Pathwise Kallianpur-Robbins laws for Brownian motion in the plane

Peter Mörters
Universität Kaiserslautern, Fachbereich Mathematik, 67663 Kaiserslautern, Germany.
E-Mail: peter@mathematik.uni-kl.de

Summary. The Kallianpur-Robbins law describes the long term asymptotic behaviour of the distribution of the occupation measure of a Brownian motion in the plane. In this paper we show that this behaviour can be seen at every typical Brownian path by choosing either a random time or a random scale according to the logarithmic laws of order three. We also prove a ratio ergodic theorem for small scales outside an exceptional set of vanishing logarithmic density of order three.

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1 Introduction and statement of the main theorems

Since the discovery of pathwise, or almost sure, central limit theorems independently by Fisher (1987, 1990), Brosamler (1988) and Schatte (1988) there has been a great deal of interest in strong versions of the classical theorems on limit laws associated with stochastic processes. Interesting further progress on this subject can be found, for example, in the papers of Berkes and Dehling (1993), Fahrner and Stadtmüller (1998), Ibragimov and Lifshits (1998), Berkes, Csaki and Csörgö (1999) or the survey of Berkes (1998).

The basic idea of these pathwise limit theorems is the following: There is a classical theorem describing the limit of the laws of random variables depending on a stochastic process with respect to the distribution of the process. The corresponding pathwise law describes, for almost every sample of the process, the limit of the laws of the random variables with respect to a random time parameter, which is typically chosen according to the parametrized family

$$\left\{ \frac{1}{\log t} \int_1^t \delta_{\{s\}} \, \frac{ds}{s} \ , \, t > 1 \right\}$$

of laws, which we call the *logarithmic laws of order two* as they are associated with the order two logarithmic averages of Hardy and Riesz. A closely related idea are limit theorems for time averages taken with respect to the logarithmic laws, the so-called logarithmic averages. They have been studied for a long time, an early example can be found in Erdös and Hunt (1953).

The Kallianpur-Robbins law for linear Brownian motion provides a simple example of a limit law that allows a pathwise version of this kind: Look at a linear Brownian motion $\{B_t\}_{t\geq 0}$ started at the origin and denote by ℓ the Lebesgue measure. We define the occupation measure $\mu[t]$ as the random measure given by

$$\mu[t](A) = \ell\{s \in [0,t] \, : \, B_s \in A\} \text{ for } A \subseteq \mathbb{R} \text{ Borel.}$$

Writing $\delta_{\{x\}}$ for the Dirac measure concentrated at x allows us to describe the distribution of a random variable X on a probability space $(\Omega, \mathcal{A}, P)$ as $\int \delta_{\{X(\omega)\}} \, dP(\omega)$ and most of the time we omit the ω. The classical Kallianpur-Robbins law, proved first by Kallianpur and Robbins (1953), states that, for the Wiener measure $\mathbb{W}$,

$$w - \lim_{t\to\infty} \int \delta_{\left\{\frac{\mu[t]}{\sqrt{t}}\right\}} \, d\mathbb{W} = \sqrt{\frac{2}{\pi}} \int_0^\infty \delta_{\{a\cdot\ell\}} \, e^{-a^2/2} \, da \, ,$$

where $w - \lim$ denotes weak convergence of distributions on the Polish space $\mathcal{M}(\mathbb{R})$ of locally finite Borel measures on the real line, see Section 2 for measure theoretic preliminaries. The following pathwise version of this theorem for random times chosen with respect to the second order logarithmic laws can be proved easily exploiting the existence of the local time process and using its scaling property and the ergodic theorem, see Section 3.

Theorem 1.1 (Random time Kallianpur-Robbins law for linear Brownian motion) $\mathbb{W}$-*almost surely, the distributions*

$$w - \lim_{t\to\infty} \frac{1}{\log t} \int_1^t \delta_{\left\{\frac{\mu[s]}{\sqrt{s}}\right\}} \, \frac{ds}{s} = \sqrt{\frac{2}{\pi}} \int_0^\infty \delta_{\{a\cdot\ell\}} \, e^{-a^2/2} \, da \, ,$$

and for all bounded $A \subset \mathbb{R}$ Borel and $k \geq 1$, the moments

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \Big(\frac{\mu[s](A)}{\sqrt{s}}\Big)^k \, \frac{ds}{s} = \sqrt{\frac{2}{\pi}} \int_0^\infty (a\ell(A))^k \, e^{-a^2/2} \, da \, ,$$

converge.

It is the principal aim of this paper to prove pathwise Kallianpur Robbins laws for the case of a planar Brownian motion, where no local time process is available and the scaling properties of Brownian motion are more difficult to exploit. In fact, it turns out that in order to prove a random time Kallianpur-Robbins law for planar Brownian motion we have to choose the random time according to a logarithmic law of higher order.

Writing $\log^{(n)}$ and $\exp^{(n)}$ for the n-th iterate of the logarithm or exponential, we can define, following again Hardy and Riesz, for every $n \geq 1$, the *logarithmic laws of order* n as

$$\frac{1}{\log^{(n-1)}(t)} \int_{\exp^{(n-1)}(0)}^t \delta_{\{s\}} \, \frac{ds}{\prod_{j=0}^{n-2} \log^{(j)}(s)} \, , \text{ for } t > \exp^{(n-1)}(0).$$

Note that this definition allows us to build a hierarchy of theorems: the limit law for the random times with respect to the order n logarithmic law implies analogous limit laws for random times with respect to all higher order logarithmic laws, but not conversely, see Fisher (1990). Limit laws for averages with respect to logarithmic laws of higher order were studied, for example, by Brosamler (1973), Földes (1992) or Marcus and Rosen (1995).

Let $\{B_t\}_{t\geq 0}$ be a planar Brownian motion started at the origin. For every $t > 0$ the *occupation measure* $\mu[t]$ is the random measure on the path of the Brownian motion defined by

$$\mu[t](A) = \ell\{s \in [0,t] \, : \, B_s \in A\} \text{ for } A \subseteq \mathbb{R}^2 \text{ Borel.}$$

The classical two-dimensional Kallianpur-Robbins law, also due to Kallianpur and Robbins (1953), states that, for the Wiener measure $\mathbb{W}^2$,

$$w - \lim_{t\to\infty} \int \delta_{\left\{\frac{\mu[t]}{\log\sqrt{t}}\right\}} \, d\mathbb{W}^2 = \int_0^\infty \delta_{\left\{\frac{a}{\pi}\cdot\ell^2\right\}} e^{-a}\, da\,,$$

where ℓ^2 is Lebesgue measure in the plane and $w - \lim$ denotes the weak convergence of distributions on the Polish space $\mathcal{M}(\mathbb{R}^2)$ of locally finite Borel measures on the plane with the vague topology.

In order to get a pathwise version of this theorem for a randomly chosen time we look at a *fixed* Brownian path and study the distribution of the measure $\mu[t]$ with respect to a random choice of the time parameter t picked according to the logarithmic laws of order three.

Theorem 1.2 (Random time Kallianpur-Robbins law for planar Brownian motion) $\mathbb{W}^2$-*almost surely, the distributions*

$$w - \lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t \delta_{\left\{\frac{\mu[s]}{\log\sqrt{s}}\right\}} \, \frac{ds}{s\,\log s} = \int_0^\infty \delta_{\left\{\frac{a}{\pi}\cdot\ell^2\right\}} e^{-a}\, da\,,$$

and for all bounded $A \subset \mathbb{R}^2$ Borel and $k \geq 1$, the moments

$$\lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t \Big(\frac{\mu[s](A)}{\log\sqrt{s}}\Big)^k \frac{ds}{s\,\log s} = k!\Big[\frac{\ell(A)}{\pi}\Big]^k$$

converge.

The proof of Theorem 1.2 is based on an approximation of the occupation measure up to time t by weighted occupation measures. The moments of these weighted occupation measures can be calculated explicitly and the statement of Theorem 1.2 can be derived from analogous statements about the random choice of a weighting parameter. The case of first moments in Theorem 1.2 was proved before by Brosamler (1973) with a different method.

It is natural to ask whether a pathwise Kallianpur Robbins law can also be formulated for the random choice of a *scale* instead of time parameter. To motivate this question further we recall the scaling invariance of Brownian motion,

$$\int \delta_{\left\{\{B_s\}_{s\geq 0}\right\}} \, d\mathbb{W}^2 = \int \delta_{\left\{\{\sqrt{1/r}B_{rs}\}_{s\geq 0}\right\}} \, d\mathbb{W}^2 \quad \text{for all } r > 0.$$

By the scaling invariance the measure $\nu[r]$ given by

$$\nu[r](A) = \ell\{s \in [0,1] \,:\, B_s \in rA\}$$

satisfies

$$\int \delta_{\left\{\mu[t]\right\}} \, d\mathbb{W}^2 = \int \delta_{\left\{t\cdot\nu[1/\sqrt{t}]\right\}} \, d\mathbb{W}^2\,.$$

This allows a different view of the Kallianpur-Robbins law: Instead of the long term behaviour we can also study the local behaviour of the occupation measure at the origin and the Kallianpur Robbins law now reads

$$w-\lim_{r\downarrow 0}\int \delta_{\left\{\frac{\nu[r]}{r^2\log(1/r)}\right\}}\,d\mathbb{W}^2=\int_0^\infty \delta_{\left\{\frac{a}{\pi}\cdot\ell^2\right\}}\,e^{-a}\,da\,. \tag{1}$$

It is interesting to observe that, although this statement is completely equivalent to the Kallianpur-Robbins law in the classical formulation, it is impossible to derive a random scale Kallianpur Robbins law from the random time Kallianpur-Robbins law: The pathwise Kallianpur-Robbins law is a statement about the *process* $\{\mu[s]/\log\sqrt{s})\}_{s\geq e}$, which has a different distribution from the process $\{s\cdot\nu[1/\sqrt{s}]/\log\sqrt{s}\}_{s\geq e}$. However, a pathwise Kallianpur-Robbins law for random scales, which is completely independent of Theorem 1.2, can be formulated and proved by means of an entirely different method. We again look at a fixed typical path of planar Brownian motion and now we study the distribution of the measure $\nu[r]$ with a random choice of the scale parameter $1/r$ with respect to a logarithmic law of order three.

Theorem 1.3 (Random scale Kallianpur-Robbins law for planar Brownian motion) $\mathbb{W}^2$*-almost surely, all distributions*

$$w-\lim_{\varepsilon\downarrow 0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\delta_{\left\{\frac{\nu[r]}{r^2\log(1/r)}\right\}}\,\frac{dr}{r\,\log(1/r)}=\int_0^\infty \delta_{\left\{\frac{a}{\pi}\cdot\ell^2\right\}}\,e^{-a}\,da\,, \tag{2}$$

and for all bounded $A\subset\mathbb{R}^2$ Borel with $\ell^2(\partial A)=0$ and $k\geq 1$, the moments

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\Big(\frac{\nu[r](A)}{r^2\log(1/r)}\Big)^k\,\frac{dr}{r\log(1/r)}=k!\Big[\frac{\ell(A)}{\pi}\Big]^k$$

converge.

