Average densities, tangent measures and rectifiability

Peter Mörters
Universität Kaiserslautern, Fachbereich Mathematik, 67663 Kaiserslautern, Germany.
E-Mail: peter@mathematik.uni-kl.de

1 Densities and average densities of measures

We consider a nonnegative, nonzero Radon measure μ on $\mathbb{R}^d$ and let $\alpha \geq 0$. The measure μ is called *α-rectifiable* if α is an integer and μ is absolutely continuous with respect to α-Hausdorff measure restricted to a countable family of smooth α-manifolds. In the case of $\alpha = 1$ we say that μ is *linearly rectifiable*. An important problem in geometric measure theory is to find local criteria for the rectifiability of measures. In this paper we informally discuss some aspects of this problem related to the notions of average densities and tangent measures.

Many of the classical criteria for rectifiability are linked to the notion of the density of the measure: We define the *lower α-density* of μ at x as the number

$$\underline{d}^\alpha(\mu, x) = \liminf_{r \downarrow 0} \frac{\mu(U(x,r))}{r^\alpha} ,$$

where $U(x,r)$ denotes the open Euclidean ball centred in x of radius r, and the *upper α-density* of μ at x as the number

$$\overline{d}^\alpha(\mu, x) = \limsup_{r \downarrow 0} \frac{\mu(U(x,r))}{r^\alpha} .$$

It is not hard to show that for an α-rectifiable measure the lower and upper α-density of μ are positive and coincide at μ-almost every point (see e.g. [Mat95, 16.2]). Much harder is the question whether a converse statement holds. A crucial step in this direction was accomplished in the early sixties by Marstrand, who showed that the equality

$$0 < \underline{d}^\alpha(\mu, x) = \overline{d}^\alpha(\mu, x) < \infty \ \mu\text{-almost everywhere} \tag{1}$$

implies that α must be an integer, see [Mar64] or [Mat95, Chapter 14]. If $\alpha = 1$ the question whether (1) implies linear rectifiability was answered earlier by Besicovitch [Bes38] in the case of Hausdorff measures in the plane, and by Moore [Moo50] in the case of general Radon measures in arbitrary Euclidean spaces. The affirmative answer to this question in the much harder case of arbitrary integers α was given only in the eighties in a very deep paper by Preiss [Pre87], see also [Mat95, Chapter 17] for an accessible account of the proof.

The results of Marstrand and Preiss imply that for measures of fractal nature the lower and upper densities cannot coincide. As it would be very desirable to describe the "lacunarity" of a fractal measure about its points by means of a density parameter, Bedford and Fisher introduce in [BF92] the notion of *average densities* into fractal geometry. Applying classical summation techniques of Hardy and Riesz to the density functions they define the *lower* and *upper average α-densities of order two* of μ at x as

$$\underline{D}^\alpha(\mu, x) = \liminf_{\varepsilon \downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\mu(U(x,r))}{r^\alpha} \frac{dr}{r} ,$$

and

$$\overline{D}^\alpha(\mu,x) = \limsup_{\varepsilon\downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 \frac{\mu(U(x,r))}{r^\alpha}\frac{dr}{r}\,.$$

Observing that the choice of a positive upper bound in the integral is irrelevant to the value of the average density we get the following inequalities:

$$\underline{d}^\alpha(\mu,x) \le \underline{D}^\alpha(\mu,x) \le \overline{D}^\alpha(\mu,x) \le \overline{d}^\alpha(\mu,x).$$

Several authors have shown that for measures with some self-similarity properties the lower and upper average densities in the appropriate dimension coincide and define an interesting fractal parameter. Examples include the natural measures on hyperbolic Cantor sets, see [BF92], random self-similar sets, see [PZ94], mixing repellers, see [KF92], paths of stable processes, see [FX95], and intersections of Brownian paths in $\mathbb{R}^3$, see [NS98]. However there are also natural examples of measures where this approach fails to give a density parameter, for example in the case of the paths of Brownian motion in the plane, see [Mör98b].

