# On One-Sided Average Densities of Fractal Measures on the Line

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By a theorem of A.S. Besicovitch (see [Bes29] or [Bes68]) the lower one-sided $\alpha$–densities

$$\underline{d}_+^\alpha(\mu, x) = \liminf_{t\to 0} \frac{\mu([x, x+t])}{t^\alpha} \quad \text{and} \quad \underline{d}_-^\alpha(\mu, x) = \liminf_{t\to 0} \frac{\mu([x-t, x])}{t^\alpha}$$

of a measure of fractional dimension $\alpha$ on the real line vanish $\mu$-almost everywhere if the circular $\alpha$–densities

$$\overline{d}^\alpha(\mu, x) = \limsup_{t\to 0} \frac{\mu([x-t, x+t])}{t^\alpha} \quad \text{and} \quad \underline{d}^\alpha(\mu, x) = \liminf_{t\to 0} \frac{\mu([x-t, x+t])}{t^\alpha}$$

are finite and positive $\mu$-almost everywhere. It is the aim of this paper to show that the lower one-sided average densities of these measures do not vanish, and therefore the concept of average densities is able to reveal some of the local symmetry a measure with finite and positive densities necessarily possesses.

Average densities were introduced by Bedford and Fisher in [BF92] for the study of measures of fractional dimension. For these measures the density functions $t \mapsto \mu([x-t, x+t])/t^\alpha$ fluctuate as $t$ tends to 0 and therefore the limit does not exist. Bedford and Fisher apply a logarithmic average and define the lower and upper circular average densities as

$$\underline{D}^\alpha(\mu, x) = \liminf_{\varepsilon\to 0} (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu([x-t, x+t])}{t^\alpha} \frac{dt}{t} ,$$

and

$$\overline{D}^\alpha(\mu, x) = \limsup_{\varepsilon\to 0} (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu([x-t, x+t])}{t^\alpha} \frac{dt}{t} .$$

They also define the lower and upper right-sided average densities

$$\underline{D}_+^\alpha(\mu, x) = \liminf_{\varepsilon\to 0} (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu([x, x+t])}{t^\alpha} \frac{dt}{t} ,$$

and

$$\overline{D}_+^\alpha(\mu, x) = \limsup_{\varepsilon\to 0} (|\log\varepsilon|)^{-1} \int_\varepsilon^1 \frac{\mu([x, x+t])}{t^\alpha} \frac{dt}{t} ,$$

and analogously the lower and upper left-sided average densities $\underline{D}_-^\alpha(\mu, x)$ and $\overline{D}_-^\alpha(\mu, x)$. Average densities are also known as order-two densities.

Bedford and Fisher show that the lower and upper average density agree almost everywhere for Hausdorff measure on hyperbolic Cantor sets and zero-sets of Brownian motion. Recently, many other authors (see for example [Fal92], [FX93] and [PZ93]) have shown that for particular classes of fractal measures the lower and upper average densities agree. This also applies to the one-sided average densities.

But average densities can also be used for the investigation of more general measures. For example, Falconer and Springer in [FS95] generalize a classical inequality of Marstrand involving average densities. In this paper we show that for a measure with bounded densities the one-sided average densities are bounded away from 0. More precisely:

**Theorem** *Suppose $\mu$ is a Radon measure on the real line such that for some $0 < c \leq C < \infty$, $0 < \alpha < 1$ we have*

$$c < \underline{d}^\alpha(\mu, x) \leq \overline{d}^\alpha(\mu, x) < C$$

*for $\mu$-almost every $x$.*
*Then there is a number $\tau > 0$, depending only on $c$, $C$ and $\alpha$, such that*

$$\underline{D}_-^\alpha(\mu, x)\,,\, \underline{D}_+^\alpha(\mu, x) > \tau$$

*for $\mu$-almost every $x$.*

The proof of the theorem will be prepared in chapter 1 and finished in chapter 2. An easy consequence of the theorem is:

**Corollary** *Suppose $\mu$ is a Radon measure on the real line such that for some $0 < \alpha < 1$ we have*

$$0 < \underline{d}^\alpha(\mu, x) \leq \overline{d}^\alpha(\mu, x) < \infty$$

*for $\mu$-almost every $x$.*
*Then we have*

$$\underline{D}_-^\alpha(\mu, x)\,,\, \underline{D}_+^\alpha(\mu, x) > 0$$

*and*

$$\overline{D}_-^\alpha(\mu, x)\,,\, \overline{D}_+^\alpha(\mu, x) < \overline{D}^\alpha(\mu, x)$$

*for $\mu$-almost every $x$.*

**Proof** To prove the first statement of the corollary apply the theorem to the measure $\mu$ restricted to the sets

$$\{x \in \mathbb{R} \,:\, 1/n < \underline{d}^\alpha(\mu, x) \leq \overline{d}^\alpha(\mu, x) < n\}$$

for $n \in \mathbb{N}$, and apply the density theorem (see e.g. [Mat95, 2.14]) to these sets in order to check the hypothesis of the theorem.
The second statement of the corollary follows using

$$\overline{D}_+^\alpha(\mu, x) < \overline{D}_+^\alpha(\mu, x) + \underline{D}_-^\alpha(\mu, x) \leq \overline{D}^\alpha(\mu, x)$$

$\mu$-almost everywhere and the analogous statement for the left-sided densities. $\square$

## 1. Some Lemmas on the Geometry of Measures with Bounded Densities

We suppose that a Radon-measure $\mu$ is given such that the assumptions on the densities, as formulated in the theorem, are fulfilled. Without losing generality we can assume that $\mu(\mathbb{R}) < 1$ . Due to the symmetry of the problem, we can concentrate on the investigation of the right-sided average densities.

In the following the constants $\mathcal{C}_1, \mathcal{C}_2, \ldots$ are numbers which may depend on $c, C$ or $\alpha$ but not on anything else, especially not on the particular geometry of $\mu$.

Throughout the proof $|\cdot|$ is used to denote length of an interval and $d(\cdot,\cdot)$ to denote the distance of points or sets. $<$ and $>$ are used for the natural orderings of disjoint intervals and $I^c$ is used to denote the complement of a set $I \subseteq \mathbb{R}$.

For every $1 > \delta > 0$ we can use the density theorem to find a family of disjoint Borel sets $(B_i)_{i\in\mathbb{N}}$ with $|B_i| < 1$ and a family of compact sets $(E_i)_{i\in\mathbb{N}}$ with $E_i \subseteq B_i$, such that

$$\mu\Big(\mathbb{R} \setminus \bigcup_{i=1}^{\infty} E_i\Big) < \delta \,,$$

and for every $i \in \mathbb{N}$ there is a number $\varepsilon_0(i) > 0$ such that, for all $x \in B_i$, $t \geq 0$, we have $\mu([x-t, x+t] \cap B_i) \leq Ct^\alpha$, and for all $x \in E_i$, $0 \leq t \leq \varepsilon_0(i)$, we have $\mu([x-t, x+t] \cap B_i) \geq ct^\alpha$. Again by the density theorem we have

$$\overline{D}^\alpha_+(\mu, x) = \overline{D}^\alpha_+(\mu|_{B_i}, x)$$

for $\mu$-almost every $x \in E_i$. Therefore it suffices to prove the inequality of the theorem for the restricted measures $\mu|_{B_i}$ and $\mu$-almost every point $x \in E_i$.