It is unclear whether one can remove the assumption $\ell^2(\partial A)=0$ from the theorem. A weaker result of this kind can be found in Mörters (1998), where it was shown that, $\mathbb{W}^2$-almost surely,

$$w-\lim_{\varepsilon\downarrow 0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\delta_{\left\{\frac{\nu[r](B(0,1))}{r^2\log(1/r)}\right\}}\,\frac{dr}{r\,\log(1/r)}=\int_0^\infty \delta_{\{a\}}\,e^{-a}\,da\,, \tag{3}$$

where $B(0,r)$ denotes the open ball of radius r. Compared to Theorem 1.3 this statement is weaker mainly because of the restriction to a rotationally symmetric situation. In the random time case such a restriction would not be a problem: The ratio ergodic theorem of Maruyama and Tanaka (1959) states that

$$\lim_{t\to\infty}\frac{\mu[t](A)}{\mu[t](B)}=\frac{\ell^2(A)}{\ell^2(B)}\ ,\ \mathbb{W}^2\text{-almost surely,}$$

for $A,B\subseteq\mathbb{R}^2$ bounded Borel sets with $\ell^2(B)>0$. But no such statement holds for the small scale limit. In order to close the gap between (3) and (2) we prove a ratio ergodic theorem for small scales which holds in logarithmic density of order three. This is also interesting in its own right.

Theorem 1.4 (Ratio ergodic theorem for small scales) *For all bounded ℓ^2-continuity Borel sets $A, B \subseteq \mathbb{R}^2$ with $\ell^2(B) > 0$, $\mathbb{W}^2$-almost surely there is a set $N \subseteq (0,\infty)$ such that*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} 1_N(r) \, \frac{dr}{r \, \log(1/r)} = 0 ,$$

and

$$\lim_{\substack{r\to 0 \\ r \notin N}} \frac{\nu[r](A)}{\nu[r](B)} = \frac{\ell^2(A)}{\ell^2(B)} .$$

The proofs of Theorems 1.3 and 1.4 are based on a generalization of a method of Ray (1963). Ray showed that the occupation measure of small balls can be approximated by means of the number of inward crossings of small annuli performed by the Brownian motion before it leaves the unit ball. We generalize this approach and approximate the occupation measure of small sets (of arbitrary shape) by means of these crossing numbers. Our theorems can then be derived from easy statements about the behaviour of the crossing numbers. A similar technique was used by Le Gall (1987) to study the exact Hausdorff function of the set of multiple points of a planar Brownian motion.

The random scale Kallianpur-Robbins law can also be motivated from the point of view of fractal geometry. The study of limits of local enlargements of fractal measures with respect to a deterministic or random scale parameter is a common theme in fractal geometry and I briefly explain how the above result fits into the framework of tangent measure distributions. This concept and the related notion of average densities were studied, for example, in Bandt (1992), Bedford and Fisher (1992), Falconer (1997), Graf (1995) or Mörters and Preiss (1998).

Suppose $\mu \in \mathcal{M}(\mathbb{R}^d)$ is any measure on $\mathbb{R}^d$ and φ a gauge function, i.e. a monotonically increasing function with $\varphi(0) = 0$. Define, for every $x \in \mathbb{R}^d$ and $s > 0$, the measure $\mu_{x,s}$ by $\mu_{x,s}(A) = \mu(x + sA)$. The set

$$\mathrm{Tan}_\varphi(\mu, x) = \{\nu : \nu = \lim_{n\to\infty} \frac{\mu_{x,t_n}}{\varphi(t_n)} \text{ for } t_n \to 0\} \subseteq \mathcal{M}(\mathbb{R}^d)$$

is the set of *φ-tangent measures* of μ at x, which was introduced by Preiss, for a survey see Falconer (1997). With the appropriate choice of the gauge function φ the φ-tangent measures at x describe, roughly speaking, the scenery of μ in small neighbourhoods of x. As the set of tangent measures is often quite big, it is natural to look for canonical probability distributions on this set. If, for some $n \geq 2$,

$$P = w - \lim_{\varepsilon \downarrow 0} \frac{1}{\log^{(n-1)}(1/\varepsilon)} \int_\varepsilon^{1/\exp^{(n-1)}(0)} \delta_{\left\{\frac{\mu_{x,s}}{\varphi(s)}\right\}} \, \frac{ds}{s^2 \prod_{j=0}^{n-2} \log^{(j)}(1/s)} ,$$

exists, it defines a probability measure on $\mathrm{Tan}_\varphi(\mu, x)$, which is called the *$\varphi$-tangent measure distribution of order n of μ at x.*

In the case of the occupation measure $\mu[1]$ of a planar Brownian motion we have $\mu[1]_{0,s} = \nu[s]$ and hence the random scale Kallianpur-Robbins law states that, for $\varphi(t) = t^2 \log(1/t)$, $\mathbb{W}^2$-almost surely, the φ-tangent measure distribution of $\mu[1]$ at the origin exists and is equal to the distribution of a standard exponential multiple of planar Lebesgue measure. Using the idea of Palm distributions as in Mörters (1998), Section 3, this statement can be extended easily to the statement of the following theorem.

Theorem 1.5 *Let $\mu = \mu[T]$ be the occupation measure of a planar Brownian motion run for a finite time interval of arbitrary length $T > 0$ and define the gauge function $\varphi(t) = t^2 \log(1/t)$. Then, $\mathbb{W}^2$-almost surely, the φ–tangent measure distribution of order three of μ exists at μ–almost every x and is equal to the distribution of a random constant multiple of planar Lebesgue measure. The constant is gamma distributed with parameter two, which is the distribution of the sum of two independent standard exponentially distributed random variables. More explicitly,*

$$w - \lim_{\varepsilon \downarrow 0} \frac{1}{\log|\log \varepsilon|} \int_{\varepsilon}^{1/e} \delta_{\left\{\frac{\mu_{x,t}}{\varphi(t)}\right\}} \frac{dt}{t \log(1/t)} = \int_0^\infty \delta_{\{a \cdot \ell^2\}}\, a e^{-a}\, da\,.$$

Note that there is no random scale analogy to the pathwise Kallianpur-Robbins law for linear Brownian motion because almost surely $\lim_{r\to 0} \frac{\nu[r]}{r} = \ell \cdot L(0,1)$, where $L(0,t)$ is the local time at the origin, see Chapter VI in Revuz and Yor (1994).

This paper is organized as follows. In Section 2 we collect some measure theoretic preliminaries and in Section 3 we give a proof of the pathwise Kallianpur-Robbins law in the linear case, which is based on the ergodic theorem. In the following two sections we deal with the case of planar Brownian motion, in Section 4 we prove the random time Kallianpur-Robbins law and in Section 5 we prove the ratio ergodic theorem for small scales and the random scale Kallianpur Robbins law. In Section 6 we briefly indicate why corresponding results fail to hold for logarithmic averages of order two. We finish the paper with a selection of brief remarks.

Note: A shorter paper on the present subject is to appear in *Probability theory related fields.*

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2 Some preliminaries

In this section we collect some measure theoretic preliminaries, a reference for these facts is Kallenberg (1983). By $\mathcal{M}(\mathbb{R}^d)$ we denote the set of all measures on the Borel σ-algebra of the Euclidean space $\mathbb{R}^d$ that are finite on bounded sets, in other words the set of locally finite Borel measures. We equip $\mathcal{M}(\mathbb{R}^d)$ with the *vague topology*, this is the smallest topology, which makes the functionals $\nu \mapsto \int f\, d\nu$, f continuous with compact support, continuous. Then the mapping $\nu \mapsto \nu(A)$ is upper semicontinuous if A is open and lower semicontinuous if A is compact. The vague topology makes $\mathcal{M}(\mathbb{R}^d)$ a separable space and may also be generated by a complete metric. In other words, $\mathcal{M}(\mathbb{R}^d)$ is a *Polish space.*

A set $M \subseteq \mathcal{M}(\mathbb{R}^d)$ is precompact if and only if the set $\{\mu(K) : \mu \in M\}$ is bounded for all compact sets $K \subseteq \mathbb{R}^d$. We equip $\mathcal{M}(\mathbb{R}^d)$ with the Borel σ-algebra coming from the weak topology. For probability distributions P_n on $\mathcal{M}(\mathbb{R}^d)$ we now have the usual notion of *weak convergence*, namely $w - \lim_{n\to\infty} P_n = P$ if and only if

$$\lim_{n\to\infty} \int F(\mu)\, dP_n(\mu) = \int F(\mu)\, dP(\mu) \text{ for all } F : \mathcal{M}(\mathbb{R}^d) \to \mathbb{R} \text{ continuous, bounded.}$$

As a particular consequence of the weak convergence of P_n to P we note that, for all bounded continuity sets $A \subseteq \mathbb{R}^2$ of $\int \nu\, dP(\nu)$, we have, in the sense of weak convergence of probability measures on $\mathbb{R}$,

$$w - \lim_{n\to\infty} \int \delta_{\{\mu(A)\}}\, dP_n(\mu) = \int \delta_{\{\mu(A)\}}\, dP(\mu)\,.$$

An important criterion for the existence of weakly convergent subsequences in a sequence (P_n) of probability measures is provided by the classical theorem of Prohorov.

Lemma 2.1 *Suppose $\mathcal{P}$ is a family of probability measures on a Polish space X, then every sequence in $\mathcal{P}$ has a weakly convergent subsequence if and only if the family $\mathcal{P}$ is uniformly tight, i.e. for every $\delta > 0$ there is a compact $C \subseteq X$ such that $P(C) > 1 - \delta$ for all $P \in \mathcal{P}$.*

In order to take advantage of this theorem we need the following simple tightness criterion for families of probability distributions on $\mathcal{M}(\mathbb{R}^d)$. We denote by $\mathcal{A}$ the countable collection of all open or closed cubes with rational vertices in $\mathbb{R}^d$.

Lemma 2.2 *Suppose $\{P_n\}_{n\geq 1}$ is a sequence of probability measures on $\mathcal{M}(\mathbb{R}^d)$ such that, for all closed cubes $A \in \mathcal{A}$,*

$$\lim_{M\to\infty} \limsup_{n\to\infty} P_n\{\nu(A) \geq M\} = 0\,. \tag{4}$$

Then the family $\{P_n : n \geq 1\}$ is uniformly tight.

Proof. Let $\{A_1, A_2, A_3, \ldots\}$ be an enumeration of the close cubes $A \in \mathcal{A}$. By (4) we can find for every integer k and $\delta > 0$, an $M(k,\delta) > 0$ such that

$$P_n\{\nu(A_k) \geq M(k,\delta)\} \leq \delta 2^{-k}\ , \text{ for all } n.$$

Hence

$$P_n\Big\{\nu(A_k) < M(k,\delta)\ , \text{ for all } k\Big\} \geq 1 - \delta \text{ for all } n.$$

This set is precompact in $\mathcal{M}(\mathbb{R}^d)$ and hence the family is uniformly tight. ■

We now get a sufficiently strong technical tool to deal with the convergence of random measures.

