Symmetry Properties of Average Densities and Tangent Measure Distributions of Measures on the Line

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Abstract: Answering a question by Bedford and Fisher in [4] we show that for the circular and one-sided average densities of a Radon measure μ on the line with positive lower and finite upper α-densities the following relations hold μ-almost everywhere

$$\underline{D}^{\alpha}_{-}(\mu,x) = \underline{D}^{\alpha}_{+}(\mu,x) = (1/2)\cdot\underline{D}^{\alpha}(\mu,x) \text{ and } \overline{D}^{\alpha}_{-}(\mu,x) = \overline{D}^{\alpha}_{+}(\mu,x) = (1/2)\cdot\overline{D}^{\alpha}(\mu,x)\,.$$

We infer the result from a more general formula, which is proved by means of a detailed study of the structure of the measure and which involves the notion of tangent measure distributions introduced by Bandt ([2]) and Graf ([9]). We show that for μ-almost every point x the formula

$$\int\int G(\nu,u)\,d\nu(u)\,dP(\nu) = \int\int G(T^u\nu,-u)\,d\nu(u)\,dP(\nu)$$

holds for every tangent measure distribution P of μ at x and all Borel functions $G : \mathcal{M}(\mathbb{R})\times\mathbb{R} \longrightarrow [0,\infty)$. Here $T^u\nu$ is the measure defined by $T^u\nu(E) = \nu(u+E)$ and $\mathcal{M}(\mathbb{R})$ is the space of Radon measures with the vague topology. By this formula the tangent measure distributions are Palm distributions and thus define α-self similar random measures in the sense of U.Zähle ([22]).

1 Introduction

In this paper we study nonnegative Radon measures on the real line such that, for some $0 < \alpha < 1$, μ has positive lower and finite upper α-densities, i.e.

$$0 < \liminf_{t\downarrow 0} \frac{\mu([x-t,x+t])}{t^\alpha} \le \limsup_{t\downarrow 0} \frac{\mu([x-t,x+t])}{t^\alpha} < \infty \quad \text{for } \mu\text{-almost every } x.$$

Examples of measures fulfilling these conditions are Hausdorff measures on many α-sets including self-similar sets and statistically self-similar sets, measures arising in dynamical systems theory and many more. Typically these measures do *not* have obvious self-similarity properties.

In [4] Bedford and Fisher introduce average or order-two densities for the study of measures of fractional dimension. For these measures the density functions $t \mapsto \mu([x-t,x+t])/t^\alpha$ fluctuate as t tends to 0 and therefore the limit does not exist (see [3]). Bedford and Fisher apply a logarithmic average and define the *lower* and *upper circular average densities* as

$$\underline{D}^{\alpha}(\mu,x) = \liminf_{\varepsilon\to 0} (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu([x-t,x+t])}{t^\alpha}\frac{dt}{t}\ ,$$

and

$$\overline{D}^{\alpha}(\mu,x)=\limsup_{\varepsilon\to 0}\,(|\log\varepsilon|)^{-1}\int_{\varepsilon}^{1}\frac{\mu([x-t,x+t])}{t^{\alpha}}\frac{dt}{t}\,.$$

The *lower* and *upper left-sided average densities* $\underline{D}^{\alpha}_{-}(\mu,x)$ and $\overline{D}^{\alpha}_{-}(\mu,x)$ are defined in the same way replacing the symmetric interval $[x-t,x+t]$ by $[x-t,x]$, and the *lower* and *upper right-sided average densities* $\underline{D}^{\alpha}_{+}(\mu,x)$ and $\overline{D}^{\alpha}_{+}(\mu,x)$ are defined replacing $[x-t,x+t]$ by $[x,x+t]$. The average density of μ at x exists if $\underline{D}^{\alpha}(\mu,x)=\overline{D}^{\alpha}(\mu,x)$ and in this case the joint value is denoted by $D^{\alpha}(\mu,x)$.

Bedford and Fisher show that the average density exists almost everywhere for Hausdorff measure on hyperbolic Cantor sets and zero-sets of Brownian motion. Recently other authors (see for example [5], [19] and [7]) have extended this result to various other classes of fractal measures with self–similarity properties. Average densities have also been used for the investigation of general measures with positive lower and finite upper densities. For example, Falconer and Springer in [6] and Marstrand in [11] generalize a classical inequality of Marstrand using average densities and in [16] it is shown that the lower one-sided average densities do not vanish.

In [4] Bedford and Fisher ask whether the left-sided and right-sided average densities always agree. An answer to this question can be given in the following form.

Theorem 1.1 *Suppose μ is a Radon measure on the line with positive lower and finite upper α-densities. Then at μ-almost every point x the following equations hold*

$$\underline{D}^{\alpha}_{-}(\mu,x)=\underline{D}^{\alpha}_{+}(\mu,x)=(1/2)\cdot\underline{D}^{\alpha}(\mu,x)\quad\textit{and}\quad\overline{D}^{\alpha}_{-}(\mu,x)=\overline{D}^{\alpha}_{+}(\mu,x)=(1/2)\cdot\overline{D}^{\alpha}(\mu,x)\,.$$

In order to get a more detailed analysis of the local geometry Bandt in [2] and Graf in [9] suggested the investigation of random tangent measures based on the same averaging principle. These random measures or, equivalently, probability distributions on the space $\mathcal{M}(\mathbb{R})$ of non-negative Radon measures with the vague topology are called tangent measure distributions.

For every $x\in\mathbb{R}$ define the family of measures $(\mu_{x,t})_{t>0}$, the *enlargements of μ about x*, by $\mu_{x,t}(A)=\mu(x+tA)$ for all Borel sets $A\subseteq\mathbb{R}$. Define probability distributions P^x_ε on $\mathcal{M}(\mathbb{R})$ by

$$P^x_\varepsilon(M)=(|\log\varepsilon|)^{-1}\int_\varepsilon^1 1_M\Big(\frac{\mu_{x,t}}{t^\alpha}\Big)\frac{dt}{t}\quad\text{for Borel sets } M\subseteq\mathcal{M}(\mathbb{R}).$$

$\mathcal{P}(\mu,x)$ is defined as the set of all limit points in the weak topology of $(P^x_\varepsilon)_{\varepsilon>0}$ as $\varepsilon\downarrow 0$. The elements of $\mathcal{P}(\mu,x)$ are the *tangent measure distributions* of μ at x. We also define measures $\nu^x_\varepsilon\in\mathcal{M}(\mathbb{R})$ by

$$\nu^x_\varepsilon(A)=(|\log\varepsilon|)^{-1}\int_\varepsilon^1\frac{\mu_{x,t}(A)}{t^\alpha}\frac{dt}{t}\ \text{for Borel sets } A\subseteq\mathbb{R}.$$

$\mathcal{A}(\mu,x)$ is defined as the set of all limit points in the vague topology of $(\nu^x_\varepsilon)_{\varepsilon>0}$ as $\varepsilon\downarrow 0$. The elements of $\mathcal{A}(\mu,x)$ are called *average tangent measures* of μ at x. If $\mathcal{P}(\mu,x)$ is a singleton, we say that μ has a unique tangent measure distribution at x; if $\mathcal{A}(\mu,x)$ is a singleton, we say that μ has a unique average tangent measure at x.

The following formula is a generalization of Theorem 1.1 involving tangent measure distributions. For every $u\in\mathbb{R}$ we define the shift operator $T^u:\mathcal{M}(\mathbb{R})\to\mathcal{M}(\mathbb{R})$ by $T^u\nu(E)=\nu(u+E)$.

Theorem 1.2 *Suppose that μ is a nonnegative Radon measure on the line with positive lower and finite upper α-densities. Then at μ-almost all points x every tangent measure distribution $P \in \mathcal{P}(\mu, x)$ fulfills*

$$\int\int G(\nu, u)\, d\nu(u)\, dP(\nu) = \int\int G(T^u\nu, -u)\, d\nu(u)\, dP(\nu)$$

$$\text{for all Borel functions } G : \mathcal{M}(\mathbb{R}) \times \mathbb{R} \longrightarrow [0, \infty). \tag{1}$$

This theorem not only implies the statement on one-sided average densities, but it is also the key to a surprising self-similarity property of the tangent measure distributions described in Section 2. The paper is organized as follows: In Section 2 we show some of the consequences of Theorem 1.2, in Section 3 we attempt to give a global description of the measure μ, and in the final section we give the proof of Theorem 1.2 using the results of Section 3.

2 The Local Geometry of the Measure

In this section let $0 < \alpha < 1$ and μ be a nonnegative Radon measure on the line with positive lower and finite upper α-densities. We analyse μ using its tangent measure distributions (see the introduction for the definition). We start with some immediate observations: For all $x \in \mathbb{R}$ all tangent measure distributions $P \in \mathcal{P}(\mu, x)$ are concentrated on the set

$$\mathrm{Tan}(\mu, x) = \{\nu = \lim_{n\to\infty} \frac{\mu_{x,t_n}}{t_n^\alpha} \text{ in the vague topology for some } t_n \downarrow 0 \ \}$$

of *tangent measures*, which were introduced in [21]. For every x such that the upper α-density is finite, the closure of the set

$$\left\{ \frac{\mu_{x,t}}{t^\alpha} \ : \ t \in (0,1) \right\} \subseteq \mathcal{M}(\mathbb{R})$$

is compact. Hence every sequence $(P^x_{r_n})_{n\in\mathbb{N}}$ has a convergent subsequence and $\mathcal{P}(\mu, x)$ is compact in the weak topology. The average tangent measures can be described as

$$\mathcal{A}(\mu, x) = \left\{ \int \nu\, dP \ : \ P \in \mathcal{P}(\mu, x) \right\} \tag{2}$$

and the average densities are given by

$$\overline{D}^\alpha(\mu, x) = \sup \left\{ \int \nu([-1,1])\, dP \ : \ P \in \mathcal{P}(\mu, x) \right\} \quad \text{and}$$

$$\underline{D}^\alpha(\mu, x) = \inf \left\{ \int \nu([-1,1])\, dP \ : \ P \in \mathcal{P}(\mu, x) \right\}. \tag{3}$$

The following scaling-invariance property of tangent measure distributions is easy to check.

Proposition 2.1 *For every $\lambda > 0$ define the rescaling operator $S^\alpha_\lambda : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ by $S^\alpha_\lambda \nu(E) = (1/\lambda^\alpha) \cdot \nu(\lambda E)$. For all $x \in \mathbb{R}$ and $P \in \mathcal{P}(\mu, x)$ we have*

$$P = P \circ (S^\alpha_\lambda)^{-1} \quad \textit{for all } \lambda > 0.$$

The next proposition contains a useful localization principle.