2 Average densities and rectifiability of measures

It is natural to ask whether one can get statements about the geometric regularity of the measure μ from weaker inequalities than (1) by using the average densities. The examples mentioned above show that we cannot expect to infer regularity of μ from the equality of the lower and upper average densities, but it is interesting to ask for the consequences of equality of the lower and lower average densities or of the upper and upper average densities. This problem was first studied by Falconer and Springer in [FS95] and their results were recently improved by Marstrand in [Mar96]. They showed the following theorem:

Theorem 2.1 *Suppose μ is a nonnegative, nonzero Radon measure on $\mathbb{R}^d$ and $\alpha \ge 0$ such that*

(i) $0 < \overline{D}^\alpha(\mu,x) = \overline{d}^\alpha(\mu,x) < \infty$ for μ-almost every x, or

(ii) $0 < \underline{d}^\alpha(\mu,x) = \underline{D}^\alpha(\mu,x) < \infty$ for μ-almost every x,

then α must be an integer.

An elegant proof of this result, which also removes a technical restriction necessary to carry out Marstrand's proof, can be found in [MP98] and we sketch the argument in Section 3 of this paper.

Do the inequalities above also imply α-rectifiability of μ? The answer is no for the first inequality, even in the case $\alpha = 1$. The first example of this type can be found in [O'N95]. Let us sketch a simpler example here, more details can be found in [Mör95]. Denote by $B(x,r)$ the closed ball of radius x centred in r.

Example: We construct a sequence (I_n) of compact sets $I_n \subseteq \mathbb{R}^2$ as follows: Let $I_0 = B(0,1)$. In the first step inscribe 2 touching, closed balls of radius $1/2$ with centres on the vertical diameter in I_0. Denote the resulting set I_1. In the next step inscribe 4 touching, closed balls of radius $1/8$ with centres on the horizontal diameter in each ball of I_1. Call the resulting set

I_2. Having constructed I_n, we inscribe 2^{n+1} touching, closed balls of radius $(\prod_{i=1}^{n+1} 2^i)^{-1}$ with centres on the horizontal diameter (if n odd) or on the vertical diameter (if n even) in each ball of I_n and denote the resulting set I_{n+1}. We denote by $\mathcal{B}$ the family of balls appearing in the construction and let

$$I := \bigcap_{n=0}^{\infty} I_n .$$

Obviously I is compact. There is a natural measure μ supported by I which is characterised by the property that for every ball $B \in \mathcal{B}$ we have $\mu(B \cap I) = \operatorname{diam}(B)$.

For μ-almost every x, we have

$$\frac{1}{2} \le \underline{d}^1(\mu, x) < \underline{D}^1(\mu, x) = \overline{D}^1(\mu, x) = \overline{d}^1(\mu, x) = 2 .$$

To see this, observe that every small ball centred in I of radius r must contain disjoint balls $B \in \mathcal{B}$ whose summed diameters exceed $r/2$ and may be covered by a family of balls $B \in \mathcal{B}$ whose summed diameter does not exceed $2r$. Hence $\underline{d}^1(\mu, x) \ge 1/2$ and $\overline{d}^1(\mu, x) \le 2$. By a more detailed analysis, carried out in [Mör95] one finds that there is a $Z \subseteq (0,1)$ of vanishing logarithmic density, i.e. such that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\log(1/\varepsilon)} \int_{\varepsilon}^{1} 1_Z(r)\, \frac{dr}{r} = 0 ,$$

with the property that

$$\lim_{\substack{r \downarrow 0 \\ r \notin Z}} \frac{\mu(U(x,r))}{r} = 2 .$$

Hence $\underline{D}^1(\mu, x) = \overline{D}^1(\mu, x) = 2$ and, finally, it is not hard to find sequences $r_n \downarrow 0$ such that

$$\lim_{n \to \infty} \frac{\mu(U(x, r_n))}{r_n} < 2 .$$

We will return to this example in the next section.