Lemma 2.3 *Let $\{P_t\}_{t>0}$ be a family of probability measures on $\mathcal{M}(\mathbb{R}^d)$ and P a probability measure on $\mathcal{M}(\mathbb{R}^d)$. Suppose that, for all $A \in \mathcal{A}$, $\nu(\partial A) = 0$ for P-almost every ν and, for all rational $\kappa_1, \ldots, \kappa_m > 0$ and all $\{A_1, \ldots, A_m\} \subseteq \mathcal{A}$,*

$$\lim_{t\to\infty} \int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(A_i)\Big)\, dP_t(\nu) = \int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(A_i)\Big)\, dP(\nu)\,,$$

then $w - \lim_{t\to\infty} P_t = P$.

Proof. We first use Lemma 2.2 to show that every sequence $\{P_n\}_{n\geq 1}$ with $P_n = P_{t_n}$ for some $t_n \to \infty$ is uniformly tight. Namely, if $A \in \mathcal{A}$ is closed then, by Lévy's continuity theorem, $\int \delta_{\{\nu(A)\}} dP_n(\nu) \to \int \delta_{\{\nu(A)\}} dP$ and hence

$$\limsup_{n\to\infty} P_n\{\nu(A) \geq M\} \leq P\{\nu(A) \geq M\} \longrightarrow 0\ , \text{ as } M \to \infty.$$

By Lemma 2.1 and the tightness it suffices show that every limit point $\tilde{P}$ of $\{P_n\}_{n\geq 1}$ coincides with P. If $\{A_1, \dots, A_m\} \subseteq \mathcal{A}$ are open cubes, the mapping $\nu \mapsto \exp\big(-\sum_{i=1}^m \kappa_i \nu(A_i)\big)$ is lower semicontinuous and hence

$$\begin{aligned}\int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(A_i)\Big)\, dP(\nu) &= \lim_{n\to\infty} \int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(A_i)\Big)\, dP_n(\nu) \\ &\geq \int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(A_i)\Big)\, d\tilde{P}(\nu)\,.\end{aligned}$$

For every $\varepsilon > 0$ we pick closed cubes B_i, concentric with A_i but slightly larger, such that

$$\int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(B_i)\Big)\, dP(\nu) \geq \int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(A_i)\Big)\, dP(\nu) - \varepsilon\,.$$

Using now upper semicontinuity we infer that

$$\begin{aligned}\int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(A_i)\Big)\, dP(\nu) &\leq \lim_{n\to\infty} \int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(B_i)\Big)\, dP_n(\nu) + \varepsilon \\ &\leq \int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(B_i)\Big)\, d\tilde{P}(\nu) + \varepsilon \\ &\leq \int \exp\Big(-\sum_{i=1}^m \kappa_i \nu(A_i)\Big)\, d\tilde{P}(\nu) + \varepsilon\,.\end{aligned}$$

The continuity of the Laplace transform ensures that all finite dimensional marginal distributions of P and $\tilde{P}$ coincide, i.e., for all open cubes $A_1, \dots, A_m \in \mathcal{A}$,

$$\int \delta_{\{\nu(A_1),\dots,\nu(A_m)\}}\, dP(\nu) = \int \delta_{\{\nu(A_1),\dots,\nu(A_m)\}}\, d\tilde{P}(\nu)\,.$$

As the system of all open cubes in $\mathcal{A}$ is closed under finite intersections and is a base of the topology on $\mathbb{R}^d$, this implies that $P = \tilde{P}$. ∎

3 The random time Kallianpur-Robbins law for linear Brownian motion

The proof relies heavily on the $\mathbb{W}$-almost sure existence of the local time process

$$L(0,t) = \lim_{r\to 0} \frac{\mu[t]((-r,r))}{r}$$

at the origin of the linear Brownian motion. Recall the fact that the Brownian motion is invariant and ergodic with respect to rescaling $\{B_t\} \mapsto \{\sqrt{1/s}\, B_{st}\}$, see for example Fisher (1987), and that the process $\{L(0,s)\}_{s\geq 0}$ has the same distribution as the process $\{\max_{0\leq t\leq s} B_t\}_{s\geq 0}$, see

Chapter VI in Revuz and Yor (1994). By means of the ergodic theorem we infer that, $\mathbb{W}$-almost surely, for every $\kappa > 0$,

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \exp\Big(- \kappa \frac{L(0,s)}{\sqrt{s}}\Big) \frac{ds}{s} = \int \exp\big(- \kappa L(0,1)\big)\, d\mathbb{W}\,. \tag{5}$$

The ratio ergodic theorem, see Chapter X in Revuz and Yor (1994), states that for every Borel set A

$$\lim_{s\to\infty} \frac{\mu[s](A)}{L(0,s)} = \ell(A) \;\; , \mathbb{W}\text{-almost surely.} \tag{6}$$

Hence, $\mathbb{W}$-almost surely, for every finite family $A_1, \dots, A_n$ of rational intervals in $\mathbb{R}$ and every tuple $(\kappa_1, \dots, \kappa_n)$ of positive reals, we infer from (5)

$$\lim_{t\to\infty} \frac{1}{\log t} \int_1^t \exp\Big(- \sum_{i=1}^n \kappa_i \frac{\mu[s](A_i)}{\sqrt{s}}\Big) \frac{ds}{s} = \int \exp\Big(- \sum_{i=1}^n \kappa_i \ell(A_i) \cdot L(0,1)\Big)\, d\mathbb{W}\,.$$

Thus, by Lemma 2.3, we infer that, $\mathbb{W}$-almost surely,

$$w - \lim_{t\to\infty} \frac{1}{\log t} \int_1^t \delta_{\left\{\frac{\mu[s]}{\sqrt{s}}\right\}} \frac{ds}{s} = \int \delta_{\{L(0,1)\cdot\ell\}}\, d\mathbb{W}\,.$$

Finally, the distribution of $L(0,1)$ is known from the classical Kallianpur-Robbins law, as

$$\int \delta_{\{L(0,1)\}}\, d\mathbb{W} = \lim_{r\to 0} \int \delta_{\left\{\frac{\mu[1](-r,r)}{2r}\right\}}\, d\mathbb{W} = \lim_{t\to\infty} \int \delta_{\left\{\frac{\mu[t](-1,1)}{2\sqrt{t}}\right\}}\, d\mathbb{W} = \sqrt{\frac{2}{\pi}} \int_0^\infty \delta_{\{x\}} e^{-x^2/2}\, dx\,.$$

The analogous argument gives convergence of the moments and thus Theorem 1.1 follows.

4 The random time Kallianpur-Robbins law for planar Brownian motion

In their elegant proof of the Kallianpur-Robbins law Darling and Kac (1957) calculate the asymptotics at 0 of the Laplace transforms of the moments of $\mu[t](A)$ in order to study the behaviour of the distribution of $\mu[t](A)$ as $t \to \infty$. In a somewhat similar spirit we look at occupation measures with respect to the whole time axis, which are discounted by means of exponential functions. It is possible to determine the moments of these measures explicitly and this calculation is carried out in Lemma 4.1. This is used to prove the almost sure convergence of the moments of the weighted occupation measures with respect to a random choice of a weighting parameter λ. This result is then applied via approximation to the occupation measures up to time t.

Define for every $\lambda > 0$ the measure $\tilde{\mu}[\lambda]$ by

$$\tilde{\mu}[\lambda](A) = \int_0^\infty e^{-\lambda t} \mathbf{1}_{\{B_t \in A\}}\, dt \;\; \text{for every } A \subseteq \mathbb{R}^2 \text{ Borel.}$$

Note that throughout this paper the symbols $\mathbb{E}$ and Var are reserved for expectation and variance with respect to the Wiener measure.

Lemma 4.1 *For every bounded Borel set $A \subseteq \mathbb{R}^2$ we have*

$$\mathbb{E}\Big\{ \prod_{n=1}^k \frac{\tilde\mu[\lambda_n](A)}{\log\sqrt{1/\lambda_n}} \Big\} = \Big(\frac{\ell^2(A)}{\pi}\Big)^k \sum_{\gamma\in\Gamma} \prod_{n=1}^k \frac{\log(1/\sum_{j=n}^k \gamma_j)}{\log(1/\gamma_n)} + O\Big(\frac{1}{\log(1/\lambda_k)}\Big),$$

as $0 < \lambda_1 < \ldots < \lambda_k \to 0$, where Γ denotes the set of all permutations of $(\lambda_1, \ldots, \lambda_k)$.

Proof. For every tuple $(x_1, \ldots, x_k)$ we denote $x_1' = x_1$ and $x_k' = x_k - x_{k-1}$ for $k > 1$. With this notation we can calculate

$$\begin{aligned}
&\mathbb{E}\Big\{ \prod_{n=1}^k \tilde\mu[\lambda_n](A) \Big\} \\
&= \int_0^\infty \cdots \int_0^\infty e^{-\sum_{n=1}^k \lambda_n t_n} P(\{B_{t_1} \in A, \ldots, B_{t_k} \in A\})\, dt_k \ldots dt_1 \\
&= \sum_{\gamma\in\Gamma} \int_0^\infty \int_{t_1}^\infty \cdots \int_{t_{k-1}}^\infty \Big[\int_A \cdots \int_A \prod_{n=1}^k \frac{1}{2\pi t_n'} e^{-\gamma_n t_n} e^{\frac{-\|x_n'\|^2}{2t_n'}}\, dx_k \cdots dx_1 \Big] dt_k \ldots dt_1 \\
&= \frac{1}{(2\pi)^k} \sum_{\gamma\in\Gamma} \int_A \cdots \int_A \int_0^\infty \frac{1}{t_1'} e^{-\gamma_1 t_1} e^{\frac{-\|x_1'\|^2}{2t_1'}} \int_{t_1}^\infty \cdots \int_{t_{k-1}}^\infty \frac{1}{t_k'} e^{-\gamma_k t_k} e^{\frac{-\|x_k'\|^2}{2t_k'}}\, dt_k \ldots dt_1\, dx_k \ldots dx_1.
\end{aligned}$$

For the innermost integral we find

$$\int_{t_{k-1}}^\infty \frac{1}{t_k'} e^{-\gamma_k t_k} e^{\frac{-\|x_k'\|^2}{2t_k'}}\, dt_k = e^{-\gamma_k t_{k-1}} \int_0^\infty e^{-\gamma_k t} e^{\frac{-\|x_k'\|^2}{2t}} \frac{dt}{t} = e^{-\gamma_k t_{k-1}} \cdot 2K_0(\sqrt{2\gamma_k}\|x_k'\|),$$

where K_0 is the Bessel function of imaginary argument, which satisfies, see e.g. Watson (1966), p.183,

$$K_0(z) = \frac{1}{2} \int_0^\infty e^{-t-\frac{z^2}{4t}} \frac{dt}{t}, \text{ for } z \in \mathbb{R}.$$

