact Hausdorff measure of Brownian multiple points I. *In: Seminar on Stochastic Processes 1986*, 107–137, Birkhäuser, Boston (1987).

[LeG89] J.-F. Le Gall. The exact Hausdorff measure of Brownian multiple points II. *In: Seminar on Stochastic Processes 1988*, 193–197, Birkhäuser, Boston (1989).

[LeG91] J.-F. Le Gall. Brownian excursions, trees and measure-valued branching processes. *Ann. Probab.* **19**, 1399–1439, (1991).

[LeG92] J.-F. Le Gall. Some properties of planar Brownian motion. In: M. Freidlin, J.-F. Le Gall, *Ecole d'été de probabilités XX*, Springer LNM **1527**, (1992).

[LeG93] J.-F. Le Gall. A class of path-valued Markov processes and its applications to superprocesses. *Probab. Theory Related Fields* **95**, 25–46, (1993).

[LeG99] J.-F. Le Gall. *Spatial branching processes, random snakes and partial differential equations.* Birkhäuser, Basel, (1999).

[LGP95] J.-F. Le Gall and E.A. Perkins. The Hausdorff measure of the support of two-dimensional super-Brownian motion. *Ann. Probab.* **23**, 1719–1747, (1995).

[LL97] E.H. Lieb and M. Loss. *Analysis.* American Mathematical Society Graduate Studies, Volume 14 (1997).

[Liu96] Q.S. Liu. The exact Hausdorff dimension of a branching set. *Prob. Theory and Rel. Fields.* **104**, 515-538, (1996).

[Liu99] Q.S. Liu. Local dimension of the branching measure on a Galton-Watson tree. *Ann. Inst. H. Poincaré, Probab. et Stat.* **37(2)**, 195–222, (2001).

[LyR90] R. Lyons. Random walks and percolation on trees. *Ann. Probab.* **18**, 931–958, (1990).

[LyT86] T.J. Lyons. The critical dimension at which quasi-every Brownian motion is self-avoiding. *Adv. in Appl.Probab.* (Special Suppl. 1986), 87–99, (1986).

[Mar64] J.M. Marstrand. The (ϕ, s)-regular subsets of n-space. *Trans. Amer. Math. Soc.* **113**, 369–392, (1964).

[Mar96] J.M. Marstrand. Order-two density and the strong law of large numbers, *Mathematika* **43**, 1–22, (1996).

[Mat95] P. Mattila. *The Geometry of Sets and Measures in Euclidean Spaces.* Cambridge University Press, Cambridge (1995).

[Mec67] J. Mecke. Stationäre zufällige Maße auf lokalkompakten abelschen Gruppen. *Zeitschr. Wahrsch. verw. Gebiete* **9**, 36–58, (1967).

[Moo50] E.F. Moore. Density ratios and $(\Phi, 1)$ rectifiability in n-space. *Trans. Amer. Math. Soc.* **69**, 324–334, (1950).

[Mör95] P. Mörters. *Tangent Measure Distributions and the Geometry of Measures.* PhD thesis, University College London, (1995).

[Mör98a] P. Mörters. Symmetry properties of average densities and tangent measure distributions of measures on the line. *Adv. Appl. Math.* **21**, 146–179, (1998).

[Mör98b] P. Mörters. The average density of the path of planar Brownian motion. *Stoch. Proc. Appl.* **74**, 133–149, (1998).

[Mör99] P. Mörters. A uniform dimension result for the Brownian snake. *WIAS Berlin, Preprint No. 478*, (1999)

[Mör00] P. Mörters. Almost sure Kallianpur–Robbins laws for Brownian motion in the plane. *Probab. Theory rel. Fields* **118**, 49–64, (2000).

[Mör01a] P. Mörters. The average density of super-Brownian motion. *Ann. Inst. Henri Poincaré, Probabilités et Statistiques* **37**(1), 71–100, (2001).

[Mör01b] P. Mörters. A pathwise version of Spitzer's theorem. In: *Limit Theorems in Probability and Statistics*, Balatonlelle, Hungary, 1999. To appear in Bolyai Society Math. Studies (2001).

[MöP95] P. Mörters and D. Preiss. On one–sided average densities of fractal measures on the line. *Sem. d'initiation à l'analyse, Publ. Univ. Pierre et Marie Curie, Paris* **117**: Exposé 4, (1994/5).

[MöP98] P. Mörters and D. Preiss. Tangent measure distributions of fractal measures. *Math. Ann.* **312**, 53–93, (1998).

[MöS99] P. Mörters and N.R. Shieh. Small scale limit theorems for intersection local times of Brownian motion. *El. Journal of Probab.* **4**, Paper No. 9 , pp 1-23, (1999).

[MöS01] P. Mörters and N.R. Shieh. Thin and thick points for branching measure on a Galton-Watson tree. Preprint Universität Kaiserslautern (2001).