Fix such a measure $\mu|_{B_i}$ and let us also denote it by $\mu$. Let $E = E_i$ and $\varepsilon_0 = \varepsilon_0(i)$. We now have a measure $\mu$ with support contained in a compact interval $D$ of length less than one and a compact set $E$ such that, without loss of generality, $\mu(E) > 0$ and such that, for all $x \in E$,

$$\mu([x-r, x+r]) \leq Cr^\alpha \quad \text{if } r \geq 0, \tag{1}$$

and

$$\mu([x-r, x+r]) \geq cr^\alpha \quad \text{if } 0 \leq r \leq \varepsilon_0. \tag{2}$$

We can write

$$D \setminus E = \bigcup_{I \in \mathcal{A}} I \,,$$

where $\mathcal{A}$ is the collection of connected components of $D \setminus E$. We can assume that $|I| \leq \varepsilon_0$ for all $I \in \mathcal{A}$, since otherwise we could split $E$ suitably and argue for each set separately.
We can also write

$$D \setminus \bigcup_{\substack{I\in\mathcal{A} \\ |I|\geq\varepsilon}} I = \bigcup_{K \in \mathcal{N}_\varepsilon} K \,,$$

where $\mathcal{N}_\varepsilon$ is the collection of connected components of the set $D \setminus \bigcup_{\substack{I\in\mathcal{A} \\ |I|\geq\varepsilon}} I$.
$\mathcal{N}_\varepsilon$ is a collection of disjoint compact intervals.

Before we give an upper bound to the length of the intervals in $\mathcal{N}_\varepsilon$, let us introduce some useful notation. For every interval $I \subseteq \mathbb{R}$ and every $\kappa \geq 0$ let

$$I^-(\kappa) = \{x \in \mathbb{R} : \text{ there is } y \in I \text{ such that } 0 \leq y - x \leq \kappa \cdot |I|\} \setminus I$$

and

$$I^+(\kappa) = \{x \in \mathbb{R} : \text{ there is } y \in I \text{ such that } 0 \leq x - y \leq \kappa \cdot |I|\} \setminus I$$

and also

$$I^0(\kappa) = I^-(\kappa) \cup I \cup I^+(\kappa)\,.$$

**Lemma 1** *There is a constant $\mathcal{C}_1 > 1$ such that, for all $0 < \varepsilon \leq \varepsilon_0$ and all $K \in \mathcal{N}_\varepsilon$, we have*

$$|K| < \mathcal{C}_1 \cdot \varepsilon\,.$$

**Proof** For $0 < \varepsilon \leq \varepsilon_0$ denote $r = r(\varepsilon) = \max\{|N| : N \in \mathcal{N}_\varepsilon\}$ and pick $N \in \mathcal{N}_\varepsilon$ such that $|N| = r$. Let $\tilde{N} = N^-(1) \cup N$, in other words $\tilde{N}$ is the closed interval of diameter $2r$ with centre at the left endpoint of $N$. Then, by (1),

$$\mu(\tilde{N}) \leq C \cdot r^\alpha\,.$$

Look at the intervals $I_1, I_2, I_3, \ldots \in \mathcal{A}$ that fulfill $I_i \subseteq N$. Obviously $|I_i| \leq \varepsilon$ and thus

$$\varepsilon^{\alpha-1} \cdot |I_i| \leq |I_i|^\alpha.$$

Define $\tilde{I}_i = I_i \cup I_i^-(1) \subseteq \tilde{N}$. $E$ is a set of Lebesgue measure 0, and therefore the sets $\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \ldots$ cover almost all of $N$ in the sense of Lebesgue measure. By Vitali's covering theorem (see for example [Mat95, 2.1]) we can pick a disjoint subsequence

$$\tilde{I}_{k_1}, \tilde{I}_{k_2}, \tilde{I}_{k_3}, \ldots$$

covering at least $1/10$ of the length of $N$. Now we can use (2) to see

$$C \geq \frac{\mu(\tilde{N})}{r^\alpha} \geq \frac{1}{r^\alpha}\sum_{i=1}^{\infty} \mu(\tilde{I}_{k_i}) \geq \frac{c}{r^\alpha}\sum_{i=1}^{\infty} |I_{k_i}|^\alpha \geq \frac{c}{r^\alpha}\varepsilon^{\alpha-1}\sum_{i=1}^{\infty} |I_{k_i}| \geq \frac{c}{10}\left(\frac{\varepsilon}{r}\right)^{\alpha-1}$$

and, defining $\mathcal{C}_1 = 2(10 \cdot C/c)^{1/(1-\alpha)}$, we have $r(\varepsilon) < \mathcal{C}_1 \varepsilon$, as required. □

**Lemma 2** *There is a constant $\mathcal{C}_2 > 1$ such that, for all $0 < \varepsilon \leq \varepsilon_0$,*

$$\sum_{N\in\mathcal{N}_\varepsilon} |N|^\alpha \leq \mathcal{C}_2.$$

**Proof** Fix $0 < \varepsilon \leq \varepsilon_0$. We first show that for all $N \in \mathcal{N}_\varepsilon$ we have

$$|N|^\alpha \leq (\mathcal{C}_1^\alpha / c) \cdot \mu(N^*)\,,$$

where $N^* = N^-(1/\mathcal{C}_1)$. For this purpose look at the closed interval $B$ of diameter $2|N|/\mathcal{C}_1$ centred at the left endpoint of $N$. We have $B \subseteq N^*$ and thus, using (2),

$$\mu(N^*) \geq \mu(B) \geq c\,(|N|/\mathcal{C}_1)^\alpha\ .$$

Because, by lemma 1, the intervals $I$ separating the $N \in \mathcal{N}_\varepsilon$ fulfill $|I| \geq \varepsilon > |N|/\mathcal{C}_1$, the collection $\{N^* : N \in \mathcal{N}_\varepsilon\}$ is disjoint. Therefore we can put $\mathcal{C}_2 = (\mathcal{C}_1^\alpha/c)$ and get

$$\sum_{N \in \mathcal{N}_\varepsilon} |N|^\alpha \leq (\mathcal{C}_1^\alpha/c) \cdot \sum_{N \in \mathcal{N}_\varepsilon} \mu(N^*) \leq (\mathcal{C}_1^\alpha/c) = \mathcal{C}_2\,.$$

$\square$

**Lemma 3** *There is a constant $\mathcal{C}_3 > 1$ such that, for all $0 < \varepsilon \leq (1/e)$, we have*

$$\sum |I|^\alpha \leq \mathcal{C}_3 \cdot |\log \varepsilon|\ ,$$

*where the sum extends over all $I \in \mathcal{A}$ such that $|I| \geq \varepsilon$.*