Iterating this argument we get the following exact expression

$$\mathbb{E}\Big\{ \prod_{n=1}^k \tilde\mu[\lambda_n](A) \Big\} = \frac{1}{\pi^k} \sum_{\gamma\in\Gamma} \int_A \cdots \int_A \prod_{n=1}^k K_0\Big(\sqrt{2\textstyle\sum_{j=n}^k \gamma_j}\|x_n'\|\Big)\, dx_k \ldots dx_1. \tag{7}$$

We now use the asymptotic formula, see e.g. Watson (1966), p.80,

$$K_0(s) = \log(1/s) + O(1),\ s \to 0,$$

to conclude

$$\begin{aligned}
&\mathbb{E}\Big\{ \prod_{n=1}^k \frac{\tilde\mu[\lambda_n](A)}{\log\sqrt{1/\lambda_n}} \Big\} \\
&= \frac{1}{\pi^k} \sum_{\gamma\in\Gamma} \int_A \cdots \int_A \prod_{n=1}^k \frac{\log(1/\sum_{j=n}^k \gamma_j)}{\log(1/\lambda_n)}\, dx_k \ldots dx_1 + O\Big(\frac{1}{\log(1/\lambda_k)}\Big) \\
&= \Big(\frac{\ell^2(A)}{\pi}\Big)^k \sum_{\gamma\in\Gamma} \prod_{n=1}^k \frac{\log(1/\sum_{j=n}^k \gamma_j)}{\log(1/\gamma_n)} + O\Big(\frac{1}{\log(1/\lambda_k)}\Big).
\end{aligned}$$

■

Lemma 4.2 *Let $A \subseteq \mathbb{R}^2$ be a bounded Borel set, k,l nonnegative integers and $a_1,\dots,a_N \in \mathbb{R}$. Then, for $0<\kappa\le\lambda\to 0$,*

$$\mathbb{E}\Big\{\Big(\sum_{n=1}^N \frac{a_n\tilde\mu[n\kappa](A)}{\log\sqrt{1/n\kappa}}\Big)^k\Big(\sum_{n=1}^N \frac{a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/n\lambda}}\Big)^l\Big\}$$
$$= \Big(\frac{\ell^2(A)}{\pi}\sum_{n=1}^N a_n\Big)^{k+l}\cdot k!l!\cdot\sum_{i=l}^{k+l}\binom{i-1}{l-1}\Big(\frac{\log(1/\lambda)}{\log(1/\kappa)}\Big)^{i-l}+O\Big(\frac{1}{\log(1/\lambda)}\Big).$$

Proof. From Lemma 4.1 we get that

$$\mathbb{E}\Big\{\Big(\sum_{n=1}^N \frac{a_n\tilde\mu[n\kappa](A)}{\log\sqrt{1/n\kappa}}\Big)^k\Big(\sum_{n=1}^N \frac{a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/n\lambda}}\Big)^l\Big\}$$
$$= \Big(\frac{\ell^2(A)}{\pi}\cdot\sum_{n=1}^N a_n\Big)^{k+l}\cdot k!l!\cdot\sum_{\gamma\in\Gamma}\frac{\prod_{n=1}^{k+l}\log(1/\sum_{j=n}^{k+l}\gamma_j)}{\log(1/\kappa)^k\log(1/\lambda)^l}+O\Big(\frac{1}{\log(1/\lambda)}\Big),$$

as $0<\kappa<\lambda\to 0$, where Γ denotes the set of all *distinct* permutations of $(\kappa,\overset{k}{\dots},\kappa,\lambda,\overset{l}{\dots},\lambda)$. Observe now that, as $\kappa\le\lambda\to 0$,

$$\log(1/\textstyle\sum_{j=n}^{k+l}\gamma_j)=\begin{cases}\log(1/\kappa)+O(1) & \text{if } \gamma_j=\kappa \text{ for all } j\in\{n,\dots,k+l\},\\ \log(1/\lambda)+O(1) & \text{if } \gamma_j=\lambda \text{ for some } j\in\{n,\dots,k+l\},\end{cases}$$

and hence

$$\sum_{\gamma\in\Gamma}\prod_{n=1}^{k+l}\log(1/\textstyle\sum_{j=n}^{k+l}\gamma_j)$$
$$=\sum_{i=l}^{k+l}\binom{i-1}{l-1}\Big(\log(1/\kappa)\Big)^{k+l-i}\Big(\log(1/\lambda)\Big)^i+O\Big((\log(1/\kappa))^k(\log(1/\lambda))^{l-1}\Big).$$

Therefore, for $0\le\kappa\le\lambda\to 0$,

$$\mathbb{E}\Big\{\Big(\sum_{n=1}^N \frac{a_n\tilde\mu[n\kappa](A)}{\log\sqrt{1/n\kappa}}\Big)^k\Big(\sum_{n=1}^N \frac{a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/n\lambda}}\Big)^l\Big\}$$
$$=\Big(\frac{\ell^2(A)}{\pi}\sum_{n=1}^N a_n\Big)^{k+l}\cdot k!l!\cdot\sum_{i=l}^{k+l}\binom{i-1}{l-1}\frac{\log(1/\lambda)^i\log(1/\kappa)^{k+l-i}}{\log(1/\lambda)^l\log(1/\kappa)^k}+O\Big(\frac{1}{\log(1/\lambda)}\Big).$$

■

We now use Lemma 4.2 to prove the almost sure convergence of the moments of $\tilde\mu[\lambda](A)/\log\sqrt{1/\lambda}$ with respect to a random choice of λ.

Lemma 4.3 *Let $A \subseteq \mathbb{R}^2$ be a bounded Borel set. Then, $\mathbb{W}^2$-almost surely, for all $a_1, \ldots, a_N \in \mathbb{R}$, we have*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\sum_{n=1}^N \frac{a_n\, \tilde{\mu}[n\lambda](A)}{\log \sqrt{1/n\lambda}} \right]^k \frac{d\lambda}{\lambda\, \log(1/\lambda)} = k! \left[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \right]^k . \tag{8}$$

Proof. We first fix $a_1, \ldots, a_N$ and look at the expectations. We use the previous lemma to see that

$$\mathbb{E}\Big\{ \sum_{n=1}^N \frac{a_n\, \tilde{\mu}[n\lambda](A)}{\log\sqrt{1/\lambda}} \Big\}^k = k! \left[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \right]^k + O\Big(\frac{1}{\log(1/\lambda)} \Big) .$$

Because

$$\frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \frac{1}{\log(1/\lambda)} \frac{d\lambda}{\lambda\, \log(1/\lambda)} \le \frac{1}{\log\log(1/\varepsilon)} , \tag{9}$$

we conclude that, for some $C > 0$,

$$\left| \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \mathbb{E}\Big\{ \sum_{n=1}^N \frac{a_n \tilde{\mu}[n\lambda](A)}{\log\sqrt{1/n\lambda}} \Big\}^k \frac{d\lambda}{\lambda\, \log(1/\lambda)} - k! \left[\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \right]^k \right| \le \frac{C}{\log\log(1/\varepsilon)} .$$

Using Lemma 4.2, we get as $0 < \kappa \le \lambda \to 0$, for some constant $C > 0$,

$$\begin{aligned}
&\mathrm{Var}\Big\{ \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\sum_{n=1}^N \frac{a_n \tilde{\mu}[n\lambda](A)}{\log\sqrt{1/n\lambda}} \right]^k \frac{d\lambda}{\lambda\, \log(1/\lambda)} \Big\} \\
&\le \; 2\Big(\frac{\ell^2(A)}{\pi} \cdot \sum_{n=1}^N a_n \Big)^{2k} \Big(\frac{1}{\log\log(1/\varepsilon)} \Big)^2 \int_\varepsilon^{1/e} \int_\kappa^{1/e} (k!)^2 \Bigg(\sum_{i=k+1}^{2k} \binom{i-1}{k-1} \Big(\frac{\log(1/\lambda)}{\log(1/\kappa)} \Big)^{i-k} \\
&\qquad + \frac{C}{\log(1/\lambda)} \Bigg) \frac{d\lambda}{\lambda\, \log(1/\lambda)} \frac{d\kappa}{\kappa\, \log(1/\kappa)} .
\end{aligned}$$

Recalling from (9) that

$$\Big(\frac{1}{\log\log(1/\varepsilon)} \Big)^2 \int_\varepsilon^{1/e} \int_\kappa^{1/e} \frac{1}{\log(1/\lambda)} \frac{d\lambda}{\lambda\, \log(1/\lambda)} \frac{d\kappa}{\kappa\, \log(1/\kappa)} \le \frac{1}{\log\log(1/\varepsilon)}$$

and observing that

$$\begin{aligned}
\Big(\frac{1}{\log\log(1/\varepsilon)} \Big)^2 \int_\varepsilon^{1/e} \int_\kappa^{1/e} \frac{\log(1/\lambda)}{\log(1/\kappa)} \frac{d\lambda}{\lambda\, \log(1/\lambda)} \frac{d\kappa}{\kappa\, \log(1/\kappa)} \;&\le\; \Big(\frac{1}{\log\log(1/\varepsilon)} \Big)^2 \int_\varepsilon^{1/e} \frac{d\kappa}{\kappa\, \log(1/\kappa)} \\
&\le\; \frac{1}{\log\log(1/\varepsilon)} ,
\end{aligned}$$

we conclude that, for a suitable constant $C > 0$,

$$\mathrm{Var}\Big\{ \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\sum_{n=1}^N \frac{a_n \tilde{\mu}[n\lambda](A)}{\log\sqrt{1/n\lambda}} \right]^k \frac{d\lambda}{\lambda\, \log(1/\lambda)} \Big\} \le \frac{C}{\log\log(1/\varepsilon)} .$$

By Chebyshev's inequality, for every $\delta > 0$ and sufficiently small ε,

$$\mathbb{W}\Big\{\Big|\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\Big[\sum_{n=1}^N \frac{a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/n\lambda}}\Big]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)} - k!\,\Big[\frac{\ell^2(A)}{\pi}\cdot\sum_{n=1}^N a_n\Big]^k\Big| > 2\delta\Big\}$$
$$\le \frac{C}{\delta^2\cdot\log\log(1/\varepsilon)}\,.$$

Let $\varepsilon_j = \exp(-\exp(j^2))$. Then, by the Borel-Cantelli lemma, $\mathbb{W}^2$-almost surely,

$$\lim_{j\to\infty}\frac{1}{\log\log(1/\varepsilon_j)}\int_{\varepsilon_j}^{1/e}\Big[\frac{\sum_{n=1}^N a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}}\Big]^k\frac{d\lambda}{\lambda\,\log(1/\lambda)} = k!\,\Big[\frac{\ell^2(A)}{\pi}\cdot\sum_{n=1}^N a_n\Big]^k.$$