Proposition 2.2 *If $E \subseteq \mathbb{R}$ is μ-measurable and $\mu|_E$ is defined by $\mu|_E(A) = \mu(E \cap A)$ for all $A \subseteq \mathbb{R}$ Borel, then $\mathcal{P}(\mu, x) = \mathcal{P}(\mu|_E, x)$ for μ-almost every $x \in E$.*

Proof: This is an application of the Density Theorem [8, 2.9.11]. ∎

We shall prove our main theorem (Theorem 1.2) in Section 4. In the remainder of this section we show how Theorem 1.2 can be used to derive interesting properties of average densities and tangent measure distributions. The description of one-sided average densities in terms of circular average densities (Theorem 1.1) is an immediate consequence of the following symmetry principle:

Theorem 2.3 *At μ-almost every point x we have*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{|\log \varepsilon|} \int_\varepsilon^1 \frac{\mu([x-t, x]) - \mu([x, x+t])}{t^\alpha} \frac{dt}{t} = 0.$$

Proof: Let x be such that (1) holds and the upper α-density of μ at x is finite. Suppose $\varepsilon_n \downarrow 0$ is given. Then there is a subsequence (r_n) of (ε_n) such that there is $P = \lim_{n\to\infty} P^x_{r_n}$. Define $G(\nu, x) = 1_{[0,1]}(x)$. Then we have

$$\begin{aligned} & \lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu([x, x+t])}{t^\alpha} \frac{dt}{t} \\ = & \lim_{n\to\infty} \int\int G(\nu, y)\, d\nu(y)\, dP^x_{r_n}(\nu) = \int\int G(\nu, y)\, d\nu(y)\, dP(\nu)\,, \end{aligned}$$

and

$$\begin{aligned} & \lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu([x-t, x])}{t^\alpha} \frac{dt}{t} \\ = & \lim_{n\to\infty} \int\int G(T^y\nu, -y)\, d\nu(y)\, dP^x_{r_n}(\nu) = \int\int G(T^y\nu, -y)\, d\nu(y)\, dP(\nu)\,, \end{aligned}$$

and this implies, by means of (1),

$$\lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu([x, x+t])}{t^\alpha} \frac{dt}{t} = \lim_{n\to\infty} (|\log r_n|)^{-1} \int_{r_n}^1 \frac{\mu([x-t, x])}{t^\alpha} \frac{dt}{t}\,,$$

which implies the statement. ∎

In the next corollary we give a reformulation of this symmetry principle in the language of singular integrals. For $0 < s \leq 1$ consider the kernel

$$\begin{aligned} K_s : \ \mathbb{R} \setminus \{0\} & \longrightarrow \mathbb{R} \\ x & \mapsto \frac{\text{sign } (x)}{|x|^s}\,. \end{aligned}$$

K_s is a natural generalization of the kernel $1/x$ of the classical Hilbert transform

$$\begin{aligned} Hf(x) &= \lim_{\varepsilon\downarrow 0} \int_{\{y:|x-y|>\varepsilon\}} \frac{f(t)}{t-x}\, dt \\ &= \lim_{\varepsilon\downarrow 0} \int_{\{y:|x-y|>\varepsilon\}} K_1(t-x) f(t)\, dt\,. \end{aligned}$$

The question whether for $0<\alpha<1$ the limits

$$\lim_{\varepsilon\downarrow 0} \int_{\{y:|x-y|>\varepsilon\}} K_\alpha(y-x)\, d\mu(y)$$

exist on a set of positive measure has been answered in the negative by Mattila and Preiss in [13] (see also [12]). Our symmetry principle yields:

Corollary 2.4 *For μ-almost all x we have*

$$\lim_{\varepsilon\downarrow 0} (|\log\varepsilon|)^{-1} \int_{\{y:|x-y|>\varepsilon\}} K_\alpha(y-x)\, d\mu(y) = 0\,.$$

Proof: We can assume without loss of generality that μ is finite. Fix x such that the upper density of μ at x if finite and the symmetry principle holds. Integration by parts yields

$$\int_{\{y:\, y-x>\varepsilon\}} \frac{1}{(y-x)^\alpha}\, d\mu(y) = \left[-\frac{\mu([x,x+\varepsilon])}{\varepsilon^\alpha} + \alpha\cdot \int_\varepsilon^\infty \frac{\mu([x,x+t])}{t^{\alpha+1}}\, dt \right],$$

and thus, for some constant $C>0$,

$$\begin{aligned} -\frac{C}{|\log\varepsilon|} + \alpha\cdot(|\log\varepsilon|)^{-1}\int_\varepsilon^\infty \frac{\mu([x,x+t])}{t^\alpha}\frac{dt}{t} &\le (|\log\varepsilon|)^{-1} \int_{\{y:\, y-x>\varepsilon\}} \frac{1}{(y-x)^\alpha}\, d\mu(y) \\ &\le \alpha\cdot(|\log\varepsilon|)^{-1}\int_\varepsilon^\infty \frac{\mu([x,x+t])}{t^{\alpha+1}}\, dt\,, \end{aligned}$$

and analogously we get

$$\begin{aligned} -\frac{C}{|\log\varepsilon|} + \alpha\cdot(|\log\varepsilon|)^{-1}\int_\varepsilon^\infty \frac{\mu([x-t,x])}{t^\alpha}\frac{dt}{t} &\le (|\log\varepsilon|)^{-1} \int_{\{y:\, x-y>\varepsilon\}} \frac{1}{(x-y)^\alpha}\, d\mu(y) \\ &\le \alpha\cdot(|\log\varepsilon|)^{-1}\int_\varepsilon^\infty \frac{\mu([x-t,x])}{t^{\alpha+1}}\, dt\,. \end{aligned}$$

As $\varepsilon\downarrow 0$ we thus have

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} (|\log\varepsilon|)^{-1} \int_{\{y:|x-y|>\varepsilon\}} K_\alpha(y-x)\, d\mu(y) \\ &= \alpha\cdot \lim_{\varepsilon\downarrow 0} (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu([x,x+t])-\mu([x-t,x])}{t^\alpha}\frac{dt}{t} = 0. \end{aligned}$$

■

Another remarkable fact is that on the real line the average tangent measures are completely determined by the average densities. Note that by an example of O'Neil (see [17]) this is different in higher dimensions.

Corollary 2.5

(a) *For μ-almost every x all average tangent measures $\bar{\nu}$ of μ at x are symmetric around* 0.

(b) *Suppose the average densities of μ exist μ-almost everywhere. Then μ has unique average tangent measures $\bar{\nu}^x$ at μ-almost every x. Moreover, $\bar{\nu}^x$ is given by*

$$\bar{\nu}^x(A) = (1/2) \cdot D^\alpha(\mu, x) \cdot \int_A \alpha |t|^{\alpha-1}\, dt \quad \textit{for every } A \subseteq \mathbb{R} \textit{ Borel.}$$

Proof: Let x be such that the upper density of μ at x is finite and (1) holds. If $\bar{\nu}$ is an average tangent measure at x, then there is a tangent measure distribution P such that $\bar{\nu} = \int \nu \, dP(\nu)$. Using (1) for the function $G(\nu, y) = 1_A(y)$ we get, for every Borel set $A \subseteq \mathbb{R}$,

$$\bar{\nu}(A) = \int \nu(A)\, dP(\nu) = \int \nu(-A)\, dP(\nu) = \bar{\nu}(-A)\,,$$

which is the first statement.
Suppose now that the average density at x exists. For $\lambda > 0$ and any half-open interval $[0, \lambda)$ we have, by Proposition 2.1, $\bar{\nu}([0,\lambda)) = \lambda^\alpha \cdot \bar{\nu}([0,1))$, and using the symmetry and $\bar{\nu}(\{0\}) = 0$ we have

$$\bar{\nu}([0,1)) = (1/2) \cdot \bar{\nu}((-1,1)) = (1/2) \cdot D^\alpha(\mu, x)\,,$$

and similarly for intervals $[-\lambda, 0)$. Therefore the measure $\bar{\nu}^x$ defined using the formula in the corollary and the measure $\bar{\nu}$ agree on all (right-)half-open intervals and hence they are identical. This implies the uniqueness of the average tangent measures as well as the formula. ■

Studying the relation between the existence of average densities and uniqueness of average tangent measures and tangent measure distributions we get the following picture:

Theorem 2.6 *For the following statements the implications $(a) \Rightarrow (b)$ and $(b) \Leftrightarrow (c)$ hold. $(b) \Rightarrow (a)$ does not hold.*

(a) *The measure μ has a unique tangent measure distribution μ-almost everywhere.*

(b) *The measure μ has a unique average tangent measure μ-almost everywhere.*

(c) *The average density of μ at x exists μ-almost everywhere.*

Proof: Implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ follow immediately from formulas (2) and (3). $(c) \Rightarrow (b)$ follows from Corollary 2.5, and it remains to give an example that the implication $(b) \Rightarrow (a)$ fails. Consider the following construction: Fix a sequence (a_k) of integers with $a_0 = 0$ and $a_k \uparrow \infty$ such that

$$\frac{a_k}{a_{k+1}} \longrightarrow 0\,.$$

Define the code-space $\Sigma = \prod_{i=1}^\infty \{0,1,2\}$ and define a measure $\bar{\mu}$ on Σ by

$$\bar{\mu}\left(\{(x_i)_{i\in\mathbb{N}} \,:\, x_1 = y_1, \ldots, x_n = y_n\}\right) = (1/3)^n \quad \text{for } y \in \Sigma.$$

Define sets I_1, I_2, I by

$$I_1 = \left\{ x = \sum_{i=1}^{\infty} \frac{x_i}{7^i} \, : \, x_i \in \{0,2,6\} \right\} , \quad I_2 = \left\{ x = \sum_{i=1}^{\infty} \frac{x_i}{7^i} \, : \, x_i \in \{0,4,6\} \right\}$$

and

$$I = \left\{ x = \sum_{i=1}^{\infty} \frac{x_i}{7^i} \quad : \quad x_i \in \{0,2,6\} \text{ if } a_{2k} < i \le a_{2k+1} \text{ and } \right.$$
$$\left. x_i \in \{0,4,6\} \text{ if } a_{2k+1} < i \le a_{2k+2} \right\},$$

and mappings ϕ_1, ϕ_2, ϕ by

$$\phi_k : \Sigma \longrightarrow I_k \quad , \quad x \mapsto \sum_{i=1}^{\infty} \frac{\varphi_k(x_i)}{7^i} \qquad \text{for } k = 1, 2,$$

and

$$\phi : \Sigma \longrightarrow I \quad , \quad x \mapsto \sum_{i=1}^{\infty} \frac{\varphi_3^i(x_i)}{7^i},$$

where

$$\varphi_1(x) = \begin{cases} 0 & \text{if } x = 0 \ , \\ 2 & \text{if } x = 1 \ , \\ 6 & \text{if } x = 2 \ , \end{cases} \qquad \text{and} \qquad \varphi_2(x) = \begin{cases} 0 & \text{if } x = 0 \ , \\ 4 & \text{if } x = 1 \ , \\ 6 & \text{if } x = 2 \ , \end{cases}$$

and

$$\varphi_3^i(x) = \begin{cases} 0 & \text{if } x = 0 \ , \\ 2 & \text{if } x = 1 \text{ and } a_{2k} < i \le a_{2k+1}, \\ 4 & \text{if } x = 1 \text{ and } a_{2k+1} < i \le a_{2k+2}, \\ 6 & \text{if } x = 2 \ . \end{cases}$$

Let $\mu = \bar{\mu} \circ \phi^{-1}$, $\mu_1 = \bar{\mu} \circ \phi_1^{-1}$ and $\mu_2 = \bar{\mu} \circ \phi_2^{-1}$. μ, μ_1 and μ_2 can be extended in a natural way to Radon measures on $\mathbb{R}$. Let $\alpha = \frac{\log 3}{\log 7}$. μ_1, μ_2 and μ have positive lower and finite upper α-densities for all $x \in I_1$, I_2, I. As μ_1 and μ_2 are self-similar measures fulfilling the strong separation condition they have unique tangent measure distributions P_1, P_2 almost everywhere and, as in [9], we can describe P_1, P_2 by

$$P_{1,2}(E) = \frac{1}{\log 7} \int_{I_{1,2}} \int_{\eta/7}^{\eta} 1_E \left(\frac{(\mu_{1,2})_{x,t}}{t^\alpha} \right) \frac{dt}{t} \, d\mu_{1,2}(x) \, ,$$

where $E \in \mathcal{M}_b$, the σ-algebra on $\mathcal{M}(\mathbb{R})$ generated by the mappings $\nu \mapsto \nu(B)$ for all Borel sets $B \subseteq B(0,b)$, and $\eta < (1/(7b))$. It is easy to see that $\int \nu([-1,1]) \, dP_1(\nu) = \int \nu([-1,1]) \, dP_2(\nu)$. Using messy but straightforward calculations (see [15] for details) we can also see that for μ-almost all points $x \in I$ the set of tangent measure distributions of μ at x is given by

$$\{\lambda P_1 + (1-\lambda) P_2 \, : \, \lambda \in [0,1]\} \, .$$

This not only shows that the set of tangent measure distributions of μ at x is not a singleton but (with the help of (3)) also implies that the average densities of μ exist and thus μ has a unique average tangent measure at μ-almost all points. ■

We now show that formula (1) relates tangent measure distributions to Palm distributions. Palm distributions originate from queuing theory and they are nowadays widely used in the theory of point processes where they play the rôle of conditional distributions of stationary point processes given a point at the origin (see [10]). A probability distribution P on $\mathcal{M}(\mathbb{R})$ is a *Palm distribution* if there is a stationary σ-finite measure Q on $\mathcal{M}(\mathbb{R})$ with finite intensity $\lambda > 0$ and

$$\int_M \nu(B)\, dQ(\nu) = \lambda \cdot \int_B P \circ (T^u)(M)\, du \quad \text{for all } M \subseteq \mathcal{M}(\mathbb{R}),\ B \subseteq \mathbb{R} \text{ Borel.}$$

The link between our theorem and Palm distributions is the following classical characterization due to Mecke (see [14]):

Lemma 2.7 *A probability measure P on $\mathcal{M}(\mathbb{R})$ is a Palm distribution if and only if $P(\{\phi\}) = 0$, where ϕ is the zero-measure, and (1) holds.*

Theorem 1.2 and Proposition 2.1 yield an interesting connection of tangent measure distributions to the theory of self-similar random measures. In [22] U. Zähle suggested the following axiomatic concept of statistical self-similarity.

