A set with finite curvature and projections of zero length

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1 Introduction

A compact subset E of the complex plane is called *removable* if all bounded analytic functions on its complement are constant or, equivalently, if its analytic capacity vanishes. The problem of finding a geometric characterization of the removable sets is more than a hundred years old and still not completely solved (see [5] for an introduction and [7] for a recent survey). However, there has been enormous progress in the last couple of years and this is to a large extent due to the introduction of the notion of the curvature of a set by Melnikov in [10]. We say that a compact subset E of the plane has *finite (Melnikov) curvature* if there is a nonzero measure μ supported by E, such that $\mu(B) \leq |B|$ for every closed ball B of diameter $|B|$ and such that

$$c^2(\mu) = \int_E \int_E \int_E \frac{1}{R(x,y,z)^2} \, d\mu(x) \, d\mu(y) \, d\mu(z) < \infty \,,$$

where $R(x,y,z)$ denotes the radius of the ball whose circumference passes through the points x, y and z. $c^2(\mu)$ is the *Melnikov curvature* of the measure μ. The importance of the notion of Melnikov curvature is on the one hand due to the fact that it is linked to the L^2-norm of the Cauchy operator on E and hence to the existence of bounded analytic functions on the complement of E, see [10] where this relationship was first established, and on the other hand due to the connection between curvature and rectifiability properties of E, that is projection properties of E, a fact which was first exploited by Mattila, Melnikov and Verdera in [9]. The latter relation found its final form in the following theorem, the harder 'only if' part of which was shown recently by David and Lger, see [3].

Theorem 1.1 *A compact set in the plane of finite length has finite Melnikov curvature if and only if it projects onto a set of positive measure in almost all directions.*

The development of these ideas culminated in the geometric characterization of the removable sets amongst the sets of finite length. In 1997 David finished in [1] the proof of the following statement, known as Vitushkin's conjecture.

Theorem 1.2 *A compact subset of the plane of finite length is removable if and only if it projects onto a set of length zero in almost all directions.*

This theorem does not completely solve the problem of a geometric characterization of sets of zero analytic capacity. Already in 1986 Mattila [6] showed that Vitushkin's conjecture fails to

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hold if the assumption of finite length is removed and, about two years later, Jones and Murai [2] came up with an explicit example of a set of positive analytic capacity which projects onto a set of zero length in almost all directions. The purpose of our research was to construct an example of a compact set of finite curvature which projects onto sets of zero length in all directions, thus showing that the theorem of David and Lger fails without the finite length assumption. As every set of finite curvature has positive analytic capacity, see [10], our example also serves as a counterexample to Vitushkin's conjecture in the case of infinite length and we believe that it is easier than the construction of Jones and Murai.

There is another interesting aspect to our example. Recall that a set of dimension strictly larger than one automatically projects onto sets of positive length in almost all directions, a fact first proved by Marstrand. Hence every counterexample to the above theorems is necessarily of dimension one but must have infinite, in fact non-σ-finite, length. To explore this margin a bit more we need to measure the size of a set by means of gauge functions φ. Let φ be a nondecreasing function on an interval $[0, \varepsilon)$ with $\varphi(0) = 0$. The *(generalized) Hausdorff measure* $\mathcal{H}^\varphi$ is defined by

$$\mathcal{H}^\varphi(E) = \sup_{\delta>0} \inf \Big\{ \sum_{i=1}^{\infty} \varphi(|U_i|) \ : \ E \subseteq \bigcup U_i, \ |U_i| \le \delta \Big\} ,$$

where the infimum runs through all coverings of E by sets U_i of diameter $|U_i| \le \delta$. The concept of generalized Hausdorff measures allows a stronger formulation of Marstrand's Theorem, which follows by means of a careful rereading of the proof given in [5].

Theorem 1.3 *If $E \subseteq \mathbb{R}^2$ fulfills $0 < \mathcal{H}^\varphi(E) < \infty$ for some φ with $\int_0^\varepsilon \varphi(r)/r^2 < \infty$, then E projects onto a set of positive length in almost all directions.*

Our example shows that this theorem is, in a sense, the strongest possible. More precisely we show:

Theorem 1.4 *There is a compact set $E \subseteq \mathbb{R}^2$ such that the projection of E in every direction has length zero and $0 < \mathcal{H}^\varphi(E) < \infty$ for a gauge function φ fulfilling $\int_0^\varepsilon \varphi(r)^\alpha/r^{\alpha+1} < \infty$ for every $\alpha > 1$.*

A construction similar to ours was used by Martin and Mattila in [4] to construct s-sets with projections of vanishing s-dimensional Hausdorff measure.