In particular, we get

$$\begin{aligned}
&\limsup_{j\to\infty}\frac{1}{\log\log(1/\varepsilon_j)}\int_{\varepsilon_{j+1}}^{\varepsilon_j}\Big|\frac{\sum_{n=1}^N a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}}\Big|^k\frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
\le\;& \lim_{j\to\infty}\frac{\log\log(1/\varepsilon_{j+1})}{\log\log(1/\varepsilon_j)}\cdot\lim_{j\to\infty}\frac{1}{\log\log(1/\varepsilon_{j+1})}\int_{\varepsilon_{j+1}}^{1/e}\Big[\frac{\sum_{n=1}^N |a_n|\tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}}\Big]^k\frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
&-\lim_{j\to\infty}\frac{1}{\log\log(1/\varepsilon_j)}\int_{\varepsilon_j}^{1/e}\Big[\frac{\sum_{n=1}^N |a_n|\tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}}\Big]^k\frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
=\;& 0\,.
\end{aligned}$$

Hence, for $\varepsilon_{j+1}\le\varepsilon\le\varepsilon_j$,

$$\begin{aligned}
&\limsup_{\varepsilon\downarrow 0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\Big[\frac{\sum_{n=1}^N a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}}\Big]^k\frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
\le\;& \lim_{j\to\infty}\frac{1}{\log\log(1/\varepsilon_j)}\int_{\varepsilon_j}^{1/e}\Big[\frac{\sum_{n=1}^N a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}}\Big]^k\frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
&+\limsup_{j\to\infty}\frac{1}{\log\log(1/\varepsilon_j)}\int_{\varepsilon_{j+1}}^{\varepsilon_j}\Big|\frac{\sum_{n=1}^N a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}}\Big|^k\frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
=\;& k!\,\Big[\frac{\ell^2(A)}{\pi}\cdot\sum_{n=1}^N a_n\Big]^k.
\end{aligned}$$

Similarly, we get

$$\liminf_{\varepsilon\downarrow 0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\Big[\frac{\sum_{n=1}^N a_n\tilde\mu[n\lambda](A)}{\log\sqrt{1/\lambda}}\Big]^k\frac{d\lambda}{\lambda\,\log(1/\lambda)}\ge k!\,\Big[\frac{\ell^2(A)}{\pi}\cdot\sum_{n=1}^N a_n\Big]^k,$$

and hence equality (8). Finally, we observe that (8) holds $\mathbb{W}^2$-almost surely simultaneously for all tuples with rational entries and thus, by approximation, $\mathbb{W}^2$-almost surely, for all $a_1,\ldots,a_N\in\mathbb{R}$. ∎

We now go back to the study of the measures $\mu[t]$, using an approximation argument. The case $k = 1$ of the following lemma is Brosamler's Theorem, which was proved in Brosamler (1973) by a different method.

Lemma 4.4 *Let $A \subseteq \mathbb{R}^2$ be a bounded Borel set. Then, $\mathbb{W}^2$-almost surely,*

$$\lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t \left[\frac{\mu[s](A)}{\log\sqrt{s}}\right]^k \frac{ds}{s\,\log s} = k!\left[\frac{\ell^2(A)}{\pi}\right]^k. \tag{10}$$

Proof. We look at the continuous function $g:[0,1]\to[0,1]$ with $g(x)=1$ if $x\geq 1/e$ and $g(x)=0$ if $x<1/e^2$, which is linear on the interval $(1/e^2,1/e)$. We can now find, for every $\eta>0$, polynomials $p=\sum_{n=1}^N p_n x^n$ and $P=\sum_{n=1}^N P_n x^n$ such that, for every $0\leq x\leq 1$,

$$p(x)\leq g(x)\leq P(x)\leq p(x)+\eta\,.$$

Observe that $g(e^{-\lambda t})=1$ if $t\leq 1/\lambda$ and hence

$$\mu[1/\lambda](A)\leq \int_0^\infty g(e^{-\lambda t})\mathbf{1}_{\{B_t\in A\}}\,dt \leq \int_0^\infty P(e^{-\lambda t})\mathbf{1}_{\{B_t\in A\}}\,dt = \sum_{n=1}^N P_n\tilde{\mu}[n\lambda](A).$$

As $\sum P_n\leq 1+\eta$ we may use Lemma 4.3 to conclude that, for all k,

$$\limsup_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\mu[1/\lambda](A)}{\log\sqrt{1/\lambda}}\right]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)} \leq k!\left[\frac{\ell^2(A)}{\pi}\cdot(1+\eta)\right]^k.$$

Similarly, we have $g(e^{-\lambda t})=0$ if $t\geq 2/\lambda$ and hence

$$\mu[2/\lambda](A)\geq \int_0^\infty g(e^{-\lambda t})\mathbf{1}_{\{B_t\in A\}}\,dt \geq \int_0^\infty p(e^{-\lambda t})\mathbf{1}_{\{B_t\in A\}}\,dt = \sum_{n=1}^N p_n\tilde{\mu}[n\lambda](A)\,.$$

Upon observing that $\sum p_n\geq 1-\eta$, we infer that, for all k,

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\mu[1/\lambda](A)}{\log\sqrt{1/\lambda}}\right]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
=\;& \liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\mu[2/\lambda](A)}{\log\sqrt{1/\lambda}}\right]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)}\\
\geq\;& k!\left[\frac{\ell^2(A)}{\pi}(1-\eta)\right]^k.
\end{aligned}$$

As these inequalities hold for every $\eta>0$ we conclude that, $\mathbb{W}^2$-almost surely,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \left[\frac{\mu[1/\lambda](A)}{\log\sqrt{1/\lambda}}\right]^k \frac{d\lambda}{\lambda\,\log(1/\lambda)} = k!\left[\frac{\ell^2(A)}{\pi}\right]^k.$$

We finally conclude

$$\lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t \left[\frac{\mu[s](A)}{\log\sqrt{s}}\right]^k \frac{ds}{s\,\log s} = k!\left[\frac{\ell^2(A)}{\pi}\right]^k, \tag{11}$$

by means of a change of variable. ∎

From the convergence of the moments we can also pass to the distributions of $\mu[s](A)/\log\sqrt{s}$ by standard methods.

Lemma 4.5 *Let $A \subseteq \mathbb{R}^2$ be a bounded Borel set. Then, $\mathbb{W}^2$-almost surely,*

$$\lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t \delta_{\left\{\frac{\mu[s](A)}{\log\sqrt{s}}\right\}} \frac{ds}{s\,\log s} = \int_0^\infty \delta_{\left\{\frac{a}{\pi}\cdot\ell^2(A)\right\}}\, e^{-a}\, da\,. \tag{12}$$

Proof. We apply the method of moments. The limits we get in (11) are the moments of an exponential distribution with parameter $\ell^2(A)/\pi$. Moreover, the exponential distribution is the only distribution with these moments, as can be checked easily by means of the criterion given in Breiman (1968), Proposition 8.49. Therefore we get, using for example Theorem 8.48 of Breiman, $\mathbb{W}^2$-almost surely,

$$\lim_{t\to\infty} \frac{1}{\log\log(t)} \int_e^t \delta_{\left\{\frac{\mu[s](A)}{\log\sqrt{s}}\right\}} \frac{ds}{s\,\log s} = \int_0^\infty \delta_{\left\{\frac{a}{\pi}\cdot\ell^2(A)\right\}}\, e^{-a}\, da\,, \tag{13}$$

which finishes the proof. ∎

Proof of Theorem 1.2. Recall the ratio ergodic theorem in the case of planar Brownian motion, see e.g. Chapter X of Revuz and Yor (1994) for a proof. For all bounded Borel sets A and B in the plane with $\ell^2(B) > 0$ we have, $\mathbb{W}^2$-almost surely,

$$\lim_{t\to\infty} \frac{\mu[t](A)}{\mu[t](B)} = \frac{\ell^2(A)}{\ell^2(B)}\,.$$

From Lemma 4.5 and the ratio ergodic theorem we get that, $\mathbb{W}^2$-almost surely, for every family $\{A_1, \dots, A_m\} \subseteq \mathcal{A}$ of open or closed cubes with rational vertices,

$$\lim_{t\to\infty} \frac{1}{\log\log t} \int_e^t \delta_{\left\{\frac{\mu[s](A_1)}{\log\sqrt{s}},\dots,\frac{\mu[s](A_m)}{\log\sqrt{s}}\right\}} \frac{ds}{s\,\log s} = \int_0^\infty \delta_{\left\{\frac{a}{\pi}\cdot\ell^2(A_1),\dots,\frac{a}{\pi}\cdot\ell^2(A_m)\right\}}\, e^{-a}\, da\,.$$

The statement of the random time Kallianpur-Robbins law follows now from Lemma 2.3. ∎

5 The random scale Kallianpur-Robbins law for planar Brownian motion

The random scale Kallianpur-Robbins law for planar Brownian motion will follow from the same sources as the ratio ergodic theorem for small scales and we first concentrate on the proof of the latter theorem. The main tool for the proof is the approximation of the occupation measure of small sets by means of crossing numbers. To this end we introduce a random decomposition of the time axis as in Ray (1963). Let τ be the stopping time given by

$$\tau = \inf\{t > 0 \,:\, |B_t| \geq 1\}\,.$$

Denote by μ the occupation measure $\mu = \mu[\tau]$ of the stopped Brownian motion. For the moment we fix a number $b > 0$ and define $a_n = e^{-bn}$. By N_n we denote the number of inward crossings of the annulus with radii a_{n-1} and a_n the Brownian motion performs before it hits the sphere of radius 1. These *crossing numbers* play a crucial rôle in the proof. We quote the following essential fact from Mörters (1998), Section 4.

Lemma 5.1 *The distribution of* $\{N_n\}_{n\geq 1}$ *satisfies*

$$\mathbb{W}^2\{N_n = k\} = \frac{1}{n}\Big(1-\frac{1}{n}\Big)^k \text{ for } k \geq 0,$$

and, $\mathbb{W}^2$*-almost surely,*

$$w - \lim_{n\to\infty} \frac{1}{\log n}\sum_{i=1}^n \frac{1}{i}\delta_{\{N_i/i\}} = \int \delta_{\{a\}}\, e^{-a}\, da\,,$$

and, for all $k \geq 1$,

$$w - \lim_{n\to\infty} \frac{1}{\log n}\sum_{i=1}^n \frac{1}{i}\Big(\frac{N_i}{i}\Big)^k = k!\,.$$

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and write $B_t = |B_t|\cdot\exp(i\vartheta_s)$ with $\vartheta_s \in [0,2\pi)$. For every n we define stopping times

$$\begin{aligned} t_0 &= 0\,,\\ t_{2k+1} &= \inf\{t > t_{2k} \,:\, |B_t| > a_{n-1}\}\,,\\ t_{2k} &= \inf\{t > t_{2k-1} \,:\, |B_t| < a_n\}\,. \end{aligned}$$

Note that $N_n = \max\{k \geq 0 : t_{2k} < \tau\}$. We define random variables M_k^n with values in $[0,2\pi)\times[0,2\pi)$ by

$$M_k^n := (\Theta_{2k}, \Theta_{2k+1}) := (\vartheta_{t_{2k}}, \vartheta_{t_{2k+1}})\,.$$

We have omitted the dependence of the stopping times t_k and angles Θ_k on n for notational convenience. The processes $\{M_k^n\}_{k\geq 1}$ are Markov chains and the necessary information about the distribution of these chains is given in the following lemma.