Definition
A probability distribution P on $\mathcal{M}(\mathbb{R})$ defines an *α-self similar random measure* if P is a Palm distribution and invariant under the rescaling group $(S^\alpha_\lambda)_{\lambda>0}$.

The heuristic idea of this definition is the following: A random measure is statistically self-similar if it is scaling invariant with respect to a "typical point" of the random measure. We can interpret Palm distributions as those distributions which have the origin as a "typical point of their realizations" (see [22] for details). This concept of statistical self-similarity has been studied by Patzschke, U. Zähle and M. Zähle for example in [18], [20] where also its relation to statistically self-similar measures in the constructive sense was investigated.

We get the following theorem (recall that we did not require μ to be self-similar in any sense).

Theorem 2.8 *At μ-almost all points x every tangent measure distribution $P \in \mathcal{P}(\mu, x)$ defines an α-self similar random measure.*

Proof: Clearly $P(\{\phi\}) = 0$ for all $P \in \mathcal{P}(\mu, x)$ if the lower density of μ at x is positive. Thus Theorem 1.2 together with Lemma 2.7 implies that for μ-almost every x every $P \in \mathcal{P}(\mu, x)$ is a Palm distribution. This fact and the scaling invariance of tangent measure distributions imply the statement. ■

3 The Global Geometry of the Measure

Let $0 < \alpha < 1$ and μ a finite nonnegative Radon measure on the real line with positive and finite α-densities μ-almost everywhere. Using the inner regularity of μ we can find for every $\delta > 0$ a compact set $E \subseteq \mathbb{R}$ such that $\mu(\mathbb{R} \setminus E) < \delta$ and there are $0 < c \leq C < \infty$ and $t_0 > 0$ with

$$\mu([x-t, x+t]) \leq Ct^\alpha \text{ and } \mu([x-t, x+t]) \geq ct^\alpha \text{ for all } x \in E \text{ and } 0 \leq t \leq t_0. \tag{4}$$

In this section we study the geometry of the set E. The constants in the following lemmas may depend on the measure μ. $|U|$ denotes the diameter of a set $U \subseteq \mathbb{R}$.

Proposition 3.1 *E is an α-set, i.e. E has positive and finite α-Hausdorff measure.*

Proof: For every $t_0 > t > 0$ we can cover E with a family $\mathcal{U} = \{[x-t, x+t] : x \in S\}$ of intervals such that $S \subseteq E$ and every $y \in \mathbb{R}$ is contained in at most two sets $U \in \mathcal{U}$. Then

$$\sum_{U \in \mathcal{U}} |U|^\alpha \leq (2^\alpha/c) \cdot \sum_{U \in \mathcal{U}} \mu(U) \leq 2(2^\alpha/c) \cdot \mu(\mathbb{R}) < \infty$$

and thus $\mathcal{H}^\alpha(E) < \infty$. Now let $t_0 > t > 0$ and let $\mathcal{U}$ be an arbitrary cover of E such that $|U| \leq t$ and $U \cap E \neq \emptyset$ for all $U \in \mathcal{U}$. Then

$$\sum_{U \in \mathcal{U}} |U|^\alpha \geq (1/C) \sum_{U \in \mathcal{U}} \mu(U) \geq \mu(E)/C$$

and thus $\mathcal{H}^\alpha(E) > 0$. ■

Let D be the convex hull of E. Then we can write

$$D \setminus E = \bigcup_{I \in \mathcal{A}} I,$$

where $\mathcal{A}$ is the collection of connected components of $D \setminus E$. $\mathcal{A}$ is a collection of disjoint open intervals. Let

$$\mathcal{A}_\varepsilon = \{I \in \mathcal{A} : |I| \geq \varepsilon\} \quad \text{and} \quad E_\varepsilon = D \setminus \bigcup_{I \in \mathcal{A}_\varepsilon} I.$$

Then $E = \bigcap_{\varepsilon > 0} E_\varepsilon$. Let

$$E_\varepsilon = \bigcup_{N \in \mathcal{N}_\varepsilon} N,$$

where $\mathcal{N}_\varepsilon$ is the collection of connected components of the set E_ε. $\mathcal{N}_\varepsilon$ is a collection of disjoint compact (possibly degenerate) intervals.

Before we give an upper bound to the length of the intervals in $\mathcal{N}_\varepsilon$, let us introduce some useful notation. For every interval $I \subseteq \mathbb{R}$ and every $\kappa \geq 0$ let

$$I^-(\kappa) = \{x \in \mathbb{R} \setminus I : \text{ there is } y \in I \text{ such that } 0 \leq y - x \leq \kappa \cdot |I|\},$$

and

$$I^+(\kappa) = \{x \in \mathbb{R} \setminus I : \text{ there is } y \in I \text{ such that } 0 \leq x - y \leq \kappa \cdot |I|\},$$

and also

$$I^0(\kappa) = I^-(\kappa) \cup I \cup I^+(\kappa).$$

Proposition 3.2 *There are constants $\mathcal{C}_1 > 1$ and $0 < \varepsilon_0 < t_0$ with $\varepsilon_0 < 1/e$ such that, for all $0 < \varepsilon \le \varepsilon_0$ and all $K \in \mathcal{N}_\varepsilon$, we have $|K| < \mathcal{C}_1 \cdot \varepsilon$.*

Proof: For $\varepsilon > 0$ denote $r = r(\varepsilon) = \max\{|N| : N \in \mathcal{N}_\varepsilon\}$ and choose $0 < \varepsilon_0 < t_0$ such that $\varepsilon_0 < 1/e$ and $r(\varepsilon_0) < t_0$. Fix $0 < \varepsilon \le \varepsilon_0$ and pick $N \in \mathcal{N}_\varepsilon$ such that $|N| = r$. Let $\tilde{N} = N^-(1) \cup N$, in other words $\tilde{N}$ is the closed interval of diameter $2r$ with centre at the left endpoint of N. Consider the intervals $I_1, I_2, I_3, \ldots \in \mathcal{A}$ that fulfill $I_i \subseteq N$ and observe that $|I_i| \le \varepsilon$. Define $\tilde{I}_i = I_i \cup I_i^-(1) \subseteq \tilde{N}$. The sets $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \ldots$ cover almost all of N in the sense of Lebesgue measure. By Vitali's Covering Theorem (see for example [12, Theorem 2.1]) we can pick a disjoint subsequence $\tilde{I}_{k_1}, \tilde{I}_{k_2}, \tilde{I}_{k_3}, \ldots$ covering at least $1/5$ of the length of N. Now we can use (4) to infer

$$C \ge \frac{\mu(\tilde{N})}{r^\alpha} \ge \frac{1}{r^\alpha} \sum_{i=1}^{\infty} \mu(\tilde{I}_{k_i}) \ge \frac{c}{r^\alpha} \sum_{i=1}^{\infty} |I_{k_i}|^\alpha \ge \frac{c}{r^\alpha} \varepsilon^{\alpha-1} \sum_{i=1}^{\infty} |I_{k_i}| > \frac{c}{10} \left(\frac{\varepsilon}{r}\right)^{\alpha-1}$$

and, defining $\mathcal{C}_1 = (10C/c)^{1/(1-\alpha)}$, we have $r(\varepsilon) < \mathcal{C}_1 \varepsilon$, as required. ∎

Proposition 3.3 *There is a constant $\mathcal{C}_2 > 1$ such that, for all $0 < \varepsilon \le \varepsilon_0$ and every interval K with endpoints in E and $|K| \le t_0$, we have*

$$\sum_{\substack{N \in \mathcal{N}_\varepsilon \\ N \subseteq K}} |N|^\alpha \le \mathcal{C}_2 \cdot |K|^\alpha \quad \text{and} \quad \sum_{N \in \mathcal{N}_\varepsilon} |N|^\alpha \le \mathcal{C}_2.$$

In particular, we have

$$\mathcal{H}^\alpha(E) \le \liminf_{\varepsilon \downarrow 0} \sum_{N \in \mathcal{N}_\varepsilon} |N|^\alpha \le \limsup_{\varepsilon \downarrow 0} \sum_{N \in \mathcal{N}_\varepsilon} |N|^\alpha \le \mathcal{C}_2 \cdot \mathcal{H}^\alpha(E).$$

Proof: We fix K and $0 < \varepsilon < \varepsilon_0$. For all $N \in \mathcal{N}_\varepsilon$ we let $N^* = N^-(1/\mathcal{C}_1) \cup N$. Consider the closed interval B of diameter $2|N|/\mathcal{C}_1$ centred at the left endpoint of N. We have $B \subseteq N^*$ and thus we get, using (4),

$$\mu(N^*) \ge \mu(B) \ge c\,(|N|/\mathcal{C}_1)^\alpha \ .$$

Because, by Proposition 3.2, the intervals I separating the $N \in \mathcal{N}_\varepsilon$ fulfill $|I| \ge \varepsilon > |N|/\mathcal{C}_1$, the collection $\{N^* : N \in \mathcal{N}_\varepsilon \text{ and } N \subseteq K\}$ is disjoint. Also $N^* \subseteq K^*$ for all N^* in the collection. Therefore

$$\sum_{\substack{N \in \mathcal{N}_\varepsilon \\ N \subseteq K}} |N|^\alpha \le (\mathcal{C}_1^\alpha / c) \cdot \sum_{\substack{N \in \mathcal{N}_\varepsilon \\ N \subseteq K}} \mu(N^*) \le (\mathcal{C}_1^\alpha / c) \cdot \mu(K^*)\,. \tag{5}$$

Now the first inequality follows using $\mu(K^*) \le C|K|^\alpha$ and summing (5) for all intervals $K \in \mathcal{N}_{\varepsilon_0}$ yields the second inequality. The first inequality involving Hausdorff-measure is immediate since, by Proposition 3.2, $\mathcal{N}_\varepsilon$ is a covering of E by sets of diameter less than $\mathcal{C}_1\varepsilon$. Given an arbitrary cover $\mathcal{U}$ of E with intervals of length less than t_0 we can assume, by expanding each $U \in \mathcal{U}$ slightly and using the compactness of E, that $\mathcal{U}$ is finite and every $U \in \mathcal{U}$ has endpoints in $D \setminus E$. By shrinking each U slightly we can now get a covering $\mathcal{U}'$ such that there is $\varepsilon_0 > \varepsilon' > 0$

such that every set $U \in \mathcal{U}'$ is the convex hull of a collection of intervals from $\mathcal{N}_{\varepsilon'}$. The first inequality asserts that for $\varepsilon < \varepsilon'$ the coverings $\mathcal{N}_\varepsilon$ fulfill

$$\sum_{N \in \mathcal{N}_\varepsilon} |N|^\alpha \le \mathcal{C}_2 \cdot \sum_{U \in \mathcal{U}} |U|^\alpha .$$

This yields the last inequality. ∎

Proposition 3.4 *There is a constant $\mathcal{C}_3 > 1$ such that, for all $0 < \varepsilon \le \varepsilon_0$ and all intervals K with endpoints in E and $|K| \le \varepsilon_0$, we have*

$$\sum_{\substack{I \in \mathcal{A}_\varepsilon \\ I \subseteq K}} |I|^\alpha \le \mathcal{C}_3 \cdot |K|^\alpha \cdot |\log \varepsilon| \quad \textit{and} \quad \sum_{I \in \mathcal{A}_\varepsilon} |I|^\alpha \le \mathcal{C}_3 \cdot |\log \varepsilon| \, .$$