**Proof** For $I \in \mathcal{A}$ with $|I| \geq \varepsilon$ we define $\tilde{I} = I^-(1) \cup I$ and observe that, by (2), $\mu(\tilde{I}) \geq c|I|^\alpha$. For $x \in \tilde{K}$ let

$$I_1 < I_2 < \ldots < I_n$$

be the collection of intervals $I \in \mathcal{A}$ such that $|I| \geq \varepsilon$ and $x \in \tilde{I}$, in their natural order. For $3 \leq k \leq n$ we have $|I_k| \geq \varepsilon 2^{k-3}$ and, since $|I_n| \leq |D|$, we get

$$n \leq (3 + 1/\log 2) \cdot |\log \varepsilon|\,.$$

For the indicator functions $1_{\tilde{I}}$ we thus have

$$\sum |I|^\alpha \leq \sum (1/c) \cdot \mu(\tilde{I}) = (1/c) \cdot \int \sum \mathbf{1}_{\tilde{I}}\, d\mu \leq (n/c)\,,$$

where the sum extends over all $I \in \mathcal{A}$ such that $|I| \geq \varepsilon$. To finish the proof put $\mathcal{C}_3 = (3 + 1/\log 2) \cdot (1/c)$. $\square$

Define measures $\psi_\varepsilon$ on $\mathbb{R}$ by

$$\psi_\varepsilon(A) = (|\log \varepsilon|)^{-1} \int_\varepsilon^1 (\mathbf{1}_A(t) + \mathbf{1}_A(-t))\ \frac{dt}{t}$$

for all Borel sets $A \subseteq \mathbb{R}$ and measures $\psi_\varepsilon^x$ for $x \in \mathbb{R}$ by

$$\psi_\varepsilon^x(A) = \psi_\varepsilon(A - x)$$

for Borel sets $A \subseteq \mathbb{R}$. Note that the total mass of each of the measures $\psi_\varepsilon$ and $\psi_\varepsilon^x$ is 2.

**Lemma 4** *There are constants $\mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_6 \geq 1$, such that, for all intervals $I \subseteq \mathbb{R}$ with endpoints in $E$, and all $\varepsilon > 0$, $\kappa > 0$, the following estimates hold:*

**(a)** $\displaystyle\int_{(I^0(\kappa)^c)} \psi_\varepsilon^x(I)\, d\mu(x) \leq \mathcal{C}_4 \cdot \frac{|I|^\alpha}{|\log\varepsilon|} \cdot \left(\frac{1}{\kappa}\right)^{1-\alpha}$,

**(b)** $\displaystyle\int_{I^c} \psi_\varepsilon^x(I)\, d\mu(x) \leq \mathcal{C}_5 \cdot \frac{|I|^\alpha}{|\log\varepsilon|}$,

**(c)** $\displaystyle\int_{I^c} (\psi_\varepsilon^x(I))^2\, d\mu(x) \leq \mathcal{C}_6 \cdot \frac{|I|^\alpha}{|\log\varepsilon|^2}$.

**Proof** Let $\kappa \geq 0$ and observe that by Fubini's theorem

$$\int_{(I^0(\kappa))^c} \psi_\varepsilon^x(I)\, d\mu(x) = \int_0^1 \mu(\{x \notin I^0(\kappa) \,:\, \psi_\varepsilon^x(I) > t\})\, dt\,.$$

$\psi_\varepsilon^x(I) > t$ implies $d(x,I) < |I| \cdot (\varepsilon^t/(1-\varepsilon^t))$. Therefore the set

$$\{x \notin I^0(\kappa) \,:\, \psi_\varepsilon^x(I) > t\} \subseteq \{x \,:\, \kappa|I| \leq d(x,I) < (\varepsilon^t/(1-\varepsilon^t))|I|\}$$

is empty if $\kappa \geq \varepsilon^t/(1-\varepsilon^t)$, which is equivalent to $t \geq (|\log\varepsilon|)^{-1} \cdot \log(1+1/\kappa)$. Otherwise,

$$\begin{aligned}\mu(\{x \notin I^0(\kappa) \,:\, \psi_\varepsilon^x(I) > t\}) &\leq \mu(\{x \notin I \,:\, d(x,I) < |I| \cdot (\varepsilon^t/(1-\varepsilon^t))\}) \\ &\leq 2C \cdot |I|^\alpha \cdot (\varepsilon^t/(1-\varepsilon^t))^\alpha.\end{aligned}$$

To prove (a) we use

$$\int_0^1 \mu(\{x \notin I^0(\kappa) \,:\, d(x,I) < |I| \cdot \varepsilon^t/(1-\varepsilon^t)\})\, dt$$

$$\leq 2C \cdot |I|^\alpha \cdot \int_0^{\frac{\log(1+1/\kappa)}{|\log\varepsilon|}} (\varepsilon^t/(1-\varepsilon^t))^\alpha\, dt$$

and

$$\int_0^{\frac{\log(1+1/\kappa)}{|\log\varepsilon|}} (\varepsilon^t/(1-\varepsilon^t))^\alpha\, dt = (|\log\varepsilon|)^{-1} \cdot \int_1^{1+1/\kappa} \frac{d\tau}{(\tau-1)^\alpha \tau}$$

$$\leq (|\log\varepsilon|)^{-1} \cdot \int_0^{1/\kappa} \frac{d\tau}{\tau^\alpha} \leq 1/(1-\alpha) \cdot (|\log\varepsilon|)^{-1} \cdot (1/\kappa)^{1-\alpha},$$

and let $\mathcal{C}_4 = 2C/(1-\alpha)$. To prove (b) we use

$$\begin{aligned}&\int_0^1 \mu(\{x \,:\, d(x,I) < |I| \cdot \varepsilon^t/(1-\varepsilon^t)\})\, dt \\ &\leq\ 2C|I|^\alpha \int_0^1 (\varepsilon^t/(1-\varepsilon^t))^\alpha\, dt = \frac{|I|^\alpha}{|\log\varepsilon|} \cdot 2C \cdot \int_1^\infty \frac{d\tau}{(\tau-1)^\alpha \tau}\end{aligned}$$

and let $\mathcal{C}_5 = 2C \int_1^\infty \frac{d\tau}{(\tau-1)^\alpha\tau} < \infty$. Inequality (c) follows similarly. $\square$

## 2. Proof of the Theorem

For every $\varepsilon > 0$ and $I \in \mathcal{A}$ we define a function $\tilde{\varphi}_I$ describing the influence of the scales in the interval $I - x$ on the one-sided average densities by

$$\tilde{\varphi}_I(x,\varepsilon) = \begin{cases} \displaystyle\int_I \frac{\mu([x,y])}{|y-x|^\alpha}\, d\psi_\varepsilon^x(y) & \text{if } x < I \quad , \\ \displaystyle\int_I \frac{\mu([y,x])}{|y-x|^\alpha}\, d\psi_\varepsilon^x(y) & \text{if } x > I \quad , \\ 0 & \text{otherwise.} \end{cases}$$

Observe that the functions $\tilde{\varphi}_I$ fulfill $|\tilde{\varphi}_I(x,\varepsilon)| \le C \cdot \psi_\varepsilon^x(I)$ and therefore we have (using lemma 4(b)) that

$$\int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x)\ ,\ \int_{I^+(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x) \le \mathcal{C}_5 \cdot C \cdot \frac{|I|^\alpha}{|\log\varepsilon|} \tag{3}$$

for all $I \in \mathcal{A}$ and $\varepsilon > 0$. We can use (2) to get a converse inequality.