Lemma 5.2 *For every* n *the sequence* $\{M_k^n\}_{k\geq 1}$ *of random variables is a stationary, ergodic Markov chain. If* h_θ *is the distribution of the angle of the point in which a Brownian motion started in* $a_n e^{i\theta}$ *hits the sphere of radius* a_{n-1} *for the first time, then* h_θ *is independent of* n *and the invariant distribution of* $\{M_k^n\}_{k\geq 1}$ *is given by*

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi} \delta_{\{(\theta_0,\theta_1)\}}\, dh_{\theta_0}(\theta_1)\, d\theta_0\,.$$

If g_θ *is the distribution of the angle of the point in which a Brownian motion started in* $a_{n-1}e^{i\theta}$ *hits the sphere of radius* a_n *for the first time, then* g_θ *is independent of* n *and the transition probabilities of* $(M_k^n)_{k\geq 1}$ *from* (θ_0,θ_1) *are given by*

$$\int_0^{2\pi}\int_0^{2\pi} \delta_{\{(\theta_2,\theta_3)\}}\, dh_{\theta_2}(\theta_3)\, dg_{\theta_1}(\theta_2)\,.$$

Proof. This is immediate from the strong Markov property and the scaling invariance of planar Brownian motion. ∎

Remark. The measures h_{θ_0} and g_{θ_1} are the *harmonic measures on* $\partial B(0,1)$. Denoting the *Poisson kernel* for the ball $B(0,1)$ by

$$P(x,y) = \frac{|1-|x|^2|}{|x-y|^2} \text{ for } x \notin \partial B(0,1) \text{ and } y \in \partial B(0,1),$$

they are given by

$$dh_{\theta_0}(\theta_1) = (1/2\pi) P(e^{-b}e^{i\theta_0}, e^{i\theta_1})\, d\theta_1 \text{ and } dg_{\theta_1}(\theta_2) = (1/2\pi) P(e^{b}e^{i\theta_1}, e^{i\theta_2})\, d\theta_2 .$$

See, for example, Chapter II in Bass (1995).

We require some basic facts about Brownian motions conditioned to exit the ball $B(0,e^b)$ at a certain angle. These facts are provided by well known statements from probabilistic potential theory, see Bass (1995) for an excellent introduction.

Lemma 5.3 *For every* $(\theta_1,\theta_2) \in [0,2\pi) \times [0,2\pi)$ *there is a process* $\{X_t(\theta_1,\theta_2)\}_{t\geq 0}$ *on a probability space* $(\Omega, \mathcal{A}, P)$ *such that*

(i) if $(B_t)_{t\geq 0}$ *is a Brownian motion started in* $e^{i\theta_1}$, ρ *is the first exit time of* (B_t) *from* $B(0,e^b)$ *and* $\Theta \in [0,2\pi)$ *is the exit angle defined by* $B_\rho = e^b e^{i\Theta}$, *then the distribution of the process* $\{X_t(\theta_1,\theta_2)\}_{t\geq 0}$ *is a conditional distribution of* $(B_t)_{t\geq 0}$ *given* $\Theta = \theta_2$,

(ii) if ρ *is the first exit time of* $\{X_t(\theta_1,\theta_2)\}$ *from* $B(0,e^b)$, *then, for every Borel set* $A \subseteq B(0,1)$ *the function*

$$\begin{array}{rccc} f_A(\,\cdot\,;b): & [0,2\pi)\times[0,2\pi) & \longrightarrow & [0,\infty) \\ & (\theta_1,\theta_2) & \mapsto & \mathbb{E}_P\Big\{ \int_0^\rho \mathbf{1}_A(X_t(\theta_1,\theta_2))\, dt \Big\} \end{array}$$

is continuous, where $\mathbb{E}_P$ *denotes the expectation with respect to* P,

(iii) for every Borel set A *in a compact subset of the open ball* $B(0,1)$ *we have*

$$\frac{1}{2\pi b} \int_0^{2\pi} \int_0^{2\pi} f_A(\theta_1,\theta_2;b)\, dh_{\theta_1}(\theta_2)\, d\theta_1 = \frac{\ell^2(A)}{\pi},$$

(iv) for every $m \geq 1$ *and Borel set* $A \subseteq B(0,1)$ *the moments* $\mathbb{E}_P\big\{ \big(\int_0^\rho \mathbf{1}_A(X_t(\theta_1,\theta_2))\, dt \big)^m \big\}$ *are bounded in* (θ_1,θ_2).

Proof. Let $(\theta_1,\theta_2) \in [0,2\pi)\times[0,2\pi)$. We only have to construct the process $\{X_t\} = \{X_t(\theta_1,\theta_2)\}$ up to the first exit time ρ from $B(0,e^b)$. Define the harmonic function

$$h(x) = \frac{e^{2b} - |x|^2}{|e^b e^{i\theta_2} - x|^2} \text{ for } x \in B(0,e^b).$$

Let $\{X_t\}_{t\geq 0}$ on $(\Omega, \mathcal{A}, \tilde{P})$ be a Brownian motion started in $X_0 = e^{i\theta_1}$ and killed upon exiting the domain $B(0,e^b)$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration for this Brownian motion with $\mathcal{A} = \bigcup_{t\geq 0} \mathcal{F}_t$ and recall that $\{h(X_{t\wedge\rho})\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}$-martingale. The *h-path transform* of $\{X_t\}$ is defined by

$$P_h(M) = \int_M \frac{h(X_{t\wedge\rho})}{h(X_0)}\, d\tilde{P} \text{ for } M \in \mathcal{F}_t.$$

For properties of the h-path transform we refer the reader to Bass (1995). The required process is $\{X_t\}_{0\leq t\leq\rho}$ on the probability space $(\Omega,\mathcal{A},P_h)$. Property (i) is Proposition (2.7) in Bass (1995). To check property (ii) we look at a sequence $b_n\uparrow b$ and denote the first exit time from $B(0,e^{b_n})$ by ρ_n. Then

$$\begin{aligned} f_A(\theta_1,\theta_2,;b) &= \lim_{n\to\infty}\int_0^\infty\int \mathbf{1}_{\{s<\rho_n\}}\mathbf{1}_A(X_s)\,dP_h\,ds \\ &= \lim_{n\to\infty}\int_0^\infty\int \mathbf{1}_{\{s<\rho_n\}}\mathbf{1}_A(X_s)\frac{h(X_s)}{h(e^{i\theta_1})}\,d\tilde{P}\,ds \\ &= \lim_{n\to\infty}\int\int_0^{\rho_n}\mathbf{1}_A(X_s)\frac{h(X_s)}{h(e^{i\theta_1})}\,ds\,d\tilde{P} \\ &= \lim_{n\to\infty}\int G_{b_n}(e^{i\theta_1},y)\mathbf{1}_A(y)\frac{h(y)}{h(e^{i\theta_1})}\,dy \\ &= \int_A G_b(e^{i\theta_1},y)\frac{h(y)}{h(e^{i\theta_1})}\,dy\,, \end{aligned}$$

where G_b denotes Green's function for the domain $B(0,e^b)$. As

$$G_b(e^{i\theta},y)=\frac{1}{\pi}\log\Big(\frac{|e^be^{i\theta}-e^{-b}y|}{|e^{i\theta}-y|}\Big)\,,$$

we can see that the integrand of the last expression is uniformly equicontinuous in (θ_1,θ_2) for all $y\in A$ and this implies property (ii).

To check property (iii) note that

$$\int_0^{2\pi} f_A(\theta_1,\theta_2;b)\,dh_{\theta_1}(\theta_2)=\int_A G_b(e^{i\theta_1},y)\,dy$$

and $\int_0^{2\pi}G_b(e^{i\theta_1},y)\,d\theta_1$ is a harmonic function inside $B(0,1)$, which is constant on the boundary. Hence this function is constant and equal to its value at 0, which is $2b$.

Finally, the boundedness of the moments of ρ follows from the inequality

$$P_h\{\rho>t\}\leq c_1\exp(-c_2t)\,,$$

for positive constants c_1,c_2 not depending on (θ_1,θ_2), see Bañuelos and Davis (1989). ■

Our aim is to approximate the occupation measure in terms of the Markov chain $\{N_n\}_{n\geq1}$. This is done in the following lemma, which constitutes the main ingredient of the proofs of Theorems 1.4 and 1.3. It was proved by Ray (1963) for the special case $A=B(0,1)$.

Lemma 5.4 *Suppose $A\subseteq B(0,1)$ is a Borel set. Then, $\mathbb{W}^2$-almost surely,*

$$\mu\Big(a_nA\Big)=a_n^2\cdot N_nb\cdot\left[\frac{\ell^2(A)}{\pi}+o(1)\right]+o(na_n^2)\ ,\ as\ n\to\infty. \tag{14}$$

Proof. We fix n and let

$$T_k^n = \int_{t_{2k}}^{t_{2k+1}} \mathbf{1}_{\{B_s \in a_n A\}}\, ds\,.$$

Observe that

$$\mu(a_n A) = \sum_{k=0}^{N_n} T_k^n\,. \tag{15}$$

Conditional on the sequence $M^n = \{M_k^n\}_{k\geq 1}$ the T_k^n are independent and T_k^n has the same distribution as $a_n^2 T(\Theta_{2k}, \Theta_{2k+1}; b)$, where

$$T(\theta_1, \theta_2; b) = \int_0^\rho \mathbf{1}_{\{X_t(\theta_1,\theta_2)\in A\}}\, dt\,,$$

and $\{X_t(\theta_1, \theta_2)\}_{t\geq 0}$ is as in Lemma 5.3. Hence,

$$\begin{aligned}
&\mathbb{E}\Big\{ \Big(\sum_{k=1}^{N} (T_k^n - f_A(\Theta_{2k};\Theta_{2k+1};b)\cdot a_n^2) \Big)^4 \,\Big|\, M^n \Big\} \\
&= \; a_n^8 \sum_{k=1}^{N} \mathbb{E}_P \Big\{ \Big(T(\Theta_{2k}, \Theta_{2k+1}, b) - f_A(\Theta_{2k}, \Theta_{2k+1}; b) \Big)^4 \Big\} \\
&\quad + 6a_n^8 \sum_{k=1}^{N} \sum_{j=k+1}^{N} \mathbb{E}_P \Big\{ \Big(T(\Theta_{2k}, \Theta_{2k+1}; b) - f_A(\Theta_{2k}, \Theta_{2k+1}; b) \Big)^2 \Big\} \\
&\quad \times \mathbb{E}_P \Big\{ \Big(T(\Theta_{2j}, \Theta_{2j+1}; b) - f_A(\Theta_{2j}, \Theta_{2j+1}; b) \Big)^2 \Big\}\,.
\end{aligned}$$