Proof: Fix K and denote $\tilde{K} = K^-(1) \cup K$. Observe that $\mu(\tilde{K}) \le C|K|^\alpha$. Similarly, for $I \in \mathcal{A}$ with $|I| \ge \varepsilon$, $I \subseteq K$, we define $\tilde{I} = I^-(1) \cup I$ and observe that $\tilde{I} \subseteq \tilde{K}$ and $\mu(\tilde{I}) \ge c|I|^\alpha$. For $x \in \tilde{K}$ denote by $I_1, I_2, \ldots, I_n$ the collection of intervals $I \in \mathcal{A}_\varepsilon$ such that $I \subseteq K$ and $x \in \tilde{I}$, ordered from left to right. For $3 \le k \le n$ we have $|I_k| \ge |I_{k-1}| + |I_{k-2}| + \ldots + |I_2|$ and thus $\varepsilon 2^{n-3} \le |I_n| \le |D|$. Hence we get $n \le 3 + \log|D|/\log 2 + |\log \varepsilon|/\log 2$ and thus

$$\sum_{\substack{I \in \mathcal{A}_\varepsilon \\ I \subseteq K}} |I|^\alpha \le \sum_{\substack{I \in \mathcal{A}_\varepsilon \\ I \subseteq K}} (1/c)\cdot\mu(\tilde{I}) \le (1/c)\cdot \int \sum_{\substack{I \in \mathcal{A}_\varepsilon \\ I \subseteq K}} 1_{\tilde{I}} \, d\mu \le (n/c)\cdot\mu(\tilde{K}) \le (C/c)\cdot n \cdot |K|^\alpha \le \mathcal{C}_3 \cdot |\log \varepsilon| \cdot |K|^\alpha$$

as required to prove the first inequality. To prove the second inequality observe

$$\sum_{I \in \mathcal{A}_\varepsilon} |I|^\alpha \le \sum_{I \in \mathcal{A}_{\varepsilon_0}} |I|^\alpha + \sum_{K \in \mathcal{N}_{\varepsilon_0}} \sum_{\substack{I \in \mathcal{A}_\varepsilon \\ I \subseteq K}} |I|^\alpha$$

and use Proposition 3.3 and the first part. ∎

We shall now derive some useful estimates involving the averaging procedure of Bedford and Fisher. For this purpose define for every $x \in \mathbb{R}$ and $\varepsilon > 0$ a measure ψ_ε^x by

$$\psi_\varepsilon^x(A) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 (1_A(x+t) + 1_A(x-t)) \, \frac{dt}{t} \quad \text{for } A \subseteq \mathbb{R} \text{ Borel}.$$

Let $E_0 \subseteq E$ be a subset of diameter less than ε_0 and define $\bar{\mu} = \mu|_{E_0}$. Then

$$\bar{\mu}(B) \le C \cdot |B|^\alpha \quad \text{for all } B \subseteq \mathbb{R}. \tag{6}$$

Lemma 3.5 *There are constants $\mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_7 > 0$ such that, for all intervals $I \subseteq \mathbb{R}$ and all $\varepsilon > 0$, the following estimates hold:*

(a) $\displaystyle\int_{I^-(\kappa) \cup I^+(\kappa)} \psi_\varepsilon^x(I) \, d\bar{\mu}(x) \le \mathcal{C}_4 \cdot \frac{|I|^\alpha}{|\log \varepsilon|} \cdot \left(\log\left(\frac{\kappa+1}{\kappa}\right) \cdot \kappa^\alpha \right)$ *for all* $0 < \kappa \le 1$,

(b) $\displaystyle\int_{\mathbb{R} \setminus I^0(\kappa)} \psi_\varepsilon^x(I) \, d\bar{\mu}(x) \le \mathcal{C}_5 \cdot \frac{|I|^\alpha}{|\log \varepsilon|} \cdot \left(\frac{1}{\kappa}\right)^{1-\alpha}$ *for all* $\kappa > 0$,

(c) $\displaystyle\int_{\mathbb{R}\setminus I} \psi_\varepsilon^x(I)\, d\bar\mu(x) \le \mathcal{C}_6 \cdot \frac{|I|^\alpha}{|\log\varepsilon|}$,

(d) $\displaystyle\int_{\mathbb{R}\setminus I} (\psi_\varepsilon^x(I))^2\, d\bar\mu(x) \le \mathcal{C}_7 \cdot \frac{|I|^\alpha}{|\log\varepsilon|^2}$.

Proof: Denote the left endpoint of I by a and let $R(t) = \bar\mu([a-t,a])$. By (6) we have $R(t) \le Ct^\alpha$. We use integration by parts to see, for all $0 < \kappa \le 1$,

$$\begin{aligned}
\int_{I^-(\kappa)} \psi_\varepsilon^x(I)\, d\bar\mu(x) &\le (|\log\varepsilon|)^{-1} \cdot \int_0^{\kappa|I|} \log\Big(\frac{t+|I|}{t}\Big)\, dR(t) \\
&\le (|\log\varepsilon|)^{-1} \cdot \Big[\log\Big(\frac{\kappa+1}{\kappa}\Big) \cdot R(\kappa\cdot|I|) + \int_0^\kappa \frac{R(t|I|)}{(t+1)t}\, dt\Big] \\
&\le \frac{C|I|^\alpha}{|\log\varepsilon|} \cdot \Big[\log\Big(\frac{\kappa+1}{\kappa}\Big) \cdot \kappa^\alpha + \int_0^\kappa \frac{dt}{t^{1-\alpha}}\Big] \\
&\le (1 + 1/(\alpha\cdot\log 2)) \cdot C \cdot \frac{|I|^\alpha}{|\log\varepsilon|} \cdot \Big(\log\Big(\frac{\kappa+1}{\kappa}\Big) \cdot \kappa^\alpha\Big)
\end{aligned}$$

and, for all $\lambda > 0$ and $\kappa > 0$,

$$\begin{aligned}
\int_{I^-(\lambda)\setminus I^-(\kappa)} \psi_\varepsilon^x(I)\, d\bar\mu(x) &\le (|\log\varepsilon|)^{-1} \cdot \int_\kappa^\infty \log\Big(\frac{t+1}{t}\Big)\, dR(t|I|) \\
&\le (|\log\varepsilon|)^{-1} \cdot \Big[\int_\kappa^\infty \frac{R(t|I|)}{(t+1)t}\, dt\Big] \\
&\le \frac{C|I|^\alpha}{|\log\varepsilon|} \cdot \int_\kappa^\infty \frac{dt}{t^{2-\alpha}} \\
&\le 1/(1-\alpha) \cdot C \cdot \frac{|I|^\alpha}{|\log\varepsilon|} \cdot \Big(\frac{1}{\kappa}\Big)^{1-\alpha}.
\end{aligned}$$

Analogous calculations can be performed for $I^+(\kappa)$ and thus (a) and (b) follow. (c) follows by adding (a) and (b) for $\kappa = 1$. To prove (d) observe that, for every $\lambda > 0$,

$$\begin{aligned}
\int_{I^-(\lambda)} (\psi_\varepsilon^x(I))^2\, d\bar\mu(x) &\le (|\log\varepsilon|)^{-2} \cdot \int_0^\infty \log^2\Big(\frac{t+1}{t}\Big)\, dR(t|I|) \\
&\le \frac{C|I|^\alpha}{|\log\varepsilon|^2} \cdot \int_0^\infty \frac{2t^\alpha}{t(t+1)} \cdot \log\Big(\frac{t+1}{t}\Big)\, dt.
\end{aligned}$$

An analogous estimate for $I^+(\lambda)$ completes the proof. ■

Lemma 3.6 *There is $\mathcal{C}_8 > 1$ such that, for all $0 < \varepsilon \le \delta \le \varepsilon_0/\mathcal{C}_1$, all $\kappa > 1$ and for every $K \in \mathcal{N}_\delta$, we have*

$$\int \Big(\psi_\varepsilon^x(K) - \sum_{\substack{I\in\mathcal{A}_\varepsilon, I\subseteq K\\ x\in I^0(\kappa)}} \psi_\varepsilon^x(I)\Big)\, d\bar\mu(x) \le \mathcal{C}_8 \cdot |K|^\alpha \cdot \Big(\Big(\frac{1}{\kappa}\Big)^{1-\alpha} + \frac{1}{|\log\varepsilon|}\Big),$$

and

$$\int \Big(\psi_\varepsilon^x(\mathbb{R}) - \sum_{\substack{I\in\mathcal{A}_\varepsilon\\ x\in I^0(\kappa)}} \psi_\varepsilon^x(I)\Big)\, d\bar\mu(x) \le \mathcal{C}_8 \cdot \Big(\Big(\frac{1}{\kappa}\Big)^{1-\alpha} + \frac{1}{|\log\varepsilon|}\Big).$$

Proof: We start with the first inequality. For all $x \in \mathbb{R}$ we have

$$\psi_\varepsilon^x(K) - \sum_{\substack{I \in \mathcal{A}_\varepsilon, I \subseteq K \\ x \in I^0(\kappa)}} \psi_\varepsilon^x(I) = \sum_{\substack{N \in \mathcal{N}_\varepsilon \\ N \subseteq K}} \psi_\varepsilon^x(N) + \sum_{\substack{I \in \mathcal{A}_\varepsilon, I \subseteq K \\ x \notin I^0(\kappa)}} \psi_\varepsilon^x(I) .$$

To estimate the integral of the first summand, we use $|N| \leq \mathcal{C}_1 \varepsilon$ to infer that

$$\int_N \psi_\varepsilon^x(N)\, d\bar{\mu}(x) \leq \frac{2 \cdot \log \mathcal{C}_1}{|\log \varepsilon|} \cdot \bar{\mu}(N) \leq 2C \cdot \log \mathcal{C}_1 \cdot \frac{|N|^\alpha}{|\log \varepsilon|} .$$

By Lemma 3.5(c) we know that

$$\int_{\mathbb{R} \setminus N} \psi_\varepsilon^x(N)\, d\bar{\mu}(x) \leq \mathcal{C}_6 \cdot \frac{|N|^\alpha}{|\log \varepsilon|}.$$

These two estimates together with Proposition 3.3 give

$$\sum_{\substack{N \in \mathcal{N}_\varepsilon \\ N \subseteq K}} \int \psi_\varepsilon^x(N)\, d\bar{\mu}(x) \leq (2C \cdot \log \mathcal{C}_1 + \mathcal{C}_6) \cdot \sum_{\substack{N \in \mathcal{N}_\varepsilon \\ N \subseteq K}} \frac{|N|^\alpha}{|\log \varepsilon|} \leq \mathcal{C}_2 \cdot (2C \cdot \log \mathcal{C}_1 + \mathcal{C}_6) \cdot \frac{|K|^\alpha}{|\log \varepsilon|}.$$

For the second summand we use Lemma 3.5(b) and Proposition 3.4 to see

$$\sum_{\substack{I \in \mathcal{A}_\varepsilon \\ I \subseteq K}} \int_{\mathbb{R} \setminus I^0(\kappa)} \psi_\varepsilon^x(I)\, d\bar{\mu}(x) \leq \mathcal{C}_5 \cdot \sum_{\substack{I \in \mathcal{A}, |I| \geq \varepsilon \\ I \subseteq K}} \frac{|I|^\alpha}{|\log \varepsilon|} \cdot \left(\frac{1}{\kappa}\right)^{1-\alpha} \leq \mathcal{C}_5 \mathcal{C}_2 \cdot |K|^\alpha \cdot \left(\frac{1}{\kappa}\right)^{1-\alpha}.$$

This gives the first inequality. To prove the second inequality, apply the first part to the set $N_0 \in \mathcal{N}_{\varepsilon_0}$ that contains E_0. Denote $a = \min(N_0)$, $b = \max(N_0)$ and use Lemma 3.5(c) to infer

$$\int (\psi_\varepsilon^x(\mathbb{R}) - \psi_\varepsilon^x(N_0))\; d\bar{\mu}(x) \leq \int_a^b \psi_\varepsilon^x([a-1, a])\, d\bar{\mu}(x) + \int_a^b \psi_\varepsilon^x([b, b+1])\, d\bar{\mu}(x) \leq \frac{2\mathcal{C}_6}{|\log \varepsilon|} .$$