**Lemma 5** *There are constants $\mathcal{C}_7 > 0$ and $1 > \lambda > 0$, such that, for all $\varepsilon > 0$, $\kappa > 1$ and all $I \in \mathcal{A}$ with $|I| \ge \varepsilon/\lambda$ and $\mu(I) \le (c/4)\cdot|I|^\alpha$, we have*

$$\int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x)\ ,\ \int_{I^+(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x) \ge \mathcal{C}_7 \cdot \frac{|I|^\alpha}{|\log\varepsilon|}\ .$$

**Proof** Let $\lambda = \Big(c/(4C)\Big)^{1/\alpha}$ and $\mathcal{C}_7 = (1/16)\cdot\log(4/3)\cdot(c^3/C)$.
Suppose $I$ is an interval as in the hypothesis and denote its left endpoint by $a$. For all $x \in I^-(1) \setminus I^-(\lambda)$ and $y \in I$ such that $y - a \ge \lambda|I|$ we get

$$\frac{\mu([x,y])}{(y-x)^\alpha} \ge \frac{\mu([a-\lambda|I|, a+\lambda|I|])}{(2|I|)^\alpha} \ge \frac{c\lambda^\alpha|I|^\alpha}{(2|I|)^\alpha} \ge \frac{c^2}{8C}\,.$$

Denote $R(t) = \mu([a-t,a])$ and observe that our assumptions imply

$$|I|^\alpha C t^\alpha \ge R(t|I|) \ge |I|^\alpha(ct^\alpha - (c/4))$$

for $0 \le t \le 1$. These inequalities yield

$$\begin{aligned} \int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x) &\ge \frac{c^2}{8C}\cdot\int_{I^-(1)\setminus I^-(\lambda)} \psi_\varepsilon^x(\{y\in I\,:\, y-a\ge\lambda|I|\})\, d\mu(x) \\ &\ge \frac{c^2}{8C}\cdot\frac{1}{|\log\varepsilon|}\cdot\int_\lambda^1 \log\left(\frac{t+1}{t+\lambda}\right)\, dR(t|I|) \\ &\ge \frac{c^2}{8C}\cdot\frac{1}{|\log\varepsilon|}\cdot\log\left(\frac{2}{1+\lambda}\right)\cdot\Big(R(|I|)-R(\lambda|I|)\Big) \\ &\ge \frac{c^2}{8C}\cdot\log(4/3)\cdot\frac{|I|^\alpha}{|\log\varepsilon|}\cdot(c-(c/4)-C\lambda^\alpha) \\ &= (1/16)\cdot\log(4/3)\cdot\Big(c^3/C\Big)\cdot\frac{|I|^\alpha}{|\log\varepsilon|}. \end{aligned}$$

This completes the proof for $I^-(\kappa)$. For $I^+(\kappa)$ the proof is analogous. $\square$

We are now able to define $\tau > 0$, depending only on $C, c$ and $\alpha$, as

$$\tau = \frac{\mathcal{C}_7}{8C \cdot \mathcal{C}_5}\,.$$

We will fix the value of $\kappa$ depending on $\varepsilon$ as

$$\kappa = \kappa(\varepsilon) = (\log|\log\varepsilon|)^{\frac{2}{1-\alpha}}\,.$$

For $\varepsilon > 0$ let

$$\mathcal{G}_\varepsilon = \{I \in \mathcal{A} \,:\, |I| \geq \varepsilon/\lambda \text{ and } \mu(I) \leq (c/4)\cdot|I|^\alpha\}\,,$$

the collection of those intervals to which lemma 5 can be applied. Also let

$$\mathcal{B}_\varepsilon = \{I \in \mathcal{A}\,,\, |I| \geq \varepsilon/\lambda \text{ and } I \notin \mathcal{G}_\varepsilon\}\,.$$

For $I \in \mathcal{G}_\varepsilon$ we have, by (3) and lemma 5, that the numbers

$$\eta_I(\varepsilon) := \frac{\int_{I^-(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x)}{\int_{I^+(\kappa)} \tilde{\varphi}_I(x,\varepsilon)\, d\mu(x)}$$

fulfill

$$8\tau \leq \eta_I(\varepsilon) \leq \frac{1}{8\tau}\,. \tag{4}$$

For $I \in \mathcal{B}_\varepsilon$ let $\eta_I(\varepsilon) = 1$. For $\varepsilon > 0$ and $I \in \mathcal{A}$ with $|I| \geq \varepsilon/\lambda$ define the function $\varphi_I$ by

$$\varphi_I(x,\varepsilon) = \begin{cases} -\eta_I(\varepsilon)\cdot\tilde{\varphi}_I(x,\varepsilon) & \text{if } x \in I^+(\kappa(\varepsilon))\ , \\ \tilde{\varphi}_I(x,\varepsilon) & \text{if } x \in I^-(\kappa(\varepsilon))\ , \\ 0 & \text{otherwise.} \end{cases}$$

The functions $\varphi_I$ have the following properties:

1. There is a constant $\mathcal{C}_8 = C/(8\tau) > 0$ such that, for all $x \in \mathbb{R}$ and $\varepsilon > 0$,

$$|\varphi_I(x,\varepsilon)| \leq \mathcal{C}_8 \cdot \psi_\varepsilon^x(I).$$

2. $\varphi_I(\cdot,\varepsilon)$ is supported by $I^-(\kappa(\varepsilon)) \cup I^+(\kappa(\varepsilon))$.
3. For $I \in \mathcal{G}_\varepsilon$ we have

$$\int \varphi_I(x,\varepsilon)\, d\mu(x) = 0.$$

We now proceed to the main step of the proof and show that the sum

$$\sum_{I\in\mathcal{A}, |I|\geq\varepsilon/\lambda} \varphi_I(x,\varepsilon)$$

has a small mean square (with respect to $\mu$). We start by showing that, for most $x$, the summands in the sum above are small.

**Lemma 6** *Define the set $S(\varepsilon)$ as the set of all $x \in \mathbb{R}$ such that there is $J \in \mathcal{A}$ with $|J| \geq \varepsilon/\lambda$ and*

$$|\varphi_J(x,\varepsilon)| \geq \left(\frac{2\mathcal{C}_8}{\alpha}\right)\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\,.$$

*Then there is a constant $\mathcal{C}_9 > 1$ such that, for all sufficiently small $\varepsilon > 0$,*

$$\mu(S(\varepsilon)) \leq \mathcal{C}_9 \cdot (|\log\varepsilon|)^{-1}.$$

**Proof** Denote

$$\delta = \frac{2\cdot\log|\log\varepsilon|}{|\log\varepsilon^\alpha|}$$

and observe

$$\mu(S(\varepsilon)) \leq \sum_{\substack{J\in\mathcal{A}\\ |J|\geq\varepsilon/\lambda}} \mu(\{x \in \mathbb{R}\setminus J \,:\, (|\log\varepsilon|)^{-1}\cdot\log\left(\frac{d(x,J)+|J|}{d(x,J)}\right) \geq \delta\}).$$