In the following we prove Theorem 1.4 and then show that the statement of the theorem automatically implies that E has finite curvature.

2 The construction of the set

We pick angles $\alpha_1, \alpha_2, \ldots$ with

$$\alpha_j = \frac{\pi}{2^n} \text{ for all } 2^n \le j < 2^{n+1} .$$

We choose a constant $M \geq 3$ such that, for all $k \geq 1$,

$$\frac{1}{M} < \frac{1}{12} \cdot (k+1) \cdot \sin \alpha_{k+1} \,. \tag{1}$$

We define $m_k = M \cdot k$ and write $m(k) = m_1 \cdots m_k$. We find an increasing sequence of numbers $1/2 < \beta_k < 1$ converging to 1 so slowly that

$$(k+1)^2 \, m(k)^{\beta_{16m(k)}-1} \longrightarrow 0 \,, \tag{2}$$

and define a sequence σ_k by

$$\sigma_k = \Big(\frac{k+1}{k}\Big)^{\beta_k} .$$

We start the construction of our set with a closed ball of diameter 1, which we denote by E_0. In the first step we place m_1 closed balls of diameter σ_1/m_1 inside E_0 such that the centres are on the diameter of E_0 which makes an angle of α_1 with the x-axis. We place these balls in such a way that they overlap as little as possible, so that two balls touch the boundary of E_0 and the distance between the centres of any two neighbouring balls is $(1-\sigma_1/m_1)/(m_1-1)$. We call these balls the *balls of the first stage* and the union of these balls is the set E_1.

We proceed like this in every ball. To construct the set E_{k+1} we put into every one of the $m(k)$ balls making E_k exactly m_{k+1} small closed balls with the centres on a diameter making an angle of $\sum_{i=1}^{k+1} \alpha_i$ with the x-axis. The diameters of these small balls are $\sigma_1 \cdots \sigma_{k+1}/m(k+1)$ and, in order to make their overlap as small as possible, we have to choose the distance between their centres equal to

$$d_{k+1} := \frac{\sigma_1 \cdots \sigma_k}{m_1 \cdots m_k} \cdot \frac{1-\sigma_{k+1}/m_{k+1}}{m_{k+1}-1} \,.$$

These balls are the *balls of* $(k+1)$*th stage* and their union is the set E_{k+1}. Finally, we define the compact set $E = \bigcap_{i=0}^{\infty} E_i$.

2.1 The size of the set

We first show that two balls of the $(k+1)$th stage can intersect only if they are in the same ball of kth stage. For this purpose we look at the two parallel lines through either centre of two neighbouring balls of kth stage that make an angle of α_{k+1} with the line through the centres of the balls. The distance between these two lines is $\sin \alpha_{k+1} \cdot d_k$. The centres of the balls of $(k+1)$th stage in the two balls are on these lines and hence there is no overlap between the balls of $(k+1)$th stage if the distance between the lines exceeds the diameter of the balls of $(k+1)$th stage, *i.e.* if

$$\sin \alpha_{k+1} \cdot d_k > \frac{\sigma_1 \cdots \sigma_{k+1}}{m_1 \cdots m_{k+1}} \,. \tag{3}$$

This follows from our choice of M in (1). Namely, as $\sigma_j/m_j \leq 2/3$ and $\sigma_j \leq 2$ for all $j \geq 1$,

$$\begin{aligned} \sin \alpha_{k+1} \cdot d_k &> \frac{12}{M \cdot (k+1)} \cdot \frac{\sigma_1 \cdots \sigma_{k-1}}{m_1 \cdots m_{k-1}} \cdot \frac{1-\sigma_k/m_k}{m_k - 1} \\ &\geq \frac{\sigma_1 \cdots \sigma_{k+1}}{m_1 \cdots m_{k+1}} \,, \end{aligned}$$

and this is (3). Hence we have shown that with this choice of M no two balls of $(k+1)$th stage which are not in the same ball of kth stage can intersect. It is worth noting that this allows a coding of our set, more explicitly there is a natural bijection

$$p : \prod_{k=1}^{\infty} \{1, \ldots, m_k\} \to E \, .$$

We now choose a gauge function φ such that $0 < \mathcal{H}^{\varphi}(E) < \infty$. We define a piecewise linear function φ by

$$\varphi(r) = r/(\sigma_1 \cdots \sigma_k) \text{ for all } d_{k+1} \le r < d_k .$$