We have seen that there are non-rectifiable measures μ such that $0 < \overline{d}^1(\mu, x) = \overline{D}^1(\mu, x) < \infty$ at μ-almost every point. For the lower and lower average densities, however, this is different.

Theorem 2.2 *Suppose μ is a nonnegative Radon measure on R^d. Then*

$$0 < \underline{d}^1(\mu, x) = \underline{D}^1(\mu, x) < \infty \quad \textit{for } \mu\textit{-almost every } x \tag{2}$$

implies that

$$0 < \underline{d}^1(\mu, x) = \overline{d}^1(\mu, x) < \infty \quad \textit{for } \mu\textit{-almost every } x$$

and hence μ is linearly rectifiable.

This result was proved in [Mör97] and we sketch the ideas of this proof in the following three sections. It is an open problem whether an analogue of this result holds for arbitrary integers α and we shall try to point out in the sequel which parts of our proof remain valid for integers α different from 1.

3 Tangent measures and flatness

To formulate the first step in our proof we introduce the language of tangent measures. This language was introduced by Preiss in his seminal paper [Pre87] and has turned out to be an extremely useful tool in geometric measure theory. See [Mat95] for an exposition and [O'N95], [Huo97] for recent progress.

Definition Let $\mathcal{M}(\mathbf{R}^d)$ be the set of nonnegative Radon measures on $\mathbf{R}^d$. Equipped with the vague topology, which is generated by the mappings $\mu \mapsto \int \varphi \, d\mu$, φ continuous with compact support, $\mathcal{M}(\mathbf{R}^d)$ is a Polish space, see [Mat95, Chapter 14].

Let $0 \le \alpha \le d$ and $\mu \in \mathcal{M}(\mathbf{R}^d)$. For $r > 0$ define $\mu_{x,r} \in \mathcal{M}(\mathbf{R}^d)$ to be the *enlargement of* μ *at* x *of factor* $1/r$, i.e. the measure defined by $\mu_{x,r}(A) = \mu(x + rA)$. The set of α*-dimensional tangent measures of* μ *at* x is defined as the set

$$\mathrm{Tan}_\alpha(\mu, x) = \Big\{ \lim_{n\to\infty} \mu_{x,r_n}/r_n^\alpha \text{ in the vague topology for some } r_n \downarrow 0 \Big\}.$$

A (tangent) measure ν is called α*-flat* if α is an integer and there is a linear space $V \subseteq \mathbf{R}^d$ of dimension α and some $c > 0$ such that $\nu = c \cdot \mathcal{H}^\alpha|_V$, a multiple of the restriction of α-Hausdorff measure to the space V.

For an α-rectifiable measure at almost all points the set of tangent measures is a singleton consisting of an α-flat measure (this is not hard to prove, see e.g. [Mat95, 16.7]). For fractal measures of dimension α however the set of tangent measures is typically very rich and a converse to the statement above holds. We shall discuss a partial strengthening of this converse in the final section, but at the moment our aim is to show the *existence* of certain flat tangent measures using our hypothesis on μ.

Theorem 3.1 *Let* μ *be a Radon measure on* $\mathbf{R}^d$ *and let* $\alpha \ge 0$.

(a) Suppose $0 < \overline{d}^\alpha(\mu, x) = \overline{D}^\alpha(\mu, x) < \infty$ *for* μ*-almost every* x*. Then, at* μ*-almost every* x*, there is an* α*-flat measure* $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ *such that*

$$\nu(U(0,1)) = \overline{d}^\alpha(\mu, x) .$$

(b) Suppose $0 < \underline{d}^\alpha(\mu, x) = \underline{D}^\alpha(\mu, x) < \infty$ *for* μ*-almost every* x*. Then, at* μ*-almost every* x*, there is an* α*-flat measure* $\nu \in \mathrm{Tan}_\alpha(\mu, x)$ *such that*

$$\nu(U(0,1)) = \underline{d}^\alpha(\mu, x) .$$

We remark that at this stage of the proof we still have symmetry in the statements about lower and upper densities. Also the result applies to arbitrary α, only in the final section we will specialize to the case $\alpha = 1$. Observe that this theorem implies Theorem 2.1. The proof uses an idea, which could be called a "probabilistic method". This idea will be explained in the next section.