This finishes the proof of the second inequality. ■

4 Proof of Theorem 1.2

We may assume without loss of generality that μ is a finite measure. Recall the definition of the set E from the beginning of Section 3 and, as before, fix a set $E_0 \subseteq E$ with $|E_0| < \varepsilon_0$ and let $\bar{\mu} = \mu|_{E_0}$. Since, by Proposition 2.2,

$$\mathcal{P}(\mu, x) = \mathcal{P}(\bar{\mu}, x) \quad \text{for } \mu\text{-almost every } x \in E_0,$$

it suffices to prove (1) for the restricted measures $\bar{\mu}$ and $\bar{\mu}$-almost every point x. We fix a continuous function

$$\begin{array}{rccc} G: & \mathcal{M}(\mathbb{R}) \times \mathbb{R} & \longrightarrow & [0, \infty), \\ & (\nu, y) & \mapsto & g(y) \cdot h(\nu(f)) \end{array}$$

where $f : \mathbb{R} \longrightarrow [0, \infty)$, $g : \mathbb{R} \longrightarrow [0, \infty)$ are Lipschitz functions with compact support and $h : [0, \infty) \longrightarrow [0, 1]$ is a Lipschitz function. We start by showing that (1) holds for G and all

$P \in \mathcal{P}(\bar\mu, x)$ at $\bar\mu$-almost all points x. To this end we shall introduce a family (φ_I) of functions, the sum of which approximates the difference of the two sides of the formula (see Lemma 4.1) and show that the set of points where the approximating function has large modulus has small measure (see Lemma 4.2). In the following, we allow the constants $\mathcal{C}_9, \mathcal{C}_{10}, \ldots$ to depend on the choice of G. Define $G_1, G_2 : \mathcal{M}(\mathbb{R}) \longrightarrow [0,\infty)$ by

$$\begin{aligned} G_1(\nu) &= \int G(\nu, y)\, d\nu(y) = \int g(y) \cdot h\Big(\int f(w)\, d\nu(w) \Big)\, d\nu(y)\,, \\ G_2(\nu) &= \int G(T^y\nu, -y)\, d\nu(y) = \int g(-y) \cdot h\Big(\int f(w-y)\, d\nu(w) \Big) d\nu(y)\,. \end{aligned}$$

G_1 and G_2 are continuous and there is $\mathcal{C}_9 > 0$ such that, for all $x \in \mathbb{R}$ and $t > 0$,

$$G_1\Big(\frac{\bar\mu_{x,t}}{t^\alpha}\Big), G_2\Big(\frac{\bar\mu_{x,t}}{t^\alpha}\Big) \le \|g\|_{\sup} \cdot \frac{\bar\mu([x-tR, x+tR])}{t^\alpha} \le \|g\|_{\sup} \cdot CR^\alpha = \mathcal{C}_9, \tag{7}$$

choosing $R \in \mathbb{N}$ such that $\operatorname{supp} g \subseteq [-R, R]$. Define a signed measure φ^x_ε by

$$\varphi^x_\varepsilon(A) = \int_{A\cap[x,\infty)} G_1\left(\frac{\bar\mu_{x,z-x}}{(z-x)^\alpha}\right) d\psi^x_\varepsilon(z) - \int_{A\cap(-\infty,x]} G_2\left(\frac{\bar\mu_{x,x-z}}{(x-z)^\alpha}\right) d\psi^x_\varepsilon(z)$$

for $A \subseteq \mathbb{R}$ Borel and observe that the total variation measure $|\varphi^x_\varepsilon|$ is dominated by $\mathcal{C}_9 \cdot \psi^x_\varepsilon$. Define the function $\kappa(\varepsilon) = (\log|\log\varepsilon|)^{6/(1-\alpha)}$. For every interval $I \subseteq \mathbb{R}$ define a function $\varphi_I : \mathbb{R} \times (0,\infty) \to \mathbb{R}$ by

$$\varphi_I(x,\varepsilon) = \begin{cases} \varphi^x_\varepsilon(I) & \text{if } x \in I^-(\kappa(\varepsilon)) \cup I^+(\kappa(\varepsilon)), \\ 0 & \text{otherwise.} \end{cases}$$

Observe that for all intervals $I \subseteq \mathbb{R}$, $\varepsilon > 0$ and for all $x \in \mathbb{R}$,

$$|\varphi_I(x,\varepsilon)| \le |\varphi^x_\varepsilon(I)| \le \mathcal{C}_9 \cdot \psi^x_\varepsilon(I), \tag{8}$$

and therefore, noting that $\psi^x_\varepsilon(\mathbb{R}) = 2$,

$$\sum_{I \in \mathcal{A}_\varepsilon} |\varphi_I(x,\varepsilon)| \le 2\mathcal{C}_9\,. \tag{9}$$

From the definition of G_1 and G_2 we get

$$G_1\Big(\frac{\bar\mu_{x,t}}{t^\alpha}\Big) = \int G\Big(\frac{\bar\mu_{x,t}}{t^\alpha}, y\Big)\, d\frac{\bar\mu_{x,t}}{t^\alpha}(y) = \int G\Big(\frac{\bar\mu_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar\mu(y)\,,$$

and

$$G_2\Big(\frac{\bar\mu_{x,t}}{t^\alpha}\Big) = \int G\Big(T^y\frac{\bar\mu_{x,t}}{t^\alpha}, -y\Big)\, d\frac{\bar\mu_{x,t}}{t^\alpha}(y) = \int G\Big(\frac{\bar\mu_{y,t}}{t^\alpha}, \frac{x-y}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar\mu(y)\,.$$

and hence we can derive the following expression for φ^x_ε

$$\begin{aligned} \varphi^x_\varepsilon(A) &= \int_{A\cap[x,\infty)} \int G\Big(\frac{\bar\mu_{x,z-x}}{(z-x)^\alpha}, \frac{y-x}{z-x}\Big)\, d\bar\mu(y)\, \frac{d\psi^x_\varepsilon(z)}{(z-x)^\alpha} \\ &\quad - \int_{A\cap(-\infty,x]} \int G\Big(\frac{\bar\mu_{y,x-z}}{(x-z)^\alpha}, \frac{x-y}{x-z}\Big)\, d\bar\mu(y)\, \frac{d\psi^x_\varepsilon(z)}{(x-z)^\alpha}\,. \end{aligned} \tag{10}$$

For small $\varepsilon > 0$ the function $\sum_{I\in\mathcal{A}_\varepsilon} \varphi_I(x,\varepsilon)$ is a good approximation of

$$\begin{aligned}\varphi_\varepsilon^x(\mathbb{R}) &= \int_x^\infty \int G\Big(\frac{\bar\mu_{x,z-x}}{(z-x)^\alpha}, \frac{y-x}{z-x}\Big)\, d\bar\mu(y)\, \frac{d\psi_\varepsilon^x(z)}{(z-x)^\alpha} - \int_{-\infty}^x \int G\Big(\frac{\bar\mu_{y,x-z}}{(x-z)^\alpha}, \frac{x-y}{x-z}\Big)\, d\bar\mu(y)\, \frac{d\psi_\varepsilon^x(z)}{(x-z)^\alpha} \\ &= (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\, \frac{dt}{t}\end{aligned}$$

as the following lemma shows.

Lemma 4.1 *There is a constant $\mathcal{C}_{10} > 0$ such that, for all $0 < \varepsilon \le \varepsilon_0/\mathcal{C}_1$ and $\sigma > 0$, we have*

$$\bar\mu\Big(\Big\{x \,:\, \Big|\frac{1}{|\log\varepsilon|}\int_\varepsilon^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\, \frac{dt}{t} - \sum_{I\in\mathcal{A}_\varepsilon} \varphi_I(x,\varepsilon)\Big| > \sigma\Big\}\Big) \le \frac{\mathcal{C}_{10}}{\sigma\cdot(\log|\log\varepsilon|)^2}.$$

Proof: We use $|\varphi_\varepsilon^x| \le \mathcal{C}_9 \cdot \psi_\varepsilon^x$ and Lemma 3.6 to get

$$\begin{aligned}&\sigma\cdot\bar\mu\Big(\Big\{x \,:\, |(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\, \frac{dt}{t} - \sum_{I\in\mathcal{A}_\varepsilon} \varphi_I(x,\varepsilon)| > \sigma\Big\}\Big) \\ &\le \int \Big|(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\, \frac{dt}{t} - \sum_{I\in\mathcal{A}_\varepsilon} \varphi_I(x,\varepsilon)\Big|\, d\bar\mu(x) \\ &= \int \Big|\varphi_\varepsilon^x\Big(\mathbb{R}\setminus \bigcup_{\substack{I\in\mathcal{A}_\varepsilon\\ x\in I^0(\kappa)}} I\Big)\Big|\, d\bar\mu(x) \\ &\le \mathcal{C}_9\cdot\int \Big(\psi_\varepsilon^x(\mathbb{R}) - \sum_{\substack{I\in\mathcal{A}_\varepsilon\\ x\in I^0(\kappa)}} \psi_\varepsilon^x(I)\Big)\, d\bar\mu(x) \\ &\le \mathcal{C}_9\mathcal{C}_8\cdot\Big(\Big(\frac{1}{\kappa(\varepsilon)}\Big)^{1-\alpha} + \frac{1}{|\log\varepsilon|}\Big) \le \frac{2\mathcal{C}_9\mathcal{C}_8}{(\log|\log\varepsilon|)^2}\,.\end{aligned}$$

■

We now show that the set of points $x\in E$ where the function $\sum_{I\in\mathcal{A}_\varepsilon} \varphi_I(x,\varepsilon)$ has large modulus is small. This is the main step in the proof.

Lemma 4.2 *For $\sigma > 0$ and $\varepsilon > 0$ denote*

$$B_\varepsilon = \Big\{x\in E \,:\, \Big|\sum_{I\in\mathcal{A}_\varepsilon} \varphi_I(x,\varepsilon)\Big| \ge \sigma\Big\}\,.$$

Then there is a constant $\mathcal{C}_{11} > 0$ such that, for every $\sigma > 0$ and all sufficiently small $\varepsilon > 0$,

$$\mu(B_\varepsilon) \le \frac{\mathcal{C}_{11}}{\sigma^2\cdot(\log|\log\varepsilon|)^2}\,.$$

Proof: Let $\varepsilon_1 = 1/e$ and define a sequence $\varepsilon_k \downarrow 0$ such that

$$\log\varepsilon_k = \Big(1 + (1/k)^{3/4}\Big)\cdot\log\varepsilon_{k-1} \text{ for all } k > 1.$$

For $0<\varepsilon<\varepsilon_1$ define $p=p(\varepsilon)\in\mathbb{N}$ such that $\varepsilon_{p-1}>\varepsilon\geq\varepsilon_p$, and define $\lambda=\lambda(\varepsilon)$ as the largest integer such that $\lambda(\varepsilon)\leq(\log|\log\varepsilon|)^2$. We establish some inequalities which hold for sufficiently small $\varepsilon>0$: Using

$$\log|\log\varepsilon|\leq\sum_{k=1}^{p}\log(1+(1/k)^{3/4})\leq\sum_{k=1}^{p}(1/k)^{3/4}\leq\int_0^p\frac{dx}{x^{3/4}}=4\sqrt[4]{p}$$

and a similar chain of reverse inequalities we get constants $c_1,c_2>0$ such that

$$\text{(a)}\qquad c_1\cdot\Big(\log|\log\varepsilon|\Big)^4\leq p(\varepsilon)\leq c_2\cdot\Big(\log|\log\varepsilon|\Big)^4.$$

We also get a constant $c_3>0$ such that

$$\text{(b)}\qquad\Big|\frac{\log\varepsilon_\lambda}{\log\varepsilon}\Big|\leq\prod_{k=\lambda+1}^{p-1}\Big|\frac{\log\varepsilon_{k-1}}{\log\varepsilon_k}\Big|\leq\prod_{k=\lambda+1}^{p-1}\frac{k}{k+1}=\frac{\lambda+1}{p}\leq\frac{c_3}{(\log|\log\varepsilon|)^2}.$$