If $\varepsilon$ is sufficiently small, the condition

$$(|\log\varepsilon|)^{-1}\cdot\log\left(\frac{d(x,J)+|J|}{d(x,J)}\right) \geq \frac{2\cdot\log|\log\varepsilon|}{|\log\varepsilon^\alpha|}$$

implies $d(x,J) \leq 2|J|\cdot|\log\varepsilon|^{-2/\alpha}$. Thus, denoting $\mathcal{C}_9 = 4C\cdot\mathcal{C}_3$, we have, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
\mu(S(\varepsilon)) &\leq \sum_{\substack{J\in\mathcal{A}\\ |J|\geq\varepsilon/\lambda}} \mu(\{x\in\mathbb{R}\setminus J \,:\, d(x,J) \leq 2|J|\cdot|\log\varepsilon|^{-2/\alpha}\}) \\
&\leq 2C\cdot 2^\alpha \cdot \sum_{\substack{J\in\mathcal{A}\\ |J|\geq\varepsilon/\lambda}} \frac{|J|^\alpha}{|\log\varepsilon|^2} \leq \frac{\mathcal{C}_9}{|\log\varepsilon|}\,.
\end{aligned}$$

□

**Lemma 7** *Denote*

$$B_\varepsilon = \{x \in E \,:\, |\sum_{\substack{I\in\mathcal{A}\\ |I|\geq\varepsilon/\lambda}} \varphi_I(x,\varepsilon)| \geq \tau\}.$$

*Then there is a constant $\mathcal{C}_{10} > 1$ such that, for all sufficiently small $\varepsilon > 0$,*

$$\mu(B_\varepsilon) \leq \frac{\mathcal{C}_{10}}{\sqrt{|\log\varepsilon|}}\,.$$

**Proof** We fix $0 < \varepsilon < \varepsilon_0$, sufficiently small in the sense of the preceding lemma, and sufficiently small to fulfill several computational conditions, specified as they appear in the proof. We begin the proof by estimating $\mu(B_\varepsilon)$ by means of suitable sums of integrals

$$\begin{aligned}\mu(B_\varepsilon)\cdot\tau^2 &\le \int\Big(\sum_{I\in\mathcal{A},|I|\ge\varepsilon/\lambda}\varphi_I(x,\varepsilon)\Big)^2\,d\mu(x)\\ &= \sum_{I\in\mathcal{B}_\varepsilon \text{ or } J\in\mathcal{B}_\varepsilon}\sum\int\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\,d\mu(x) \qquad (5)\\ &\quad+\sum_{I\in\mathcal{G}_\varepsilon}\sum_{J\in\mathcal{G}_\varepsilon}\int\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\,d\mu(x). \qquad (6)\end{aligned}$$

In order to give an estimate of (5), we observe

$$\sum_{I\in\mathcal{B}_\varepsilon}|I|^\alpha \le \sum_{\substack{I\in\mathcal{A}\\ \mu(I)>(c/4)|I|^\alpha}}|I|^\alpha \le (4/c)\cdot\sum_{I\in\mathcal{A}}\mu(I)\le(4/c)\,.$$

Therefore we can use lemma 4(b) to show

$$\begin{aligned}&\sum_{I\in\mathcal{B}_\varepsilon \text{ or } J\in\mathcal{B}_\varepsilon}\sum\int\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\,d\mu(x)\\ &\le 2\sum_{I\in\mathcal{B}_\varepsilon}\int\Big(|\varphi_I(x,\varepsilon)|\sum_{\substack{J\in\mathcal{A}\\ |J|\ge\varepsilon/\lambda}}|\varphi_J(x,\varepsilon)|\Big)\,d\mu(x)\\ &\le 2\mathcal{C}_8^2\cdot\sum_{I\in\mathcal{B}_\varepsilon}\int_{I^c}\Big(\psi_\varepsilon^x(I)\cdot\sum_{\substack{J\in\mathcal{A}\\ |J|\ge\varepsilon/\lambda}}\psi_\varepsilon^x(J)\Big)\,d\mu(x)\\ &\le 4\mathcal{C}_8^2\cdot\sum_{I\in\mathcal{B}_\varepsilon}\int_{I^c}\psi_\varepsilon^x(I)\,d\mu(x)\\ &\le 4\mathcal{C}_5\mathcal{C}_8^2\cdot\sum_{I\in\mathcal{B}_\varepsilon}\frac{|I|^\alpha}{|\log\varepsilon|}\\ &\le 4\mathcal{C}_5\mathcal{C}_8^2(4/c)\cdot\frac{1}{|\log\varepsilon|}\le\frac{1}{\sqrt{|\log\varepsilon|}}\end{aligned}$$

for sufficiently small $\varepsilon>0$, as required.
For (6) we have to work harder. Observe that

$$\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\le 0$$

unless $x\in I^-(\kappa)\cap J^-(\kappa)$ or $x\in I^+(\kappa)\cap J^+(\kappa)$. We can therefore split the sum again

$$\begin{aligned}\sum_{I\in\mathcal{G}_\varepsilon}\sum_{J\in\mathcal{G}_\varepsilon}\int\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\,d\mu(x) &\le 2\sum\sum\int\varphi_I(x,\varepsilon)\varphi_J(x,\varepsilon)\,d\mu(x) \qquad (7)\\ &\quad+\sum_{I\in\mathcal{G}_\varepsilon}\int\varphi_I^2(x,\varepsilon)\,d\mu(x)\,, \qquad (8)\end{aligned}$$

where the double sum on the right hand side extends over those intervals $I,J\in\mathcal{G}_\varepsilon$ with $I<J$ that fulfill $I^-(\kappa)\cap J^-(\kappa)\neq\emptyset$ or $I^+(\kappa)\cap J^+(\kappa)\neq\emptyset$.

In order to handle (8), we can use lemmas 4(c) and 3 and get

$$\sum_{I\in\mathcal{G}_\varepsilon}\int \varphi_I^2(x,\varepsilon)\,d\mu(x) \le \mathcal{C}_6\mathcal{C}_8^2\cdot\sum_{I\in\mathcal{G}_\varepsilon}\frac{|I|^\alpha}{|\log\varepsilon|^2} \le \mathcal{C}_8^2\mathcal{C}_6\mathcal{C}_4\cdot\frac{1}{|\log\varepsilon|} \le \frac{1}{\sqrt{|\log\varepsilon|}}$$

for sufficiently small $\varepsilon > 0$, as required. The investigation of (7) constitutes the main part of the proof. We can concentrate our effort on the case $I^-(\kappa)\cap J^-(\kappa)\neq\emptyset$. We fix an interval $I=(a,b)\in\mathcal{G}_\varepsilon$ and look at the intervals $J\in\mathcal{G}_\varepsilon$ that fulfill $I<J$ and $I^-(\kappa)\cap J^-(\kappa)\neq\emptyset$. We order them from left to right

$$I < J_1 < J_2 < \ldots < J_N$$

and denote $J_i=(c_i,d_i)$ for $1\le i\le N$ and $c_{N+1}=\infty$. Because $I^-(\kappa)\cap J_i^-(\kappa)\neq\emptyset$, we have $c_i-a\le\kappa(d_i-c_i)$, and thus

$$c_i-a \le \frac{\kappa}{\kappa+1}(d_i-a) \le \frac{\kappa}{\kappa+1}(c_{i+1}-a).$$