4 Tangent measure distributions and a probabilistic method

The idea of the probabilistic method is to introduce a probability distribution P on the set $\mathrm{Tan}_\alpha(\mu,x)$ of tangent measures and show that P-almost every tangent measure ν fulfills $\nu(U(y,r)) = c \cdot r^\alpha$ for every $r > 0$ and $y \in \mathrm{supp}\ \nu$ and $c = \underline{d}^\alpha(\mu,x)$ or $c = \overline{d}^\alpha(\mu,x)$. This implies in particular that such tangent measures exist, and this almost immediately implies Theorem 3.1.

Definition Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $0 \le \alpha \le d$. For $x \in \mathbb{R}^d$ and $\varepsilon > 0$ define probability distributions P_ε^x on $\mathcal{M}(\mathbb{R}^d)$ by

$$P_\varepsilon^x(M) = \frac{1}{\log(1/\varepsilon)} \int_\varepsilon^1 1_M\Big(\frac{\mu_{x,r}}{r^\alpha}\Big)\, \frac{dr}{r} \quad \text{for Borel sets } M \subseteq \mathcal{M}(\mathbb{R}^d).$$

$\mathcal{P}^\alpha(\mu,x)$ is the set of all limit points of $(P_\varepsilon^x)_{\varepsilon>0}$ as $\varepsilon \downarrow 0$ in the weak topology, i.e. the topology generated by the mappings $P \mapsto \int F\, dP$, F continuous and bounded. Elements of $\mathcal{P}^\alpha(\mu,x)$ are called *α-dimensional tangent measure distributions* of μ at x. It is easy to see that they are supported by the set $\mathrm{Tan}_\alpha(\mu,x)$.

Tangent measure distributions were introduced by Bandt in [Ban92] and Graf in [Gra95] and used as a tool for the investigation of self-similar sets. Tangent measure distributions of more general measures were investigated in [Mör95],[Mör98a] and [MP98]. There it was shown that they have interesting invariance properties. To describe them, define, for every $\lambda > 0$, the rescaling operator

$$S_\lambda^\alpha : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)\ , S_\lambda^\alpha \nu(E) = (1/\lambda^\alpha) \cdot \nu(\lambda E)$$

and for every $u \in \mathbb{R}^d$ the shift operator

$$T^u : \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)\ , T^u\nu(E) = \nu(u+E)\,.$$

The invariance of the tangent measure distributions $P \in \mathcal{P}^\alpha(\mu,x)$ with respect to the rescaling operators S_λ^α is an easy calculation based on the invariance properties of the measure dr/r. However, the tangent measure distributions are in general *not* invariant with respect to the shift operators T^u. The obstacle to this invariance is the fact that, under mild assumptions on the measure μ, such as $\underline{d}^\alpha(\mu,x) > 0$, all tangent measures $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ have the origin in their support. This observation leads us to the notion of a *Palm distribution*, a well known concept in stochastic geometry, random measures, queuing theory and other branches of probability.

Definition Suppose Q is a σ-finite measure on $\mathcal{M}(\mathbb{R}^d)$, which is invariant with respect to the family of shift operators $\{T^u\}$. The *intensity of* Q is the number $\lambda = \int \nu(B)\, dQ(\nu)/\ell^d(B)$, which is independent of the choice of a Borel set B of positive and finite Lebesgue measure. With every Q of finite intensity λ we associate the *Palm distribution* P, which is defined by

$$P(M) = \frac{1}{\ell^d(B)} \int\int_B 1_M(T^u\nu)\, d\nu(u)\, dQ(\nu)\,, \tag{3}$$

for all Borel sets $M \subseteq \mathcal{M}(\mathbb{R}^d)$ and an arbitrary choice of a Borel set $B \subseteq \mathbb{R}^d$ of positive and finite Lebesgue measure. Heuristically, one could say that a sample of P is a sample of Q with a typical point of its support shifted to the origin.