From

$$\log(\varepsilon_{k-1}/\varepsilon_k)=(-\log\varepsilon_{k-1})/k^{3/4}=\prod_{i=1}^{k-1}\Big(1+(1/i)^{3/4}\Big)\cdot(1/k)^{3/4}\geq k^{1/4}$$

we infer that $\log(\varepsilon_{k-1}/\varepsilon_k)$ is monotonically increasing and from the estimates on $\lambda(\varepsilon)$ and $p(\varepsilon)$ we thus get constants $c_4,c_5>0$ such that, for all $p(\varepsilon)\geq k\geq\lambda(\varepsilon)$,

$$\text{(c)}\qquad c_4\cdot\sqrt{\log|\log\varepsilon|}\leq\log\Big(\frac{\varepsilon_{k-1}}{\varepsilon_k}\Big)\leq c_5\cdot\frac{|\log\varepsilon|}{(\log|\log\varepsilon|)^3},$$

and, in particular, for all $p(\varepsilon)-1\geq k\geq\lambda(\varepsilon)$,

$$\text{(d)}\qquad\frac{\varepsilon_k}{\varepsilon_{k+1}}\geq\exp\Big[c_4\cdot\sqrt{\log|\log\varepsilon|}\Big]\geq(\log|\log\varepsilon|)^{\frac{6}{1-\alpha}}=\kappa(\varepsilon).$$

Fix $0<\varepsilon<\varepsilon_0/\mathcal{C}_1$ such that (a), (b), (c) and (d) hold. Denote $\kappa=\kappa(\varepsilon)$. Define $\mathcal{I}_1=\{I\in\mathcal{A}_\varepsilon:|I|\geq\varepsilon_1\}$ and $\mathcal{I}_k=\{I\in\mathcal{A}_\varepsilon:\varepsilon_{k-1}>|I|\geq\varepsilon_k\}$ for $k\geq2$. Then

$$\mathcal{A}_\varepsilon=\bigcup_{k=1}^{p}\mathcal{I}_k\,.$$

We estimate $\mu(B_\varepsilon)$ by means of the mean square of $\sum_{I\in\mathcal{A}_\varepsilon}\varphi_I$. Observing that $\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\leq 0$ unless $x\in I^-(\kappa)\cap J^-(\kappa)$ or $x\in I^+(\kappa)\cap J^+(\kappa)$ and using the natural partial order on the set of intervals we can write

$$\begin{aligned}\mu(B_\varepsilon)\cdot\sigma^2&\leq\int\Big(\sum_{I\in\mathcal{A}_\varepsilon}\varphi_I(x,\varepsilon)\Big)^2d\bar\mu(x)\\&=\int\Big(\sum_{k=1}^{p}\sum_{I\in\mathcal{I}_k}\varphi_I(x,\varepsilon)\Big)^2d\bar\mu(x)&(11)\\&\leq2\cdot\sum_{k=1}^{\lambda}\sum_{j=1}^{p}\sum_{I\in\mathcal{I}_j}\sum_{J\in\mathcal{I}_k}\int\Big|\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\Big|d\bar\mu(x)&(12)\end{aligned}$$

$$+ 2 \cdot \sum_{k=\lambda+1}^{p-1} \sum_{I \in \mathcal{I}_k \cup \mathcal{I}_{k+1}} \sum_{\substack{J \in \mathcal{I}_k \cup \mathcal{I}_{k+1} \\ I < J}} \int_{I^-(\kappa) \cap J^-(\kappa)} |\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)| \, d\bar{\mu}(x) \tag{13}$$

$$+ 2 \cdot \sum_{k=\lambda+1}^{p-1} \sum_{I \in \mathcal{I}_k \cup \mathcal{I}_{k+1}} \sum_{\substack{J \in \mathcal{I}_k \cup \mathcal{I}_{k+1} \\ I < J}} \int_{I^+(\kappa) \cap J^+(\kappa)} |\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)| \, d\bar{\mu}(x) \tag{14}$$

$$+ \sum_{k=\lambda+1}^{p} \sum_{I \in \mathcal{I}_k} \int |\varphi_I(x,\varepsilon)|^2 \, d\bar{\mu}(x) \tag{15}$$

$$+ 2 \cdot \int \sum_{k=\lambda+1}^{p-2} \sum_{j=k+2}^{p} \Big(\sum_{I \in \mathcal{I}_j} \sum_{J \in \mathcal{I}_k} \varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon) \Big) \, d\bar{\mu}(x) \, . \tag{16}$$

To estimate (12) we observe that, by (9), $\sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \Big|\varphi_I(x,\varepsilon)\Big| \le 2\mathcal{C}_9$, and therefore we have, using also (8), Lemma 3.5(c), Proposition 3.4 and (b),

$$\begin{aligned}
\sum_{k=1}^{\lambda} \sum_{J \in \mathcal{I}_k} \sum_{j=1}^{p} \sum_{I \in \mathcal{I}_j} \int \Big|\varphi_I(x,\varepsilon) \cdot \varphi_J(x,\varepsilon)\Big| \, d\bar{\mu}(x) &\le 2\mathcal{C}_9 \cdot \sum_{k=1}^{\lambda} \sum_{J \in \mathcal{I}_k} \int |\varphi_J(x,\varepsilon)| \, d\bar{\mu}(x) \\
\le \quad 2\mathcal{C}_9^2 \mathcal{C}_6 \cdot \sum_{k=1}^{\lambda} \sum_{J \in \mathcal{I}_k} \frac{|J|^\alpha}{|\log \varepsilon|} \le 2\mathcal{C}_9^2 \mathcal{C}_6 \mathcal{C}_3 \cdot \Big|\frac{\log \varepsilon_\lambda}{\log \varepsilon}\Big| &\le \frac{2\mathcal{C}_9^2 \mathcal{C}_6 \mathcal{C}_4 \cdot c_3}{(\log|\log \varepsilon|)^2} \, .
\end{aligned}$$

Let us look at (13) and fix an interval $I = (a,b) \in \mathcal{I}_k \cup \mathcal{I}_{k+1}$. If $J \in \mathcal{I}_k \cup \mathcal{I}_{k+1}$ then $|J| \le \varepsilon_{k-1}$ and if, moreover, $I < J$ and $I^-(\kappa) \cap J^-(\kappa) \ne \emptyset$, we infer

$$J \subseteq [b, a + (\kappa+1) \cdot \varepsilon_{k-1}] \, .$$

For all $x \in I^-(\kappa)$ we thus get, using (8) and $|I| \ge \varepsilon_{k+1}$,

$$\begin{aligned}
\sum |\varphi_J(x,\varepsilon)| \quad &\le \quad \mathcal{C}_9 \cdot \psi_\varepsilon^x([a + \varepsilon_{k+1}, a + (\kappa+1) \cdot \varepsilon_{k-1}]) \\
&\le \quad \mathcal{C}_9 \cdot \frac{\log(\kappa+1) + \log \varepsilon_{k-1} - \log \varepsilon_{k+1}}{|\log \varepsilon|},
\end{aligned}$$

where the sum extends over all $J \in \mathcal{I}_k \cup \mathcal{I}_{k+1}$ such that $I < J$ and $I^-(\kappa) \cap J^-(\kappa) \ne \emptyset$. We use this inequality and (9) to estimate

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p-1} \sum_{I \in \mathcal{I}_k \cup \mathcal{I}_{k+1}} \sum_{\substack{J \in \mathcal{I}_k \cup \mathcal{I}_{k+1} \\ I < J}} \int_{I^-(\kappa) \cap J^-(\kappa)} |\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)| \, d\bar{\mu}(x) \\
\le \quad &\mathcal{C}_9 \cdot \max_{k=\lambda+1}^{p-1} \frac{\log(\kappa+1) + \log \varepsilon_{k-1} - \log \varepsilon_{k+1}}{|\log \varepsilon|} \cdot \int 2 \cdot \sum_{I \in \mathcal{A}_\varepsilon} |\varphi_I(x,\varepsilon)| \, d\bar{\mu}(x) \\
\le \quad &4\mathcal{C}_9^2 \cdot \bar{\mu}(\mathbb{R}) \cdot \max_{k=\lambda+1}^{p-1} \Big(\frac{\log(\kappa(\varepsilon)+1)}{|\log \varepsilon|} + \frac{\log(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}})}{|\log \varepsilon|} \Big) \le \frac{1}{(\log|\log \varepsilon|)^2} \, ,
\end{aligned}$$

for sufficiently small $\varepsilon > 0$ by (c), finishing the estimate of term (13). Term (14) can be estimated in exactly the same manner.

Let us now consider (15), and estimate using (8), Lemma 3.5(d) and Proposition 3.4,

$$\sum_{k=\lambda+1}^{p}\sum_{I\in\mathcal{I}_k}\int|\varphi_I(x,\varepsilon)|^2\,d\bar{\mu}(x)\le\mathcal{C}_9^2\mathcal{C}_7\cdot\sum_{I\in\mathcal{A}_\varepsilon}\frac{|I|^\alpha}{|\log\varepsilon|^2}\le\frac{\mathcal{C}_9^2\mathcal{C}_7\mathcal{C}_3}{|\log\varepsilon|}\le\frac{1}{(\log|\log\varepsilon|)^2}$$

for sufficiently small $\varepsilon>0$.

We now look at term (16). Given $J\in\mathcal{I}_k$ we denote by $\mathcal{K}_J^-$, respectively $\mathcal{K}_J^+$, the collection of all $K\in\mathcal{N}_{\varepsilon_{k+1}}$ such that

$$K\cap J^-(\kappa)\neq\emptyset\ \text{, respectively }\ K\cap J^+(\kappa)\neq\emptyset.$$

Recall again that $\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\le 0$ unless $x\in I^-(\kappa)\cap J^-(\kappa)$ or $x\in I^+(\kappa)\cap J^+(\kappa)$.
Observe that, whenever $p-1\ge k\ge\lambda$, $J\in\mathcal{I}_k$, $j\ge k+2$, $I\in\mathcal{I}_j$ and $I^-(\kappa)\cap J^-(\kappa)\neq\emptyset$, there is a $K\in\mathcal{K}_J^-$ such that $I\subseteq K$. To see this we suppose the contrary. Since I is contained in some $K\in\mathcal{N}_{\varepsilon_{k+1}}$ we must have $I>J$ and hence $\varepsilon_k\le|J|<\kappa(\varepsilon)|I|\le\kappa(\varepsilon)\cdot\varepsilon_{k+1}$. This contradicts (d) and therefore our statement holds.
Also, by an analogous argument, if $J\in\mathcal{I}_k$, $I\in\mathcal{I}_j$ and $I^+(\kappa)\cap J^+(\kappa)\neq\emptyset$, there is $K\in\mathcal{K}_J^+$ such that $I\subseteq K$. Therefore we have

$$\sum_{k=\lambda+1}^{p-2}\sum_{j=k+2}^{p}\sum_{J\in\mathcal{I}_k}\sum_{I\in\mathcal{I}_j}\int\Big(\varphi_I(x,\varepsilon)\cdot\varphi_J(x,\varepsilon)\Big)\,d\bar{\mu}(x)$$

$$\le\sum_{k=\lambda+1}^{p-2}\sum_{j=k+2}^{p}\sum_{J\in\mathcal{I}_k}\sum_{K\in\mathcal{K}_J^-}\sum_{\substack{I\in\mathcal{I}_j\\ I\subseteq K}}\int\Big(\varphi_I(x,\varepsilon)\cdot\varphi_J(x,\varepsilon)\Big)\,d\bar{\mu}(x)\tag{17}$$

$$+\sum_{k=\lambda+1}^{p-2}\sum_{j=k+2}^{p}\sum_{J\in\mathcal{I}_k}\sum_{K\in\mathcal{K}_J^+}\sum_{\substack{I\in\mathcal{I}_j\\ I\subseteq K}}\int\Big(\varphi_I(x,\varepsilon)\cdot\varphi_J(x,\varepsilon)\Big)\,d\bar{\mu}(x)\,.\tag{18}$$

We can concentrate our investigation on one of these expressions, say (17), since the other one can be treated analogously. For $J\in\mathcal{I}_k$ and $K\in\mathcal{K}_J^-$ define

$$\tilde{K}=\big(K^-(\kappa)\cup K\big)\cap J^-(\kappa)$$

and

$$\tilde{\varphi}_K(x,\varepsilon)=\sum_{j=k+2}^{p}\sum_{\substack{I\in\mathcal{I}_j\\ I\subseteq K}}\varphi_I(x,\varepsilon)\,.$$