[Mol94] S.A. Molchanov. Lectures on random media. In: D. Bakry, R.D. Gill and S.A. Molchanov, *Ecole d'été de probabilités XXII*, Springer LNM **1581**, pp.242-411, (1994).

[MuP92] C. Mueller and E.A. Perkins. The compact support property for solutions to the heat equation with noise. *Probab. Theory Related Fields* **93(3)**, 325–358, (1992).

[ONe95] T. C. O'Neil. *A Local Version of the Projection Theorem and Other Results in Geometric Measure Theory.* PhD Thesis, University College London, (1995).

[ONe96] T. C. O'Neil. A local version of the projection theorem. *Proc. London Math. Soc.(3)* **73**, 68–104, (1996).

[OrT74] S. Orey and S.J. Taylor. How often on a Brownian path does the law of the iterated logarithm fail? *Proc. London Math. Soc.(3)* **28**, 174–192, (1974).

[PZU90] N. Patzschke and U. Zähle. Self similar random measures IV. The recursive construction model of Falconer, Graf, Mauldin and Williams. *Math. Nachr.* **149**, 285–302, (1990).

[PZM94] N. Patzschke and M. Zähle. Self similar random measures are locally scale invariant. *Prob. Theory Rel. Fields* **97**, 559–574, (1993).

[Pen89] M.D. Penrose. On the existence of self-intersections for quasi-every Brownian path in space *Ann. Probab.* **17(2)**, 482–502, (1989).

[Per96] Y. Peres. Remarks on intersection-equivalence and capacity equivalence. *Ann. Inst. H. Poincaré (Physique Théorique)*, **64** 339–347, (1996).

[Per83] E.A. Perkins. On the Hausdorff dimension of the Brownian slow points. *Zeitschr. Wahr. verw. Gebiete*, **64**, 369–399, (1983).

[PoS78] S. Port and C. Stone. *Brownian Motion and Classical Potential Theory.* Academic Press, New York (1978).

[Pre87] D. Preiss. Geometry of measures in $\mathbb{R}^n$: Distribution, rectifiability and densities. *Ann. Math.* **125**, 537–643, (1987).

[Ray63] D. Ray. Sojourn times and the exact Hausdorff measure of the sample paths of planar Brownian motion. *Trans. Amer. Math. Soc.* **108**, 436–444, (1963).

[ReY98] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer Verlag (1998).

[Sch73] L. Schwartz. *Radon measures on arbitrary topological spaces and cylindrical measures.* Oxford University Press (1973).

[Ser93] L. Serlet. Quelques propriétés du super-mouvement brownien. *Thèse, Université Paris VI*, (1993).

[Ser95] L. Serlet. Some dimension results for super-Brownian motion. *Probab. Theory Related Fields* **101**, 371-391, (1995).

[ShT98] N.R. Shieh and S.J. Taylor. Logarithmic multifractal spectrum of stable occupation measure. *Stoch. Proc. Appl.* **75(2)**, 249–261, (1998).

[ShT00] N.R. Shieh and S.J. Taylor. Multifractal spectra of branching measure on a Galton-Watson tree. Preprint Taiwan National University (2000).

[Sin82] Ya.G. Sinai. The limiting behaviour of a one-dimensional random walk in a random medium. *Theory Probab. Appl.* **27**, 256–268, (1982).

[Sla99] G. Slade. Lattice trees, percolation and super-Brownian motion. In: *Perplexing problems in probability*, Eds. M. Bramson, R. Durrett, pp. 35–53. Birkhäuser, Basel (1999).

[Ste70] E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton University Press (1970).

[Sul87] D. Sullivan. Differentiable structures on fractal–like sets, determined by intrinsic scaling functions on dual Cantor sets. *AMS Proc. Symp. Pure Math.*, **48**, 15–23, (1987).

[Tay66] S.J. Taylor. Multiple points for the sample paths of the symmetric stable process. *Zeitschr. Wahr. verw. Gebiete* **5**, 247–264, (1966).

[Top70] F. Topsøe. *Topology and Measure.* Springer LNM **133**, (1970).

[Tri89] R. Tribe. *Path properties of superprocesses.* Ph.D.-thesis, Univ. of British Columbia, (1989).

[Val89] M. Valadier. Young measures. In: *Methods of Nonconvex Analysis*, Springer LNM **1446**, pp.152–188, (1989).

[Ver95] J. Verzani. The slow points in the support of historical super-Brownian motion. *Ann. Probab.* **23**, 56–70, (1995).

[Wil98] S. Wilhelm. *Die Tangentialmaßverteilungen des Aufenthaltsmaßes und des Lokalzeitmaßes der Brownschen Bewegung.* Diplomarbeit Fachbereich Mathematik, Universität Kaiserlautern (1998).

[Zäh88] U. Zähle. Self similar random measures I. Notion, carrying Hausdorff dimension and hyperbolic distribution. *Prob. Theory Rel. Fields* **80**, 79–100, (1988).