There is a characterization of those probability measures P which are the Palm distributions of some Q, which is due to Mecke [Mec67]. Using (3) we infer that, for every $G : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \to [0,\infty]$ Borel,

$$\begin{aligned}
&\iint G(T^u\nu, -u)\, d\nu(u)\, dP(\nu) \\
&= \frac{1}{\ell^d(B)} \iint_B \int G(T^{v+u}\nu, -u)\, dT^v\nu(u)\, d\nu(v)\, dQ(\nu) \\
&= \frac{1}{\ell^d(B)} \iiint_B G(T^u\nu, v-u)\, d\nu(v)\, d\nu(u)\, dQ(\nu) \\
&= \frac{1}{\ell^d(B)} \iiint_{B-u} G(T^u\nu, v)\, dT^u\nu(v)\, d\nu(u)\, dQ(\nu) \\
&= \frac{1}{\ell^d(B)} \iiint_{B-u} G(\nu, v)\, d\nu(v)\, du\, dP(\nu) \\
&= \iint G(\nu, u)\, d\nu(u)\, dP(\nu)\,.
\end{aligned}$$

Mecke showed that every probability measure P on $\mathcal{M}(\mathbb{R}^d)$ which does not charge the zero measure and fulfills this equality is the Palm distribution of a $\{T^u\}$–invariant, σ-finite measure Q of finite intensity.

The desired shift-invariance result for the tangent measure distributions can now be formulated conveniently: every tangent measure distribution is a Palm distribution. The (non-trivial) proof of this can be found — together with an interpretation and a number of applications in [MP98]. Altogether we have the following invariance properties of the tangent measure distributions:

Theorem 4.1 *Let $0 \le \alpha \le d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$.*

(i) At every $x \in \mathbb{R}^d$ every tangent measure distribution $P \in \mathcal{P}^\alpha(\mu, x)$ fulfills

$$P = P \circ (S^\alpha_\lambda)^{-1} \quad \text{for all } \lambda > 0. \tag{4}$$

(ii) At μ-almost every x every tangent measure distribution $P \in \mathcal{P}^\alpha(\mu, x)$ fulfills

$$\iint G(\nu, u)\, d\nu(u)\, dP(\nu) = \iint G(T^u\nu, -u)\, d\nu(u)\, dP(\nu) \tag{5}$$

for all Borel functions $G : \mathcal{M}(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow [0,\infty)$.

We can now finish the proof of Theorem 3.1 and concentrate on part (b) — the proof of part (a) is analogous. We first show that under the conditions of the theorem, at μ-almost every x, there is $P \in \mathcal{P}^\alpha(\mu, x)$ such that P-almost every ν satisfies $\nu(U(0,1)) = \underline{d}^\alpha(\mu, x)$.

Fix $x \in \mathbb{R}^d$ such that the density condition is fulfilled and choose $\varepsilon_n \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{1}{\log(1/\varepsilon_n)} \int_{\varepsilon_n}^1 \frac{\mu(U(x,r))}{r^\alpha}\, \frac{dr}{r} = \underline{D}^\alpha(\mu, x)\,.$$

After checking the conditions of Prohorov's theorem, which is easy, we can choose a convergent subsequence of $(P^x_{\varepsilon_n})$ and denote the limit P. As the mapping $\nu \mapsto \nu(U(0,1))$ is lower semicontinuous we have

$$\int \nu(U(0,1))\,dP(\nu) \le \lim_{n\to\infty} \frac{1}{\log(1/\varepsilon_n)} \int_{\varepsilon_n}^1 \frac{\mu(U(x,r))}{r^\alpha}\,\frac{dr}{r} = \underline{D}^\alpha(\mu,x) = \underline{d}^\alpha(\mu,x).$$

But $\nu(U(0,1)) \ge \underline{d}^\alpha(\mu,x)$ for every $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ and therefore we must have

$$\nu(U(0,1)) = c := \underline{d}^\alpha(\mu,x) \text{ for } P\text{-almost every } \nu. \tag{6}$$