Choosing the covering of E by the balls of kth stage gives $\mathcal{H}^{\varphi}(E) < \infty$. To show that $\mathcal{H}^{\varphi}(E) > 0$ we denote $\varepsilon = d_1$ and construct a probability measure μ on E such that, for all closed balls $B(x,r)$ with radius $r < \varepsilon$,

$$\mu(B(x,r)) \le 84 \cdot \varphi(r) \text{ for all } x \in E.$$

Let μ be the measure on E that we get from pushing forward the natural measure on the code space, more explicitly,

$$\mu(p(\{(x_i) \,:\, x_1 = a_1, \ldots, x_k = a_k\})) = 1/(m_1 \cdots m_k) \, .$$

In particular, the mass assigned to a set A of m adjacent balls of $(k+1)$th stage within a single ball of kth stage is bounded by

$$\mu(A) \le \frac{m+2}{m_1 \cdots m_{k+1}} \, .$$

Suppose $d_{k+1} \le r < d_k$ and denote $r = \rho \cdot \sigma_1 \cdots \sigma_k$. Then $B(x,r)$ hits at most 4 balls of kth stage and hence at most

$$4 \cdot \Big(3 + \frac{2r}{d_{k+1}}\Big)$$

balls of $(k+1)$th stage. Hence

$$\begin{aligned} \mu(B(x,r)) &\le \frac{4}{m_1 \cdots m_{k+1}} \Big(5 + \frac{2\rho m_1 \cdots m_k (m_{k+1}-1)}{1 - \sigma_{k+1}/m_{k+1}}\Big) \\ &\le \frac{20}{m_1 \cdots m_{k+1}} + \rho \cdot \frac{8}{1 - \sigma_{k+1}/m_{k+1}} \, . \end{aligned}$$

As

$$\rho \ge \frac{1}{m_1 \cdots m_k} \cdot \frac{1 - \sigma_{k+1}/m_{k+1}}{m_{k+1} - 1} \ge \frac{1}{3\, m_1 \cdots m_{k+1}} \, ,$$

we infer that, for all $r \le d_1$,

$$\mu(B(x,r)) \le 84 \cdot \rho = 84 \cdot \varphi(r) \, .$$

This implies, by a standard covering argument, that $\mathcal{H}^{\varphi}(E) > 0$.

We now show that the function φ fulfills the integral conditions. For this purpose let $\alpha > 1$. Observe that

$$\int_0^{\varepsilon} \frac{\varphi(r)^{\alpha}}{r^{1+\alpha}}\, dr = \sum_{k=1}^{\infty} \int_{d_{k+1}}^{d_k} \frac{\varphi(r)^{\alpha}}{r^{1+\alpha}}\, dr = \sum_{k=1}^{\infty} \frac{1}{(\sigma_1 \cdots \sigma_k)^{\alpha}} \cdot \int_{d_{k+1}}^{d_k} \frac{dr}{r} \, .$$

The last expression is, for some constant $C > 0$ and every $k > 1$,

$$\int_{d_{k+1}}^{d_k} \frac{dr}{r} = \log\Big(\frac{d_k}{d_{k+1}}\Big) = \log\Big(\frac{m(k)(m_{k+1}-1)(1-\sigma_k/m_k)}{m(k-1)\sigma_k(m_k-1)(1-\sigma_{k+1}/m_{k+1})}\Big) \le C \cdot \log k\,.$$

Fix an n such that $\beta_n \ge \beta$ for some $\beta > 1/\alpha$. Then, for a constant $C > 0$ and all $k \ge n$,

$$\frac{1}{(\sigma_1 \cdots \sigma_k)^\alpha} = \prod_{j=1}^{k} \Big(\frac{j}{j+1}\Big)^{\beta_j \alpha} \le C \cdot \Big(\frac{1}{k}\Big)^{\beta\alpha}.$$

Putting these facts together we get that, for any $\alpha > 1$ and some constant $C > 0$,

$$\int_0^\varepsilon \frac{\varphi(r)^\alpha}{r^{1+\alpha}}\, dr \le C \cdot \sum_{k=1}^{\infty} \frac{\log k}{k^{\beta\alpha}} < \infty\,.$$

2.2 The projections of the set

We denote the orthogonal projection on a line orthogonal to the direction θ by p_θ and the Lebesgue measure on a line by $\mathcal{L}$. To show that all projections of E on lines are of zero length we fix a projection direction $\theta \in [0, \pi)$ and a large integer k. Let N be the integer such that

$$\frac{\pi}{2^N} < \frac{\sigma_1 \cdots \sigma_k}{m(k)} \le \frac{\pi}{2^{N-1}}\,.$$