Observing that $c_1-a\ge|I|$, we therefore get by induction, for all $1\le i\le N$,

$$c_i-a \ge |I|\cdot\left(\frac{\kappa+1}{\kappa}\right)^{i-1}.$$

Let $\xi=\sup(I^+(\kappa))$ and denote by $k$ the smallest integer such that

$$c_k-\xi \ge |\log\varepsilon|^{1/\alpha}\cdot|I|\,. \tag{9}$$

We have

$$|I|\left(\frac{\kappa+1}{\kappa}\right)^{k-2} \le (c_{k-1}-\xi)+(\xi-a) \le |\log\varepsilon|^{1/\alpha}|I|+(\kappa+1)|I|\,,$$

and therefore we can put $\beta=3/(1-\alpha)$ and get, provided $\varepsilon$ is small enough,

$$k \le (2/\alpha)\cdot\frac{\log|\log\varepsilon|}{\log((\kappa+1)/\kappa)} \le (\log|\log\varepsilon|)^\beta\,. \tag{10}$$

Now we use the moderate growth of $k$ and the fact that outside the set $S(\varepsilon)$ (see lemma 6) the $|\varphi_{J_i}(x,\varepsilon)|$ are small, to see

$$\begin{aligned}
&\sum_{i=1}^k\int\varphi_I(x,\varepsilon)\varphi_{J_i}(x,\varepsilon)\,d\mu(x)\\
\le\ & \int_{S(\varepsilon)}\Big(|\varphi_I(x,\varepsilon)|(\sum_{i=1}^k|\varphi_{J_i}(x,\varepsilon)|)\Big)\,d\mu(x)+\int_{S(\varepsilon)^c}\Big(|\varphi_I(x,\varepsilon)|(\sum_{i=1}^k|\varphi_{J_i}(x,\varepsilon)|)\Big)\,d\mu(x)\\
\le\ & \mathcal{C}_8\int_{S(\varepsilon)}|\varphi_I(x,\varepsilon)|\cdot\psi_\varepsilon^x(\bigcup_{i=1}^k J_i)\,d\mu(x)+\int_{S(\varepsilon)^c}|\varphi_I(x,\varepsilon)|\cdot k\cdot\max_{i=1}^k\{|\varphi_{J_i}(x,\varepsilon)|\}\,d\mu(x)\\
\le\ & 2\mathcal{C}_8\cdot\int_{S(\varepsilon)}|\varphi_I(x,\varepsilon)|\,d\mu(x)+(2\mathcal{C}_8/\alpha)\cdot\frac{(\log|\log\varepsilon|)^{1+\beta}}{|\log\varepsilon|}\cdot\int_{S(\varepsilon)^c}|\varphi_I(x,\varepsilon)|\,d\mu(x)\,.
\end{aligned}$$

Summing over all $I \in \mathcal{G}_\varepsilon$ we get, using $\sum_{I\in\mathcal{G}_\varepsilon} |\varphi_I(x,\varepsilon)| \le 2\mathcal{C}_8$ and lemma 6,

$$\sum_{I\in\mathcal{G}_\varepsilon} \sum_{i=1}^{k} \int \varphi_I \varphi_{J_i}\, d\mu \le 4\mathcal{C}_8^2 \Big( \mu(S(\varepsilon)) + \frac{(\log|\log\varepsilon|)^{1+\beta}}{\alpha|\log\varepsilon|} \cdot \mu(\mathbb{R}) \Big) \le \frac{1}{\sqrt{|\log\varepsilon|}}$$

for sufficiently small $\varepsilon > 0$, as required.
In order to give an upper bound for the influence ot the intervals $J_{k+1}, \ldots, J_N$ we first calculate that on the domain of $\varphi_I(\cdot,\varepsilon)$ the variation of the function $\tilde{\varphi}_{J_i}(\cdot,\varepsilon)$ is small, namely there is a constant $\mathcal{C}_{11} > 0$ such that, for

$$c(I, J_i) = \sup_{x,y \in I^-(\kappa)\cup I^+(\kappa)} |\tilde{\varphi}_{J_i}(x,\varepsilon) - \tilde{\varphi}_{J_i}(y,\varepsilon)|\,,$$

we have

$$c(I, J_i) \le \mathcal{C}_{11} \cdot \psi_\varepsilon^\xi(J_i) \cdot \left( \frac{|I|}{c_{i-1} - \xi} \right)^\alpha \cdot \kappa^{2\alpha}\,. \tag{11}$$

In order to prove (11) we pick $x, y \in I^-(\kappa) \cup I^+(\kappa)$ with $x < y$. For $z \in J_i$ we have

$$1 \le \frac{z-x}{z-y} \le 1 + \frac{y-x}{z-y} \le 1 + \frac{(2\kappa+1)|I|}{c_i - \xi}\,,$$

and, if $\varepsilon$ is small enough to ensure

$$\frac{(2\kappa+1)|I|}{c_i - \xi} \le \frac{(2\kappa+1)}{|\log\varepsilon|^{1/\alpha}} < 1\,,$$

we have

$$\left| 1 - \left( \frac{z-x}{z-y} \right)^{1+\alpha} \right| \le \left( 1 + \frac{(2\kappa+1)|I|}{c_i - \xi} \right)^{1+\alpha} - 1 \le 3 \left( \frac{(2\kappa+1)|I|}{c_i - \xi} \right)^\alpha\,. \tag{12}$$

Because $[y, c_i] \supseteq [c_{i-1} - (c_{i-1} - \xi)/\kappa, c_{i-1} + (c_{i-1} - \xi)/\kappa]$, we have

$$\mu([y, c_i]) \ge (c/\kappa^\alpha)(c_{i-1} - \xi)^\alpha\,.$$

Thus, for all $z \in J_i$,

$$1 \le \frac{\mu([x,z])}{\mu([y,z])} \le 1 + \frac{\mu([x,y])}{\mu([y,c_i])} \le 1 + \frac{\kappa^\alpha C|I|^\alpha (2\kappa+1)^\alpha}{c(c_{i-1} - \xi)^\alpha}\,. \tag{13}$$

Let $\mathcal{C}_{11} = 12(C^2/c)$. (12) and (13) together imply

$$\begin{aligned}
&\left| \frac{\mu([x,z])}{(z-x)^{1+\alpha}} - \frac{\mu([y,z])}{(z-y)^{1+\alpha}} \right| \\
&\le \frac{\mu([x,z])}{(z-x)^{1+\alpha}} \cdot \left| 1 - \left( \frac{z-x}{z-y} \right)^{1+\alpha} \right| + \frac{\mu([y,z])}{(z-y)^{1+\alpha}} \cdot \left| 1 - \frac{\mu([x,z])}{\mu([y,z])} \right| \\
&\le \mathcal{C}_{11} \cdot \frac{1}{z-\xi} \cdot \kappa^{2\alpha} \left( \frac{|I|}{c_{i-1} - \xi} \right)^\alpha\,,
\end{aligned}$$

and thus

$$
\begin{aligned}
|\tilde{\varphi}_{J_i}(x,\varepsilon)-\tilde{\varphi}_{J_i}(y,\varepsilon)| &\leq (|\log\varepsilon|)^{-1}\int_{J_i}\left|\frac{\mu([x,z])}{(z-x)^{1+\alpha}}-\frac{\mu([y,z])}{(z-y)^{1+\alpha}}\right| dz \\
&\leq \mathcal{C}_{11}\cdot\psi_\varepsilon^\xi(J_i)\cdot\kappa^{2\alpha}\left(\frac{|I|}{c_{i-1}-\xi}\right)^\alpha ,
\end{aligned}
$$