This equation together with the invariance properties of P has far reaching consequences. From (4) we infer that, for every $r>0$,

$$\begin{aligned} P(\{\nu : \nu(U(0,r)) = cr^\alpha\}) &= P\circ(S_r^\alpha)^{-1}(\{\nu : \nu(U(0,1)) = c\}) \\ &= P(\{\nu : \nu(U(0,1)) = c\}) = 1\,. \end{aligned}$$

Thus, using (5) in the last step,

$$\begin{aligned} 1 &= P(\{\nu : \nu(U(0,r)) = cr^\alpha\}) = \frac{1}{cs^\alpha}\int\int_{U(0,s)} 1_{\{\nu : \nu(U(0,r))=cr^\alpha\}}(\nu)\,d\nu(u)\,dP(\nu) \\ &= \frac{1}{cs^\alpha}\int\int_{U(0,s)} 1_{\{\nu : \nu(U(u,r))=cr^\alpha\}}(\nu)\,d\nu(u)\,dP(\nu). \end{aligned}$$

As this holds for arbitrary r and s we infer that, for P-almost every ν, we have that $\nu(U(u,r)) = cr^\alpha$ for ν-almost every u and every rational $r>0$. By continuity we get, for P-almost every ν,

$$\nu(U(u,r)) = cr^\alpha \text{ for every } u \in \mathrm{supp}\,\nu \text{ and every } r>0. \tag{7}$$

Measures with the property (7) are called *α-uniform* measures and although they are not necessarily α-flat (if $\alpha \ge 3$) we can still conclude, using Marstrand's Theorem, that α is an integer and, using Preiss' Theorem, that ν is α-rectifiable. Of course we do not need the full strength of these theorems to draw our conclusions, only much simplified arguments are needed, see [Mat95, Chapter 14].

To finish the proof fix an α-rectifiable tangent measure ν, which fulfills $\nu(U(y,r)) = c\cdot r^\alpha$ for all $r>0$ and y in the support of ν. Using the rectifiability of ν we may fix y in its support such that all tangent measures of ν at y are α-flat. Since, for μ-almost every x, $\mathrm{Tan}_\alpha(\nu,y) \subseteq \mathrm{Tan}_\alpha(\mu,x)$ for all $\nu \in \mathrm{Tan}_\alpha(\mu,x)$ and y in the support of ν (see e.g. [Mat95, Theorem 14.16]), any tangent measure φ of ν at y is a also a tangent measure to μ at x, which is flat and inherits the property $\varphi(U(0,1)) = c = \underline{d}^\alpha(\mu,x)$ from property (7) of ν. This finishes the proof of our theorem.

Example: In the example described in Section 2, at a typical point x, the set $\mathrm{Tan}_1(\mu,x)$ contains both 1-flat and non-flat tangent measures. It is shown in [Mör95] that for sequences $r_n \downarrow 0$ outside the set Z of vanishing logarithmic density the only possible tangent measures are the 1-flat measures

$$h = \mathcal{H}^1|_{\{(x,0)\,:\,x\in\mathsf{R}\}} \text{ and } v = \mathcal{H}^1|_{\{(0,x)\,:\,x\in\mathsf{R}\}}\,.$$

Moreover, the only tangent measure distribution is

$$P = \frac{1}{2}\cdot\delta_h + \frac{1}{2}\cdot\delta_v\,,$$

where δ_h, δ_v are the Dirac distributions with mass concentrated at h, v, respectively. As the set $\mathrm{Tan}_1(\mu, x)$ is always connected in the vague topology there must be a lot of other tangent measures coming from sequences $r_n \downarrow 0$ in Z. It is easy to see that they are non-flat.