Let $K_J^0\in\mathcal{K}_J^-$ be the element of $\mathcal{K}_J^-$ adjacent to J, i.e. $K<K_J^0$ for all $K\in\mathcal{K}_J^-\setminus\{K_J^0\}$. Since $\tilde{\varphi}_K(x,\varepsilon)\cdot\varphi_J(x,\varepsilon)\le 0$ for all $x\in\mathbb{R}\setminus\tilde{K}$, we have

$$\sum_{k=\lambda+1}^{p-2}\sum_{j=k+2}^{p}\sum_{J\in\mathcal{I}_k}\sum_{K\in\mathcal{K}_J^-}\sum_{\substack{I\in\mathcal{I}_j\\ I\subseteq K}}\int\Big(\varphi_I(x,\varepsilon)\cdot\varphi_J(x,\varepsilon)\Big)\,d\bar{\mu}(x)$$

$$\le\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k}\int_{\tilde{K}_J^0}\Big|\tilde{\varphi}_{K_J^0}(x,\varepsilon)\varphi_J(x,\varepsilon)\Big|\,d\bar{\mu}(x)\tag{19}$$

$$+\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k}\sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}}\int_{\tilde{K}}\Big(\tilde{\varphi}_K(x,\varepsilon)\varphi_J(x,\varepsilon)\Big)\,d\bar{\mu}(x)\,.\tag{20}$$

Let us give the estimate for (19) first. Observe that, by Proposition 3.2, $|\tilde{K}_J^0| \leq (\kappa+1)\cdot \mathcal{C}_1 \cdot \varepsilon_{k+1}$ and therefore, denoting $J=(a,b)$,

$$\tilde{K}_J^0 \subseteq [a - (\mathcal{C}_1 \varepsilon_{k+1}(1+\kappa)), a] \subseteq [a - \delta|J|, a]\,,$$

where

$$\delta := \mathcal{C}_1(1+\kappa(\varepsilon)) \frac{\varepsilon_{\lambda+2}}{\varepsilon_{\lambda+1}} \geq \mathcal{C}_1(1+\kappa(\varepsilon)) \cdot \frac{\varepsilon_{k+1}}{|J|}.$$

Since, by (9), $|\tilde{\varphi}_{K_J^0}(x,\varepsilon)| \leq 2\mathcal{C}_9$ we get, using also (8), Lemma 3.5(a) and Proposition 3.4,

$$\begin{aligned}
\sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \int_{\tilde{K}_J^0} \left|\tilde{\varphi}_{K_J^0}(x,\varepsilon)\varphi_J(x,\varepsilon)\right| d\bar{\mu}(x) &\leq 2\mathcal{C}_9^2 \cdot \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \int_{J^-(\delta)} \psi_\varepsilon^x(J)\, d\bar{\mu}(x) \\
&\leq 2\mathcal{C}_4\mathcal{C}_9^2 \cdot \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \frac{|J|^\alpha}{|\log\varepsilon|} \log\Big(\frac{\delta+1}{\delta}\Big)\cdot \delta^\alpha \\
&\leq 2\mathcal{C}_4\mathcal{C}_9^2\mathcal{C}_3 \cdot \log 2 \cdot \Big[\mathcal{C}_1(1+\kappa(\varepsilon)) \cdot \frac{\varepsilon_{\lambda+2}}{\varepsilon_{\lambda+1}}\Big]^\alpha \\
&\leq \frac{1}{(\log|\log\varepsilon|)^2}\,,
\end{aligned}$$

for sufficiently small $\varepsilon > 0$ by (c), finishing the estimate of term (19).

It remains to investigate (20). This is the crucial part. In order to carry out the estimate we first observe that for every $J \in \mathcal{I}_k$ and $K \in \mathcal{K}_J^- \setminus \{K_J^0\}$, $x \in \tilde{K}$ we have

$$\varphi_J(x,\varepsilon) = (|\log\varepsilon|)^{-1} \int_J \int g\Big(\frac{y-x}{z-x}\Big) \cdot \frac{d\bar{\mu}(y)}{(z-x)^\alpha} \cdot \Big(h\Big[\int f\Big(\frac{w-x}{z-x}\Big)\,\frac{d\bar{\mu}(w)}{(z-x)^\alpha}\Big]\Big)\,\frac{dz}{z-x}.$$

A straightforward calculation shows that there is a constant $\mathcal{C}_{12} > 0$ such that $x \mapsto \varphi_J(x,\varepsilon)$ is Lipschitz on the domain $\tilde{K}$ with a Lipschitz constant $c(K,J)$ such that

$$c(K,J) \leq \frac{\mathcal{C}_{12}}{|\log\varepsilon|} \cdot \log\Big(\frac{d(K,J)+|J|}{d(K,J)}\Big) \cdot \frac{1}{d(K,J)}\,, \tag{21}$$

where $d(K,J)$ denotes the distance of K and J. We now show, and this is a crucial step in our proof, that here is a constant $\mathcal{C}_{13} > 0$ such that, for all $K \in \mathcal{K}_J^-$,

$$\left|\int_{\tilde{K}} \varphi_\varepsilon^x(K)\, d\bar{\mu}(x)\right| \leq \mathcal{C}_{13} \cdot \frac{(\kappa(\varepsilon)+1)^\alpha}{|\log\varepsilon|} \cdot |K|^\alpha\,. \tag{22}$$

For this purpose recall equation (10) and observe that

$$\begin{aligned}
\int_{\tilde{K}} \varphi_\varepsilon^x(K)\, d\bar{\mu}(x) &= (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \Big[\int_{\tilde{K}\cap(K-t)} \Big(\int G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}(y)\Big)\, d\bar{\mu}(x) \\
&\quad - \int_{\tilde{K}\cap(K+t)} \Big(\int G\Big(\frac{\bar{\mu}_{y,t}}{t^\alpha}, \frac{x-y}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}(y)\Big)\, d\bar{\mu}(x)\Big]\,\frac{dt}{t} \\
&= (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \Big[\int_{K(t)} G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}^2(x,y) \\
&\quad - \int_{K(-t)} G\Big(\frac{\bar{\mu}_{x,t}}{t^\alpha}, \frac{y-x}{t}\Big) \cdot \frac{1}{t^\alpha}\, d\bar{\mu}^2(x,y)\Big]\,\frac{dt}{t}\,,
\end{aligned}$$

where, choosing R such that supp $g \subseteq [-R,R]$,

$$K(t) = \Big\{(x,y)\ :\ x \in (K-t)\cap \tilde{K}\,,\ \frac{y-x}{t} \in [-R,R]\Big\}.$$

Thus we can use the cancelation and get

$$\int_{\tilde{K}} \varphi_\varepsilon^x(K)\, d\bar{\mu}(x) \le (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \|g\|_{\sup} \cdot \bar{\mu}^2(K(t)\setminus K(-t))\, \frac{dt}{t^{1+\alpha}}, \tag{23}$$

and

$$-\int_{\tilde{K}} \varphi_\varepsilon^x(K)\, d\bar{\mu}(x) \le (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \|g\|_{\sup} \cdot \bar{\mu}^2(K(-t)\setminus K(t))\, \frac{dt}{t^{1+\alpha}}, \tag{24}$$

recalling that G is bounded by $\|g\|_{\sup}$. For $t \in \mathbb{R}$ we take a closer look at the sets

$$K(t)\setminus K(-t) = \Big\{(x,y)\ :\ x \in (K-t)\cap\tilde{K}\,,\ \frac{y-x}{t} \in [-R,R]\,,\ y \notin (K+t)\cap\tilde{K}\Big\}.$$

First observe that if $|t| > |K|(\kappa(\varepsilon)+1)$ we have $(K-t)\cap\tilde{K} = \emptyset$ and thus $K(t)\setminus K(-t) = \emptyset$. Otherwise if $(x,y) \in K(t)\setminus K(-t)$ then, denoting the distance of z and K by $d(z,K)$,

$$y \in S := \{z+t\ :\ d(z,K) \le R|t| + 2|t| \ \text{ and } \ z \notin K\} \cup \{z\ :\ d(z,\tilde{K}) \le R|t| \ \text{ and } \ z \notin \tilde{K}\}.$$

Then, by (6), $\bar{\mu}(S) \le 2C\cdot |t|^\alpha \cdot [(R+2)^\alpha + R^\alpha]$ and thus

$$\bar{\mu}^2(K(t)\setminus K(-t)) \le \int_S \bar{\mu}([y-|t|R, y+|t|R])\, d\bar{\mu}(y) \le 2C^2\cdot |t|^{2\alpha}\cdot R^\alpha \cdot [R^\alpha + (R+2)^\alpha]. \tag{25}$$

Let $\mathcal{C}_{13} = 2C^2/\alpha \cdot \|g\|_{\sup}\cdot R^\alpha(R^\alpha + (R+2)^\alpha)$. Then (23),(24) and (25) give

$$\Big|\int_{\tilde{K}} \varphi_\varepsilon^x(K)\, d\bar{\mu}(x)\Big| \le (|\log\varepsilon|)^{-1}\cdot\alpha\cdot\mathcal{C}_{13}\cdot \int_\varepsilon^{|K|(\kappa+1)} t^{\alpha-1}\, dt \le \mathcal{C}_{13}\cdot \frac{|K|^\alpha\cdot(\kappa(\varepsilon)+1)^\alpha}{|\log\varepsilon|},$$

which is the required estimate (22).

We can now split (20) again, writing ζ_K for the right endpoint of K and using the Lipschitz property of φ_J,

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}} \int_{\tilde{K}} \Big(\tilde{\varphi}_K(x,\varepsilon)\varphi_J(x,\varepsilon)\Big)\, d\bar{\mu}(x) \\
&\le \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}} c(K,J)\cdot|\tilde{K}|\cdot \int_{\tilde{K}} |\tilde{\varphi}_K(x,\varepsilon)|\, d\bar{\mu}(x) && (26)\\
&\quad + \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}} |\varphi_J(\zeta_K,\varepsilon)|\cdot \int_{\tilde{K}} |\varphi_\varepsilon^x(K) - \tilde{\varphi}_K(x,\varepsilon)|\, d\bar{\mu}(x) && (27)\\
&\quad + \sum_{k=\lambda+1}^{p-2} \sum_{J\in\mathcal{I}_k} \sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}} |\varphi_J(\zeta_K,\varepsilon)|\cdot \Big|\int_{\tilde{K}} \varphi_\varepsilon^x(K)\, d\bar{\mu}(x)\Big|. && (28)
\end{aligned}$$

To finish the proof we have to give estimates for (26) to (28).
Let us start by considering (26). Using (8), Lemma 3.5(c) and Propositions 3.4 and 3.2 we get

$$\int_{\tilde{K}} |\tilde{\varphi}_K(x,\varepsilon)|\, d\bar{\mu}(x) \;\leq\; 2\mathcal{C}_9\mathcal{C}_6 \cdot \sum_{\substack{I\in\mathcal{A}_\varepsilon\\ I\subseteq K}} \frac{|I|^\alpha}{|\log\varepsilon|} \leq 2\mathcal{C}_9\mathcal{C}_6\mathcal{C}_3 \cdot |K|^\alpha \leq 2\mathcal{C}_9\mathcal{C}_6\mathcal{C}_3\mathcal{C}_1^\alpha \cdot \varepsilon_{k+1}^\alpha\,. \tag{29}$$

Using $\kappa\cdot|J| \geq d(K,J) \geq \varepsilon_{k+1} \geq |K|/\mathcal{C}_1$ for all $K\in\mathcal{K}_J^-\setminus\{K_J^0\}$, we get

$$\begin{aligned}
\sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}} \frac{|K|}{d(K,J)} &= \sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}} \frac{|K|+d(K,J)}{d(K,J)}\cdot\frac{|K|}{|K|+d(K,J)} \leq (1+\mathcal{C}_1)\cdot\int_A \frac{dt}{t}\\
&\leq (1+\mathcal{C}_1)\cdot\log\Big(\kappa\cdot\Big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}}\Big)+\mathcal{C}_1\Big).
\end{aligned} \tag{30}$$

where the domain of integration is $A=\{t\,:\,\varepsilon_{k+1}\leq t\leq(\kappa|J|+\mathcal{C}_1\varepsilon_{k+1})\}$. We can now use (21), (29), (30) and Proposition 3.4 and get, abbreviating $\mathcal{C}_{14}=2\mathcal{C}_9\mathcal{C}_6\mathcal{C}_3\mathcal{C}_1^\alpha\mathcal{C}_{12}(1+\mathcal{C}_1)$,

$$\begin{aligned}
&\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k}\sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}} c(K,J)\cdot|\tilde{K}|\cdot\int_{\tilde{K}}|\tilde{\varphi}_K(x,\varepsilon)|\,d\bar{\mu}(x)\\
&\leq\; 2\mathcal{C}_9\mathcal{C}_6\mathcal{C}_3\mathcal{C}_1^\alpha\mathcal{C}_{12}\cdot\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k}\varepsilon_{k+1}^\alpha\cdot\frac{\kappa(\varepsilon)+1}{|\log\varepsilon|}\cdot\Bigg(\sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}}\frac{|K|}{d(K,J)}\cdot\log\Big(\frac{|J|}{d(K,J)}+1\Big)\Bigg)\\
&\leq\; \mathcal{C}_{14}\cdot\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k}\Bigg(\frac{(\kappa+1)\cdot\varepsilon_{k+1}^\alpha}{|\log\varepsilon|}\cdot\log\Big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}}+1\Big)\cdot\log\Big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}}\cdot\kappa+\mathcal{C}_1\Big)\Bigg)\\
&\leq\; \mathcal{C}_{14}\cdot\Big(\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k}\frac{|J|^\alpha}{|\log\varepsilon|}\Big)\cdot\max_{k=\lambda+1}^{p-2}\Big[\Big(\frac{\varepsilon_{k+1}}{\varepsilon_k}\Big)^\alpha\cdot\log^2\Big(\frac{\varepsilon_{k-1}}{\varepsilon_{k+1}}\cdot\kappa(\varepsilon)+\mathcal{C}_1\Big)\cdot(\kappa(\varepsilon)+1)\Big]\\
&\leq\; \frac{1}{(\log|\log\varepsilon|)^2}\,,
\end{aligned}$$

for sufficiently small $\varepsilon>0$ by (c), and this finishes the estimate of term (26).