We now denote by $k_0 = k_0(k)$ the largest integer with $2^N \le k_0 < 2^{N+1}$ such that

$$\Big(\sum_{j=1}^{k_0} \alpha_j - \pi N\Big) - \theta < \frac{\sigma_1 \cdots \sigma_k}{m(k)}\,.$$

Note that $\alpha_{k_0} = \pi/2^N$ and

$$\pi \cdot \frac{m(k)}{\sigma_1 \cdots \sigma_k} \le k_0 < k_0 + 1 \le 4\pi \cdot \frac{m(k)}{\sigma_1 \cdots \sigma_k}\,. \tag{4}$$

Observe that

$$\sum_{j=k_0+1}^{k_0+k} \alpha_j \le k \cdot \frac{\pi}{2^N} < k \cdot \frac{\sigma_1 \cdots \sigma_k}{m(k)}\,.$$

We now look at any ball B of k_0th stage. The distance from the centre of a ball of $(k_0 + k)$th stage to the diameter of B in the direction θ is now bounded by

$$|B| \cdot \sin\Big[\sum_{j=k_0+1}^{k_0+k} \alpha_j + \Big(\sum_{j=1}^{k_0} \alpha_j - \pi N - \theta\Big)\Big]$$

and hence the projection of B in direction θ is contained in an interval of length

$$\frac{\sigma_1 \cdots \sigma_{k_0+k}}{m(k_0+k)} + \frac{\sigma_1 \cdots \sigma_{k_0}}{m(k_0)} \cdot \sin\Big[\sum_{j=k_0+1}^{k_0+k} \alpha_j + \Big(\sum_{j=1}^{k_0} \alpha_j - \pi N - \theta\Big)\Big]$$

$$\leq \frac{\sigma_1 \cdots \sigma_{k_0+k}}{m(k_0+k)} + (k+1) \cdot \frac{\sigma_1 \cdots \sigma_{k_0} \cdot \sigma_1 \cdots \sigma_k}{m(k_0)m(k)}.$$

Therefore, for every k, we may find a covering of $p_\theta(E)$ by intervals $I_1, \ldots, I_{m(k_0)}$ with

$$\begin{aligned}
\sum_{j=1}^{m(k_0)} |I_j| &\leq \frac{m(k_0)}{m(k_0+k)} \cdot \sigma_1 \cdots \sigma_{k_0+k} + (k+1) \cdot \frac{\sigma_1 \cdots \sigma_{k_0} \cdot \sigma_1 \cdots \sigma_k}{m(k)} \\
&\leq \frac{m(k_0)}{m(k_0+k)} \cdot (k_0+k+1) + \frac{k+1}{m(k)} \cdot (k_0+1)^{\beta_{k_0}} \cdot (k+1)^{\beta_k} \\
&\overset{(4)}{\leq} \frac{m(k_0)}{m(k_0+k)} \cdot \Big(\frac{4\pi\, m(k)}{\sigma_1 \cdots \sigma_k} + k\Big) + 4\pi \cdot \frac{(k+1)^{1+\beta_k}}{(\sigma_1 \cdots \sigma_k)^{\beta_{k_0}}}\, m(k)^{\beta_{k_0}-1} \\
&\leq (4\pi) \cdot \Big(\frac{m(k_0)}{m(k_0+k)} \cdot \Big(\frac{m(k)}{\sqrt{k}} + k\Big) + (k+1)^2\, m(k)^{\beta_{16m(k)}-1}\Big),
\end{aligned}$$

using that $k_0 \leq 16m(k)$. As $k \to \infty$, the first term clearly goes to 0 and, by (2), so does the second term. Therefore we have $\mathcal{L}(p_\theta(E)) = 0$ for all $\theta \in [0, \pi]$, as required.

2.3 The curvature of the set

We now show that any set fulfilling the condition of Theorem 1.4 has finite curvature. To see this we use an estimate which is due to Mattila [8].