which proves (11).
We can now use the third property of the functions $\varphi_I$ to see that, for an arbitrarily chosen fixed $y\in I^-(\kappa)\cup I^+(\kappa)$, we have

$$
\int \varphi_I(x,\varepsilon)\tilde{\varphi}_{J_i}(y,\varepsilon)\,d\mu(x)=0,
$$

and use this together with (11) and (9) to estimate

$$
\begin{aligned}
\sum_{i=k+1}^{N}\int \varphi_I\varphi_{J_i}\,d\mu &\leq \sum_{i=k+1}^{N}\int \varphi_I(x,\varepsilon)\tilde{\varphi}_{J_i}(x,\varepsilon)\,d\mu(x) \\
&= \sum_{i=k+1}^{N}\int \varphi_I(x,\varepsilon)\left(\tilde{\varphi}_{J_i}(x,\varepsilon)-\tilde{\varphi}_{J_i}(y,\varepsilon)\right)d\mu(x) \\
&\leq \sum_{i=k+1}^{N} c(I,J_i)\cdot\int|\varphi_I(x,\varepsilon)|\,d\mu(x) \\
&\leq \mathcal{C}_{11}\mathcal{C}_8\mathcal{C}_5\cdot\sum_{i=k+1}^{N}\psi_\varepsilon^\xi(J_i)\cdot\kappa^{2\alpha}\cdot\left(\frac{|I|}{c_{i-1}-\xi}\right)^\alpha\cdot\frac{|I|^\alpha}{|\log\varepsilon|} \\
&\leq \mathcal{C}_{11}\mathcal{C}_8\mathcal{C}_5\cdot\frac{1}{|\log\varepsilon|}\sum_{i=k+1}^{N}\psi_\varepsilon^\xi(J_i)\cdot\frac{|I|^\alpha}{|\log\varepsilon|}\cdot\kappa^{2\alpha} \\
&\leq \mathcal{C}_{11}\mathcal{C}_8\mathcal{C}_5\cdot\frac{|I|^\alpha}{|\log\varepsilon|^2}\cdot\kappa^{2\alpha} ,
\end{aligned}
$$

and finally, summing over all $I\in\mathcal{G}_\varepsilon$ and using lemma 3,

$$
\sum_{I\in\mathcal{G}_\varepsilon}\sum_{i=k+1}^{N}\int\varphi_I(x,\varepsilon)\varphi_{J_i}(x,\varepsilon)\,d\mu(x)\leq\mathcal{C}_{11}\mathcal{C}_8\mathcal{C}_5\mathcal{C}_4\cdot\frac{\kappa^{2\alpha}}{|\log\varepsilon|}\leq\frac{1}{\sqrt{|\log\varepsilon|}}
$$

for sufficiently small $\varepsilon>0$.
In the same manner we get the analogous estimates for intervals $I$, $J$ with $I^+(\kappa)\cap J^+(\kappa)\neq\emptyset$. This finishes the proof of our main lemma. □

The function $\sum_{\substack{I\in\mathcal{A}\\|I|\geq\varepsilon/\lambda}}\varphi_I(x,\varepsilon)$ is an approximation of

$$
D(x,\varepsilon)=(|\log\varepsilon|)^{-1}\int_\varepsilon^1\frac{\mu([x,x+t])-\eta(x-t,\varepsilon)\cdot\mu([x-t,x])}{t^\alpha}\,\frac{dt}{t},
$$

where

$$
\eta(x,\varepsilon)=\begin{cases}\eta_I(\varepsilon) & \text{if } x\in I \text{ for some } I\in\mathcal{G}_\varepsilon,\\ 1 & \text{otherwise.}\end{cases}
$$

**Lemma 8** *Define the approximation error*

$$E(x,\varepsilon) = \left| D(x,\varepsilon) - \sum_{I\in\mathcal{A},|I|\geq\varepsilon/\lambda} \varphi_I(x,\varepsilon) \right| .$$

*There is a constant* $\mathcal{C}_{12} > 0$ *such that, for all* $0 < \varepsilon \leq \varepsilon_0$, *we have*

$$\mu(\{x \in E \,:\, E(x,\varepsilon) > \tau\}) \leq \frac{\mathcal{C}_{12}}{(\log|\log\varepsilon|)^2} .$$

**Proof** We have for all $x \in E$

$$E(x,\varepsilon) \leq \mathcal{C}_8 \cdot \Big( \psi_\varepsilon^x(\mathbb{R}) - \sum_{\substack{I\in\mathcal{A},|I|\geq\varepsilon/\lambda \\ x\in I^0(\kappa)}} \psi_\varepsilon^x(I) \Big) .$$

We therefore have to show that $\int_E \left(\psi_\varepsilon^x(\mathbb{R}) - \sum \psi_\varepsilon^x(I)\right) \, d\mu(x)$ is small, where the sum extends over all $I \in \mathcal{A}$ such that $|I| \geq \varepsilon/\lambda$ and $x \in I^0(\kappa)$. To do this observe that for all $x \in \mathbb{R}$ we have

$$\psi_\varepsilon^x(D) - \sum \psi_\varepsilon^x(I) = \sum_{N\in\mathcal{N}_{\varepsilon/\lambda}} \psi_\varepsilon^x(N) + \sum_{\substack{I\in\mathcal{A},|I|\geq\varepsilon/\lambda \\ x\notin I^0(\kappa)}} \psi_\varepsilon^x(I) \, .$$

Therefore we have to give appropriate estimates for

$$\sum_{N\in\mathcal{N}_{\lambda\varepsilon}} \int \psi_\varepsilon^x(N) \, d\mu(x) \quad \text{and} \quad \sum_{I\in\mathcal{A},|I|\geq\lambda\varepsilon} \int_{(I^0(\kappa))^c} \psi_\varepsilon^x(I) \, d\mu(x) \, .$$

To estimate the first expression, we use $|N| \leq \mathcal{C}_1\varepsilon/\lambda$ to see

$$\int_N \psi_\varepsilon^x(N) \, d\mu(x) \leq \frac{\log \mathcal{C}_1/\lambda}{|\log\varepsilon|} \cdot \mu(N) \leq C \cdot \log \mathcal{C}_1/\lambda \cdot \frac{|N|^\alpha}{|\log\varepsilon|} \, .$$

By lemma 4(b) we know that

$$\int_{N^c} \psi_\varepsilon^x(N) \, d\mu(x) \leq \mathcal{C}_5 \cdot \frac{|N|^\alpha}{|\log\varepsilon|}$$

holds. These two estimates together with lemma 2 give

$$\begin{aligned} \sum \int \psi_\varepsilon^x(N) \, d\mu(x) \quad &\leq \quad (C \cdot \log \mathcal{C}_1/\lambda + \mathcal{C}_5) \cdot \sum \frac{|N|^\alpha}{|\log\varepsilon|} \\ &\leq \quad \mathcal{C}_2 \, (C \cdot \log \mathcal{C}_1/\lambda + \mathcal{C}_5) \cdot \frac{1}{|\log\varepsilon|} \end{aligned}$$