The heuristic message of this example is the following: if an unrectifiable measure has flat tangent measures, the flat tangent measures typically come from sequences of scales r_n of small lacunarity, or, equivalently, large density. Therefore we may have flat tangent measures ν with $\nu(U(0,1)) = \overline{d}^1(\mu, x)$. Sequences r_n of scales with high lacunarity, or low density, however correspond to those scales where the "breaking" or fractal nature of the measure becomes apparent and hence they do not lead to flat tangent measures. Information about rectifiability of μ can therefore only be obtained from an analysis of sequences of scales with low density. In other words, only the conclusion of part (b) in Theorem 3.1 can lead to rectifiability of μ.

5 Flat tangent measures with minimal mass in the unit ball

We now study the case $\alpha = 1$. The following theorem constitutes the second step in our proof. It is an open problem, whether an analogous theorem holds for integers α different from 1.

Theorem 5.1 *If $\mu \in \mathcal{M}(\mathbf{R}^d)$ is a measure such that at μ-almost every x there is a 1-flat measure $\nu \in \mathrm{Tan}_1(\mu, x)$ with $\nu(U(0,1)) = \underline{d}^1(\mu, x)$, then*

$$0 < \underline{d}^1(\mu, x) = \overline{d}^1(\mu, x) < \infty \quad \textit{for } \mu\textit{-almost every } x$$

and hence μ is linearly rectifiable.

Note that the assumption $0 < \underline{d}^1(\mu, x) < \infty$ is implicit in our assumption on the tangent measures. Clearly, Theorems 3.1 and 5.1 together imply Theorem 2.2. The proof of Theorem 5.1 is based on a lemma about measures on the line, which we call the Monotonicity Lemma and which may also be of some independent interest. Roughly speaking, this lemma says that a measure which is "similar" to Lebesgue measure on a compact interval cannot be supported by a smaller set. Slightly more precisely, if a measure is supported by a small subset spread over the interval and if the mass of small intervals centred in this set is no smaller than their length, the total mass of the interval is strictly larger than its length (this formulation justifies the name of the lemma). The Monotonicity Lemma is our main ingredient in the proof of Theorem 5.1.

Lemma 5.2 *There is an absolute constant $\xi > 0$ such that for every $0 \le p < 1$ there is an $0 < \varepsilon_0 = \varepsilon_0(p) < 1$ such that, for all $0 < \varepsilon < \varepsilon_0$, the following holds:*
If E is a compact subset of the interval $[a, b]$ such that

(i) $\mathcal{H}^1(E) \le p \cdot (b - a)$ *and*

(ii) every connected component of the set $[a, b] \setminus E$ has length at most $\varepsilon \cdot (b - a)$,

and whenever ν is a measure on the line such that

(iii) $\nu([a, b] \setminus E) \le \varepsilon \cdot (b - a)$ *and*

(iv) $\nu((x-t,x+t)) \geq 2t$ *for all* $x \in E$, $0 < t \leq 2\varepsilon \cdot (b-a)$,

then we have

$$\nu([a,b]) \geq \Big[1 + \xi(1-p)\Big] \cdot (b-a) > b-a \,.$$

The proof of the Monotonicity Lemma is given in [Mör97], it is completely elementary. Condition *(iv)* ensures the existence of intervals whose ν-measure is at least their length and conditions *(i)*, *(ii)* and *(iii)* allow to arrange these intervals in such a way that they overlap very little in the support of ν and quite a lot outside the support. This leads to a favorable estimate of the total mass of $[a,b]$ compared to its length. It is quite possible that an analogous lemma holds in higher dimensions and this would allow to carry our proof over to higher dimensions. However, it is very unclear how such an analogous lemma could be proved.

Having the Monotonicity Lemma at our disposal, the strategy of the proof of Theorem 5.1 is to use the knowledge about the existence of flat tangent measures with minimal mass in the unit ball to project the measure μ restricted to a ball of small density onto the tangent and thus construct a measure on an interval, which is "similar" to Lebesgue measure. If the lower and upper density of μ do not agree, it turns out that this measure is supported by a smaller set and thus the conditions of our Monotonicity Lemma are fulfilled. The conclusion of the Monotonicity Lemma then leads to an excess of measure in this ball and thus to a contradiction. Again, a proof including all technical details can be found in [Mör97].

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