Let us give an estimate for (27). We use $|\varphi_\varepsilon^x|\leq\mathcal{C}_9\cdot\psi_\varepsilon^x$ and Lemma 3.6 to infer

$$\begin{aligned}
\int_{\tilde{K}}|\varphi_\varepsilon^x(K)-\tilde{\varphi}_K(x,\varepsilon)|\,d\bar{\mu}(x) &= \int_{\tilde{K}}\Big|\varphi_\varepsilon^x\Big(K\setminus\bigcup_{\substack{I\in\mathcal{A}_\varepsilon,I\subseteq K\\ x\in I^0(\kappa)}} I\Big)\Big|\,d\bar{\mu}(x)\\
&\leq \mathcal{C}_9\cdot\int\Big(\psi_\varepsilon^x(K)-\sum_{\substack{I\in\mathcal{A}_\varepsilon,I\subseteq K\\ x\in I^0(\kappa)}}\psi_\varepsilon^x(I)\Big)\,d\bar{\mu}(x)\\
&\leq 2\mathcal{C}_9\mathcal{C}_8\cdot\frac{|K|^\alpha}{(\log|\log\varepsilon|)^6}\,.
\end{aligned}$$

Recall from (9) that, for every $K\in\mathcal{N}_{\varepsilon_{k+1}}$, $\sum_{J\in\mathcal{I}_k}|\varphi_J(\zeta_K,\varepsilon)|\leq 2\mathcal{C}_9$. For (27) we get, using Proposition 3.3,

$$\sum_{k=\lambda+1}^{p-2}\sum_{J\in\mathcal{I}_k}\sum_{K\in\mathcal{K}_J^-\setminus\{K_J^0\}}|\varphi_J(\zeta_K,\varepsilon)|\cdot\int_{\tilde{K}}|\varphi_\varepsilon^x(K)-\tilde{\varphi}_K(x,\varepsilon)|\,d\bar{\mu}(x)$$

$$\leq \sum_{k=\lambda+1}^{p-2} \sum_{K \in \mathcal{N}_{\varepsilon_{k+1}}} (2\mathcal{C}_9) \cdot \Big(2\mathcal{C}_9 \mathcal{C}_8 \cdot \frac{|K|^\alpha}{(\log|\log\varepsilon|)^6} \Big)$$

$$\leq (4\mathcal{C}_9^2 \mathcal{C}_8 \mathcal{C}_2) \cdot \frac{p(\varepsilon)}{(\log|\log\varepsilon|)^6} \leq (4\mathcal{C}_9^2 \mathcal{C}_8 \mathcal{C}_2 c_2) \cdot \frac{1}{(\log|\log\varepsilon|)^2},$$

for sufficiently small $\varepsilon > 0$ by (a). This finishes the estimate of (27).

Finally consider (28). Use (22), (9) and Proposition 3.3 to estimate

$$\sum_{k=\lambda+1}^{p-2} \sum_{J \in \mathcal{I}_k} \sum_{K \in \mathcal{K}_J^- \setminus \{K_J^0\}} |\varphi_J(\zeta_K, \varepsilon)| \cdot \Big| \int_{\tilde{K}} \varphi_\varepsilon^x(K)\, d\bar{\mu}(x) \Big|$$

$$\leq \sum_{k=\lambda+1}^{p-2} \sum_{K \in \mathcal{N}_{\varepsilon_{k+1}}} (2\mathcal{C}_9) \cdot \Big(\mathcal{C}_{13} \cdot \frac{(\kappa(\varepsilon)+1)^\alpha}{|\log\varepsilon|} \cdot |K|^\alpha \Big)$$

$$\leq (2\mathcal{C}_9 \mathcal{C}_{13} \mathcal{C}_2) \cdot \frac{(\kappa(\varepsilon)+1)^\alpha}{|\log\varepsilon|} \cdot p(\varepsilon) \leq \frac{1}{(\log|\log\varepsilon|)^2},$$

for sufficiently small $\varepsilon > 0$ by (a), finishing the estimate of (28).

We have thus finished the proof of Lemma 4.2 by showing that all the sums, in which we have split the original expression (11), are bounded by a constant multiple of $1/(\log|\log\varepsilon|)^2$. ■

Lemma 4.3 *The set*

$$\Big\{ x : \int\int G(\nu, y)\, d\nu(y)\, dP(\nu) = \int\int G(T^y\nu, -y)\, d\nu(y)\, dP(\nu) \text{ for all } P \in \mathcal{P}(\bar{\mu}, x) \Big\}$$

has full $\bar{\mu}$-measure.

Proof: To begin with, fix $s > 1$ and let $\delta_k = \exp(-s^k)$. Let $1 \geq \sigma > 0$. We have

$$\Big| (|\log\delta_n|)^{-1} \int_{\delta_n}^1 \Big(G_1(\frac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\frac{\bar{\mu}_{x,t}}{t^\alpha}) \Big) \frac{dt}{t} \Big|$$

$$\leq \Big| (|\log\delta_n|)^{-1} \int_{\delta_n}^1 \Big(G_1(\frac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\frac{\bar{\mu}_{x,t}}{t^\alpha}) \Big) \frac{dt}{t} - \sum_{I \in \mathcal{A}_{\delta_n}} \varphi_I(x, \delta_n) \Big| + \Big| \sum_{I \in \mathcal{A}_{\delta_n}} \varphi_I(x, \delta_n) \Big|.$$

Lemma 4.1 and Lemma 4.2 therefore give, for sufficiently large $n \in \mathbb{N}$,

$$\bar{\mu} \left(\left\{ x : \Big| (|\log\delta_n|)^{-1} \int_{\delta_n}^1 \Big(G_1(\frac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\frac{\bar{\mu}_{x,t}}{t^\alpha}) \Big) \frac{dt}{t} \Big| > 2\sigma \right\} \right) \leq \frac{\mathcal{C}_{10} + \mathcal{C}_{11}}{\sigma^2 \cdot (\log s)^2 \cdot n^2}.$$

Since $\sum_{n=1}^\infty (1/n)^2 < \infty$, the Borel-Cantelli-Lemma yields

$$\bar{\mu} \left(\left\{ x : \limsup_{n\to\infty} \Big| (|\log\delta_n|)^{-1} \int_{\delta_n}^1 \Big(G_1(\frac{\bar{\mu}_{x,t}}{t^\alpha}) - G_2(\frac{\bar{\mu}_{x,t}}{t^\alpha}) \Big) \frac{dt}{t} \Big| > 0 \right\} \right) = 0. \tag{31}$$

For every $\delta_n < \varepsilon \le \delta_{n-1}$ we have

$$(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\,\frac{dt}{t} = (|\log\varepsilon|)^{-1}\int_\varepsilon^{\delta_n} \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\,\frac{dt}{t}$$
$$+\frac{|\log\delta_n|}{|\log\varepsilon|}\cdot(|\log\delta_n|)^{-1}\int_{\delta_n}^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\,\frac{dt}{t}$$

Now $|\log\delta_n|/|\log\varepsilon| \le s$ and thus

$$\left|(|\log\varepsilon|)^{-1}\int_\varepsilon^{\delta_n} \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\,\frac{dt}{t}\right| \le \mathcal{C}_9\cdot\left|\frac{\log(\delta_n/\varepsilon)}{\log\varepsilon}\right| \le \mathcal{C}_9\cdot(s-1).$$

This and (31) together imply, for $\bar\mu$-almost every x,

$$\limsup_{\varepsilon\downarrow 0}\left|(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\,\frac{dt}{t}\right| \le \mathcal{C}_9\cdot(s-1)\,.$$

Since this holds for all $s>1$, we get

$$\lim_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\int_\varepsilon^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\,\frac{dt}{t} = 0$$

for $\bar\mu$-almost all x. By Proposition 2.2(2) the closure of the set $\{\frac{\bar\mu_{x,t}}{t^\alpha} : t\in(0,1)\}$ is compact, hence the continuous functions G_1 and G_2 are bounded on the set and, for $\bar\mu$-almost every x, every tangent measure distribution $P=\lim_{n\to\infty}P^x_{\varepsilon_n}$ of $\bar\mu$ at x fulfills

$$\begin{aligned}
\int\!\!\int G(\nu,y)\,d\nu(y)\,dP(\nu) &= \lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 G_1(\frac{\bar\mu_{x,t}}{t^\alpha})\,\frac{dt}{t}\\
&= \lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 \Big(G_1(\frac{\bar\mu_{x,t}}{t^\alpha}) - G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\Big)\,\frac{dt}{t} + \lim_{n\to\infty}(|\log\varepsilon_n|)^{-1}\int_{\varepsilon_n}^1 G_2(\frac{\bar\mu_{x,t}}{t^\alpha})\,\frac{dt}{t}\\
&= \int\!\!\int G(T^y\nu,-y)\,d\nu(y)\,dP(\nu)
\end{aligned}$$

as required. ∎

To finish the proof it remains to show that the set of all x where the Palm formula (1) holds for *all* Borel measurable functions $G:\mathcal{M}(\mathbb{R})\times\mathbb{R}\longrightarrow[0,\infty)$ has full measure. We use a straightforward approximation argument to show that there is a set $A\subseteq E$ with $\bar\mu(A)=\bar\mu(E)$ such that (1) holds for all $P\in\mathcal{P}(\bar\mu,x)$, $x\in A$ and all Borel functions $G:\mathcal{M}(\mathbb{R})\times\mathbb{R}\longrightarrow\mathbb{C}$ of the form $G(\nu,x)=g(x)\cdot\exp(i\nu(f))$ for g,f continuous with compact support and $g\ge 0$. Now fix such a function g and define two finite measures Λ_1, Λ_2 on the Borel σ-algebra of $\mathcal{M}(\mathbb{R})$ by means of

$$\begin{aligned}
\Lambda_1(M) &= \int\!\!\int g(y)1_M(\nu)\,d\nu(y)dP(\nu),\\
\Lambda_2(M) &= \int\!\!\int g(-y)1_M(T^y\nu)\,d\nu(y)dP(\nu)
\end{aligned}$$

for all Borel sets $M\subseteq\mathcal{M}(\mathbb{R})$. Λ_1 and Λ_2 coincide because their Fourier transforms coincide and hence the Palm formula holds for all $x\in A$, $P\in\mathcal{P}(\bar\mu,x)$ and all bounded functions G of the form $G(\nu,y)=g(y)\cdot F(\nu)$ for Borel functions $F:\mathcal{M}(\mathbb{R})\to[0,\infty)$ and for $g:\mathbb{R}\to[0,\infty)$ continuous with compact support. Again by a standard approximation argument this can be extended to the full statement of Theorem 1.2.

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