By Lemma 5.3(iv), for every positive integer m, $\mathbb{E}_P\{T(\theta_1,\theta_2;b)^m\}$ is bounded in (θ_1,θ_2) and hence the expression above is bounded by a constant multiple of $N^2 a_n^8$. We now apply Markov's inequality and get, for every $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{W}^2\Big\{ \Big| \sum_{k=1}^{N} \Big(T_k^n - f_A(\Theta_{2k}, \Theta_{2k+1}, b)\cdot a_n^2 \Big) \Big| > \varepsilon\cdot n a_n^2 \,\Big|\, M^n \Big\} \\
&\leq \; \frac{1}{\varepsilon^4 n^4 a_n^8}\, \mathbb{E}\Big\{ \Big[\sum_{k=1}^{N} \Big(T_k^n - f_A(\Theta_{2k}, \Theta_{2k+1}, b)\cdot a_n^2 \Big) \Big]^4 \Big| M^n \Big\} \leq C\cdot \frac{N^2}{n^4\varepsilon^4}\,.
\end{aligned}$$

Because $\{T_k^n\}_{k\geq 1}$ and N_n are independent conditional on $\{M_k^n\}_{k\geq 1}$ we have

$$\mathbb{W}^2\Big\{ \Big| \sum_{k=1}^{N_n} \Big(T_k^n - f_A(\Theta_{2k}, \Theta_{2k+1}; b)\cdot a_n^2 \Big) \Big| > \varepsilon\cdot n a_n^2 \,\Big|\, M^n \Big\} \leq C\cdot \frac{\mathbb{E}\{N_n^2 | M^n\}}{n^4\varepsilon^4}\,,$$

and, together with (15), this gives

$$\mathbb{W}^2\Big\{ \Big| \mu(a_n A) - T_0^n - \sum_{k=1}^{N_n} f_A(\Theta_{2k}, \Theta_{2k+1}; b)\cdot a_n^2 \Big| > \varepsilon\cdot n a_n^2 \Big\} \leq C\cdot \frac{\mathbb{E}\{N_n^2\}}{n^4\varepsilon^4} \leq \frac{4C}{n^2\varepsilon^4}\,, \tag{16}$$

using in the last step that $\mathbb{E}N_n^2 \le 4$. Also T_0^n is bounded by the first time the Brownian motion hits the sphere of radius a_{n-1}. The second moment of this stopping time is bounded by a constant multiple of a_n^4 and thus we get from Markov's inequality, for every $\varepsilon > 0$,

$$\mathbb{W}^2\{|T_0^n| > \varepsilon \cdot n a_n^2\} < \frac{C}{n^2\varepsilon^2}\,. \tag{17}$$

By the Borel-Cantelli lemma we conclude from (16) and (17) that, $\mathbb{W}^2$-almost surely,

$$\mu(a_n A) = \sum_{k=1}^{N_n} f_A(M_k^n; b) \cdot a_n^2 + o(n a_n^2)\ ,\ \text{as } n \to \infty. \tag{18}$$

To complete the proof we have to show that, for every $\varepsilon > 0$, if $N_n \ge n\varepsilon$ and n is sufficiently large, then

$$\Big| \sum_{k=1}^{N_n} f_A(M_k^n; b) - N_n \cdot \Big[\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} f_A(\theta_1, \theta_2; b)\, dh_{\theta_1}(\theta_2)\, d\theta_1 \Big] \Big| \le \varepsilon \cdot N_n\,. \tag{19}$$

Recall that

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} f_A(\theta_1, \theta_2; b)\, dh_{\theta_1}(\theta_2)\, d\theta_1 = \frac{\ell^2(A) b}{\pi}\,.$$

Due to the dependence of the Markov chain $\{M_k^n\}_{k\ge 0}$ on n the proof of (19) requires a more subtle argument than the ergodic theorem for stationary, ergodic Markov chains. We will make use of a large deviations principle of Stroock for Markov chains satisfying a uniformity condition, which can be found in Dembo and Zeitouni (1992), Chapter 6.3. The transition densities $\pi((\theta_1, \theta_2), (\theta_3, \theta_4))$ of our Markov chain, which are described in Lemma 5.2 and the following remark, satisfy

$$\pi((\theta_1, \theta_2), (\theta_3, \theta_4)) \le \left[\frac{(1+e^b)(1+e^{-b})}{(1-e^b)(1-e^{-b})} \right]^2 \cdot \pi((\theta_1', \theta_2'), (\theta_3, \theta_4)) \ \text{ for all } \theta_i,\ \theta_i' \in [0, 2\pi).$$

Hence the uniformity assumption (U) is fulfilled. Also, by Lemma 5.3(ii), the set $\{\nu : | \int f_A d\nu - \ell^2(A) b/\pi| \ge \varepsilon\}$ is closed. Hence we infer from the large deviation principle for the empirical measure of $\{M_k^n\}_{k\ge 1}$, see Theorem 6.3.8 and the following remark in Dembo and Zeitouni (1992), that, for every $\varepsilon > 0$ there is some $\delta > 0$ depending, of course, on $\varepsilon > 0$, such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \mathbb{W}^2 \Big\{ \Big| \frac{1}{N} \sum_{k=1}^{N} f_A(M_k^n; b) - \ell^2(A) b/\pi \Big| \ge \varepsilon \Big\} < -\delta\,.$$

Note that the probabilities above are independent of the choice of n, as the distributions of the Markov chains $\{M_k^n\}_{k\ge 1}$ do *not* depend on n. We conclude that, for sufficiently large integers n, denoting by $[n\varepsilon]$ the integer part of $n\varepsilon$,

$$\begin{aligned} & \mathbb{W}^2 \Big\{ \Big| \frac{1}{N_n} \sum_{k=1}^{N_n} f_A(M_k^n; b) - \ell^2(A) b/\pi \Big| \ge \varepsilon \text{ and } N_n \ge n\varepsilon \Big\} \\ \le\ & \sum_{N=[n\varepsilon]}^{\infty} \mathbb{W}^2 \Big\{ \Big| \frac{1}{N} \sum_{k=1}^{N} f_A(M_k^n; b) - \ell^2(A) b/\pi \Big| \ge \varepsilon \Big\} \\ \le\ & \sum_{N=[n\varepsilon]}^{\infty} e^{-N\delta} = e^{-\delta[n\varepsilon]} \cdot \frac{1}{1-e^{-\delta}}\,. \end{aligned}$$

Hence, by the Borel-Cantelli lemma, $\mathbb{W}^2$-almost surely, for all but finitely many n,

$$\Big|\sum_{k=1}^{N_n} f_A(\Theta_{2k},\Theta_{2k+1};b) - bN_n\cdot\ell^2(A)b/\pi\Big| \le \begin{cases} (\varepsilon\cdot n)\cdot\sup_{\theta_1,\theta_2} f_A(\theta_1,\theta_2;b) & \text{if } N_n<\varepsilon n,\\ (\varepsilon\cdot N_n) & \text{if } N_n\ge\varepsilon n.\end{cases}$$

As this holds for every $\varepsilon>0$ it implies, together with (18), the required approximation (14). ■

As a consequence of Lemma 5.4 and Lemma 5.3(iii) we get the following approximation.

Lemma 5.5 *Let $\delta>0$ and let A be an ℓ^2-continuity set inside a compact subset of $B(0,1)$. Then we can find an arbitrarily small $b>0$ such that for $a_n=e^{-bn}$ and the corresponding crossing numbers $\{N_n\}_{n\ge1}$ there is, $\mathbb{W}^2$-almost surely, an integer N such that, whenever $a_n\le r\le a_{n-1}$ and $n>N$,*

$$\Big|\frac{\nu[r](A)\cdot\pi}{bn\cdot a_n^2}-\frac{N_n}{n}\cdot\ell^2(A)\Big|<\delta\Big(1+\frac{N_n}{n}\Big).$$

Proof. We denote by A_ϑ the union of all open balls of radius ϑ centred in A. Because A is an ℓ^2-continuity set with $\ell^2(A)<\infty$ we can find a $\vartheta>1$ such that $|\ell^2(A_\vartheta)-\ell^2(A)|<\delta/2$. We can find an arbitrarily small $b>0$ such that $rA\subseteq A_\vartheta$ for all $1\le r\le e^b$. We apply Lemma 5.4 to find, $\mathbb{W}^2$-almost surely, an integer N such that, for all $n>N$,

$$\Big|\frac{\mu(a_nA_\vartheta)\cdot\pi}{bn\cdot a_n^2}-\frac{N_n}{n}\ell^2(A_\vartheta)\Big|<\frac{\delta}{2}\Big(1+\frac{N_n}{n}\Big).$$

Because a Brownian motion in the plane almost surely never returns to the origin and A is bounded we can choose N so large that $\mu(rA_\vartheta)=\nu[r](A_\vartheta)$ for all $r\le a_N$. We conclude that, whenever $a_n\le r\le a_{n-1}$ for some $n>N$, then

$$\begin{aligned}\frac{\nu[r](A)\cdot\pi}{bn\cdot a_n^2} &\le \frac{\mu(a_nA_\vartheta)\cdot\pi}{bn\cdot a_n^2}\le\frac{N_n}{n}\cdot\ell^2(A_\vartheta)+\frac{\delta}{2}\Big(1+\frac{N_n}{n}\Big)\\ &\le \frac{N_n}{n}\cdot\ell^2(A)+\delta\Big(1+\frac{N_n}{n}\Big).\end{aligned}\tag{20}$$

To get the reverse estimate we look at the complement of the union of all open balls of radius ϑ centred in the complement of A. Analogous arguments to above yield an arbitrarily small $b>0$ such that, $\mathbb{W}^2$-almost surely for all sufficiently large integers n, whenever $a_n\le r\le a_{n-1}$,

$$\frac{\nu[r](A)\cdot\pi}{bn\cdot a_n^2}\ge\frac{N_n}{n}\cdot\ell^2(A)-\delta\Big(1+\frac{N_n}{n}\Big).\tag{21}$$

Picking b so small that (20) and (21) both hold finishes the proof. ■

We now have the means to prove a ratio ergodic theorem for small scales in logarithmic probability of order three.

Lemma 5.6 *For all bounded ℓ^2-continuity sets $A,B\subseteq\mathbb{R}^2$ with $\ell^2(B)>0$, $\mathbb{W}^2$-almost surely,*

$$\lim_{\varepsilon\downarrow0}\frac{1}{\log\log(1/\varepsilon)}\int_\varepsilon^{1/e}\delta_{\{r\}}\Big\{0<r<1\,:\,\Big|\frac{\nu[r](A)}{\nu[r](B)}-\frac{\ell^2(A)}{\ell^2(B)}\Big|\le\eta\Big\}\frac{dr}{r\log(1/r)}=1 \text{ for all } \eta>0.$$

Proof. Replacing A and B, if necessary, by sA, sB for a sufficiently small $s > 0$, we may assume that $A, B \subseteq B(0, 1/2)$.