Now use lemmas 4(a) and 3 to see

$$\begin{aligned} \sum_{I\in\mathcal{A},|I|\geq\varepsilon/\lambda} \int_{(I^0(\kappa))^c} \psi_\varepsilon^x(I) \, d\mu(x) \quad &\leq \quad \mathcal{C}_4 \cdot \sum_{I\in\mathcal{A},|I|\geq\varepsilon} \frac{|I|^\alpha}{|\log\varepsilon|} \cdot \left(\frac{1}{\kappa(\varepsilon)}\right)^{1-\alpha} \\ &\leq \quad \mathcal{C}_3\mathcal{C}_4 \cdot \frac{1}{(\log|\log\varepsilon|)^2} . \end{aligned}$$

Denote $a = \min(E)$, $b = \max(E)$ and use lemma 4(b) to see

$$\begin{aligned}\int_E (\psi^x_\varepsilon(\mathbb{R}) - \psi^x_\varepsilon(D))\, d\mu(x) &\leq \int_a^b \psi^x_\varepsilon([a-1,a])\, d\mu(x) + \int_a^b \psi^x_\varepsilon([b,b+1])\, d\mu(x) \\ &\leq \frac{2\mathcal{C}_5}{|\log \varepsilon|}\,.\end{aligned}$$

We can now finish the proof by observing

$$\begin{aligned}&\mu(\{x \in E \,:\, E(x,\varepsilon) > \tau\}) \\ &\leq \frac{1}{\tau}\int_E E(x,\varepsilon)\, d\mu(x) \\ &\leq (\mathcal{C}_8/\tau)\cdot \int_E \Big(\psi^x_\varepsilon(\mathbb{R}) - \sum_{\substack{I\in\mathcal{A},|I|\geq\varepsilon/\lambda \\ x\in I^0(\kappa)}} \psi^x_\varepsilon(I)\Big)\, d\mu(x) \\ &\leq \frac{\mathcal{C}_{12}}{(\log|\log\varepsilon|)^2}\,,\end{aligned}$$

defining $\mathcal{C}_{12} = (\mathcal{C}_8/\tau)\,(\mathcal{C}_2(C\cdot\log\mathcal{C}_1\lambda + \mathcal{C}_5) + \mathcal{C}_3\mathcal{C}_4 + 2\mathcal{C}_5)$. □

We can now finish the proof of our theorem.
Observe that

$$|D(x,\varepsilon)| \leq |E(x,\varepsilon)| + \Big|\sum_{\substack{I\in\mathcal{A}\\ |I|\geq\varepsilon/\lambda}} \varphi_I(x,\varepsilon)\Big|.$$

We use lemmas 7 and 8 to see that, for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}&\mu(\{x\in E \,:\, |D(x,\varepsilon)| > 2\tau\}) \\ &\leq \mu(\{x\in E \,:\, |E(x,\varepsilon)| > \tau\}) + \mu(\{x\in E \,:\, |\sum_{I\in\mathcal{A},|I|\geq\varepsilon/\lambda} \varphi_I(x,\varepsilon)| > \tau\}) \\ &\leq \frac{\mathcal{C}_{12}}{(\log|\log\varepsilon|)^2} + \frac{\mathcal{C}_{10}}{\sqrt{|\log\varepsilon|}}\,.\end{aligned}$$

Let $\delta_k = \exp(-\exp k)$. Then

$$\mu(\{x\in E \,:\, |D(x,\delta_k)| > 2\tau\}) \leq \frac{\mathcal{C}_{12}+\mathcal{C}_{10}}{k^2}\,,$$

and, since $\sum_{k=1}^\infty \frac{1}{k^2} < \infty$, we have, by the Borel-Cantelli-lemma,

$$\mu(\{x\in E \,:\, \limsup_{k\to\infty} |D(x,\delta_k)| > 2\tau\}) = 0\,.$$

We have

$$\begin{aligned}&(|\log\delta_k|)^{-1}\int_{\delta_k}^1 \frac{\mu([x,x+t])}{t^\alpha}\,\frac{dt}{t} \\ &= \frac{1}{2}\left((|\log\delta_k|)^{-1}\int_{\delta_k}^1 \frac{\mu([x,x+t]) + \eta(x-t,\varepsilon)\cdot\mu([x-t,x])}{t^\alpha}\,\frac{dt}{t} + D(x,\delta_k)\right)\,,\end{aligned}$$

and

$$\int_{\delta_k}^1 \frac{\mu([x,x+t]) + \eta(x-t,\varepsilon)\cdot\mu([x-t,x])}{t^\alpha}\,\frac{dt}{t} \geq (8\tau/c)\cdot\int_{\delta_k}^1 \frac{\mu([x-t,x+t])}{t^\alpha}\,\frac{dt}{t}\,.$$

Therefore, for $\mu$-almost every point $x \in E$,

$$\begin{aligned} &\liminf_{\varepsilon\downarrow 0}(|\log\varepsilon|)^{-1}\cdot\int_\varepsilon^1 \frac{\mu([x,x+t])}{t^\alpha}\,\frac{dt}{t} \\ \geq\ & \liminf_{k\to\infty}\frac{1}{e}\cdot(|\log\delta_k|)^{-1}\int_{\delta_k}^1\frac{\mu([x,x+t])}{t^\alpha}\,\frac{dt}{t} \\ \geq\ & (1/2e)\cdot((8\tau/c)\cdot c - 2\tau) > \tau\,, \end{aligned}$$

as required to finish the proof of the theorem.

## 3. Final Remark

Our theorem leaves a lot of questions about one-sided average densities open. For example:

- If the average density exists, does this imply that the left-sided and right-sided average densities exist?
- Do the values of the left-sided and right-sided lower (or upper) average densities always agree?
- Is the value of the one-sided lower (or upper) average density determined by the value of the lower (or upper) average density?

We shall answer all these questions in a forthcoming paper.

# References


[Bes29] A.S. Besicovitch. On linear sets of points of fractional dimension. *Mathematische Annalen*, 101:161–193, 1929.

[Bes68] A.S. Besicovitch. On linear sets of points of fractional dimension (II). *Journal of the London Mathematical Society*, 43:548–550, 1968.

[BF92] T. Bedford and A. Fisher. Analogues of the Lebesgue density theorem for fractal sets of reals and integers. *Proc.London.Math. Soc.(3)*, 64:95–124, 1992.

[Fal92] K.J. Falconer. Wavelet transforms and order-two densities of fractals. *Journal Stat.Physics*, 67:781–793, 1992.

[FS95] K.J. Falconer and O.B. Springer. Order-two density of sets and measures with non-integral dimension. *Mathematika*, 42:1-14, 1995.

[FX93] K.J. Falconer and Y.M Xiao. Order-two densities of the image and zero set of stable processes. *Preprint, University of St.Andrews*, 1993.

[Mat95] P. Mattila. *The Geometry of Sets and Measures in Euclidean Spaces.* Cambridge University Press, Cambridge, 1995.

[PZ93] N. Patzschke and M. Zähle. Fractional differentiation in the self-affine case III – The density of the Cantor set. *Proc. Amer. Math. Soc.*, 117:137–144, 1993.

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