Lemma 2.1 *Suppose E is a compact set in the plane and $0 < \mathcal{H}^\varphi(E) < \infty$ for some φ such that $\int_0^\varepsilon \varphi(r)^2/r^3\, dr < \infty$. Then E has finite Melnikov curvature.*

Proof: We first find a number $0 < \delta < \varepsilon/2$ and a compact set $E_0 \subset E$ such that $0 < \mathcal{H}^\varphi(E_0) < \infty$ and

$$\mathcal{H}^\varphi(E \cap B(x,r)) \leq 2\varphi(2r) \text{ for all } 0 < r < \delta \text{ and } x \in E_0. \tag{5}$$

This is possible as assumption of the contrary allows us to find, for $\mathcal{H}^\varphi$-almost every $x \in E$, a sequence $r_n \downarrow 0$ with $\mathcal{H}^\varphi(E \cap B(x, r_n)) \geq 2\varphi(2r_n)$. By Besicovitch's Covering Theorem (see e.g. [5, 2.8]) we can cover $\mathcal{H}^\varphi$-almost all of E with a disjoint collection of such balls and this leads to the contradiction $\mathcal{H}^\varphi(E) \geq 2\mathcal{H}^\varphi(E)$.

Observe that $\varphi(r)/r \to 0$ and hence we may assume that $\varphi(2r) \leq 3r$ for all $0 < r < \delta$. We may also assume that $|E_0| \leq \delta$. Let $\mu = (1/6)\mathcal{H}^\varphi|_{E_0}$. Then, for every closed ball B, we have $\mu(B) \leq (1/3)\varphi(2|B|) \leq |B|$. The Melnikov curvature $c^2(\mu)$ may now be estimated using an estimate from [8], which we repeat here for completeness. By elementary geometry we observe that, for all $x, y, z \in \mathbb{C}$,

$$\frac{1}{R(x,y,z)} \leq \frac{2}{|y-z|}.$$

Denote $A = \{(x,y,z) \in \mathbb{C}^3 : |x-y| \leq |x-z| \text{ and } |x-y| \leq |y-z|\}$ and estimate

$$\begin{aligned}
c^2(\mu) &\leq 3 \iiint_A \frac{1}{R(x,y,z)^2}\, d\mu(x)\, d\mu(y)\, d\mu(z) \\
&\leq 12 \int\int\int_{B(y,|y-z|)} \frac{1}{|y-z|^2}\, d\mu(x)\, d\mu(y)\, d\mu(z)
\end{aligned}$$

$$
\begin{aligned}
&= 12 \int\int \frac{\mu(B(y,|y-z|)}{|y-z|^2}\, d\mu(z)\, d\mu(y) \\
&= 12 \int \int_0^\infty \frac{\mu(B(y,r))}{r^2}\, dF_y(r)\, d\mu(y)\,,
\end{aligned}
$$

where $F_y(r) = \mu(B(y,r))$ and the inner integral is a Riemann-Stieltjes integral. Integrating by parts and recalling that $F_y(r)/r$ tends to 0 as $r \to 0$ or $r \to \infty$, we infer that

$$
c^2(\mu) \le 12 \int \int_0^\infty \frac{\mu(B(y,r))^2}{r^3}\, dr\, d\mu(y) \le 2\mathcal{H}^\varphi(E_0)\Big(\frac{1}{3}\int_0^\delta \varphi(2r)^2/r^3\, dr + \int_\delta^\infty \delta^2/r^3\, dr\Big) < \infty\,,
$$

and hence E has finite curvature. ■

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References

[1] G. David. Unrectifiable 1-sets have vanishing analytic capacity. *Prpublication, Universit Paris-Sud* (1997).

[2] P.W. Jones and T. Murai. Positive analytic capacity but zero Buffon needle probability. *Pacific J. Math.* **133** (1988) 99–114.

[3] J.C. Lger. Menger curvature and rectifiability. *Prpublication, Universit Paris-Sud* (1997).

[4] M.A. Martin and P. Mattila. k-dimensional regularity classifications for s-fractals. *Trans. Amer. Math. Soc.* **305** (1988) 293–315.

[5] P. Mattila. "Geometry of sets and measures in Euclidean spaces", Cambridge studies in advanced mathematics 44, Cambridge University Press, 1995.

[6] P. Mattila. Smooth maps, null-sets for integralgeometric measures and analytic capacity. *Ann. Math.* **123** (1986) 303–309.

[7] P. Mattila. Singular integrals, analytic capacity and rectifiability. *To appear in Proceedings of the Conference on Harmonic Analysis and Partial Differential Equations, El Escorial* (1996).

[8] P. Mattila. On the analytic capacity and curvature of some Cantor sets of non-σ-finite length. *Publ. Math.* **40** (1996) 303–309.

[9] P. Mattila, M.S. Melnikov and J. Verdera. The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. Math.* **144** (1996) 127–136.

[10] M.S. Melnikov. Analytic capacity: discrete approach and curvature of measures. *Math. Sbornik* **186:6** (1995) 827-846.