Suppose that $\eta > 0$ and $\kappa > 0$ are given. We can find a small $\delta > 0$ such that, whenever $|x - \ell^2(A)| < \delta$ and $|y - \ell^2(B)| < \delta$, then $|x/y - \ell^2(A)/\ell^2(B)| < \eta$. By Lemma 5.1 there is a $\zeta > 0$ such that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}_{\{N_n/n > \zeta\}} > 1 - \kappa\,.$$

Using Lemma 5.5 we can find $b > 0$ such that, $\mathbb{W}^2$-almost surely, there is an integer N such that, for every $n > N$ with $N_n/n > \zeta$ and all r with $a_n \le r \le a_{n-1}$,

$$\Big| \frac{\nu[r](A) \cdot \pi}{bN_n \cdot a_n^2} - \ell^2(A) \Big| < \delta\,.$$

Hence, upon observing that

$$\int_{a_{n+1}}^{a_n} \frac{dr}{r \cdot \log(1/r)} = \log(n+1) - \log n \sim \frac{1}{n}\,,$$

we get that

$$\begin{aligned}
&\liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\{r\}}\Big\{ 0 < r < 1 \,:\, \Big| \frac{\nu[r](A)\cdot\pi}{bN_n \cdot a_n^2} - \ell^2(A) \Big| < \delta \Big\} \frac{dr}{r\log(1/r)} \\
\ge\;& \liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\{r\}}\Big\{ r \,:\, a_n \le r \le a_{n-1} \text{ for } N_n/n \ge \zeta \Big\} \frac{dr}{r\log(1/r)} \\
=\;& \liminf_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{1}_{\{N_n/n > \zeta\}} > 1 - \kappa\,.
\end{aligned}$$

Using the analogous argument for the set B we infer that

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\{r\}}\Big\{ 0 < r < 1 \,:\, \Big| \frac{\nu[r](B)\cdot\pi}{bN_n \cdot a_n^2} - \ell^2(B) \Big| \le \delta \Big\} \frac{dr}{r\log(1/r)} > 1 - \kappa\,.$$

This implies, by definition of δ, that

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\{r\}}\Big\{ 0 < r < 1 \,:\, \Big| \frac{\nu[r](A)}{\nu[r](B)} - \frac{\ell^2(A)}{\ell^2(B)} \Big| \le \eta \Big\} \frac{dr}{r\log(1/r)} > 1 - 2\kappa\,.$$

As $\kappa > 0$ was arbitrary this finishes the proof. ■

In order to get the desired ratio ergodic theorem it remains to observe that convergence in logarithmic probability of order three is equivalent to convergence in logarithmic density of order three. More precisely, the following lemma of Fisher (1990), see also Berkes and Dehling (1993), holds true:

Lemma 5.7 *Let $\delta > 0$ and $f : (0, \delta) \to [0, \infty)$ be a locally integrable function with nonnegative values. The following statements are equivalent:*

(i) For every $\eta > 0$,

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \delta_{\{r\}}\Big\{0 < r < \delta \, : \, f(r) \le \eta\Big\} \frac{dr}{r\log(1/r)} = 1 \,.$$

(ii) There is a set $N \subseteq (0,\infty)$ such that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} 1_N(r) \, \frac{dr}{r\,\log(1/r)} = 0 \,,$$

and such that

$$\lim_{\substack{r\to 0\\ r\notin N}} f(r) = 0 \,.$$

Proof. See Fisher (1990), Lemma 4.9. ■

Proof of Theorem 1.4. The statement of the ratio ergodic theorem follows directly by applying Lemma 5.7 to the situation of Lemma 5.6. ■

For the sake of completeness we give the proof of the random scale Kallianpur-Robbins law by means of Lemma 5.5 directly instead of referring to (3). The convergence of the moments follows more or less directly form Lemma 5.5 and the last statement in Lemma 5.1. Denote by $\mathcal{A}$, as before, the collection of open or closed cubes with rational vertices.

Lemma 5.8 *$\mathbb{W}^2$-almost surely, for all finite families $\{A_1, \dots, A_m\} \subseteq \mathcal{A}$ and all $\kappa_1, \dots, \kappa_m > 0$,*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{1/e} \exp\Big[- \sum_{i=1}^m \kappa_i \frac{\nu[r](A_i)}{r^2\log(1/r)}\Big] \, \frac{dr}{r\,\log(1/r)} = \frac{1}{1 + \sum_{i=1}^m (\kappa_i/\pi)\ell^2(A_i)} \,.$$

Proof. It suffices to prove the lemma for a fixed family $\{A_1, \dots, A_m\} \subseteq \mathcal{A}$ of subsets of the open ball $B(0,1/2)$ and fixed $\kappa_1, \dots, \kappa_m > 0$. Using Lemma 5.5 we get, for every $\eta > 0$, an arbitrarily small $b > 0$ such that, $\mathbb{W}^2$-almost surely, there is an integer N such that, for all $n \ge N$ and all $a_{n+1} \le r \le a_n$,

$$\begin{aligned}
\Big(\frac{a_{n+1}}{a_n}\Big)^2 \cdot \Big[\sum_{i=1}^m \kappa_i \frac{\ell^2(A_i)}{\pi} - \eta\Big] \cdot \frac{N_{n+1}}{n+1} - \eta \;&\le\; \sum_{i=1}^m \kappa_i \Big[\frac{\nu[r](A_i)}{r^2\log(1/r)}\Big] \\
&\le\; \Big(\frac{a_n}{a_{n+1}}\Big)^2 \cdot \Big[\sum_{i=1}^m \kappa_i \frac{\ell^2(A_i)}{\pi} + \eta\Big] \cdot \frac{N_n}{n} + \eta \,.
\end{aligned}$$

Hence, for all $a_n \le \varepsilon \le a_{n+1}$,

$$\frac{1}{\log((n+1)b)} \sum_{k=N}^n \frac{1}{k} \exp\Big(- e^{2b}\Big[\sum_{i=1}^m \kappa_i \frac{\ell^2(A_i)}{\pi} - \eta\Big] \cdot \frac{N_k}{k} + \eta\Big) \tag{22}$$

$$\le \frac{1}{\log\log(1/\varepsilon)} \int_\varepsilon^{a_N} \exp\Big(- \sum_{i=1}^m \kappa_i \Big[\frac{\mu(rA_i)}{r^2\log(1/r)}\Big]\Big) \frac{dt}{t\cdot\log(1/t)}$$

$$\le \frac{1}{\log(nb)} \sum_{k=N}^{n+1} \frac{1}{k} \exp\Big(- e^{-2b}\Big[\sum_{i=1}^m \kappa_i \frac{\ell^2(A_i)}{\pi} - \eta\Big] \cdot \frac{N_{k+1}}{k+1} + \eta\Big) \,. \tag{23}$$

From Lemma 5.1 we know that, for every $\kappa > 0$,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=N}^{n} \frac{\exp\big(-\kappa N_k/k\big)}{k} = \frac{1}{1+\kappa}\,.$$

Hence the lower bound (22) and the upper bound (23) both converge, as $n \to \infty$ to limits which, by Lemma 5.3(iii), converge as first $b \to 0$ and then $\eta \to 0$ to $(1+\sum_{i=1}^{m}(\kappa_i/\pi)\ell^2(A_i))^{-1}$. ∎

Proof of Theorem 1.3. The first statement of the random scale Kallianpur-Robbins law follows by means of Lemma 2.3 from Lemma 5.8 . ∎

6 On the divergence of order-two averages

We briefly comment on the necessary steps to prove the divergence of the order-two averages in the random time case. In the random scale case this was proved in Mörters (1998).

First note that, by means of Lemma 4.2, we can see that, whenever $\sum_{n=1}^{N} a_n = 0$ and A is a bounded Borel set, $\mathbb{W}^2$-almost surely,

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{\varepsilon}^{1} \Big(\frac{\sum_{n=1}^{N} a_n\tilde{\mu}[n\lambda](A)}{(1/2)\log(1/\lambda)}\Big)^k \frac{d\lambda}{\lambda} = 0\,. \tag{24}$$

Assume now that, for some bounded Borel set $A \subset \mathbb{R}^2$, with positive probability,

$$\lim_{t\to\infty} \frac{1}{\log t} \int_{1}^{t} \frac{\mu[s](A)}{\log\sqrt{s}} \frac{ds}{s}$$

exists. As the limit depends only on the tail behaviour of Brownian motion, the limit exists almost surely and is necessarily constant, say equal to $C \geq 0$. We use (24) to reverse our approximation step. More precisely, as in Lemma 4.4, for every $0 < \eta < 1$, there are polynomials $\sum_{n=1}^{N} p_n x^n$ and $\sum_{n=1}^{N} P_n x^n$ with $\sum_{n=1}^{N} p_n = 0$ and $\sum_{n=1}^{N} P_n = 0$ such that, for all $\lambda > 0$,

$$\frac{1}{1+\eta}\mu[1/\lambda](B) \leq \sum_{n=1}^{N} P_n\tilde{\mu}[n\lambda](B) + \tilde{\mu}[\lambda](B)$$

and

$$\frac{1}{1-\eta}\mu[1/\lambda](B) \geq \sum_{n=1}^{N} p_n\tilde{\mu}[n\lambda](B) + \tilde{\mu}[\lambda](B)\,.$$

Thus, by (24) and our assumption

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{1/\varepsilon}^{1} \frac{\tilde{\mu}[\lambda](B)}{(1/2)\log(1/\lambda)} \frac{d\lambda}{\lambda} = C\,.$$

As, by Lemma 4.1, the third moments

$$\mathbb{E}\Big[\frac{1}{\log(1/\varepsilon)} \int_{1/\varepsilon}^{1} \frac{\tilde{\mu}[\lambda](B)}{(1/2)\log(1/\lambda)} \frac{d\lambda}{\lambda}\Big]^3$$

are bounded, we can infer that

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\Big[\frac{1}{\log(1/\varepsilon)} \int_{1/\varepsilon}^{1} \frac{\tilde\mu[\lambda](B)}{(1/2)\log(1/\lambda)}\, \frac{d\lambda}{\lambda}\Big]^2 = 0\,.$$

However, one can see from our moment calcualtions in Lemma 4.1, that this is not the case and hence we have a contradiction.

7 Final remarks

- In this paper we have presented the theory from the measure theoretic viewpoint and formulated pathwise limit theorems which hold uniformly for bounded weak functionals of the occupation *measure*. An alternative, largely equivalent, viewpoint describes the limit behaviour of *integrable additive functionals*. This point of view is adopted in the paper published in *Prob. Theory rel. Fields.*

- An alternative approach to the random time Kallianpur-Robbins law is based on the skew-product representaion of Brownian motion. In Mörters (1999) this approach is used to prove a pathwise version of Spitzer's theorem on the windings of planar Brownian motion.

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